ON INVERSE FUNCTION THEOREMS FOR SET-VALUED MAPS

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PREFACE

In this paper, the authors prove several generalizations of the inverse function theorem which they apply to optimization theory (Lipschitz properties of maps defined by constraints) and to the local controllability of differential inclusions. The generalizations are mainly concerned with inverse function theorems for smooth maps defined on closed subsets and for set-valued maps. An extension of the implicit function theorem is also provided.

This research, which was motivated partly by the need for analytic methods capable of tackling the local controllability of differential inclusions, was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

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ABSTRACT

We prove several equivalent versions of the inverse function theorem: an inverse function theorem for smooth maps on closed subsets, one for set-valued maps, a generalized implicit function theorem for set-valued maps. We provide applications to optimization theory and local controllability of differential inclusions.

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ON INVERSE FUNCTION THEOREMS FOR SET-VALUED MAPS

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1. Introduction

An "inverse function" theorem was proved in Aubin [1982] and Rockafellar [to appear (d)] for set-valued maps F from a finite dimensional space X to a finite dimensional space Y. It stated that if x_o is a solution to the inclusion $y_o \in F(x_o)$ and if the "derivative" $C\,F(x_o,y_o)$ of F(1) at $(x_o,y_o) \in$ Graph (F) is surjective from X to Y, then the inclusion $y \in F(x)$ can be solved for any y in a neighborhood of y_o and F^{-1} displays a Lipschitzean behavior around y_o. The purpose of this paper is

(a) to extend this theorem when X is any Banach space (the dimension of Y being still finite)

(b) to provide a simpler proof

(c) to extend Rockafellar's result [to appear(d)] on the Lipschitz continuity properties for set-valued maps G defined by relations of the type

$$(1) \qquad G(y) := \{x \in L \mid F(x,y) \cap M \neq \emptyset\}$$

where F is a set-valued map from X x Y to Z and $L \subset X$ and $M \subset Z$ are closed subsets. These maps play an important role in optimization theory. We shall also estimate the derivative of G in terms of the derivative of F and the tangent cones to L and M.

(d) to apply it for studying local controllability of a differential inclusion in the following sense: Let $R(T,\xi)$ denote the reachable set at time T by trajectories starting at ξ of the differential inclusion $x' \in F(x)$ and $M \subset \mathbb{R}^n$ be a target. Let $x_o(\cdot)$ be a trajectory such that $x_o(T) \in M$. We shall give sufficient conditions for proving that for all u in a neighborhood of o, there exists a trajectory x issued from ξ such that $x(T) \in M+u$. Furthermore, if $\mathcal{K}$ denotes the set of trajectories such that $x(T) \in M$, there exists a neighborhood of $\mathcal{K}$ such that, for any trajectory x in this neighborhood, we have the estimate

(2) $$d(x,\mathcal{K}) \leq \ell d(x(T),M)$$

(e) Naturally, the application to the Lipschitz behavior of optimal solutions and Lagrange multipliers of convex minimization problems

(3) $$\inf_{x \in X} (U(x) - \langle p,x \rangle + V(Ax+y))$$

studied in Aubin [1982], [1984] still holds when X is any Banach space. We do not come back to this example.

Let K be a closed subset of a Banach space X, A be a C^1 map from a neighborhood of K to a finite dimensional space Y. We assume the "surjectivity" assumption

(4) $A'(x_o)$ maps the tangent cone to K at x_o onto Y,

we can prove that a solution x to the equation

(5) $$x \in K \text{ and } A(x) = y$$

exists when y is closed to y_o and depends in a Lipschitzean way upon the right-hand side y. We then derive easily the inverse function theorem for set-valued maps from a Banach space X to a finite dimensional space Y and we study the Lipschitz continuity properties of the map G defined by (2). We conclude this paper with an application to local controllability of a dynamical system described by a differential inclusion.

2. The Inverse Function Theorem

Let X be a Banach space, $K \subset X$ be a subset of X. We recall the definition of the tangent cone to a subset K at x_o introduced in Clarke [1975].

We say that

$$C_K(x_o) := \{ v \in X \mid \lim_{\substack{h \to o+ \\ x \to x_o}} \frac{d(x+hv,K)}{h} = 0 \}$$

is the <u>tangent cone</u> to K at x and that its polar cone

$$N_K(x_o) := C_K(x_o)^- \subset X^*$$

is the <u>normal cone</u> to K at x. (See Clarke [1975], [1983]; Rockafellar [1978]; Aubin and Ekeland [1984], etc.) ▲

We state now our basic result.

<u>Theorem 2.1</u>

Let X be a Banach space, Y be a finite dimensional space, $K \subset X$ be a closed subset of X and x_o belong to K. Let A be a differentiable map from a neighborhood of K to Y. We assume that

(2.1) A' is continuous at x_o

and that

(2.2) $A'(x_o)C_K(x_o) = Y$

Then $A(x_o)$ belongs to the interior of A(K) and there exist constants ρ and ℓ such that, for all

(2.3) $\begin{cases} y_1, y_2 \in A(x_o) + \rho B \text{ and any solution } x_1 \in K \text{ to the} \\ \text{equation } A(x_1) = y_1 \text{ satisfying } \|x_o - x_1\| \leq \ell\rho, \text{ there} \\ \text{exists a solution } x_2 \in K \text{ to the equation } A(x_2) = y_2 \\ \text{satisfying } \|x_1 - x_2\| \leq \ell\|y_1 - y_2\|. \end{cases}$ ▲

We state several corollaries before proving the above theorem.

Corollary 2.2

Let K be a closed subset of a finite dimensional space. Then x_o belongs to the interior of K if and only if $C_K(x_o) = Y$. ▲

We shall derive the extension to set-valued maps of the inverse function theorem. For that purpose, we need to recall the definition of the derivative of F at a point (x_o,y_o) of its graph (see Aubin-Ekeland, Definition 7.2.4, p.413) and the definition of a pseudo-Lipschitz map introduced in Aubin [1982], [1984], (see Aubin-Ekeland, Definition 7.5.1, p.429).

The derivative $CF(x_o,y_o)$ of F at $(x_o,y_o) \in$ Graph (F) is the set-valued map from X to Y associating to any $u \in X$ elements $v \in Y$ such that (u,v) is tangent to Graph (F) at (x_o,y_o):

$$(2.4) \qquad v \in CF(x_o,y_o)(u) \Leftrightarrow (u,v) \in C_{\text{Graph}(F)}(x_o,y_o)$$

A set-valued map G from Y to Z is pseudo-Lipschitz around $(y_o,z_o) \in$ Graph (G) if there exist neighborhoods V of y_o and W of z_o and a constant ℓ such that

$$(2.5) \qquad \begin{cases} \text{i)} & \forall y \in V,\ G(y) \neq \emptyset \\ \text{ii)} & \forall y_1,y_2 \in V,\ G(y_1) \cap W \subset G(y_2) + \ell\|y_1-y_2\|B \end{cases}$$

(See Rockafellar [to appear] d) for a thorough study of pseudo-Lipschitz maps.)

Theorem 2.3

Let F be a set-valued map from a Banach space X to a finite dimensional space Y and (x_o,y_o) belong to the graph of F. If

$$(2.6) \qquad CF(x_o,y_o) \text{ is surjective,}$$

then F^{-1} is pseudo-Lipschitz around $(y_o,x_o) \in$ Graph (F^{-1}). ▲

Proof

We apply Theorem 2.1 when X is replaced by X x Y, K is the graph of F and A is the projection from X x Y to Y.

Remark

Actually, Theorem 2.3 is equivalent to Theorem 2.1, when we apply it to the set-valued map F from X to Y defined by $F(x) := \{Ax\}$ when $x \in K$ and $F(x) := \emptyset$ when $x \notin K$.

Proof of Theorem 2.1

a) Since $A'(x_o)C_K(x_o) = Y$, since $C_K(x_o)$ is a closed convex cone and since $A'(x_o)$ is a continuous linear operator, corollary 3.3.5, p.134 in Aubin-Ekeland [1984] of Robinson-Ursescu's Theorem (see Robinson [1976], Ursescu [1975]) implies the existence of a constant $k > 0$ such that

(2.7) $\quad \forall u_i \in Y,\ \exists v_i \in C_K(x_o)$ satisfying $A'(x_o)v_i = u_i$ and

$$\|v_i\| \leq k\|u_i\|$$

Let $\alpha \in]0,1[$ and γ such that $\gamma \leq \alpha/2\|A'(x_o)\|$. Since A' is continuous at x_o, there exists $\delta \leq \alpha/2(k+\gamma)$ such that for any $x \in B_K(x_o,\delta)$, $\|A'(x) - A'(x_o)\| \leq \delta$.

By the very definition of the tangent cone $C_K(x_o)$, we can associate with any $v_i \in C_K(x_o)$ constants $\eta_i \in]0,\delta]$ and $\beta_i > 0$ such that

(2.8) $\quad \forall x \in B_K(x_o,\eta_i)\ ,\ \forall h \in]0,\beta_i],\ v_i \in \frac{1}{h}(K-x) + \gamma B$

Therefore, we can associate with any u_i belonging to the unit sphere S of Y constants $\eta_i > 0$ and $\beta_i > 0$ such that

$$\forall x \in B_K(x_o,\eta_i), \forall h \in]0,\beta_i], \forall u \in (u_i+\tfrac{\alpha}{2}B) \cap S\ ,$$

$$u \in A'(x)(\tfrac{1}{h}(K-x)) + (k+\gamma)(A'(x_o) - A'(x))B + A'(x)\gamma B + \tfrac{\alpha}{2}B$$

$$\subset A'(x)(\tfrac{1}{h}(K-x)) + \alpha B.$$

The sphere S being compact because the dimension of Y is supposed to be finite, it can be covered by n balls $u_i + \alpha B$. We take $\eta := \min_{i=1,\dots,n} \eta_i$, $\beta := \min_{i=1,\dots,n} \beta_i$ and $c := k+\gamma$. These constants depend upon α only. We deduce that

$\forall u \in Y$, $\forall x \in B_K(x_o, y)$, $\forall h < \beta$, there exist $y \in K$ and $w \in Y$ satisfying

$$(2.9) \quad \begin{cases} \text{i)} \quad u = A'(x)\left(\frac{y-x}{h}\right) + w \\ \text{ii)} \quad \|y-x\| \leq ch\|u\| \ , \ \|w\| \leq \alpha\|u\| \end{cases}$$

b) We take y in the open ball $A(x_o) + r\mathring{B}$ where $r < (1-\alpha)\frac{\eta}{c}$ and ε such that $\frac{\|y-A(x_o)\|}{\eta} < \varepsilon < \frac{1-\alpha}{c}$.

We shall apply Ekeland's approximate variational principle (see Ekeland [1974] and Aubin-Ekeland [1984], Theorem 5.3.1, p.255) to the function V defined by

$$V(x) := \|y-A(x)\|$$

on the closed subset K: there exists $x_\varepsilon \in K$ satisfying

$$(2.10) \quad \begin{cases} \text{i)} \quad \|y-A(x_\varepsilon)\| + \varepsilon\|x_o-x_\varepsilon\| \leq \|y-A(x_o)\| \\ \text{ii)} \quad \|y-A(x_\varepsilon)\| \leq \|y-A(x)\| + \varepsilon\|x-x_\varepsilon\| \text{ for all } x \in K \end{cases}$$

Inequality (2.10)i) implies that

$$(2.11) \quad \|x_o-x_\varepsilon\| \leq \varepsilon^{-1}\|y-A(x_o)\| \leq \eta$$

If $y = A(x_\varepsilon)$ the result is proved. Assume that $y \neq Ax_\varepsilon$. Property (2.9) with $u = y-A(x_\varepsilon)$ imply the existence of $y_\varepsilon \in K$ such that, by setting $v_\varepsilon := \frac{y_\varepsilon - x_\varepsilon}{h}$, we have

$$(2.12) \quad y-A(x_\varepsilon) = A'(x_\varepsilon)v_\varepsilon + w_\varepsilon$$

where

$$(2.13) \quad \|v_\varepsilon\| \leq c\|y-A(x_\varepsilon)\|, \|w_\varepsilon\| \leq \alpha\|y-A(x_\varepsilon)\|$$

We observe that we can write

$$(2.14) \quad \begin{cases} y-A(y_\varepsilon) = y-A(x_\varepsilon)-hA'(x_\varepsilon)v_\varepsilon-h0(h) \\ = (1-h)(y-A(x_\varepsilon)) + h(w_\varepsilon+0(h)) \end{cases}$$

By taking $x := y_\varepsilon$ in inequality (2.10)ii), we deduce that

$$(2.15)\quad \begin{aligned} \|y-A(x_\varepsilon)\| &\le \|w_\varepsilon\| + \varepsilon\|v_\varepsilon\| + \|0(h)\| \\ &\le (\varepsilon c+\alpha)\|y-A(x_\varepsilon)\| + \|0(h)\| \end{aligned}$$

By letting h go to 0 and by observing that $\varepsilon c+\alpha < 1$, we obtain equality $y = A(x_\varepsilon)$.

c) Then there exists a solution x in $B_K(x_o,y)$ to the equation $y = A(x)$. Furthermore, inequality (2.11) implies that

$$(2.16)\quad d(x_o, A^{-1}(y) \cap K) < \frac{1}{\varepsilon}\|y-A(x_o)\|$$

By letting ε converge to $\frac{1-\alpha}{c}$, we deduce that

$$(2.17)\quad d(x_o, A^{-1}(y) \cap K) \le \frac{c}{1-\alpha}\|y-A(x_o)\|$$

Let ρ be smaller than $\frac{(1-\alpha)^2\eta}{2c+1-\alpha}$ so that there exists ε satisfying

$$(2.18)\quad \frac{2c\rho}{(1-\alpha)\eta-c\rho} < \varepsilon < \frac{1-\alpha}{c}$$

Let $y_1 \in y_o + \rho\mathring{B}$ and $x_1 \in A^{-1}(y_1) \cap K$ be a solution to the equation $y_1 = A(x_1)$ satisfying $\|x_1-x_o\| < \frac{c}{1-\alpha}\|y_1-y_o\|$. We now apply Ekeland's theorem to the function $x \to \|y_2 - A(x)\|$ where y_2 is given in $y_o + \rho\mathring{B}$: there exists $x_\varepsilon \in K$ satisfying

$$(2.19)\quad \begin{cases} \text{i)} & \|y_2-A(x_\varepsilon)\| + \varepsilon\|x_\varepsilon-x_1\| \le \|y_2-y_1\| \\ \text{ii)} & \|y_2-A(x_\varepsilon)\| \le \|y_2-A(x)\| + \varepsilon\|x-x_\varepsilon\| \text{ for all } x \in K. \end{cases}$$

Inequality (2.19)i) implies that

$$(2.20)\quad \begin{cases} \|x_\varepsilon-x_o\| < \frac{1}{\varepsilon}\|y_2-y_1\| + \|x_o-x_1\| \\ \le \frac{2\rho}{\varepsilon} + \frac{c}{1-\alpha}\rho \le \eta \end{cases}$$

so that we can use again property (2.9) for deducing from inequality (2.19)ii) that $y_2 = A(x_\varepsilon)$ as before, and prove that

$d(x_1, A^{-1}(y_2) \cap K) \le \frac{c}{1-\alpha}\|y_1-y_2\|$.

3. Applications to Non-smooth Optimization

Let X,Y,Z be three finite dimensional spaces, F be a set-valued map from X x Y to Z, $L \subset X$ and $M \subset Z$ be closed subsets and $f: X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The study of the Lipschitz continuity properties of the marginal function v of the minimization problem

(3.1) $$v(y) := \inf\{ f(x,y) \mid x \in L \text{ and } F(x,y) \cap M \neq \emptyset\}$$

is required for computing the generalized gradient of the marginal function (see Rockafellar [to appear] b)). This is the reason why we need to prove that the set-valued map G defined by

(3.2) $$G(y) := \{ x \in L \mid F(x,y) \cap M \neq \emptyset\}$$

is pseudo-Lipschitz. Let x_o belong to $G(y_o)$ and z_o be chosen in $F(x_o,y_o) \cap M$.

Theorem 3.1

We assume that

(3.3) L , M and Graph (F) are closed

and that the following transversality condition holds true:

(3.4) $$\forall v \in Y,\ CF(x_o,y_o,z_o)(C_L(y_o),v) - C_M(z_o) = Z$$

Then the derivative of G is estimated by

(3.5) $$\{ u \in C_L(x_o) \mid CF(x_o,y_o,z_o)(u,v) \cap C_M(z_o) \neq \emptyset\} \subset CG(y_o,x_o)(v)$$

and the set-valued map G defined by (3.2) is pseudo-Lipschitz around (y_o,x_o). If we assume furthermore that

(3.6) F is lower semicontinuous at (x_o,y_o) [2]

then there exist a neighborhood U of x_o and a constant $\ell > 0$ such that

(3.7) $\forall x \in U \ , \ d(x,G(y)) \leq \ell \max \ (d_L(x), \delta(F(x,y),M))$

where we set

(3.8) $\delta(A,B) := \inf\{\|x-y\|, x \in A, y \in B\}$ ▲

Remark

Let us denote by $CF(x_o,y_o.z_o)^*$ the coderivative of F at (x_o,y_o,z_o), which is the transpose of the derivative of $F(x_o,y_o,z_o)$ (see Aubin-Ekeland [1984], Definition 7.2.9, p.416). We say that $(p,q) \in CF(x_o,y_o,z_o)^*(r)$ if and only if

(3.9) $\forall (u,v) \in X \times Y, \forall w \in CF(x_o,y_o,z_o)(u,v), \langle p,u\rangle + \langle q,u\rangle \leq \langle r,w\rangle$

The transversality condition (3.5) implies constraint qualification condition

(3.10) $\begin{cases} \text{The only solution } (p,q,r) \in -N_L(x_o) \times Y^* \times N_M(z_o) \\ \text{to the inclusion } (p,q) \in CF(x_o,y_o,z_o)^*(r) \text{ is} \\ p = 0,\ q = 0 \text{ and } r = 0. \end{cases}$

When F is single-valued, we can set

(3.11) $CF(x_o,y_o) := CF(x_o,y_o,F(x_o,y_o))$

In this case, Theorem 3.1 reduces to a statement analogous to Theorem 3.2 of Rockafellar [to appear] d), where the derivative $CF(x_o,y_o)$ is replaced by the generalized Jacobian $\partial F(x_o,y_o)$ introduced by Clarke [1976] b). We do not need to assume that F is locally Lipschitz, since we do not use the generalized Jacobian. It is sufficient to assume only that the graph of F is closed.

Corollary 3.2

Let X,Y,Z be finite dimensional spaces, $L \subset X$ and $M \subset Z$ be closed subsets and F be a single-valued map from X x Y to Z with closed graph. We posit the transversality condition

$$(3.12) \qquad \forall v \in Y,\ CF(x_0,y_0)(C_L(x_0),v) - C_M(F(x_0,y_0)) = Z$$

Then

$$(3.13) \qquad \{u \in C_L(x_0) \mid CF(x_0,y_0)(u,v) \cap C_M(F(x_0,y_0) \neq \emptyset\} \subset CG(y_0,x_0)(v)$$

and G is pseudo-Lipschitz around (y_0,x_0). If F is continuous, there exists a neighborhood of x_0 and a constant $\ell > 0$ such that

$$(3.14) \qquad \forall x \in U,\ d(x,G(y)) \leq \ell \max (d_L(x),\ d_M(F(x,y)))$$

▲

Remark

Let us observe also that by taking $L = X$ and $M = \{0\}$, we obtain the usual implicit function theorem for continuous maps (instead of locally Lipschitz maps, as in Clarke [1976], Hiriart-Urruty [1979]). In this case, we can assume that X is any Banach space, Y and Z being still finite dimensional.

Corollary 3.3

Let $K := \{x \in L \mid F(x) \cap M \neq \emptyset\}$ where $L \subset X$ and $M \subset Z$ are closed subsets and where $F : X \to Z$ is a set-valued map with a closed graph. Let $x_0 \in K$ and $z_0 \in F(x_0) \cap M$ be fixed. If we assume that

$$(3.15) \qquad CF(x_0,z_0)(C_L(x_0)) - C_M(z_0) = Z \ ,$$

then the tangent cone to K at x_0 satisfies

$$(3.16) \qquad \{u \in C_L(x_0) \mid CF(x_0,z_0)(u) \cap C_M(z_0) \neq \emptyset\} \subset C_K(x_0)$$

▲

When F is a C^1 single-valued map, we obtain a result given in Aubin [1982] (see Aubin-Ekeland [1984], Proposition 7.6.3, p.440, which is true when X is a Banach space and Z a finite dimensional space).

Proof of Theorem 3.1

a) The graph of G is the projection onto Y x X of the subset $Q = H(0,0,0)$ where we set

(3.17) $\quad H(u,v,w) := \text{Graph}(F) \times M \times Y \times L \cap B^{-1}(u,v,w)$

where B is the linear map from $X \times Y \times Z \times Z \times Y \times X$ to $X \times Y \times Z$ defined by

(3.18) $\quad B(\xi,\eta,\zeta,z,y,x) = (\xi-x,\eta-y,\zeta-z)$

Let $(x_o,y_o,z_o) \in H(0,0,0)$ be chosen. We observe that the transversality condition (3.4) implies that

(3.19) $\quad B(C_{\text{Graph}(F)}(x_o,y_o,z_o) \times C_M(z_o) \times Y \times C_L(x_o)) = X \times Y \times Z$

Indeed, let $(x,y,z) \in X \times Y \times Z$ be chosen. Let z_i belong to $CF(x_o,y_o,z_o)(x,\frac{y}{2})$. By (3.4), there exist $u \in C_L(x_o)$ and $w \in C_M(z_o)$ such that $z-z_1 \in CF(x_o,y_o,z_o)(u,\frac{y}{2}) - w$. Hence, $CF(x_o,y_o,z_o)$ being a convex process, we have

$$z \in CF(x_o,y_o,z_o)(x,\tfrac{y}{2}) + CF(x_o,y_o,z_o)(u,\tfrac{y}{2}) - w$$

$$\subset CF(x_o,y_o,z_o)(x+u,y) - w$$

In other words, we have proved that

$$(x,y,z) = B(x+u,y,z+w,w,o,u) \quad \text{where}$$

$$(x+u,y,z+w,w,o,u) \in C_{\text{Graph}(F)}(x_o,y_o,z_o) \times C_M(z_o) \times Y \times C_L(x_o)$$

Then Proposition 7.6.3, p.440 of Aubin-Ekeland [1984] implies that

$$(C_{\text{Graph}(F)}(x_o,y_o,z_o) \times C_M(z_o) \times Y \times C_L(x_o)) \cap B^{-1}(0)$$

$$\subset C_Q(x_o,y_o,z_o,z_o,y_o,x_o)$$

In other words, if we take $(u,v,w) \in X \times Y \times Z$ such that

(3.20) $\quad u \in C_L(x_o), v \in Y$ and $w \in CF(x_o,y_o,z_o)(u,v) \cap C_M(z_o)$,

then (u,v,w,w,v,u) belongs to $C_Q(x_o,y_o,z_o,z_o,y_o,x_o)$. Therefore,

(v,u) belongs to the tangent cone to Graph (G) at (y_o,x_o), or $u \in CG(y_o,x_o)(v)$. (Indeed, if $(y_n,x_n) \in$ Graph(G) converges to (y_o,x_o) and $h_n > 0$ converges to 0, we deduce that there are sequences u_n,u'_n converging to u, v_n,v'_n converging to v and w_n,w'_n converging to w such that

$$(x_n+h_nu_n,y_n+h_nv_n,z_o+h_nw_n,z_o+h_nw'_n,y_n+h_nv'_n,x_n+h_nu'_n) \in Q$$

This implies that $u_n = u'_n, v_n = v'_n$, $w_n = w'_n$ and that

(3.21) $\quad x_n+h_nu_n \in C_L(x_o)$, $F(x_n+h_nu_n,y_n+h_nv_n) \cap M \neq \emptyset$,

i.e. that $x_n+h_nu_n \in G(y_n+h_nv_n)$ for all n. Hence $u \in CG(y_o,x_o)(v)$.).

b) Theorem 2.1 applied to the map B defined on the closed subset Graph (F) x M x Y x L implies that the set-valued map H defined by (3.17) is pseudo-Lipschitz around $((0,0,0),(x_o,y_o,z_o,z_o,y_o,x_o))$. In particular, there exist $\ell > 0$ and $r > 0$ such that if $\max(\|u\|, \|v\|, \|w\|) \leq r$, there exists $(x,y,z) \in X \times Y \times Z$ such that

(3.22) $\quad x \in L, z \in (F(x+u,y+v) - w) \cap M$

and

$$\max(\|x+u-x_o\| , \|y+v-y_o\| , \|z+w-z_o\|) \leq$$

$$\ell \max(\|u\| , \|v\| , \|w\|).$$

By taking $u = w = 0$ and $y = y_o$, we deduce that the map $v \rightarrow G(y_o+v)$ is pseudo-Lipschitz around (y_o,x_o).

c) Let us consider now a pair (x,y). We choose $\bar{x} \in L$ minimizing $\|\xi-x\|$ over L and $\bar{\zeta} \in F(x,y)$ and $\bar{z} \in M$ minimizing $\|\zeta-z\|$ on $F(x,y) \times M$. We set $u = x-\bar{x}, v = y-y_o$ and $w = \bar{\zeta}-\bar{z}$ so that $\|u\| = d_L(x)$ and $\|w\| = d(F(x,y),M)$. Hence

$$B(x, y, \bar{\zeta}, \bar{z}, y_o, \bar{x}) = (u,v,w)$$

Since F is lower semicontinuous at x_o, y_o, there exists a neighborhood V of (x_o, y_o) such that $d_L(x) = \|u\| \leq \rho$, $\delta(F(x,y),M) = \|w\| \leq \rho$ when $(x,y) \in V$ (because $\delta(F(x,y),M) \leq \|\zeta - z_o\| \leq \rho$ for some $\zeta \in F(x,y)$). Let $\|v\| \leq \rho$. Since H is pseudo-Lipschitz, there exists a solution $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{z}, \tilde{y}, \tilde{x}) \in H(0,0,0)$ such that

$$(3.23) \quad \begin{cases} \max (\|\tilde{x}-x\|, \|\tilde{y}-y\|, \|\tilde{z}-\zeta\|, \|\tilde{z}-\bar{z}\|, \|\tilde{y}-y_o\|, \|\tilde{x}-\bar{x}\|) \\ \leq \ell \max (\|u\|, \|v\|, \|w\|) = \ell \max (d_L(x), \|y-y_o\|, \delta(F(x,y),M)) \end{cases}$$

By taking $\tilde{y} = y = y_o$, we obtain inequality (3.7).

Remark

Let K be the map associating to $y \in Y$ the subset

$$(3.24) \quad K(y) := \{(x,z) \in L \times M \mid z \in F(x,y)\}$$

Since the graph of K is the image of $Q := H(0,0,0)$ by the map $(x,y,z,z,y,x) \to (y,x,z)$, the proof of Theorem 3.1 implies that

$$(3.25) \quad K \text{ is pseudo-Lipschitz around } (y_o, x_o, z_o)$$

and that

$$(3.26) \quad \begin{cases} \{(u,w) \in C_L(x_o) \times C_M(z_o) \mid w \in CF(x_o, y_o, z_o)(u,v)\} \\ \subset CK(y_o, x_o, z_o)(v) \end{cases}$$

If F is lower semicontinuous, inequality (3.23) implies that

$$(3.27) \quad \delta(\{x\} \times F(x,y), K(y)) \leq \ell \max (d_L(x), \delta(F(x,y),M)).$$

Remark

If we take $L = X$ (there are no constraints on x), we do not have to assume that the dimension of X is finite. We have to apply Proposition 7.6.3 of Aubin-Ekeland [1984] and Theorem 3.1

to the map B_o from (Graph (F) x M x Y to Y x Z defined by

$$B_o(\xi,\eta,\zeta,z,y) = (\eta-y,\zeta-z)$$

since the graph of G is the set of (y,x) such that

$$B_o(x,y,z,z,y) = (0,0) \text{ and } (x,y,z) \in \text{Graph }(F),\ z \in M.$$

The transversality condition (3.4) is replaced by

$$(3.28) \qquad \forall v \in Y,\ CF(x_o,y_o,z_o)(x,v) - C_M(z_o) = Z$$

and the derivative of G satisfies

$$(3.29) \qquad \begin{cases} \{u \in X \mid CF(x_o,y_o,z_o)(u,v) \cap C_M(z_o) \neq \emptyset\} \\ \subset CG(y_o,x_o)(v). \end{cases}$$

■

We shall apply corollary 3.3 to compute the epiderivative of the function $x \to V(x) + W(F(x))$ when F is a continuous single-valued map. When V is a function from X to $\mathbb{R} \cup \{+\infty\}$, we observe that the tangent cone $C_{\mathcal{E}p(v)}(x,V(x))$ to the epigraph of V at a point $(x,V(x))$ (where $x \in$ Dom V) is the epigraph of a function denoted $C_+V(x)$ and called the epiderivative of V (see Aubin-Ekeland [1984] Definition 7.3.7, p.421). When V is Lipschitz around x, we obtain for all $v \in X$

$$(3.30) \qquad C_+V(x)(v) = \limsup_{\substack{h \to 0+ \\ y \to x}} \frac{V(y+hv) - V(y)}{h} \in \mathbb{R}$$

<u>Proposition 3.4</u>

Let X and Y be finite dimensional spaces and F be a single-valued map from Dom(F) $\subset$ X to Y with closed graph, $V: X \to \mathbb{R} \cup \{+\infty\}$ and $W: Y \to \mathbb{R} \cup \{+\infty\}$ two lower semicontinuous proper functions. Let $x_o \in \text{Dom } V \cap F^{-1} \text{ Dom } W$ satisfy the transversality condition:

$$(3.31) \qquad CF(x_o)(\text{Dom } C_+V(x_o)) - \text{Dom } C_+W(x_o) = Y$$

Then

$$(3.32) \qquad C_+(V+WF)(x_o)(u) \leq C_+V(x_o)(u) + C_+W(F(x_o))(CF(x_o)(u))$$

in the sense that

$$(3.33) \qquad \begin{cases} \forall v \in CF(x_o)(u), C_+(V+WF)(x_o)(u) \leq C_+V(x_o)(u) \\ + C_+W(F(x_o))(v). \end{cases}$$

<u>Proof</u>

We consider the map G from X x ℝ x Y x ℝ x ℝ to Y x ℝ defined by

$$(3.34) \qquad G(x,a,y,b,c) = (F(x)-y, a+b-c)$$

■

We observe that the epigraph of V + WF is the image under the application $(x,a,y,b,c) \to (x,c)$ of the subset

$$(3.35) \qquad Q := (\mathcal{E}p(V) \times \mathcal{E}p(W) \times \mathbb{R}) \cap G^{-1}(0,0)$$

It is easy to check that assumption (3.31) implies that

$$CG(x_o, V(x_o), F(x_o), W(F(x_o)), V(x_o) + W(F(x_o)))$$

maps $\quad C_{\mathcal{E}p(V)}(x_o, V(x_o)) \times C_{\mathcal{E}p(W)}(F(x_o), W(F(x_o)) \times \mathbb{R}$

onto Y x ℝ. Hence Corollary 3.3 implies that the set of elements $(u,\lambda) \in C_{\mathcal{E}p(V)}(x_o, V(x_o))$, $(v,\mu) \in C_{\mathcal{E}p(W)}(F(x_o), W(F(x_o))$ and $\nu \in \mathbb{R}$ such that $CG(x_o, V(x_o), F(x_o), W(F(x_o)), V(x_o) + W(F(x_o))$ maps (u,λ,v,μ,ν) onto $(0,0)$ are contained in the tangent cone to Q at $(x_o, V(x_o), F(x_o), W(F(x_o)), V(x_o) + W(F(x_o)))$. Hence $v \in CF(x_o)(u)$, $\lambda \geq C_+V(x_o)(u)$, $\mu \geq C_+W(F(x_o))(v)$ and $\nu = \lambda + \mu$ and (u,λ,v,μ,ν) belongs to $C_Q(x_o, V(x_o), F(x_o), W(F(x_o)), (V+WF)(x_o))$. This implies that $\nu \geq C_+(V+WF)(x_o)(u)$.

4. Applications to Local Controllability

Let us consider a bounded set-valued map F from a closed subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^n$ with closed graph and convex values, satisfying

(4.1) $\qquad \forall x \in K,\ F(x) \cap T_K(x) \neq \emptyset$

By Haddad's Theorem, we know that for all $\xi \in K$, the subset $S_T(\xi)$ of viable solutions (3) to the differential inclusion

(4.2) $\qquad x'(t) \in F(x(t)),\ x(0) = \xi$

is non-empty and closed in $C(0,T;\mathbb{R}^n)$ for all $\xi \in K$.

Let $R(T,\xi) := \{x(T) \mid x \in S_T(\xi)\}$ be the reachable set and $M \subset R^n$ the target, be a closed subset. We shall say that the system is locally controllable if

(4.3) $\qquad 0 \in \text{Int}\ (R(T,\xi) - M).$

This means that there exists a neighborhood U of 0 in $\mathbb{R}^n$ such that, for all $u \in U$, there exists a solution $x(\cdot) \in S_T(\xi)$ such that $x(T) \in M + u$. We denote by $\mathcal{K} \subset S_T(\xi)$ the subset of solutions $x \in S_T(\xi)$ such that $x(T) \in M$. We denote by $C_{S_T(\xi)}(x)(T)$ the convex cone of elements $v(T)$ when v ranges over the tangent cone $C_{S_T(\xi)}(x)$ to $S_T(\xi)$ at $x(\cdot)$.

We refer to Frankowska [1984], [to appear] a) and b) for the characterization of subspaces of $C_{S_T(\xi)}(x)$ in terms of solutions to a "linearized inclusion" around the trajectory $x(\cdot)$.

Theorem 4.1

Let $x_o \in \mathcal{K}$ be a trajectory of (5.2) reaching M at time T. Assume that

(4.4) $\qquad C_{S_T(\xi)}(x_o) - C_M(x_o(T)) = \mathbb{R}^n$

Then the system is locally controllable and there exists a neighborhood U of x_o and a constant $\ell > 0$ such that, for any solution $x \in S_T(\xi)$ in U,

$$(4.5) \qquad d(x(\cdot),K) \leq \ell \, d_M(x(T))$$

Furthermore,

$$(4.6) \qquad \{v \in C_{S_T(\xi)}(x_o) \mid v(T) \in C_M(x_o(T))\} \subset C_K(x_o).$$ ▲

<u>Proof</u>

We apply Theorem 2.1 to the continuous linear map A from $C(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ to $\mathbb{R}^n$ defined by $A(x,y) := x(T) - y$, to the subset $S_T(\xi) \times M$, at $(x_o,x_o(T)) \in S_T(\xi) \times M$. We observe that $A(x_o,x_o(T)) = 0$ and that condition (4.4) can be written

$$(4.7) \qquad A\, C_{S_T(\xi)}(x_o) - C_M(x_o(T)) = \mathbb{R}^n$$

Hence 0 belongs to the interior of $A(S_T(\xi) \times M) = R(T,\xi) - M$ and there exist constants $r > 0$ and $\ell > 0$ such that $u \mapsto A^{-1}(u) \cap (S_T(\xi) \times M)$ is pseudo-Lipschitz around $(0,x_o,x_o(T))$. Let us consider now a ball U of center x_o and radius r. Let us take a solution $x \in S_T(\xi) \cap U$ of the inclusion (4.2) so that $d_M(x(T)) < \|x(T) - x_o(T)\| \leq r$. Let y belong to $\pi_M(x_o(T))$. Then $\|A(x,y)\| = d_M(x(T))$ and we deduce from the fact that $u \mapsto A^{-1}(u) \cap (S_T(\xi) \times M)$ is pseudo-Lipschitz that there exists $\tilde{x}$ such that $A(\tilde{x},\tilde{x}(T)) = 0$ (i.e., an element $\tilde{x} \in K$) such that $d(x,K) \leq \|x-\tilde{x}\| \leq \ell\|0-A(x,y)\| = \ell d_M(x(T))$. Inclusion (4.6) follows from inequality (4.5), as in the proof of Theorem 3.2. ▲

NOTES

1) The derivative of F at a point (x_o,y_o) of its graph is the set-valued map $CF(x_o,y_o)$ from X to Y whose graph is the tangent cone $C_{Graph(F)}(x_o,y_o)$ to its graph at (x_o,y_o); it is a "closed convex process" (a map whose graph is a closed convex cone), which is the "set-valued" analogue to a continuous linear operator.

2) We say that a set-valued map H from X to Y is lower semi-continuous at x_o if for any $y_o \in H(x_o)$ and any neighborhood V of y_o, there exists a neighborhood U of the x_o such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

3) A trajectory $t \to x(t)$ is viable if, for all $t \in [0,T]$, $x(t) \in K$.

REFERENCES

Aubin, J-P.

[1980] Further properties of Lagrange multipliers in non-smooth optimization. Appl. Math. Opt. 6, 79-90.

[1981] Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. In Mathematical Analysis and Applications Advances in Mathematics, Vol.7A, 159-229.

[1982] Comportement Lipschitzien des solutions de problèmes de minimisation convexes. CRAS 295, 235-238.

[1984] Lipschitz behavior of solutions to convex minimization problems. Math. Op. Res 9, 87-111.

Aubin, J-P. and F.H. Clarke

[1977] Multiplicateurs de Lagrange en optimisation non convexe et applications. CRAS 285, 451-454.

Aubin, J-P. and I. Ekeland

[1984] Applied Nonlinear Analysis, Wiley Interscience, New York.

Clarke, F.H.

[1975] Generalized gradients and applications. Trans. Am. Math. Soc. 205, 247-262.

[1976] a) A new approach to Lagrange multipliers. Math. Op. Rex. 1, 165-174.

[1976] b) On the Inverse Function Theorem. Pacific J. Math. 64, 97-102.

[1983] Optimization and non-smooth analysis, Wiley Interscience, New York.

Cornet, B. and G. Laroque

[To appear] Lipschitz properties of constrained demand functions and constrained maximizers.

Ekeland, I.

[1974] On the Variational Principle. J. Math. Anal. Appl. 47, 324-353.

Frankowska, H.

[1983] Inclusions adjointes associées aux trajectories minimales d'inclusions differentielles. CRAS 297, 461-464.

[1984] Contrôlabilité locale et proprietés des semi-groupes de correspondances. CRAS.

[To appear]a) The adjoint differential inclusion associated to a minimal trajectory of a differential inclusion. Annales de l'Institut Henri Poincaré, Analyse nonlinéaire

[To appear]b) Local controllability and infinitesimal generators of semi-groups of set-valued maps.

Hiriart-Urruty, J.B.

[1979] Tangent cones, generalized gradients and mathematical programming in Banach spaces. Math. Op. Res. 4, 79-97.

Ioffe, A.E.

[1979] Différentielles généralisées d'applications localement Lipschitziennes d'un espace de Banach dans un autre. CRAS 289, 637-640.

[1981] Non-smooth analysis: differential calculus of non-differential mappings. Trans. Am. Math. Soc. 266, 1-56.

[1982] Non-smooth analysis and the theory of Fans. In Convex Analysis and Applications. J-P Aubin and R.B. Vinter (eds). Pitman.

Lebourg, G.

[To appear] Thesis.

Robinson, S.M.

[1975] Stability theory for systems of inequalities, Part I: Linear systems. SIAM Journal on Num. Anal. 12, 754-769.

[1976] Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems. SIAM Journal on Num. Anal. 13, 497-513.

[1979] Generalized equations and their solutions, Part I: Basic Theory. Math. Prog. Study 10, 128-141.

[1980] Strongly regular generalized equations. Math. of O.R. 5, 43-62.

[1982] Generalized equations and their solutions, Part II: Applications to nonlinear programming. Math. Programming Study, 19, 200-221.

Rockafellar, R.T.

[1970] Convex Analysis. Princeton University Press.

[1978] La théorie des sous-gradients et ses applications à l'optimisation. Presses de l'Université de Montréal. English translation: Helderman Verlag Berlin [1982].

[1979]a) Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$. Nonlinear Analysis 3, 145-154.

[1979]b) Directionally Lipschitzian functions and subdifferential calculus. Proc. London Math. Soc. 3, 331-356.

[1980] Generalized directional derivatives and subgradients of non-convex functions. Can. J. Math. 2, 257-280.

[To appear] a) Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming. Math. Prog. Study No. 5. R. Wets (ed).

b) Extensions of subgradient calculus with applications to optimization.

c) Maximal monotone relations and the generalized second derivatives of non-smooth functions.

d) Lipschitzian properties of multifunctions.

Ursescu, G.

[1975] Multifunctions with closed convex graph. Czechs. Math. Journal 25, 438-49.

Wets, R.

[1973] On inf-compact mathematical programs. Fifth Conference on Optimization Techniques. Springer-Verlag, 426-436.