EVALUATION OF DEMAND UNDER RATIONING WHEN ONLY INDIRECT UTILITY FUNCTIONS ARE KNOWN: A NOTE*

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PREFACE

IIASA's Food and Agriculture Program has always been interested in seeking explanations for some of the more fundamental problems in economic decision making. Much of this work has been carried out in collaboration with the System and Decision Sciences Area at IIASA. This note is a result of such collaboration and comes from a former member of the Food and Agriculture Program who is now working at the Institute of Economic Sciences of the Polish Academy of Sciences in Warsaw, Poland. The note concentrates on the problem of evaluating consumer demand when there is a shortage of certain commodities.

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This note outlines a simple routine for evaluation of consumer demand under disequilibrium (shortage of some commodities) when the representative consumer's preferences are characterized by a cost function, or an indirect utility function consistent with the conventional theory of consumer behavior. The routine may be applied whenever the explicit form of the direct utility cannot be easily derived from the indirect utility function.

1. INTRODUCTION

The determination of consumer demand under disequilibrium (i.e. shortage or rationing of some commodities) may be easily concluded whenever the representative consumer's direct utility function is known. This involves solving a convex programming model with the utility function as maximand and constraints consisting of the conventional budget constraint and, additionally, the (upper) bounds on the values of demand for commodities subject to rationing (see L. Podkaminer, 1982). However, since the estimation of the parameters of the direct utility functions cannot be easily executed - unless quite restrictive views on the nature of underlying consumer's preferences are accepted - the current trend has been to

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characterize the consumers' preferences with indirect utility functions (or, which effectively amounts to the same thing, with the cost or expenditure functions). While both approaches are theoretically equivalent (see Diewert, 1974) the estimation of indirect utility functions is much easier to perform without postulating the oversimplified properties of the underlying preferences. However, if the estimated indirect utility function cannot be easily transformed into the corresponding direct utility function, it may be difficult to evaluate the consumer demand, should the market prices, or supplies, exhibit rigidities leading to disequilibria. Neary and Roberts (1980) have provided the formulae for this evaluation under additional condition, requiring that the consumer is "forced" to purchase quantities of the commodities that are deemed to be in short supply. This, however, leaves unanswered the question how one can know which of the commodities supplied will, given prices and intended total expenditure, eventually turn out to be effectively lacking, which will turn out to be oversupplied, and which will be neither lacking nor oversupplied.

The subject of the following text is to provide a general method for the determination of the demand under additional quantity constraints reflecting the fact that the supplies of all commodities are fixed at some level - and so are their prices, irrespective of what level and structure of the market disequilibrium this may imply.

2. THE PROBLEM AND ITS SOLUTION

Let us start with the statement of the problem in terms of the direct utility function. The representative consumer whose preferences are reflected by a continuous, nondecreasing and quasiconcave utility function U(q) is assumed to respond to the state of the market in accordance with the optimum solution q^o to the following problem:

$$\text{maximize } U(q) \tag{1}$$

$$\text{s.t.} \quad \bar{p}q \leq y \tag{2}$$

$$\text{and} \quad q \leq \bar{q} \tag{3}$$

where q is the vector of commodities demanded, $\bar{p}$ is the vector of "official" market prices, y is consumer's intended total expenditure, $\bar{q}$ is the vector of supplies. If the value of supplies (evaluated at the prices $\bar{p}$) is equal to or less than the consumer's total expenditure, then the optimum solution q^o is equal to $\bar{q}$. In this case neither direct nor indirect utility function is needed to predict the demand. If, on the other hand, $\bar{p}\bar{q} > y$, then the optimum q^o is to be derived from (1) - (3) upon the application of a suitable algorithm of mathematical programming theory. All that is known about this case before having a look at the specific numerical solution to (1) - (3) is that some goods will be oversupplied. Because of the properties of the utility function, (1) - (3) belongs to the class of convex programming models. As such it evidently possesses exactly one local (and therefore global) optimum. Moreover, any standard gradient algorithm of mathematical programming theory (see Hadley, 1964) is capable of producing q^o with any desired precision in a quite short time. The execution of the gradient algorithms proceeds in an iterative way. At each iteration step an approximation to the optimum (q^i) is computed. Also, it is usually necessary to compute the derivatives (gradients) of the maximand, $\nabla U(q^i)$, evaluated at the point of approximation q^i and the value of the maximand itself, $U(q^i)$. Now, let us consider a situation arising when the direct utility function U is not known. Instead, let us assume that the indirect utility function $g(p)$ is given. Under familiar regularity conditions for $g(p)$, including its being quasi-convex, nonincreasing and "continuous enough", there is a duality between the values of the direct and indirect utility functions (see Diewert, 1974, p. 124):

$$g(p) = \max_q \{U(q) : pq \leq 1, \quad q \geq 0\} \tag{4}$$

and

$$U(q) = \min_p \{g(p) : pq \leq 1, \quad p \geq 0\} \tag{5}$$

According to (5), the value of the direct utility function for any bundle of commodities q may be calculated upon solving a (relatively simple) convex programming model with the minimand defined in terms of the indirect utility function. At this point it may be observed that provided the formulae are devised for the determination of the derivatives of the direct utility function from the analysis of the indirect utility function, the solution of (1) - (3) with classical gradient methods is again straightforward[1]. We now proceed to establish a result linking the gradient $\nabla U(q)$ to the indirect utility function.

Lemma

Let $U(q)$ and $g(p)$ defined by (4) - (5) be a pair of direct and indirect utility functions, both quasiconvex over positive orthants (or strictly quasiconvex), continuous and once differentiable, monotonous. Then

$$\nabla U(q) = - p^* \lambda^*$$

where p^* is the solution to the optimisation problem:

$$\text{minimize } g(p) \qquad (7)$$

$$\text{s.t.} \quad pq = 1 \qquad (8)$$

and λ^* is the corresponding Lagrange multiplier for (8):

$$\lambda^* = \frac{\delta g(p^*)}{\delta (pq)}$$

The proof of Lemma is a quite elementary exercise in comparative statics. Because of the assumed strict quasiconvexity of $g(p)$, (5) can be rewritten as:

[1] According to some recent results in theory of mathematical programming, the ability to compute the values of the maximand is sufficient for the construction of quasi-gradient algorithms which are claimed to be not inferior to the classical gradient ones (see Bräuninger, 1981). The quasi-gradient methods replace exact derivatives with their approximations given by finite differences $h^{-1}(U(q + he_i) - U(q))$ where e_i is the i-th unit vector and h a positive scalar.

$$U(q) = \min_{p} \{g(p) \ : \ pq = 1 \} \qquad (9)$$

which is equivalent to (7) - (8). The unique solution to (7) - (8) satisfies the first-order conditions:

$$\nabla_p \ g(p^*) = \lambda^* q \qquad (10)$$

or

$$\frac{\delta g(p^*)}{\delta p_i} = \lambda^* q_i \qquad i = \ell, \ldots, n \qquad (11)$$

where $\lambda^* = \delta g(p^*)/\delta(pq)$ is the Lagrange multiplier for (8). Now, since all q_i are positive:

$$U(q) = g(p_1^*, p_2^*, \ldots, p_n^*) = g \left(\frac{1 - \sum_{j=2}^{n} p_j^* q_j}{q_1}, p_2^*, \ldots, p_n^* \right) \qquad (12)$$

Hence, differentiating (12) with respect to q_i we obtain:

$$\frac{\delta U(q)}{\delta q_1} = - \frac{(1 - \sum_{j=2}^{n} p_j^* q_j)}{q_1^2} \cdot \frac{\delta}{\delta p_1} g(p_1^*, p_2^*, \ldots, p_n^*)$$

$$= - \frac{p_1^*}{q_1} \cdot \frac{\delta}{\delta p_1} g(p_1^*, p_2^*, \ldots, p_n^*) \quad ,$$

and

$$\frac{\delta U(q)}{\delta q_k} = - \frac{p_k^*}{q_1} \frac{\delta}{\delta p_1} g(p_1^*, p_2^*, \ldots, p_n^*) \qquad \text{for} \quad k \geq 2$$

Since $\delta g(p_1^*, p_2^*, \ldots, p_n^*)/\delta p_1 = \lambda^* q_1$ (see (11)), we conclude

that $$\frac{\delta U(q)}{\delta q_k} = - p_k^* \lambda^* \qquad \text{for all } k.$$

Q.E.D.

3. CONCLUSIONS

We have shown that the numerical determination of the gradient of a utility function can be executed on the basis of the numerical analysis of its indirect utility function. Hence, the determination of the optimum solution to (1) - (3); i.e. the determination of the consumer demand under rationing, is not in the least impeded by the absence of the explicit formula for the direct utility function. It is interesting to note that the same qualification is true whenever there are more elaborate regulations concerning the distribution of the commodities under disequilibrium than the simple principle of "first come first served, as long as supplies are not depleted" reflected in the upper bound constraints (3). For instance, if the consumer is given some total "ration points" $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ and "ration point prices" $\Pi^{(j)}$ are prescribed, the (3) is to be complemented by the following system of linear inequalities:

$$q\,\Pi^{(j)} \leq y^{(j)} \qquad j = 1, \ldots, m \quad .$$

Similarly, the imposition of lower bound constraints ($q \geq \bar{\bar{q}}$), which may make sense with respect to the good "leisure" under involuntary unemployment, does not reduce the applicability of the approach outlined above. For, as long as the problem (1) - (3), complemented by any further constraints, continues to belong to the domain of convex programming theory, the application of the gradient algorithms necessarily results in the determination of its unique optimum.

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