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OPTIMAL CONTROL OF LINEAR ECONOMETRIC SYSTEMS WITH INEQUALITY CONSTRAINTS ON THE CONTROL VARIABLES

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Chow (1975, pp. 157) develops a series of methods to solve the following optimal tracking problem.

$$\min(y_t - a_t)'K_t(y_t - a_t)$$

subject to

$$y_t = A_t y_{t-1} + C_t x_t + B_t z_t$$

One of the methods is that of Lagrangian multipliers. K.C. Tan (1979) extends this to include the case where the instruments must satisfy

$$F_t x_t = f_t$$

The purpose of this note is to develop the corresponding solution when the instruments are constrained

$$l_t \leq x_t \leq u_t \ .$$

With the addition of these constraints the problem becomes

$$\min(y_t - a_t)'K_t(y_t - a_t)$$

subject to

$$y_t = A_t y_{t-1} + C_t x_t + B_t z_t$$

and

$$u_t - x_t \geq 0 \qquad \text{where} \quad u_t > l_t$$
$$x_t - l_t \geq 0$$

Forming the Lagrangian we get

$$\begin{aligned} L = {} & \frac{1}{2}(y_t - a_t)'K_t(y_t - a_t) \\ & - \sum_{t=1}^{T} \lambda'_t(y_t - A_t y_{t-1} - C_t x_t - B_t z_t) \\ & - \sum_{t=1}^{T} \rho'_t(u_t - x_t) \\ & - \sum_{t-1}^{T} \sigma'_t(x_t - l_t) \end{aligned} \tag{2}$$

$$\frac{\partial L}{\partial y_t} = K_t(y_t - a_t) - \lambda_t + A'_{t-1}\lambda_{t-1} = 0 \tag{3}$$

$$\frac{\partial L}{\partial x_t} = C'_t\lambda_t + \rho_t - \sigma_t = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda_t} = y_t - A_t y_{t-1} - C_t x_t - B_t z_t = 0 \tag{5}$$

$$\frac{\partial L}{\partial \sigma_t} = -x_t + l_t \leq 0, \; \sigma_t \frac{\partial L}{\partial \sigma_t} = \sigma_t(-x_t + l_t) = 0 \tag{6}$$

$$\frac{\partial L}{\partial \rho_t} = -u_t + x_t \leq 0, \; \rho_t \frac{\partial L}{\partial \rho_t} = \rho_t(-u_t + x_t) = 0 \tag{7}$$

If this is a "free endpoint" problem $\lambda_{T+1} = 0$, following Chow (1975), therefore using (3)

$$\begin{aligned} \lambda_T &= K_T y_T - K_T a_T + A_{T+1}\lambda_{T+1} \\ &= K_T y_T - K_T a_T \end{aligned} \tag{8}$$

or setting $H_T = K_T$ and $h_T = K_T a_T$

$$\lambda_T = H_T y_T - h_T \tag{9}$$

Substitute this into (4)

$$C'_T\lambda_T + \rho_T - \sigma_T = 0 \tag{10}$$

$$C'_T(H_T y_T - h_T) + \rho_T - \sigma_T = 0$$

$$C'_T H_T y_T - C'_T h_T + \rho_T - \sigma_T = 0$$

Substitute (5) into this

$$C'_T H_T A_T y_{T-1} + C'_T H_T C_T x_t + C'_T H_T B_T z_T - C'_T h_T + C'_T h_T + \rho_T - \sigma_T = 0$$

Solving for x_T

$$\begin{aligned} x_T &= G_T y_{T-1} + g_T + (C'_T H_T G)^{-1}(\rho_T - \sigma_T) \\ &= G_T y_{T-1} + g_T + \rho_T^* - \sigma_T^* \end{aligned} \tag{11}$$

where

$$G_T = -(C'_T H_T C_T)^{-1} C'_T H_T A_T$$

$$g_T = -(C'_T H_T C_T)^{-1} C'_T (H_T - B_T z_T - h_T)$$

$$\rho_T^* = (C'_T H_T C_T)^{-1} \rho_T$$

$$\sigma_t^* = (C'_T H_T C_T)^{-1} \sigma_t \ .$$

Substituting this into (5) we obtain

$$y_T = (A_T + C_T G_T) y_{T-1} + B_T z_T + C_T g_T + C_T \sigma_T^* - C_T \sigma_T^* \ .$$

Substituting this into (9)

$$\lambda_T = H_T(A_T + C_T G_T) y_{T-1} + H_T(B_T z_T + C_T g_T) + H_T C_T \rho_T^* - H_T C_T \rho_T^* - h_T \tag{12}$$

Lagging (8)

$$\lambda_{t-1} = K_{t-1} y_{t-1} - K_{t-1} a_{t-1} + A'_t \lambda_t$$

Substituting (12) into it

$$\begin{aligned} \lambda_{t-1} = {} & K_{t-1} y_{t-1} - K_{t-1} a_{t-1} + A'_t H_t (A_t + C_t G_t) y_{t-1} \\ & + A'_t H_t (B_t z_t + C_t g_t) \\ & + A'_t H_t C_t \rho_t^* - A'_t H_t C_t \sigma_t^* - A'_t h_t \end{aligned} \tag{13}$$

and

$$\lambda_{t-1} = H_{t-1} y_{t-1} - h_{t-1}$$

where

$$H_{t-1} = K_{t-1} + A'_t H_t (A_t + C_t G_t) \tag{14}$$
$$h_{t-1} = K_{t-1} a_{t-1} - A'_t H_t (B_t z_t + C_t g_t + C_t \rho_t^* - C_t \sigma_t^*) + A'_t h_t \ .$$

There are three possibilities in any given year.

A. $l_t < x_t < u_t$

Chow's unconstrained algorithm can be used to get from t to t-1.

B. $x_t = l_t$

The lower constraint is binding.

This implies $\rho_t^* = 0$, since $x_t = l_t$, therefore $x_t \neq u_t$ and $(u_t - x_t)\rho_t^* = 0$.

If the constraint is binding

$$\sigma_t^* = G_t y_{t-1} + g_t - l_t \tag{15}$$

using (11).

C. $x_t = u_t$

The upper constraint is binding.

This implies $\sigma_t^* = 0$

and

$$\rho_t^* = -G_t y_{t-1} - g_t + u_t \tag{16}$$

CASE B

For case B

$$\sigma_t^* = G_t y_{t-1} + g_t - l_t \tag{17}$$
$$\text{or } \sigma_t = (C'_t H_t C_t)^{-1}(G_t y_{t-1} + g_t - l_t) \ .$$

Substituting this into (13)

$$\lambda_{t-1} = K_{t-1} y_{t-1} - K_{t-1} a_{t-1} + A'_t H_t (A_t + C_t G_t) y_{t-1} \tag{18}$$

$$
\begin{aligned}
&+ A'_tH_t(B_tz_t + C_tg_t)\\
&- A'_tH_tC_t(G_ty_{t-1} + g_t - l_t)\\
&- A'_th_t\\
\lambda_{t-1} &= K_{t-1}y_{t-1} - K_{t-1}a_{t-1} + A'_tH_tA_ty_{t-1} + A'H_tC_tG_ty_{t-1}\\
&+A'_tH_tB_tz_t + A'_tH_tC_tg_t\\
&- A'_tH_tC_tG_ty_{t-1} - A'_tH_tC_tg_t + A'_tH_tC_tl_t - A'_th_t\\
\lambda_{t-1} &= K_{t-1}y_{t-1} - K_{t-1}a_{t-1} + A'_tH_tA_ty_{t-1} + A'_tH_tB_tz_t \qquad (19)\\
&+ A'_tH_tC_tl_t - A'_th_t\\
\lambda_{t-1} &= H^{*}_{t-1}y_{t-1} - h^{*}_{t-1}
\end{aligned}
$$

where

$$
\begin{aligned}
H^{*}_{t-1} &= K_{t-1} + A'_tH_tA_t \qquad (20)\\
h^{*}_{t-1} &= K_{t-1}a_{t-1} - A'_tH_t(B_tz_t + c_tl_t) + A'_th_t
\end{aligned}
$$

When comparing these with the normal recursion formula

$$
\begin{aligned}
H_{t-1} &= K_{t-1} + A'_tH_t(A_t + C_tG_t) \qquad (21)\\
h_{t-1} &= K_{t-1}a_{t-1} - A'_tH_t(B_tz_t + C_tg_t) + A'_th_t
\end{aligned}
$$

Notice that $x_t = l_t$ and if G_t and g_t are calculated normally and then used to calculate

$$\sigma^{*}_t = G_ty_{t-1} + g_t - l_t \qquad (21)$$

and then if G_t is set equal to 0 and g_t is set equal to l_t, then the usual recursion formulae are used then the H_{t-1} and h_{t-1} are calculated correctly. This means that after H_t, h_t, G_t, and g_t are calculated using the normal recursion and it is found that x_t would be out of the bounds set for it, then we calculate σ^{*}_t and set $G_t = 0$ and $g_t = l_t$ and calculate H_{t-1} and h_{t-1} for the given x_t and G_t and g_t.

Notice that y_{t-1} has not been calculated yet and is needed to calculate ρ^{*}_t. If one uses the nonlinear argorithm (Chow, 1975) then an estimate of y_{t-1} is available from the last iteration. At convergence this y_{t-1} will be arbitrarily close to the "actual" y_{t-1}.

CASE C

Similarly for case C:

$$x_t = u_t$$

and

$$\rho_t^* = -G_t y_{t-1} - g_t + u_t$$
$$H_{t-1}^* = K_{t-1} + A'_t H_t A_t$$
$$h_{t-1}^* = K_{t-1} a_{t-1} - A'_t H_t (B_t z_t + C_t u_t) + A'_t h_t$$

Here again if the G_t and g_t are calculated normally then $\rho_t^* = -G_t y_{t-1} - g_t + u_t$ and then set $x_t = u_t, G_t = 0$ and $g_t = u_t$. Then the normal recursion formula (21) will work correctly.

AN EXAMPLE

For example, suppose we wish to constrain the instruments to be positive, $x_t \geq 0$.

$$\min (y_t - a_t)' K_t (y_t - a_t)$$

subject to

$$y_t = A_t y_{t-1} + C_t x_t + B_t z_t$$

and

$$x_t \geq 0$$

Forming the Lagrangean we get

$$L = \frac{1}{2} \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t) - \sum_{t=1}^{T} \lambda'_t (y_t - A_t y_{t-1} - C_t x_t - B_t z_t)$$
$$- \sum_{t=1}^{T} \rho_t x_t$$

$$\frac{\partial L}{\partial y_t} = K_t (y_t - a_t) - \lambda_t + A'_{t+1} \lambda_{t+1} = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_t} = C'_t \lambda_t - \rho_t = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda_t} = y_t - A_t y_{t-1} - C_t x_t - B_t z_t = 0 \qquad (3)$$

$$\frac{\partial L}{\partial \rho_t} = -x_t \leq 0 \tag{4}$$

$$\rho_t \frac{\partial L}{\partial \rho_t} = -\rho_t x_t = 0 \tag{5}$$

Using the example in Chow (1975) we begin with period T

$$\lambda_T = K_T y_T - K_T a_T + A'_{T+1}\lambda_{T+1} \qquad \text{using} \qquad (1)$$

Setting $H_T = K_T$ and $h_T = K_T a_T$

$$\lambda_T = H_T y_T - h_T \tag{6}$$

$$C'_T\lambda_T - \rho_T = 0 \tag{2}$$

$$C'_T(H_T y_T - h_T) - \rho_T = 0 \qquad \text{using (2) and} \qquad (6)$$

$$C'_T(H_T A_T y_{T-1} + H_T C_T x_T + H_T B_T z_T - h_T) - \rho_T = 0 \qquad \text{using} \qquad (3)$$

Solving for x_T

$$C'_T H_T A_T y_{T-1} + C'_T H_T C_T x_T + C'_T H_T B_T z_T - C'_T h_T - \rho_T = 0$$

$$C'_T H_T C_T x_T = -C'_T H_T A_T y_{T-1} - C'_T H_T B_T z_T + C'_T h_T + \rho_T$$

or

$$x_T = G_T y_{T-1} + g_T + \rho_T^* \tag{7}$$

where

$$G_T = -(C'_T H_T C_T)^{-1} C_T H_T A_T$$

$$g_T = -(C'_T H_T C_T)^{-1} C'_T (H_T B_T z_T - h_T)$$

$$\rho_T^* = (C'_T H_T C_T)^{-1} \rho_T$$

Solving for y_T as a function of y_{T-1}

$$y_T = (A_T + C_T G_T) y_{T-1} + B_T z_T + C_T g_T + C_T \rho_T^*$$

$$\lambda_T = H_T(A_T + C_T G_T) y_{T-1} + H_T(B_T z_T + C_T g_T) + H_T C_T \rho_T^* - h_T \qquad \text{using} \qquad (6)$$

Substitute this into (1)

$$K_{t-1} y_{t-1} - K_{t-1} a_{t-1} - \lambda_{t-1} + A'_t \lambda_t = 0$$

$$\lambda_{t-1} = K_{t-1} y_{t-1} - K_{t-1} a_{t-1} + A'_t \lambda_t = 0$$

$$\lambda_{t-1} = K_{t-1} y_{t-1} - K_{t-1} a_{t-1} + A'_t H_t (A_t + C_t G_t) y_{t-1} + A'_t H_t \tag{8}$$

$$(B_t z_t + C_t g_t) + A'_t H_t C_t \rho_t^* - A'_t h_t$$

Or

$$\lambda_{t-1} = H_{t-1}y_{t-1} - h_{t-1} + A'_tH_tC_t\rho_t^*$$

where

$$H_{t-1} = K_{t-1} + A'_tH_t(A_t + C_tG_t) \qquad (9)$$

$$h_{t-1} = K_{t-1}a_{t-1} - A'_tH_t(B_tz_t + C_tg_t) + A'_th_t \qquad (10)$$

Using (5) the problem breaks down into two cases:

1) A. Constraint $x_t \geq 0$ is binding

$\rightarrow x_t = 0$ and $\rho_t^* = -G_ty_{t-1} - g_t$ using (7)

2) B. Constraint $x_t \geq 0$ is not binding

$\rightarrow \rho_t = 0$ and $x_t \geq 0$

In case B $\rho_t = 0$ reduces to Chow's algorithm

In case A

Substituting $\rho_t^* = -G_ty_{t-1} - g_t$ into (8), or $\rho_t = -(C'_tH_tC_t)^{-1}(G_ty_{t-1} + g_t)$,

we get

$$\begin{aligned}
\lambda_{t-1} &= K_{t-1}y_{t-1} - K_{t-1}a_{t-1} + A'_tH_t(A_t + C_tG_t)y_{t-1} + A'_tH_t \\
&\quad (B_tz_t + C_tg_t) + A'_tH_tC_t(-G_ty_{t-1} - g_t) - A'_th_t \\
&= K_{t-1}y_{t-1} - K_{t-1}a_{t-1} + A'_tH_tA_ty_{t-1} + A'_tH_tB_tz_t \\
&\quad + A'_tH_tC_tG_ty_{t-1} + A'_tH_tC_tg_t - A'_tH_tC_tG_ty_{t-1} - A'_tH_tC_tG_t - A'_th_t \\
\lambda_{t-1} &= K_{t-1}y_{t-1} - K_{t-1}a_{t-1} + A'_tH_tA_ty_{t-1} + A'_tH_tB_tz_t - A'_th_t \\
&= H_{t-1}^*y_{t-1} - h_{t-1}^*
\end{aligned}$$

where

$$H_{t-1}^* = K_{t-1} + A'_tH_tA_t$$

$$h_{t-1}^* = K_{t-1}a_{t-1} - A'_tH_tB_tz_t + A'_th_t$$

Chow (1975) shows that the two Ricatti difference equations (9) and (10) can be written as

$$H_{t-1} = K_{t-1} + (A_t + C_tG_t)'H_t(A_t + C_tG_t) \qquad (11)$$

$$h_{t-1} = K_{t-1}a_{t-1} + (A_t + C_tG_t)^1(h_t - H_tB_tz_t) \quad . \tag{12}$$

for case B

Notice that if *Case A* applies, i.e. $x_t = 0$ and the constraint is binding the recursion formulae are

$$H^{*}_{t-1} = K_{t-1} + A'_t H_t A_t$$
$$h^{*}_{t-1} = K_{t-1}a_{t-1} + A'_t(h_t - H_tB_tz_t)$$

These are exactly what (11) and (12) reduce to when G_t and g_t are set = 0. Also, since each period can be solved separately (from dynamic programming), the solution procedure for the optimal problem subject to $x \geq 0$ can be implemented as follows.

SOLUTION PROCEDURE

Steps

1. Proceed as if x_t is unconstrained
2. Calculate H_t, h_t

   then G_t, g_t

   then calculate x_t using last iterations y_{t-1}.
3. If x_t is positive, proceed as in Chow (1975) to t-1

   if x_t is negative, set $x_t = 0$, $\rho_t = -G_t y_{t-1} - g_t$

   then set $G_t = 0$ and $g_t = 0$

   then proceed as in Chow (1975) to t-1.
4. Start at step 1 with a new period t-1

Note 1: This allows not only x_{t-1} to change since y_t and y_{t-1} may be different, but also allows the coefficient feedback martices G_t and g_t to change correctly knowing that $x_t \geq 0$

Note 2: For x_t a vector only the rows of G_t and g_t corresponding to negative values are set equal to zero.

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