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STUDY OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS BY THE "PIPE METHOD"

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PREFACE

A pipe of a differential inclusion is a set-valued map associating with each time t a subset P(t) of states which contains a trajectory of the differential inclusion for any initial state x_o belonging to P(0). As in the Liapunov method, knowledge of a pipe provides information on the behavior of the trajectory. In this paper, the characterization of pipes and non-smooth analysis of set-valued maps are used to describe several classes of pipes.

This research was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

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INTRODUCTION

We propose a method analogous to Liapunov's second method for studying the asymptotic properties of solutions to differential equations or inclusions.

Let F be a set-valued map from a subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^n$ describing the dynamics of the system:

$$(0.1) \qquad x'(t) \in F(x(t)),\ x(0) = x_0 \text{ given in } K$$

Let V be a function from a neighborhood of K to $\mathbb{R}_+$.

All methods falling under the name of Liapunov's second method deal with the fact that inequalities of the type

$$(0.2) \qquad \forall x \in K,\ \inf_{v \in F(x)} \langle V'(x), v \rangle \leq -W(x,v)$$

(where W is a non-negative function on Graph(F)) imply, under reasonable assumptions, that

$$(0.3) \qquad \begin{cases} \forall x_0 \in K, \text{ there exists a solution } x(\cdot) \text{ to (0.1) such} \\ \text{that } t \to V(x(t) \text{ is decreasing} \end{cases}$$

According to the choice of the function V (and the function W), we then can deduce information upon the behavior of some solutions to (0.1); in particular, that

$$(0.4) \qquad \forall t \geq 0,\ V(x(t)) \leq V(x(t_o)).$$

(We refer to Aubin-Cellina [1984], Chapter 6, for a presentation of the Liapunov method for non-differentiable functions V and for differential inclusions.) The "pipe method" we propose aims for the same objectives. A "pipe" P is a set-valued map

$$(0.5) \qquad t \in \mathbb{R}_+ \to P(t) \subset K$$

which is related to the dynamics described by F by

$$(0.6) \qquad \forall t \geq 0,\ \forall x \in P(t),\ F(t,x) \cap DP(t,x) \neq \emptyset$$

where

$$(0.7) \qquad DP(t,x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} d(v, \frac{P(t+h) - x}{h}) = 0\}$$

denotes the "contingent derivative" of P at $(t,x) \in \mathrm{Graph}(P)$ (see Aubin-Cellina [1984], Section 4.3). Indeed, Haddad's viability theorem (see Haddad [1981], Aubin-Cellina [1984], Section 4.4) implies that under reasonable conditions, condition (0.6) implies that

$$(0.8) \qquad \begin{cases} \forall x_o \in K, \text{ there exists a solution } x(\cdot) \text{ to (0.1) such} \\ \text{that } \forall t \geq 0,\ x(t) \in P(t) \end{cases}$$

According to the choice of the pipe P, we are able to infer properties of the behavior of some solutions to (0.1). The pipe method has drawbacks analogous to Liapunov's method: when the dynamics of a system are described by a map F, how can we find its pipes?

However, we shall give several examples of pipes in the following pages. We begin with pipes derived from potential functions V, of the form

(0.9) $P(t) := \{x \in K | V(x(t)) \leq w(t)\}$

where w(t) is a function we shall construct in terms of F and V. Such pipes yield information analogous to the ones provided by Liapunov method.

We shall also characterize pipes of the form

(0.10) $P(t) := \{x \in K | w_-(t) \leq V(x-c(t)) \leq w_+(t)\}$

and more generally, of the form

(0.11) $P(t) := \{x \in K | \forall i=1,\dots,p, V_i(\Phi(t,x)) \leq w_i(t)\}$

We shall study pipes defined by constraints

(0.12) $P(t) := \{x \in K | A(t,x) \in M\}$

where A is a continuous map from $\mathbb{R}_+ \times \mathbb{R}^n$ to a vector space $\mathbb{R}^p$. Finally, we consider pipes of the form

(0.13) $P(t) := \Phi(t,C,D)$

where C and D are closed subsets of $\mathbb{R}^n$ and Φ is a differentiable map from $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$. In particular, when C is reduced to a point c, we investigate pipes of the form

(0.14) $P(t) := c + \phi(t)D$

Such pipes allow us to infer that for any $x_o \in c + \phi(0)D$, there exists a trajectory $x(\cdot)$ of (0.1) converging to c at time T when $\phi(T) = 0$.

We shall rely on results of non-smooth analysis as they are presented in Chapter 7 of Aubin-Cellina [1984] to give sufficient conditions for set-valued maps of the form (0.11), (0.12) or (0.13) to be pipes of a given set-valued map F.

1. THE PIPE METHOD: A GENERAL DESCRIPTION AND BACKGROUND NOTES

Let us consider a dynamical system described by a set-valued map $(t,x) \in [0,T] \times \mathbb{R}^n \to F(t,x) \in \mathbb{R}^n$.

We consider the initial-value problem for the differential inclusion

$$(1.1) \qquad x'(t) \in F(t,x(t)) \; ; \; x(0) = x_o$$

where $(0,x_o) \in \mathrm{Dom}(F)$ is given.

We recall the definitions of the <u>contingent cone</u> $T_K(x)$ to a subset K at $x \in K$:

$$(1.2) \qquad \begin{cases} T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} \frac{d(x+hv,K)}{h} = 0\} \\ = \bigcap_{\varepsilon,\alpha>0} \bigcup_{h\in]0,\alpha[} (\frac{1}{h}(K-x) + \varepsilon B) \end{cases}$$

(see Aubin-Cellina [1984], pp.176-177). When G is a set-valued map from X to Y and when (x,y) belongs to the graph of G, the <u>contingent derivative</u> DG(x,y) of G at (x,y) is the closed process (set-valued map whose graph is a closed cone) from X to Y defined by

$$(1.3) \qquad DG(x,y)(u) := \{v \in Y \mid \liminf_{\substack{h \to 0+ \\ u' \to u}} d(v, \frac{G(x+hu') - y}{h}) = 0\}$$

We observe that

$$(1.4) \qquad \mathrm{Graph}DG(x,y) = T_{\mathrm{Graph}(G)}(x,y)$$

When P is a set-valued map from $\mathbb{R}$ to X, it is enough to know the values of the contingent derivative DP(t,x) of P at $(t,x) \in$ Graph (P) at 1, 0 and -1. We observe that

$$(1.5) \qquad DP(t,x)(1) = \{v \in X \mid \liminf_{h \to 0+} d(v,\frac{P(t+h) - x}{h}) = 0\}$$

and we shall often set

(1.6) $DP(t,x) := DP(t,x)(1)$

We also observe that

(1.7) $$DP(t,x)(-1) = \{v \in X \mid \liminf_{h \to 0+} d(v, \frac{P(t-h) - x}{h}) = 0\}$$

and that

(1.8) $$T_{P(t)}(x) \subset DP(t,x)(0)$$

(Equality holds when P is Lipschitz around x.)

Definition 1.1: We shall say that a set-valued map P from $[0,T]$ to $\mathbb{R}^n$ is a pipe of a set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $[0,T]$ if

(1.9) $$\begin{cases} \text{(i) Graph (P) is closed and contained in Dom(F)} \\ \text{(ii) } \forall t \in [0,T[\,,\ \forall x \in P(t),\ F(t,x) \cap DP(t,x) \neq \emptyset \\ \text{(iii) } \forall x \in P(T),\ F(T,x) \cap DP(T,x)(0) \neq \emptyset \end{cases}$$

▲

Condition (1.9)iii) naturally disappears when we take $T := +\infty$. Hence Haddad's viability theorem (see Haddad [1981], Aubin-Cellina [1984], Theorem 4.1.1, p.180) can be reformulated in the following way:

Theorem 1.2: Let us assume that

(1.10) F is bounded, its graph is closed and its values are convex.

Let P be a pipe of F on $[0,T]$. Then, for any $x_o \in P(0)$, there exists a trajectory of (1.1) satisfying

(1.11) $$\begin{cases} \text{(i) } \forall t \in [0,T],\ x(t) \in P(t) \\ \text{(ii) for almost all } t \in [0,T[\,,\ x'(t) \in DP(t,x(t)). \end{cases}$$

and, when $T < +\infty$,

(1.12) $\forall t \in [T,\infty],\ x'(t) \in F(T,x(t))$ and $x(t) \in P(T)$. ▲

Proof: We introduce the following set-valued map G from Graph(P) to $\mathbb{R}_+ \times \mathbb{R}^n$ defined by

$$(1.13) \qquad G(\tau,x) := \begin{cases} \{1\} \times F(\tau,x) & \text{if } \tau \in [0,T[\\ [0,1] \times F(\tau,x) & \text{if } \tau = T \\ \{0\} \times F(T,x) & \text{if } \tau > T \end{cases}$$

When $\tau < T$, condition (1.9)ii) implies the existence of v in $F(\tau,x) \cap DP(\tau,x)$. We then deduce that (1,v) belongs to the contingent cone to Graph(P) at (τ,x). When $\tau \geq T$, condition (1.9)iii) states that (0,v) belongs to the contingent cone to Graph ($\mathbb{F}$) at (T,x). Hence, the set-valued map G satisfies

$$\forall(\tau,x) \in \text{Graph}(P),\ G(\tau,x) \cap T_{\text{Graph}(p)}(\tau,x) \neq \emptyset.$$

Since G is bounded, its graph is closed and its values are convex, Haddad's theorem states that the differential inclusion

$$(\tau'(t),x'(t)) \in G(\tau(t),x(t)) \ ; \ (\tau(0),x(0)) = (0,x_o)$$

has a viable trajectory $t \to (\tau(t),x(t))$ in Graph(P). Since $\tau(t) = t$ when $t \in [0,T]$, $\tau(t) = T$ when $t \geq T$, we infer that $x(\cdot)$ is a solution to (1.1) satisfying (1.11). ∎

In order to check that a set-valued map P is a pipe of a given set-valued map F, we need to use some calculus on contingent cones and contingent derivatives of set-valued maps. Sooner or later, we shall need regularity assumptions of the type: $x \to T_K(x)$ is lower semicontinuous at some point x_o. This motivates the introduction of the Kuratowski lim inf of the contingent cones

$$(1.14) \qquad \liminf_{y \to x} T_K(y) := \bigcap_{\varepsilon>0} \bigcup_{\alpha>0} \bigcap_{y\in B_K(x,\alpha)} (T_K(y) + \varepsilon B)$$

A theorem due to Cornet [1981], Penot [1981] (see Aubin-Ekeland [1984], Theorem 7.1.7, p.409) implies that this lim inf of the contingent cones is the tangent cone $C_K(x)$ to K at x introduced by Clarke [1975] (see also Clarke [1983], Aubin-Ekeland [1984], Definition 7.1.3, p.506) and defined by

$$(1.15)\quad \begin{cases} C_K(x) := \{v \in \mathbb{R}^n \mid \lim\limits_{\substack{h \to 0+ \\ y \to x}} \frac{d(y+hv,K)}{h} = 0\} \\ = \bigcap\limits_{\varepsilon>0} \bigcup\limits_{\substack{\alpha,\beta>0}} \bigcap\limits_{\substack{y \in B_K(x,\alpha) \\ 0<h\leq\beta}} (\frac{1}{h}(K-y) + \varepsilon B) \end{cases}$$

It is a <u>closed convex subcone</u> of the contingent cone $T_K(x)$, which coincides with it when K is a C^1- manifold, when K is convex or more generally, when $y \to T_K(y)$ is lower semicontinuous at x.

When G is a set-valued map from X to Y and when (x,y) belongs to the graph of G, we define the <u>derivative</u> CG(x,y) of G at (x,y) by

$$(1.16)\qquad v \in CG(x,y)(u) \Leftrightarrow (u,v) \in C_{\mathrm{Graph}(G)}(x,y)$$

When G is Lipschitz around x, we observe that

$$(1.17)\qquad CG(x,y)(u) = \{v \in Y \mid \lim_{\substack{h \to 0+ \\ (x',y') \to (x,y) \\ \mathrm{Graph}(G)}} d(v, \frac{G(x'+hu) - y'}{h}) = 0\}$$

When G is a C^1- single-valued map, CG(x,G(x)) coincides with the usual Jacobian G'(x). When V is a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ and when we want to take into account the order relation (for defining pipes of the form

$$P(t) := \{x \in K \mid V(x) \leq w(t)\}$$

for instance), we are led to introduce the epigraph of V defined by

$$(1.18)\qquad \mathcal{E}p(V) := \{(x,w) \in \mathrm{Dom}\, V \times \mathbb{R} \mid V(x) \leq w\}$$

We then observe the following facts:

$$T_{\mathcal{E}p(V)}(x,V(x)) = \mathcal{E}p D_+V(x)$$

is the epigraph of the <u>epi-contingent derivative</u> $D_+V(x)$ defined by

$$(1.19) \qquad D_{+}V(x)(u) = \liminf_{\substack{h \to 0+ \\ u' \to u}} \frac{V(x+hu) - Vx}{h}$$

and that

$$C_{\mathcal{E}p(V)}(x,V(x)) = \mathcal{E}_p C_{+}V(x)$$

is the epigraph of the epi-derivative $C_{+}V(x)$ of V at x, which is a lower semicontinuous, positively homogeneous convex function. We always have

$$(1.20) \qquad D_{+}V(x) \leq C_{+}V(x)$$

and equality holds when V is C^1 around x, or convex. When V is Lipschitz around x, we obtain simpler formulas

$$(1.21) \qquad \begin{cases} \text{(i) } D_{+}V(x)(u) = \liminf\limits_{h \to 0+} \dfrac{V(x+hu) - V(x)}{h} \quad \text{(a Dini derivative)} \\ \text{(ii) } C_{+}V(x)(u) = \limsup\limits_{\substack{h \to 0+ \\ y \to x}} \dfrac{V(y+hu) - V(y)}{h} \quad \text{(Clarke's directional derivative)} \end{cases}$$

We observe also that when V is upper semicontinuous at x and $w > V(x)$, then $T_{\mathcal{E}p(V)}(x,w) = C_{\mathcal{E}p(V)}(x,w) = \mathbb{R}^n \times \mathbb{R}$.

We shall also use the notation:

$$(1.22) \qquad C_{-}V(x)(u) := \liminf_{\substack{h \to 0+ \\ y \to x}} \frac{V(y+hu) - V(y)}{h}$$

whose hypograph is the tangent cone to the hypograph of V at (x,V(x)) (when V is locally Lipschitz). Finally, we define the generalized gradient $\partial V(x)$ of V at x as the (possibly empty) closed convex subset

$$(1.23) \qquad \partial V(x) := \{p \in \mathbb{R}^n \mid \forall v \in \mathbb{R}^n, \langle p,v\rangle \leq C_{+}V(x)(v)\}$$

In particular, when $V: = \psi_K$ is the indicator of a subset $K(\psi_K(x) = 0$ when $x \in K$ and $\psi_K(x) = +\infty$ when $x \notin K\}$, then

$$(1.24) \qquad \partial\psi_K(x) = (C_K(x))^- = : N_K(x)$$

where $N_K(x)$ is the normal cone to K at x. We say that a solution $\bar{x}$ to

$$(1.25) \qquad \bar{x} \in K \text{ and } 0 \in \partial V(\bar{x}) + N_K(\bar{x})$$

is a critical and stationary point of V on K. Elements $\bar{x} \in K$ minimizing V on K are critical points; the converse is true when V is convex.

2. PIPES DERIVED FROM POTENTIAL FUNCTIONS

Let $K \subset \mathbb{R}^n$ be the viability domain and let us consider a "potential function" V from $\mathbb{R}^n$ to $\mathbb{R}_+ \cup \{+\infty\}$. We shall study in this section pipes of the form

$$(2.1) \qquad P(t): = \{x \in K | V(x) \leq w(t)\}$$

where w is a non-negative function defined on $[0,T]$. We shall begin by providing sufficient conditions on K,V,w and F implying that set-valued maps P of the form (2.1) are pipes of F. We obtain in this case the following result:

Proposition 2.1: Let us assume that K is closed and that V is locally Lipschitz around K. Let w be a C^1- function defined on a neighborhood of V such that

$$(2.2) \qquad \begin{cases} \forall t \in [0,T[\text{ the elements of } P(t) \text{ are not critical} \\ \text{points of } V \text{ on } K \end{cases}$$

We posit the following condition

$$(2.3) \qquad \begin{cases} \forall t \in [0,T[\,, \ \forall x \in K \text{ such that } V(x) = w(t), \\ \exists u \in F(t,x) \cap C_K(x) \text{ such that } C_+V(x)(u) \leq w'(t) \end{cases}$$

and

$$(2.4)\qquad \begin{cases} \text{If } x \in K \text{ satisfying } V(x) = w(T) \text{ is a critical} \\ \text{point of } V \text{ on } K, \text{ then } 0 \in F(T,x). \end{cases}$$

Then the set-valued map P defined by (2.1) is a pipe of F on $[0,T]$. ▲

Therefore, if F satisfies assumption (1.10), then for all $x_o \in K$ satisfying $V(x) \leq w(0)$, there exists a trajectory of the differential inclusion (1.1) satisfying

$$(2.5)\qquad \begin{cases} \text{(i) } \forall t \in [0,T],\ x(t) \in K \text{ and } V(x(t)) \leq w(t), \\ \text{(ii) for almost all } t \geq 0, \\ x'(t) \in T_K(x(t)) \text{ and } D_+V(x(t))(x'(t)) \leq w'(t) \end{cases}$$ ▲

We recall that when V is C^1,

$$(2.6)\qquad D_+V(x)(u) = C_+V(x)(u) = \langle V'(x),u\rangle$$

and that when V is convex and continuous,

$$(2.7)\qquad D_+V(x)(u) = C_+V(x)(u) = \inf_{h > 0} \frac{V(x+hu) - V(x)}{h}\,.$$ ■

We shall also study pipes of the form

$$P(t) := \{x \in K \mid w_-(t) \leq V(x-c(t)) \leq w_+(t)\}$$

where c is a function from $[0,T]$ to K and w_- and w_+ are non-negative functions, which define some kind of neighborhood around a function $t \to c(t)$, such as periodic trajectories of the dynamical system (1.1). They are special cases of pipes associated to p potential functions V_i by the formula

$$(2.8)\qquad P(t) := \{x \in K \mid V_i(\Phi(t,x)) \leq w_i(t),\ i=1,\dots,p)\}$$

where Φ is a smooth map from $[0,T] \times K$ to Dom $\vec{V}$. We shall then provide sufficient conditions on $\Phi,\vec{w}$ and the functions V_i for a set-valued P of this type to be a pipe for a given set-valued map F.

<u>Theorem 2.2</u>: Let us assume that K is closed, that Φ is C^1 around $[0,T] \times K$, that $\vec{w}$ is C^1 around $[0,T]$ and that the p potential functions V_i are locally Lipschitz on a neighborhood of $\Phi([0,T] \times K)$. Let us set

$$(2.9) \qquad I(t,x) := \{i=1,\dots,p \mid V_i(\Phi(t,x)) = w_i(t)\}$$

We assume that

$$(2.10) \qquad \begin{cases} \forall t \in [0,T],\ \forall x \in P(t), \\ 0 \notin \text{co}[\bigcup_{i \in I(t,x)} \Phi'_x(t,x)^* \partial V_i(\Phi(t,x))] + N_K(x) \end{cases}$$

and that

$$(2.11) \qquad \begin{cases} 0 \in F(t,x) \text{ for all } x \in P(T) \text{ such that there exists} \\ i \in I(T,x) \text{ such that } 0 \in \Phi'_x(T,x)^* \partial V_i(\Phi(T,x)) + N_K(x) \end{cases}$$

We posit the following assumption

$$(2.12) \qquad \begin{cases} \text{(i)} \ \forall t \in [0,T[,\ \forall x \in P(t),\ \exists u \in F(t,x) \cap C_K(x) \text{ such that} \\ \qquad \forall i \in I(t,x), C_+V_i(\Phi(t,x))(\Phi'_t(t,x) + \Phi'_x(t,x)u) \leq w'_i(t) \\ \text{(ii)} \ \forall x \in P(T), \exists u \in F(T,x) \cap C_K(x) \text{ such that} \\ \qquad \forall i \in I(T,x), C_+V_i(\Phi(T,x))(\Phi'_x(T,x)u) \leq 0 \end{cases}$$

Then the set-valued map P defined by (2.8) is a pipe of F. ▲

If F satisfies assumption (1.10), then, for all $x_0 \in K$ satisfying $V_i(\Phi(0,x)) \leq w_i(0)\ (i=1,\dots,p)$, there exists a trajectory of the differential inclusion (1.1) satisfying

$$(2.13) \qquad \begin{cases} \text{(i)} \ \forall t \in [0,T[,\ \forall i=1,\dots,p,\ V_i(\Phi(t,x(t)) \leq w_i(t) \\ \text{(ii)} \ \text{for almost all } t \in [0,T], \text{ for all } i \in I(t,x(t)), \\ \qquad x'(t) \in T_K(x(t)) \text{ and} \\ \qquad D_+V_i(\Phi(t,x(t))(\Phi'_t(t,x(t)) + \Phi'_x(t,x(t))x'_i(t)) \leq w'_i(t) \end{cases}$$

▲

Remark: Observe that the elements $x \in K$ satisfying

$$0 \in \Phi_x'(t,x)^{*}\partial V_i(\Phi(t,x)) + N_K(x)$$

are the critical points of $x \to V_i(\Phi(t,x))$ on K. Assumption (2.11) states that critical points of some function $V_i(\Phi(T,\cdot))$ on K are equilibria of $F(T,\cdot)$. We can say that a solution to

$$(2.14) \qquad 0 \in \mathrm{co}\left(\bigcup_{i=1}^{n} \Phi_x'(t,x)^{*}\partial V_i(\Phi(t,x))\right) + N_K(x)$$

is a Pareto critical point of the functions $V_i(\Phi(t,\cdot))$, (Pareto minima do satisfy this inclusion).

Remark: Observe also that if

$$(2.15) \qquad \forall t \in [0,T[\,,\ \forall i=1,\dots,n,\ w_i'(t) < 0,$$

then property (2.10) follows from (2.12). This is usually the interesting case, since we would like the pipes P(t) to decrease when t increases.

Corollary 2.3: Let K be a closed subset, V be a C^1 function from a neighborhood of [0,T] to K, w_- and w_+ be C^1 non-negative functions satisfying

$$(2.16) \qquad \begin{cases} \forall t \in [0,T[\,,\ 0 \leq w_-(t) < w_-(T) = w_+(T) < w_+(t) \\ \text{and } w_-'(t) > 0,\ w_+'(t) < 0 \end{cases}$$

We posit the following assumption:

$$(2.17) \qquad \begin{cases} \text{(i)} \ \forall t \in [0,T[\,,\ \forall x \text{ such that } V(x-c(t)) = w_+(t), \\ \quad \text{there exists } u \in F(t,x) \cap C_K(x) \text{ such that} \\ \quad C_+V(x-c(t))(u-c'(t)) \leq w_+'(t) \\ \text{(ii)} \ \forall t \in [0,T[\,,\ \forall x \text{ such that } V(x-c(t)) = w_-(t), \\ \quad \text{there exists } u \in F(t,x) \cap C_K(x) \text{ such that} \\ \quad C_-V(x-c(t))(u-c'(t)) \geq w_-'(t) \\ \text{(iii)} \ \forall x \text{ such that } V(x-c(T)) = w_+(T) = w_-(T),\ 0 \in F(T,x) \end{cases}$$

Then the set-valued map P defined by

(2.18) $$P(t) := \{x \in K \; ; \; w_-(t) \leq V(x-c(t)) \leq w_+(t)\}$$ is a pipe of F on $[0,T]$. ▲

<u>Proof of Theorem 2.2</u>: We set $\mathrm{Dom}\,\vec{V} = \bigcap_{i=1}^{p} \mathrm{Dom}\, V_i$, $\vec{V}(x) := (V_1(x), \ldots, V_p(x))$ and $\mathcal{E}p(\vec{V}) := \{(x,w) \in \mathrm{Dom}\,\vec{V} \times \mathbb{R}^p \,|\, V_i(x) \leq w_i$ for $i=1,\ldots,p)\}$. Let A be the C^1 map from a neighborhood of $[0,T] \times K$ to $\mathbb{R}^n \times \mathbb{R}^p$ defined by

(2.20) $$A(t,x) := (\Phi(t,x), w(t))$$

Then we can write

(2.21) $$\mathrm{Graph}(P) := \{(t,x) \in [0,T] \times K \,|\, A(t,x) \in \mathcal{E}p(\vec{V})\}$$

We then use Proposition 7.6.3, p.440, of Aubin-Ekeland [1984]. It states that

(2.22) $$\begin{cases} \{\tau \in T_{[0,T]}(t), u \in T_K(x) \,|\, A'(t,x)(\tau,u) \in T_{\mathcal{E}p(\vec{V})}(A(t,x))\} \\ \supset T_{\mathrm{Graph}(P)}(t,x) \end{cases}$$

and that if the transversality condition

(2.23) $$A'(t,x)(T_{[0,T]}(t) \times C_K(x)) - C_{\mathcal{E}p(\vec{V})}(A(t,x)) = \mathbb{R}^n \times \mathbb{R}^p$$

then

(2.24) $$\begin{cases} \{\tau \in T_{[0,T]}(t) \;, \; u \in C_K(x) \,| \\ A'(t,x)(\tau,u) \in C_{\mathcal{E}p(\vec{V})}(A(t,x))\} \subset C_{\mathrm{Graph}(P)}(t,x) \end{cases}$$

Inclusion (2.22) implies that for all $t \in [0,T]$,

$$(2.25)\quad \begin{cases} DP(t,x) \\ \subset \{u \in T_K(x) \mid \forall i \in I(t,x), D_+V_i(\Phi(t,x))(\Phi_t'(t,x) + \Phi_x'(t,x)u) \\ \leq w_i'(t)\} \end{cases}$$

since

$$(2.26)\quad A'(t,x)(\tau,u) = (\Phi_t'(t,x)\tau + \Phi_x'(t,x)(u), w'(t)\tau),$$

and since

$$(2.27)\quad \begin{cases} T_{\mathcal{E}p(\vec{V})}(A(t,x)) = T_{\mathcal{E}p(\vec{V})}(\Phi(t,x), w(t)) \\ = \{(u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^p \mid \forall i \in I(t,x) \;,\; \lambda_i \geq D_+V_i(\Phi(t,x))(u)\} \end{cases}$$

In the same way, inclusion (2.24) can be rewritten in the following form

$$(2.28)\quad \begin{cases} \{u \in C_K(x) \mid \forall i \in I(t,x), C_+V_i(\Phi(t,x))(\Phi_t'(t,x)\tau + \Phi_t'(t,x)u) \\ \leq w_i'(t)\tau\} \subset CP(t,x)(\tau) \subset DP(t,x)(\tau). \end{cases}$$

This inclusion and assumption (2.12) imply that P is a pipe of F. It remains to check the transversality condition (2.23), which can be written in the following way:

$$\forall u_d \in \mathbb{R}^n, \forall \lambda_d \in \mathbb{R}^p, \exists\, u \in C_K(x), \exists\, \tau \in T_{[0,T]}(t)$$

such that

$$(2.29)\quad \begin{cases} \forall i \in I(t,x) \;,\; w_i'(t)\tau \geq \\ C_+V_i(\Phi(t,x))(\Phi'(t,x)\tau + \Phi_x'(t,x)u - u_d) + \lambda_d \end{cases}$$

By assumption (2.10) and the separation theorem, there exists $\hat{u} \in C_K(x)$ such that

$$(2.30)\quad \forall i \in I(t,x),\; C_+V_i(\Phi(t,x))(\Phi_x'(t,x)\hat{u}) < 0$$

There exists η such that $C_+V_i(\Phi(t,x))(\Phi'_x(t,x)\hat{u} + v) \leq 0$ when $v \in \eta B$. Let $\beta = 0$ if $\lambda_d \leq 0$ and

$$\beta > \lambda_d / |C_+V_i(\Phi(t,x))(\Phi'_x(t,x)\hat{u})| \text{if } \lambda_d > 0.$$

We take $\alpha = \beta + \eta|\|u_d\|$. Hence, $\tau := 0$ and $u := \alpha\hat{u}$ provide

a solution to (2.29).

Then this transversality condition holds true for all $t \in [0,T[$ and all $x \in P(t)$. When it fails to be true for some $x \in P(T)$, we then assume that such an x is an equilibrium of $F(T,\cdot)$. ∎

3. PIPES DEFINED BY TIME-DEPENDENT CONSTRAINTS

Let us consider a continuous single valued map A from a neighborhood of $[0,T] \times K$ to a vector space $\mathbb{R}^p$ and a subset M of $\mathbb{R}^p$. We shall provide sufficient conditions for a set-valued map P of the form

$$(3.1) \qquad P(t) := \{x \in K | A(t,x) \in M\}$$

to be a pipe of a set-valued map F. We begin with the case when A is continuously differentiable.

<u>Proposition 3.1</u>: Let K and M be closed subsets and A be continuously differentiable. We assume that for all $t \in [0,T], \forall x \in P(t)$,

$$(3.2) \qquad A'_x(t,x)C_K(x) - C_M(A(t,x)) = \mathbb{R}^p$$

If for any $t \in [0,T]$ and any $x \in P(t)$, there exists $v \in F(t,x) \cap C_K(x)$ satisfying

$$(3.3) \qquad \begin{cases} \text{(i) } A'_x(t,x)v \in C_M(A(t,x)) - A'_t(t,x) & \text{when } t < T \\ \text{(ii) } A'_x(T,x)v \in C_M(A(T,x)) & \text{when } t = T \end{cases}$$

then the set-valued map P defined by (3.1) is a pipe of F on $[0,T]$ and

$$(3.4) \qquad DP(t,x) \subset \{u \in T_K(x) \mid A'_x(t,x)u \in T_M(A(t,x)) - A'_t(t,x)\} \qquad \blacktriangle$$

We can relax the assumption that A is continuously differentiable and replace the Jacobian of A by the derivative $CA(t,x) := CA(t,x,A(t,x))$ whose graph is the tangent cone to the graph of A at (t,x).

Then Proposition 3.1 follows from

<u>Proposition 3.2</u>: Let K and M be closed subsets and A be a continuous map. We assume that for all $t \in [0,T], \forall x \in P(t)$,

$$(3.5) \qquad \begin{cases} \text{(i)} \ \mathrm{Dom}\, CA(t,x) = \mathbb{R} \times \mathbb{R}^n \\ \text{(ii)} \ CA(t,x)(0,C_K(x)) - C_M(A(t,x)) = \mathbb{R}^p \end{cases}$$

If for any $t \in [0,T]$ and any $x \in P(t)$ there exists $v \in F(t,x) \cap C_K(x)$ satisfying

$$(3.6) \qquad \begin{cases} \text{(i)} \ CA(t,x)(1,v) \in C_M(A(t,x)) \text{ when } t < T \\ \text{(ii)} \ CA(T,x)(0,v) \in C_M(A(T,x)) \text{ when } t = T, \end{cases}$$

then the set-valued map P defined by (3.1) is a pipe of F on $[0,T]$ and

$$(3.7) \qquad DP(t,x) \subset \{u \in T_K(x) \mid DA(t,x)(1,u) \cap T_M(A(t,u)) \neq \emptyset\} \qquad \blacktriangle$$

<u>Proof</u>: The graph of P is the projection onto $\mathbb{R} \times \mathbb{R}^n$ of the subset

$$L := ([0,T] \times K \times M) \cap \mathrm{Graph}\, A \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$$

By Proposition 7.6.3, p.440 of Aubin-Ekeland [1984], we know that

$$(3.8) \qquad \begin{cases} (T_{[0,T]}(t) \times T_K(x) \times T_M(Ax)) \cap \mathrm{Graph}\, DA(t,x)) \\ \supset T_L(t,x,A(t,x)) \end{cases}$$

This implies inclusion (3.7).

We also know that the transversality condition

(3.9) $$T_{[0,T]}(t) \times C_K(x) \times C_M(x) - \text{graph } CA(t,x) = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p,$$

implies that

(3.10) $$C_L(t,x,A(t,x)) \supset T_{[0,T]}(t) \times C_K(x) \times C_M(Ax) \cap \text{Graph } CA(t,x)$$

We then observe that

(3.11) $$\{u \in C_K(x) \mid CA(t,x)(\tau,u) \cap C_M(A(t,x))) \neq \emptyset\} \subset CP(t,x)(\tau)$$

Indeed, let w belong to $CA(t,x)(\tau,u) \cap C_M(A(t,x))$ and let $t_n \to t$, $x_n \to x$ and $h_n \to 0+$. Since (τ,u,w) belongs to $C_L(t,x,A(t,x))$, there exist sequences $\tau_n \to \tau$, $u_n \to u$ and $w_n \to w$ such that $(t_n + h_n\tau_n,\ x_n + h_n u_n,\ A(t_n,x_n) + h_n w_n) \in L$, i.e. such that $x_n + h_n u_n \in P(t_n + h_n\tau_n)$ for all n. This implies that u belongs to $C(t,x)(\tau)$. It remains to check the transversality condition (3.9).

Let τ,u,w be given in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Since $\text{Dom } CA(t,x) = \mathbb{R} \times \mathbb{R}^n$, there exists $v \in CA(t,x)(-\tau,-u)$. By assumption (3.5)ii), there exist $u_1 \in C_K(x)$ and $w_1 \in C_M(A(t,x))$ such that $-w-v \in CA(0,u_1) - w_1$. Hence $-w$ belongs to $CA(-\tau,u_1-u) - w$, i.e., $(0-\tau,u_1-u,w_1-w)$ belongs to Graph (A) and $(0,u_1,w_1)$ to $C_{[0,T] \times K \times M}(t,x,A(t,x))$.

4. HOMOTOPIC TRANSFERS

Let us consider two closed subsets C and D of $\mathbb{R}^n$ and a differentiable map Φ from a neighborhood of $[0,T] \times C \times D$ to $\mathbb{R}^n$. We consider pipes of the form

(4.1) $$P(t) := \Phi(t,C,D)$$

<u>Proposition 4.1</u>: Let us assume that

$\forall t \le T\ \forall x \in P(t)$, $\exists (y,z) \in C \times D$ such that $\Phi(t,y,z) = x$, $\exists (u,v) \in T_{C \times D}(y,z)$ such that

$$(4.2)\quad \begin{cases} \text{(i) if } t < T, \Phi'_y(t,y,z)u + \Phi'_z(t,y,z)v \in F(t,x) - \Phi'_t(t,y,z) \\ \text{(ii) if } t = T, \Phi'_y(T,y,z) + \Phi'_z(T,y,z)v \in F(T,x) \end{cases}$$

Then the set-valued map P defined by (4.1) is a pipe of F on $[0,T]$. ▲

Proof: We observe that Graph(P) is the image of $[0,T] \times C \times D$ under the map ψ defined by $\psi(t,y,z) = (t,\Phi(t,y,z))$.

By Proposition 7.6.2, p.430 of Aubin-Ekeland [1984], $\psi'(t,y,z)T_{[0,T] \times C \times D}(t,y,z) \subset T_{\text{Graph}(P)}(\psi(t,y,z))$. We deduce that condition (4.1)i) implies property (1.9)ii). We proceed in the same way to show that (4.1)ii) implies (1.9)iii) since $P(T) = \Phi(T,C,D)$. ∎

When C and D are closed and convex, we can characterize pipes of the form (4.1) through dual conditions. If K is a subset of $\mathbb{R}^n$, we denote by

$$(4.3)\qquad \sigma(K,p) := \sup_{x \in K} \langle p,x\rangle$$

its support function.

Proposition 4.2: Let us assume that the values of F are compact and convex and that the subsets C and D are closed and convex. If for any $t \in [0,T], \forall x \in P(t)$, there exists $(x,y) \in C \times D$ satisfying $\Phi(t,y,z) = x$ and for all

$$p \in \Phi'_y(t,y,z)^{*-1}N_C(y) \cap \Phi'_z(t,y,z)^{*-1}N_D(z),$$

we have

$$(4.4)\quad \begin{cases} \text{(i) } \forall t < T, \langle p,\Phi'_t(t,y,z)\rangle + \sigma(F(t,\Phi(t,y,z)),-p) \geq 0 \\ \text{(ii) for } t = T, \sigma(F(T,\Phi(T,y,z)),-p) \geq 0 \end{cases}$$

then the set-valued map P defined by (4.1) is a pipe of F on $[0,T]$. ▲

Proof: When C and D are convex, $T_{C \times D}(y,z) = T_C(y) \times T_D(z)$ so that conditions (4.2)i) and ii) can be written

$$(4.5) \quad \begin{cases} \text{(i)} \ (F(t,\Phi) - \Phi'_t(t,y,z)) \cap (\Phi'_y(t,y,z)T_C(y) + \Phi'_z(t,y,z)T_D(z)) \neq \emptyset \\ \text{(ii)} \ F(T,\Phi) \cap (\Phi'_y(T,y,z)T_C(y) + \Phi'_z(T,y,z)T_D(z)) \neq \emptyset \end{cases}$$

The separation theorem shows that they are equivalent to conditions (4.4). ∎

Corollary 4.3: Let us assume that C and D are closed convex subsets and that the values of F are convex and compact. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function satisfying either one of the following equivalent conditions:

For any $t \geq 0$, $x \in P(t)$, there exists $y \in C, z \in D$ such that $x = y + \phi(t)z$ and either

$$(4.6) \quad \begin{cases} \text{(i)} \ (F(t,y+\phi(t)z) - \phi'(t)z) \cap (T_C(y)+T_D(z)) \neq \emptyset \text{ if } t < T \\ \text{(ii)} \ F(T,y+\phi(T)z)) \cap (T_C(y)+T_D(z)) \neq \emptyset \text{ if } t = T \end{cases}$$

or

$$(4.7) \quad \begin{cases} \forall p \in N_C(y) \cap N_D(z), \\ \text{(i)} \ \phi'(t)\ \sigma_D(p) + \sigma(F(t,y+\phi(t)z),-p) \geq 0 \text{ if } t < T \\ \text{(ii)} \ \sigma(F(T,y+\phi(T)z),-p) \geq 0 \text{ if } t = T \end{cases}$$

Then the set-valued map P defined by

$$(4.8) \qquad P(t) := C + \phi(t)D$$

is a pipe of F on $[0,T]$. ▲

Let us consider the instance when $C = \{c\}$ and when 0 belongs to the interior of the closed convex subset D.

We introduce the function a_o defined by

$$(4.9)\quad \begin{cases} a_o(t,w) := \\ \sup_{z \in D} \sup_{\substack{p \in N_D(z) \\ \sigma_D(p) = 1}} \inf_{v \in F(t,c+wz)} \langle p,v\rangle \\ = \sup_{z \in D} \inf_{v \in F(t,c+wz)} \sup_{\substack{p \in N_D(z) \\ \sigma_D(p) = 1}} \langle p,v\rangle \end{cases}$$

(The last equation follows from the minimax theorem.)

Let us assume that there exists a continuous function $a : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, satisfying $a(t,0) = 0$ for all $t \geq 0$, such that

$$(4.10)\qquad \forall (t,w) \in \mathbb{R}_+ \times \mathbb{R}_+,\ a(t,w) \geq a_o(t,w)$$

Let ϕ be a solution to the differential equation

$$(4.11)\qquad \phi'(t) = a(t,\phi(t)),\ \phi(0) = \phi_o \text{ given}$$

satisfying

$$(4.12)\qquad a(T,\phi(T)) = 0$$

Since $\sigma_D(p) > 0$ for all $p \neq 0$, we deduce that for all $z \in D$ and all $p \in N_D(z)$,

$$\phi'(t)\sigma_D(p) \geq a(t,\phi(t)\sigma_D(p) \geq a_o(t,\phi(t))\sigma_D(p)$$

$$\geq -\sigma_D(p) \sup_{v \in F(t,c+\phi(t)z)} \langle -\frac{p}{\sigma_D(p)}, v\rangle$$

$$= -\sigma(F(t,c+\phi(t)z),-p)$$

Hence, condition (4.7)i) is satisfied. Also

$$0 = a(T,\phi(T)) \geq a_o(T,\phi(T)) \geq \frac{-1}{\sigma_{D(p)}} \sigma(F(T,c+\phi(T)z),-p)$$

Then

(4.13) $$P(t): = c + \phi(t)D$$

defines a pipe of F.

For instance, if $D: = B$ is the unit ball, then $\sigma_B(p) = \|p\|$ and $N_B(z) = \lambda z$ for all $z \in S: = \{x \mid \|x\| = 1\}$. Hence, in this case we have

(4.15) $$a_o(t,w): = \sup_{\|z\|=1} \inf_{v \in F(t,c+wz)} \langle v,z\rangle$$

In other words, the function a_o defined by (4.9) conceals all the information needed to check whether a given subset D can generate a pipe P.

<u>Remark</u>: When a is non-positive and satisfies $a(t,0) = 0$ for all $t \geq 0$, then there exists a non-negative non-increasing solution $\phi(\cdot)$ of the differential equation (4.11)

When $T = \infty$, we infer that $\int_0^\infty a(\tau,\phi(\tau))d\tau$ is finite. Let us assume that for 0 all $w_* \in \mathbb{R}_+$,

(4.16) $$\lim_{\substack{t \to \infty \\ w \to w_*}} a(t,w) = a_*(w_*)$$

Then the limit ϕ_* of $\phi(t)$ when $t \to \infty$ satisfies the equation

$$a_*(\phi_*) = 0$$

Otherwise, there would exist $\varepsilon > 0$ and T such that $a_*(\phi_*) + \varepsilon < 0$ and for all $t > T$, $a(t,\phi(t)) \leq a_*(\phi_*) + \varepsilon$ by definition of a_*. We deduce the contradiction

$$\phi(t) - \phi(T) = \int_T^t a(\tau,\phi(\tau))d\tau \leq (t-T)(a_*(\phi_*) + \varepsilon)$$

when t is large enough. ∎

Example: Let us consider the case when F does not depend upon t. We set

(4.17) $$\rho_o := \sup_{\lambda \in \mathbb{R}} \inf_{w > 0} (\lambda w - a_o(w))$$

Assume also that $\lambda_o \in \mathbb{R}$ achieves the supremum. We can take $\psi(w) := \lambda_o w - \rho_o$.

If $\rho_o > 0$, the function

(4.18) $$\phi_T(t) := \begin{cases} \frac{\rho_o}{\lambda_o}(1-\exp(\lambda_o(t-T)) & \text{if } \lambda_o \neq 0 \\ -\rho_o(t-T) & \text{if } \lambda_o = 0 \end{cases}$$

is such that $P(t) := \{c+\phi_T(t)D\}$ is a pipe of F such that $P(T) = \{c\}$.

If $\rho_o \leq 0$ and $\lambda_o < 0$, then the functions

(4.19) $$\phi(t) := \frac{1}{\lambda_o}(\rho_o - e^{\lambda_o t})$$

are such that $P(t) := c+\phi_c(t)D$ defines a pipe of F on $[0,\infty[$ such that P(t) decreases to the set $P_\infty := c+\frac{\rho_o}{\lambda_o}D$.

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