SOME LOCATION AND PRICE EQUILIBRIA IN FACILITY INVESTMENT WITH UNCERTAIN DEMAND

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FOREWORD

The following paper contributes to the Regional Issues Project on metropolitan change by providing new insights into the process of private location in an urban system. The authors describe a three-level game with facility customers, facility managers, and facility developers/owners as agents. The paper demonstrates the existence of a simultaneous non-cooperative equilibrium among facility managers/firms and among facility developers, for a given equilibrium behavior of customers. The three levels of the game are designed to reflect variations in speed of adjustment between the three types of agents.

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ABSTRACT

This paper describes the spatial distribution of customer demand, supply of customer services, and facility investment as the outcome of a three-level game-like interaction between customers (e.g., shoppers), suppliers (e.g., retailers) and developers (e.g., landlords). Suppliers in each center are assumed to compete with suppliers in all other centers. Similarly, the developers of each center compete with developers of all other centers. With this specification, multi-center equilibria of the Nash type are examined for suppliers on the one hand, and for developers on the other. Suppliers in a center decide about utilized floorspace and price level in the center. Developers of a center decide about available floor-space and rent level.

The uncertain customer demand is specified in probabilistic terms, representing the suppliers' and developers' perception of customer behavior. An approach to estimate customer response patterns is presented and discussed.

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1. INTRODUCTION

Much of the work on facility location in the operations research field has been confined to cases where the objectives are deterministic. In this paper we are investigating the decision problem of facility operators (suppliers) and facility developers who perceive a probabilistic component in the choice of alternative facilities by customers or users. The users are assumed to make selections according to individual trade-offs between accessibility criteria and the intrinsic advantages of the facilities themselves, usually leading to non-linear relations at the aggregate level.

The analysis focuses on location of and investments in private facilities, demand for floorspace by profit-motivated operators, and customers' demand for services supplied by facility operators. As a consequence we identify different objectives for each of these three categories of agents and examine corresponding supply/demand equilibria. Previous work on this problem is marked by Lakshmanan and Hansen (1965), and Harris and Wilson (1978).

From the viewpoint of a planning authority, different criteria apply for the location of public facilities (Leonardi, 1981a, b) as compared with private facilities (Roy and Johansson, 1981). In the latter case the authorities must contemplate the competition between developers (owners) as well as between the operators of the facilities. We examine this problem by studying the existence and nature of non-cooperative (Nash) equilibria for each of these categories of competitors.

Our analysis uses a leader-follower chain between developers and operators as well as between operators and customers. Special emphasis is devoted to customer behavior and estimation procedures to capture how customers adapt to changes in facility location, transportation costs, pricing policy, etc.

1.1 Perceived and Estimated Behavior of Customers

In the model framework presented, customers are demanding services from the facility operators whom we call suppliers; the latter are demanding floorspace which is supplied by developers. The customers are assumed to make their decisions without contemplating the effects their behavior may have on the decision-making of the suppliers and developers. The behavior of customers is estimated by means of a facility choice model based on information theory. The estimation procedure is described in section 4. It attempts to distinguish and identify "quantity" and "quality" components of attractiveness at facilities in each center. In addition, these components are functionally separated from the "macro" accessibility influences between zones of origin and destination (Roy, 1983, Roy and Lesse, 1983b).

The estimated customer model is assumed to reflect the way suppliers perceive the behavior of customers. An important feature of the customer model is a separation between (i) overall destination probabilities and (ii) the conditional probability of customers' zone of origin and income group, given each zone of destination.

1.2 A Two-Level Oligopoly Structure

The model is specified in such a way that the suppliers in each center have two decision variables; they select the size of floorspace and a price level for the goods and/or services they supply. We assume that in each given centre the suppliers are maximizing the profits in the center, given the decision made by suppliers in other centers and given their perception of how customers respond to their decisions. With these assumptions we have specified a non-cooperative competition between centers, and for this game we examine the existence of Nash equilibria, which we define and characterize contingent on prevailing perceptions (Johansson, 1978), carefully attempting to distinguish between the rules of the game and the description of how it is played (Shubik, 1959).

The suppliers anticipate the mode of reaction of customers but take the decisions of developers as given, implying customers are followers vis-a-vis suppliers, and the latter are followers in relation to developers. This means that the perceived customer behavior is embedded in the profit function of suppliers, and the perceived behavior of the latter is embedded in the objective functions of the developers (i.e., Stackelberg analysis).[1)]

Two decision variables are assigned to the developers; in each center a developer decides about the size of available floor-space and the rent level in the center. The size of a center is increased by means of investment. The character and "quality" of the infrastructure in each center is prespecified. Hence, new investments in a given center can increase the amount of infrastructure capital in the zone but not change its character.

Within the setting outlined, it becomes essential for the planning authority to evaluate states of the system which constitute simultaneous equilibria for all three categories of actors.

2. MULTI-CENTER NASH EQUILIBRIA FOR SUPPLIERS

We shall study a system with M customers who visit supplier centers in which goods and/or services are supplied from facilities located in the centers, indexed by j. The customers' zone of origin is indexed by i and their income group by k. In order to describe the customer behavior we must introduce the suppliers' decision variables:

[1)] To some extent this reflects a hierarchy in which each adjustment process is embedded in a relatively seen slower process of change.

$$W_j = \text{amount of floorspace utilized by suppliers in center } j$$
$$y_j = \text{(average) price level selected in center } j \qquad (1)$$

In addition we introduce an exogenously given factor

$$f_j = \text{measure of local attractiveness reflecting convenience and comfort in center } j \qquad (2)$$

2.1 Customer Behavior

The total number of customers is M, and the proportion of customers in origin zone i belonging to income group k is O_{ik}. The following (estimated) parameters are used to describe the behavior of customers:

- c_{ijk} and t_{ijk} = average travel cost and time respectively between zone i and center j for customer category k
- a_k = relative "quantity" of the commodities and/or services purchased by income group k (compared with that of the lowest income group)
- ψ = parameter describing customers' price and travel cost sensitivity
- β = parameter describing customers' time sensitivity
- λ_{ik} = parameter reflecting the origin constraint in the estimated customer model

Since we are only studying a sector of the whole economy we are not interested in the absolute price levels, but the relation between prices in the different centers, and the trade-off between price level and travel costs. Therefore, the following average budget relation is used:

$$b = \sum_{i,j,k} p_{ijk}(a_k y_j + c_{ijk})$$

where p_{ijk} are the purchasing probabilities given in (4) below.

The behavior of customers is described by probabilities p_{ijk}, showing the likelihood of customers of type k and origin i visiting center j. Using Bayes' formula this probability is decomposed as follows in this study (see section 4):

$$p_{ijk} = p_{ik/j} p_j \tag{4}$$

where $p_{ik/j}$ denotes the conditional probability of customers in center j coming from zone i and belonging to income group k. We express this conditional probability as

$$p_{ik/j} = f_{ijk} / \sum_{ik} f_{ijk} \tag{5}$$

where f_{ijk} can be expressed in terms of the variables and parameters introduced in (1) and (3)

$$f_{ijk} = \exp \{-\lambda_{ik} - \psi(a_k y_j + c_{ijk}) - \beta t_{ijk}\} \tag{6}$$

By p_j we denote the overall destination probability which is estimated as (see section 4):

$$p_j = \frac{W_j^{\alpha} f_j [\sum_{ik} f_{ijk}]^{\theta}}{\sum_j W_j^{\alpha} f_j [\sum_{ik} f_{ijk}]^{\theta}} \tag{7}$$

where θ and α are estimated parameters, and where α reflects how the scale of a center affects the overall destination attraction.

The probabilities in (5) and (7) are assumed to reflect the suppliers' information about customer behavior. More specifically we make the following assumption about the suppliers' perception:

> Retailers evaluate their decisions about floor-space size, W_j, and price, y_j, with the perception that the conditional probabilities $p_{ik/j}$ remain fixed. (A.1)

We may observe that a change, Δy_j, in the price brings about a change $\exp\{-\psi a_k \Delta y_j\}$ in each f_{ijk}-term. This means that the numerator and denominator in (5) will change approximately proportionally only if a_k does not vary too much or if the $p_{ik/j}$-distribution is very peaked.

2.2 Existence and Character of Nash Equilibrium

The decision problem of suppliers is conceived as a competition among centers. We assume that the supplier profits at each center j are maximized contingent on the decisions in all other centers. Let the profit function of center j be

$$\pi_j W_j = M \sum_{ik} p_{ijk} a_k (y_j - y_j^*) - (r_j + w_j) W_j \tag{8}$$

where w_j are costs proportional to the floorspace[1] and y_j^* costs proportional to the sales volume, and where π_j and r_j are profit and rent per floorspace, respectively. M, w_j and y_j^* enter as exogenous parameters, W_j and y_j are decision variables, and r_j is assumed to be fixed by developers.

Proposition 1: Identify each zone as a decision-making unit selecting W_j from a closed interval on the real line. Let for each zone the objective function be given by (8), contingent on (A.1) and let $\alpha < 1$. Then for given rent levels and prices $y_j > y_j^*$, there exists a unique Nash equilibrium $\{\hat{W}_j\}$.

Proof: A unique Nash equilibrium exists if (i) the decision sets are compact and convex, (ii) the profit function is continuous in all variables, and (iii) strictly concave in W_j (e.g., Berge, 1957). Properties (i) and (ii) are obviously satisfied. Property (iii) can be demonstrated with the help of the first and second order derivatives

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[1] In order to simplify the algebra, we frequently set $w_j = 0$.

$$\frac{\partial(\pi_j W_j)}{\partial W_j} = \alpha W_j^{-1}(p_j - p_j^2)K_j - (r_j + w_j) \tag{9}$$

$$\frac{\partial^2(\pi_j W_j)}{\partial W_j^2} = \alpha W_j^{-2} K_j (p_j - p_j^2)[\alpha(1-2p_j) - 1] \tag{10}$$

where $K_j = M\bar{a}_j(y_j - y_j^*)$, and $\bar{a}_j = \Sigma_{ik} p_{ik/j} a_k$. From (10) it follows that the second order derivative is negative for $\alpha \leq 1$, and the solution obtains for

$$\alpha(\hat{p}_j - \hat{p}_j^2)K_j = (r_j + w_j)\hat{W}_j \tag{11}$$

The situation described in Proposition 1 is illustrated by Figure 1, Case a.

When $\alpha > 1$ the function $p_j = p_j(W_j)$ is quasi-concave in W_j but the profit function is not. We observe from (10) that for $\alpha > 1$, $p_j(W_j)$ has an inflection point at

$$W_j^\alpha = B_j(\alpha-1)/f_j[\Sigma f_{ijk}]^\theta(\alpha+1)$$

where $B_j = \Sigma_{k \neq j} f_k W_k^\alpha [\Sigma fijk]^\theta$ and where f_j is defined in (2). This motivates the following proposition which is illustrated by Case b in Figure 1:

> *Proposition 2:* Let the assumptions in Proposition 1 be retained, and consider the profit function $\tilde{\pi}_j W_j = M\Sigma p_{ijk} a_k (y_j - y_j^*)(1-\rho_j)$, where $\rho_j < 1$ represents the share taken for rents and $w_j = 0$. With this function there exists a Nash equilibrium, also for $\alpha > 1$.

Outline of a Proof: We shall not show that the equilibrium is unique. Hence, instead of strict concavity (as in Proposition 1) we shall only require quasi-concavity of each profit function (Berge, 1957). $\tilde{\pi}_j W_j$ is obviously positive and monotonically increasing, with the first order derivative

$$\alpha W_j^{-1}(1-\rho_j)(p_j-p_j^2)K_j > 0$$

The function is a fraction of $K_j p_j(W_j)$ and has a similar inflection point as $p_j(W_j)$. To the left of this point it is positive, increasing and convex. To the right it is increasing and concave. Evidently, like $p_j(W_j)$ it is quasi-concave.

In order to establish the existence of a Nash equilibrium for the price decisions we need a compact (closed and bounded) decision space for each zone. As a lower bound we can select $y_j \geq y_j^*$. We can also prevent y_j from getting arbitrarily large, since $p_j \to 0$ as $y_j \to \infty$ at a faster rate than that of y_j itself.

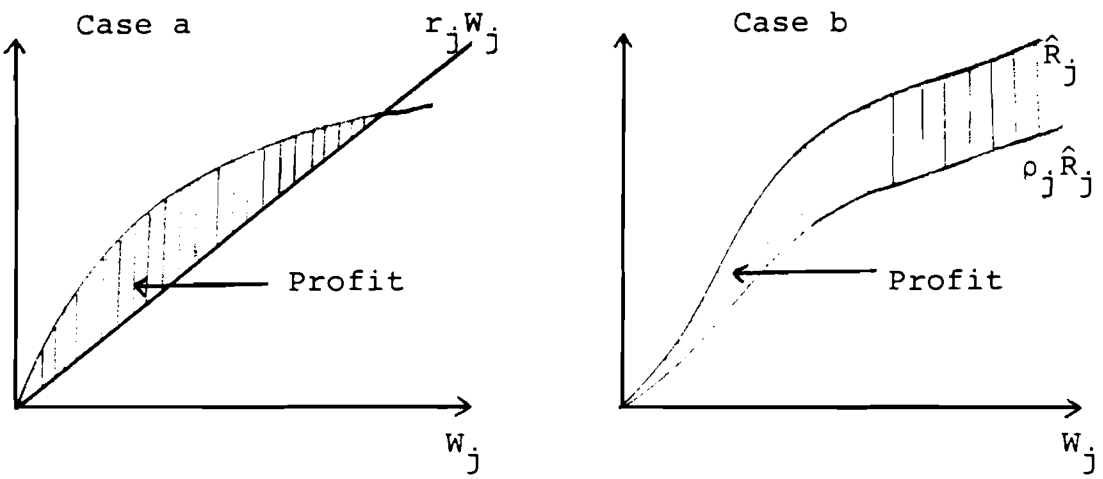

Figure 1. Illustration of Proposition 1 (Case a) and Proposition 2 (Case b).

Remark: $\hat{R}_j = M\Sigma p_{ijk} a_k (y_j - y_j^*)$, $w_j = 0$.

Proposition 3: Identify each zone as a decision-making unit selecting y_j from a compact set, and let the profit function be $\pi_j W_j$ in (8), given the assumption in (A.1). Then for given sizes W_j and given rents there exist an n-tuple (for n zones) $\{\hat{y}_j\}$ which constitutes a Nash equilibrium such that $\hat{y}_j = y_j^* + 1/\bar{a}_j \psi\theta(1-p_j)$

Proof: The profit function is continuous and the decision set compact. Then it remains to show that the profit function is quasi-concave. We shall do this by examining the first and second order derivatives with respect to y_j. Let them be denoted by Π_j' and Π_j'' respectively

$$\Pi_j' = -M\bar{a}_j^2(y_j-y_j^*)\psi\theta p_j(1-p_j) + Mp_j\bar{a}_j \tag{12}$$

$$\Pi_j'' = -\psi\theta\bar{a}_j p_j(1-p_j)\{2\bar{a}_j-(1-2p_j)\psi\theta\bar{a}_j^2(y_j-y_j^*)\} \tag{13}$$

where $\bar{a}_j = \Sigma_{ik}p_{ik/j}a_k$. For $(y_j-y_j^*)$ small $\Pi_j' > 0$ and $\Pi_j'' < 0$. Profits increase until $y_j = \hat{y}_j$. At this point $\Pi_j' = 0$ as seen by (12). By inserting $\hat{y}_j$ in (13) we can see that $\Pi_j'' < 0$ at this point. Since $(\hat{y}_j-y_j^*)\psi\theta(1-p_j)\bar{a}_j - 1 = 0$, and $\partial p_j/\partial y_j < 0$, it follows that $\Pi_j' < 0$ for $y_j > \hat{y}_j$. Observing that Π_j' is continuous, those facts imply that $\pi_j W_j$ is quasi-concave.

From Proposition 3 we have that $(\hat{y}_j-y_j^*) = 1/\bar{a}_j\psi\theta(1-p_j)$. Combining this with formula (11) one obtains the following relation between the rent level and the equilibrium solution for the choice of floorspace:

$$\begin{aligned} r_j &= \alpha\hat{p}_j M/\hat{W}_j\psi\theta \quad \text{for } w_j = 0, \text{ and} \\ \hat{W}_j &= \alpha\hat{p}_j M/[\psi\theta(r_j+w_j)] \text{ for } w_j > 0 \end{aligned} \tag{14}$$

In a dynamic context one may assume that prices can be adjusted almost instantaneously as new information becomes available while decisions about location and size cannot be adjusted at the same speed. In such cases price decisions will be based on more accurate information than decisions about siting. These observations motivate Remark 1.

Remark 1: Let all assumptions in Proposition 3 hold, except (A.1). Add the assumption that retailers also perceive the effect of price changes on the conditional probabilities $p_{ik/j}$. Then the first order derivative Π_j' becomes

$$\Pi_j' = -M(y_j-y_j^*)\psi[\bar{a}_j^2\theta p_j(1-p_j) + p_j m_2(\bar{a}_j) + Mp_j\bar{a}_j$$

where $m_2(\bar{a}_j) = \Sigma_{ik}a_k^2 p_{ik/j} - \bar{a}_j^2$ is the variance of the average quantity of goods purchased per customer at j. Also in this case, the objective function exhibits quasi-concavity.

The equilibrium price changes to be

$$y_j = y_j^* + 1/[\psi\theta\bar{a}_j(1-p_j) + \psi m_2(\bar{a}_j)/\bar{a}_j] \tag{15}$$

which will also imply a change of the rent level in formula (14).

2.3 Monopoly and Collapse of the Spatial Structure

The oligopolistic setting in the preceding subsection turns out to be essential for preserving the multi-center structure of solutions. If a single decision-maker controls all suppliers, the model generates a monopoly solution which utilizes only one center.

Assume that a monopoly has the objective to maximize the sum of profits emanating from all centers, subject to spatial constraints $W_j \leq \bar{Z}_j$. The Lagrange function corresponding to this problem is (for $w_j = 0$)

$$L = \sum_j (R_j - r_j)W_j - \sum_j \lambda_j (W_j - \bar{Z}_j) \tag{16}$$

where $\quad R_j = A_j F_j W_j^{\alpha-1}/\Sigma F_k W_k^{\alpha}$, $A_j = M\Sigma p_{ik/j} a_k (y_j - y_j^*)$,

and $\quad F_j = f_j[\Sigma f_{ijk}]^{\theta}$

The standard optimum conditions are

$$\frac{\partial L}{\partial W_j} = \frac{\alpha F_j W_j^{\alpha-1}}{\Sigma F_j W_j^{\alpha}} [A_j - \bar{A}] - r_j - \lambda_j \leq 0$$

$$\frac{\partial L}{\partial W_j} W_j = 0 \tag{17}$$

where $\bar{A} = \Sigma_k A_k F_k W_k^{\alpha}/\Sigma_k F_k W_k^{\alpha}$ has the form of an arithmetic mean.

Proposition 4: Let there be one decision-maker who maximizes the total profits over all centers as specified in (16). Moreover, let rent levels $r_j > 0$ and prices $y_j > y_j^*$ be given. Then, for non-identical centers, the maximum is obtained by selecting only one center.

Outline of a Proof: Observe that $\bar{A}$ in (17) is a weighted mean. Hence, for at least one center k we have $A_k \leq \bar{A}$. Since $r_j+\lambda_j>0$ this implies according to (17) that $W_k=0$. Having observed that $W_k=0$, we can apply the same argument for still another center, and continue to eliminate centers till only one is left. For this center the profit is $A_jW_j-r_jW_j$.

> *Remark 2:* The result in Proposition 4 can be prevented if we introduce congestion effects or a simple density constraint of the following type $p_jM/W_j \leq d$ for each j.

The statement of Remark 2 follows directly from inspection of the associate Lagrangean which becomes for $w_j=0$

$$L = \Sigma(R_j-r_j)W_j - \Sigma\lambda_j(W_j-\bar{Z}_j) - \Sigma\gamma_j(p_jM/W_j-d)$$

which yields the optimum condition

$$\frac{\partial L}{\partial W_j} = Mp_jW_j^{-1}\{\alpha(A_j-\bar{A})/M-\gamma_j(\alpha-1)W_j^{-1} +$$

$$+ \alpha\Sigma_k\gamma_kp_kW_k^{-1}\} - r_j - \lambda_j = 0$$

Remark 2 also reflects the fact that a customer density relation is lacking in the oligopoly situation. However, in that case the non-cooperative setting is enough to preserve the multi-center structure.

3. DECISION PROBLEM OF DEVELOPERS

For a system with many suppliers one might consider modeling the developers' perception of supplier behavior in probabilistic terms in analogy with the way suppliers are assumed to perceive costumer behavior in section 2. Instead we shall just illustrate the nature of the developers' decision problem in this section. We shall do this in two stages. First we consider the short term problem for which the available floorspace, $Z_j = \bar{Z}_j$, is fixed so that the rent level is the only decision variable. In a second step we allow for investments in new floorspace and associated infrastructure.

Let the profit function which summarizes the behavior of developers in center j be

$$g_j = r_j W_j - \mathcal{F}_j(Z_j - \bar{Z}_j) \tag{18}$$

where g_j denotes the profit, $\bar{Z}_j$ the already existing floorspace, and $Z_j - \bar{Z}_j$ the additional floorspace obtained through investment. By $\mathcal{F}_j$ we denote the investment costs transformed to cost per the same time unit as the one for which the profit is calculated. We do not consider neither operation and maintenance cost nor the sunk capital costs.

3.1 Short term selection of rent level

In the short term we put $Z_j - \bar{Z}_j = 0$, and assume that $\mathcal{F}_j(0) = 0$. Then assume that developers have the following perception

Suppliers perceive that (A.1) holds both with regard to W_j-decisions and y_j-decisions. (A.2)

From (A.2) and (14) we may write for $W_j \leq \bar{Z}_j$, and $w_j=0$

$$r_j W_j + r_j W_j^{1-\alpha} B_j/F_j = \alpha M/\psi\theta \quad , \tag{14'}$$

where F_j is defined in (16) and $B_j = \sum_{k\neq j} F_k W_k^{\alpha}$.

$$\frac{-dW_j}{dr_j}\left(1 + \frac{B_j}{F_j W_j^{\alpha}} - \alpha\frac{B_j}{F_j W_j^{\alpha}}\right) = \frac{W_j}{r_j}\left(1 + \frac{B_j}{F_j W_j^{\alpha}}\right)$$

which shows that the foorspace elasticity with respect to the rent level is larger than unity. This implies that a falling r_j is coupled with an increasing W_j, and W_j increases fast enough for falling r_j to bring about a total raise in profits. Hence, we may conclude

Remark 5: Given assumption (A.2), the profit maximizing rent level, for fixed $\bar{Z}_j$ and $\alpha > 0$, is from (14')

$$\hat{r}_j = F_j M/\Psi\theta [F_j \bar{Z}_j + \bar{Z}_j^{1-\alpha} B_j]$$

Observe that for rent levels below $\hat{r}_j$ in Remark 5 the profit function is $g_j = r_j \bar{Z}_j$, and for $r_j > \hat{r}_j$, g_j is monotonously falling. Hence, the developers' profit function is quasi-concave and continuous in r_j. From (14') follows that if there is a minimum W_j-value for a center (a shop), $r_j \geq 0$ will belong to a compact set. This gives us the following proposition.

Proposition 5: Let developers' perception be formed according to (A.2), let the profit function be given by (18), let $W_j \leq \bar{Z}_j$ and set $w_j = 0$. Then there exists a Nash equilibrium of rent levels for the multi-centre system.

Consider now the case when $w_j > 0$ and use (A.2) and (14) to obtain

$$(r_j + w_j) W_j [1 + B_j / F_j W_j^{\alpha}] = M\alpha/\psi\theta \qquad (14'')$$

where F_j and B_j are defined as in (14'). We observe that (14) is based on (11) and write $W_j = \hat{W}_j$. Differentiating (14'') yields

$$-\frac{dW_j}{dr_j} [1 + (1-\alpha) X_j] = \hat{W}_j / \hat{p}_j (w_j + r_j)$$

where $X_j = B_j / F_j \hat{W}_j^{\alpha}$, and thus $1 + X_j = 1/\hat{p}_j$. Observing in (18) that $dg_j/dr_j = \hat{W}_j + r_j (d\hat{W}_j/dr_j)$, we can express the first and second order derivatives of (18) as

$$\begin{aligned} g_j' &= \hat{W}_j \{1 - r_j (1 + X_j)/(w_j + r_j) G_j\} \\ g_j'' &= \frac{-\hat{W}_j \{2w_j + r_j \alpha (1 + X_j) X_j (\alpha - 1)/G_j^2\}}{(W_j + r_j)^2 G_j} \end{aligned} \qquad (19)$$

where $G_j = 1 + (1-\alpha)X_j$. From (19) g_j is positive and growing for r_j small. The function has an extremum for

$$\hat{r}_j = (w_j/\alpha)[\hat{p}_j/(1-\hat{p}_j) + (1-\alpha)] \tag{20}$$

Inserting $\hat{r}_j$ in the expression for g_j'' yields a negative value if $G_j > 0$ which is true for $\alpha < 1$. Beyond the point $\hat{r}_j$ the function is monotonically decreasing with an inflexion point at $\tilde{r}_j = [2G_j/\{(1-\alpha)(1+X_j)\}]\hat{r}_j$ if $\alpha < 1$, (see Figure 2).

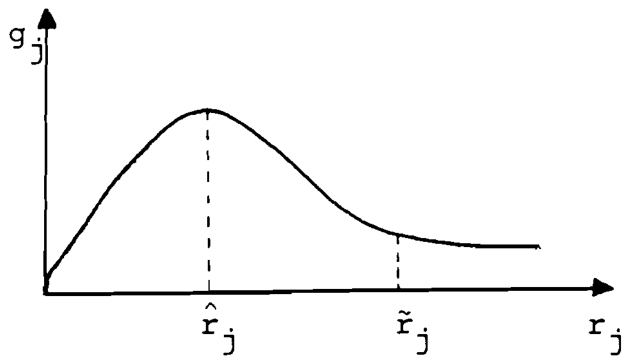

Figure 2. Developer income flow for short-term rent setting ($Z_j \leq \bar{Z}_j$)

Remark 6: Consider the same assumptions as in proposition 5 with the exception that $w_j \geq 0$. Assume in addition that $0 \leq \alpha < 1$. Then there exists a Nash equilibrium of rent levels for the multi-center system.

Proof: As stated above, $\alpha < 1$ ensures that g_j has the form illustrated in Figure 3. Such a function is quasi-concave, since it is monotonically increasing to the left of $\hat{r}_j$ and monotonically decreasing to the right. This establishes the statement in the remark.

We may finally observe that the floorspace solution adhering to (20) is

$$\hat{W}_j = M\alpha^2\hat{p}_j(1-\hat{p}_j)/\psi\theta w_j \tag{21}$$

3.2 Deciding About Floorspace in Centers

In this subsection we continue to examine the profit function in (18), and now we allow for investments which may increase the available floorspace. We do this by expressing $r_j W_j$ only in terms of floorspace by using (14) which is based on assumption (A.2). The profit function may in this case be written as

$$g_j = R_j(W_j) - F_j(Z - \bar{Z}_j) \qquad (18')$$

where
$$Z_j = \max\{W_j, \bar{Z}_j\} \text{ and}$$
$$R_j = \alpha \hat{p}_j M/\psi\theta \text{ for } w_j=0$$
$$R_j = \alpha M/\psi\theta(1+X_j) - w_j \hat{W}_j \text{ for } w_j>0$$

according to (14), (14'), and (14"). We first make the following observations

Remark 7: For $\alpha < 1$ R_j is concave both when $w_j=0$ and $w_j > 0$.[1)]

Proof: When $w_j=0$ we only have to observe from (7) that $p_j(W_j)$ is increasing and concave. Hence, R_j has the same properties. For $w_j>0$ we calculate $\partial R_j/\partial W_j = R_j'$ and $\partial^2 R_j/\partial W_j^2 = R_j''$ as follows

$$R_j' = \alpha^2 MX_j/\psi\theta W_j(1+X_j)^2 - w_j$$
$$R_j'' = \frac{-\alpha^2 MX_j[1+\alpha + (1-\alpha)X_j]}{\psi\theta(1+X_j)^3 W_j^2} < 0$$

Observing from (21) that $\hat{W}_j = \alpha^2 MX_j/w_j\psi\theta(1+X_j)^2$ we can see that $W_j < \hat{W}_j$ implies $R_j' > 0$, $W_j = \hat{W}_j$ implies $R_j' = 0$ and $W_j > \hat{W}_j$ implies $R_j' < 0$. Hence, R_j is a concave function.

With regard to investment cost function $F_j(Z_j-\bar{Z}_j)$ we can write $F_j(W_j-\bar{Z}_j)$ if we set $F_j = 0$ (or constant) for $W_j \leq \bar{Z}_j$. We can also see that if $\bar{Z}_j = 0$, development can only occur if $R_j'(0) > F_j'(0)$.

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1) From (8) we can see that if some part of the wage and overhead costs are proportional to floorspace, the $w_j>0$; otherwise these costs are included in y_j^*.

Proposition 6 below enumerates cases in which a non-cooperative equilibrium for developers exists.

Proposition 6: Let (A.2) hold and let $\alpha \leq 1$. Regard the developers in each zone as one decision maker selecting Z_j from a compact set, and let $\bar{Z}_j \geq 0$. Assume also that $R_j'(0) > F_j'(0-\bar{Z}_j)$. Then there exists a constellation of floorspace decisions, $\{Z_j\}$, which form a Nash equilibrium if for each j F_j is a monotonically increasing function which satisfies one of the following conditions:

(i) convex everywhere; or

(ii) concave with $R_j' > 0$, and $R_j' \geq F_j'$ for $W_j \geq \bar{Z}_j$; or

(iii) concave with $\bar{Z}_j > 0$, $R_j' \leq F_j'$ for $W_j \geq \bar{Z}_j$; or

(iv) S-shaped such that $F_j'' > 0$ in the convex segment of F_j, and $R_j' \leq F_j'$ in the concave segment; or

(v) concave or convex with R_j concave and peaked.[1)]

Outline of a Proof: According to Remark 7, R_j is a continuous and concave function, and F_j is continuous by assumption. Hence, g_j is continuous. The additional requirement on g_j (for an equilibrium to exist) is that g_j is quasi-concave. This is satisfied if g_j is (I) monotonically increasing or (II) monotonically decreasing, or (III) monotonically increasing to a peak and thereafter decreasing, or (IV) g_j is concave.

For all cases the assumption $R_j'(0) > F_j'(0-\bar{Z}_j)$ implies that every center considered is a potential location.

In case (i) g_j is the difference between a concave and a convex function. Hence, at least one of properties (I), (III) and (IV) is satisfied.

In case (ii) g_j is the difference between two increasing concave functions such that property (I) is satisfied.

In case (iii) g_j is monotonically increasing for $W_j \leq \bar{Z}_j$ and monotonically decreasing for $W_j > \bar{Z}_j$, since $F_j'=0$ for $W_j < \bar{Z}_j$. Hence, property (III) is satisfied.

In case (iv) property (III) is satisfied. For the convex segment of F_j we can use the result from case (i). For the subsequent

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1) Cases (i)-(iv) are illustrated in Figure 3.

segment of F_j we use the result from (iii) if $R_j' > 0$ everywhere and from (v) if R_j is concave and peaked.

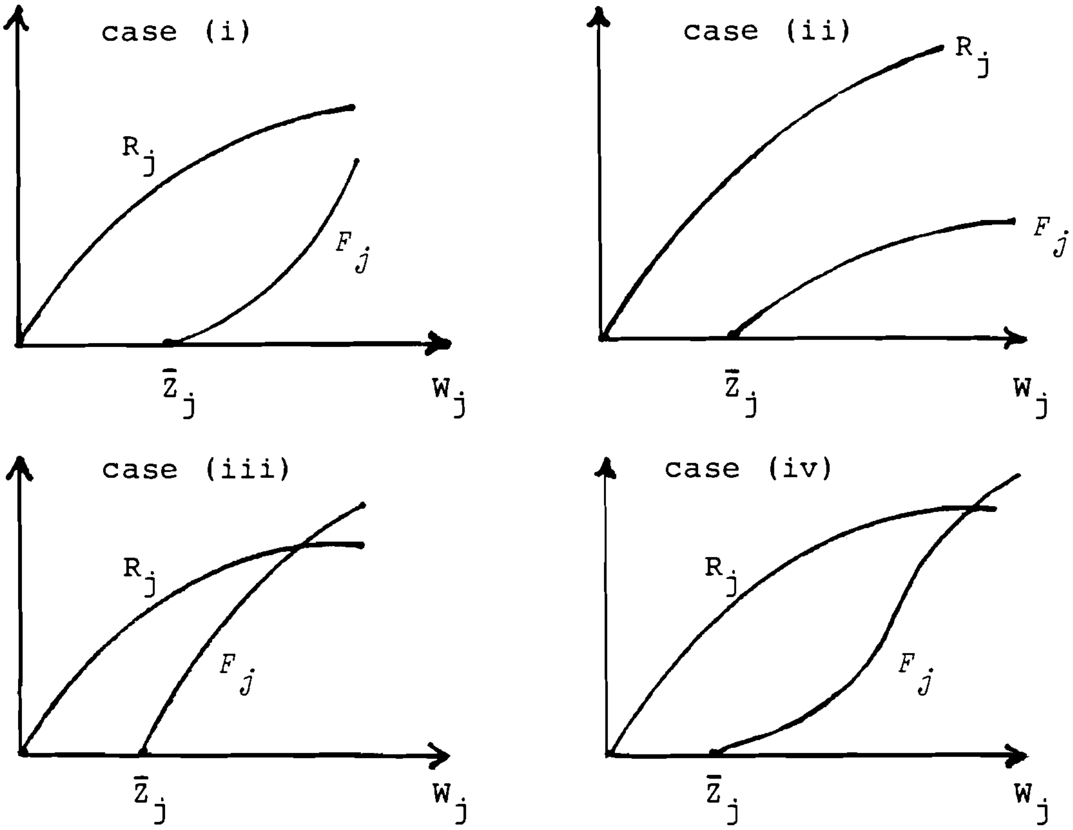

Figure 3: Illustration of case (i)-(iv) in Proposition 6

In case (v) property (III) is obviously satisfied, since g_j is the difference between a peaked concave and a monotonically increasing function. This completes the proof.

As a final exercise we apply the result in Proposition 6 to the developer cost function which is used in Section 4 in which estimation procedures are described. In this case, g_j has the form

$$g_j = R_j - o_j(\bar{Z}_j + \Delta Z_j) - e_j \Delta Z_j \tag{22}$$

where $\Delta Z_j = \min\{0, W_j - \bar{Z}_j\}$, and where o_j and e_j are positive coefficients. The current unit cost of operating established infrastructure in center j is described by o_j. The coefficient e_j reflects the annualized investment cost with regard to infrastructure of center j. In Section 4 it is assumed that a center is provided with new infrastructure of the same standard as the original one. The unit cost related to the lowest standard is denoted by e and all other levels are expressed as ratios $i_j \geq 1$ of e so that

$$e_j = i_j e$$

with $\min\{i_j\} = 1$.

It is obvious that the cost function in (22) satisfies the conditions in case (i) of Proposition 6. Hence, g_j in (22) has a form which ensures the existence of a non-cooperative equilibrium. An illustration is given in Figure 4.

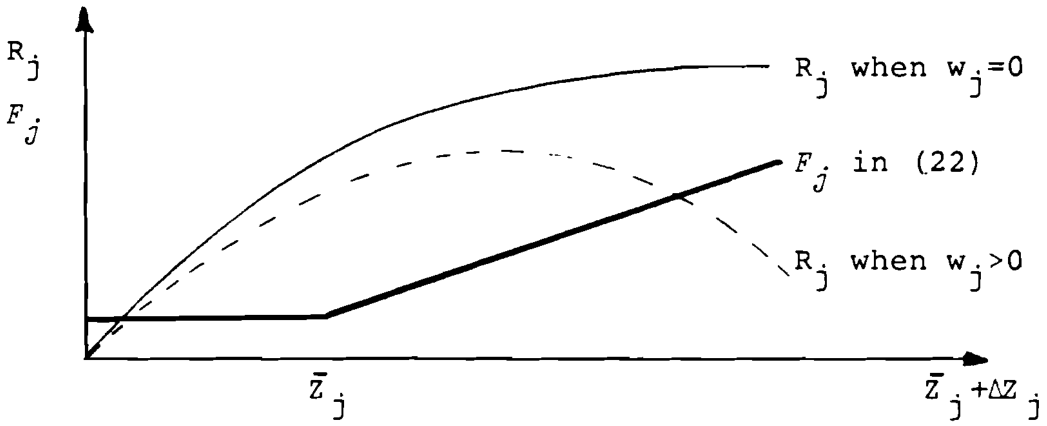

Figure 4. Illustration of the profit function components in (22)

4. MODEL ESTIMATION AND IMPLEMENTATION

With the development of the model structure in the preceding section, the model estimation and validation procedures are now examined, commencing with the customer model. Finally, suggestions for model implementation are discussed.

4.1 Estimation and Validation of Customer Model

Rather than obtaining parameters to describe the aggregate behavior of the customers via appropriate aggregation of the results of estimation of a model at the individual choice level, we estimate the aggregate model directly from aggregate data, with the inclusion of extra variance information when this can be shown to measurably improve the goodness of fit. In addition, a nested recursive approach is chosen, which separates the estimation into two phases, the first being to estimate the conditional probability $p_{ik/j}$ of customers in center j coming from zone i and being of income group k, and the second their choice probability p_j of shopping in center j. This has the advantage that the constraint information is neatly divided into two distinct sets, the first for $p_{ik/j}$ being just related to travel time and the travel plus shopping budget, and the second for p_j relating to the properties of the centers themselves (Roy, 1983). In this way, the model can be validated for $p_{ik/j}$ using the travel information, before proceeding to the p_j phase. The two phases are now treated in order.

For the estimation of $p_{ik/j}$, the average entropy $\bar{S}$ of the conditional probability distribution (Theil, 1972) is maximized in the form

$$\bar{S} = \max_{p_{ik/j}} \; - \sum_j \bar{p}_j \sum_{ik} p_{ik/j} (\log p_{ik/j} - 1) + \sum_j \omega_j \left(1 - \sum_{ik} p_{ik/j}\right)$$
$$+ \sum_{ik} \lambda_{ik} \left(O_{ik} - \sum_j \bar{p}_j \, p_{ik/j}\right) + \beta \left(t - \sum_{ijk} \bar{p}_j \, p_{ik/j} \, t_{ijk}\right)$$
$$+ \psi \left(b - \sum_{ijk} \bar{p}_j \, p_{ik/j} \, (a_k \, y_j + c_{ijk})\right)$$

where $\bar{p}_j$ is the *observed* customer distribution at j and the other terms are given in (1) to (4) of section 2.1. The solution comes out in the form of (5), and the goodness of fit may be computed as

$[(\sum_j \bar{p}_j \sum_{ik} \bar{p}_{ik/j} \log p_{ik/j})/(\sum_j \bar{p}_j \sum_{ik} \bar{p}_{ik/j} \log \bar{p}_{ik/j})-1]$, where $\bar{p}_{ik/j}$ (if available) is the observed conditional probability distribution. If the goodness fit is not satisfactory, it may be necessary to add further information on *variances* of shopping travel times and budgets.

In order to estimate the customer choice probabilities p_j, it is necessary to include all relevant information on destination *quality* via constraints, leaving the remaining effect of pure center *size* to be included as an unknown Kullback residual p_j^s, which can be shown (Roy, 1983) to be expressible as $w_j^\alpha/(\sum_j w_j^\alpha)$, where the unknown customer scale coefficient α is determined by minimizing the Kullback divergence between the model distribution p_j and the observed distribution $\bar{p}_j$. A further point is the means of most efficient aggregation of the travel time and budget information from the $p_{ik/j}$ model. As discussed in Roy and Lesse (1983b), this is achieved by evaluating the Legendre transform $\hat{S}$ or surplus form of the entropy of $p_{ik/j}$ as

$$\hat{S} = -\sum_j \bar{p}_j \sum_{ik} p_{ik/j} (\log p_{ik/j} - \log f_{ijk})$$

which upon substitution from (5) and definition of "composite" travel times $\hat{t}_j$ as

$$\hat{t}_j = - \log (\sum_{ik} f_{ijk})/\beta$$

yields the constraint to be applied to the p_j model as

$$- \beta \sum_j p_j \hat{t}_j = \hat{S}$$

The destination quality constraints should be experimented with in relation to improved goodness of fit. Typically, one may include "convenience" effects related to the amount of time m_j to park and complete the average shopping task in center j. As this time may vary considerably for different persons at different times, a constraint on the average variance measure $m_2(m)$ of m can also be applied, in terms of the observed variances $m_2(m_j)$ at each center j. As proxy for "comfort", i_j is taken as the average building infrastructure investment intensity in j (normalized to unity for the poorest center), which is the only exogenous parameter *directly* connecting *customer* choice with *developer* decisions. This parameter is related to the investment cost function in (18). Also one may consider a binary variable n_j, given as unity if undercover parking exists at j and zero otherwise. The problem for p_j then becomes

$$\bar{I} = \min_{p_j} \sum_j p_j \log (p_j/p_j^S) + \Omega(1-\sum_j p_j) - \theta (\hat{S} + \beta \sum_j p_j \hat{t}_j)$$

$$+ \zeta (m - \sum_j p_j m_j) + \mu (m_2(m) - \sum_j p_j m_2(m_j))$$

$$+ \eta (i - \sum_j p_j i_j) + \kappa (n-\sum_j p_j n_j)$$

where the average values m, i and n are evaluated using the observed choice shares $\bar{p}_j$. As $p_j^s = w_j^\alpha / (\sum_j w_j^\alpha)$ is an unknown in the above formulation, the problem must be solved simultaneously for the unknown multipliers and α together with the following

$$I^* = \underset{\alpha}{\text{Min}} \sum_j \bar{p}_j \log (\bar{p}_j / p_j)$$

when f_j in (2) is given as $\exp - (\zeta\, m_j + \mu\, m_2(m_j) + \eta\, i_j + \kappa\, n_j)$, the solution comes out as given in (7). If the goodness of fit, computed as $(I^*/(-\sum_j \bar{p}_j \log \bar{p}_j))$, is unsatisfactory, further quality constraints may be tried. The joint probabilities p_{ijk} are obtained via (4). The results may be seen to be of similar form to those arising from nested logit models.

4.2 Checks on Behavior of Retailers and Developers

With the customer model estimated as above, it is possible to obtain $\hat{w}_j$ and $\hat{n}_j$ from Remark 5, or (19) together with $\hat{y}_j$ from Proposition 3, and check these with observed short-run values. The observations should be made at a time by which the system has settled down after a change in exogenous factors. The long-run behavior should be checked after the most recent addition to infrastructure supply. A key point to investigate is whether either the more "myopic" Proposition 3, or the more complete information implied by the result in (15), is preferable for the price-setting behavior of retailers. If the customer scale coefficient α turns out to be greater than unity, the model results should be checked against observations to see if the rent setting policy over time reasonably relates to developers capturing a certain proportion ρ_j of the transaction profits of the retailers.

4.3 Some Points on Implementation

The sequence of operating the models in a forecasting context can be illustrated using a simple example. For instance, consider that undercover parking is to be introduced to one of the centers, say center g. Coefficient n_g would change to unity in (2), leading to new customer choice patterns $\hat{p}_j$ and retailer prices $\hat{y}_j$ for *all* centers. After some time, this will change the retailer floorspace demands $\hat{W}_j$ and the rents $\hat{n}_j$, which will feed back to again modify the customer demand $\hat{p}_j$ and prices $\hat{y}_j$. Finally, there may be a tendency for further changes to infrastructure supply (e.g., by centers other than j) which may be evaluated as shown in section 3.2. The effects of this would then feed back to affect retailer demand $\hat{W}_j$ and rents $\hat{n}_j$, after which customer demand $\hat{p}_j$ and prices $\hat{y}_j$ would adjust further.

5. CONCLUDING REMARKS

A three-level leader-follower model has been introduced, in which the suppliers, acting as oligopolists at each center j, set their prices and floorspace demands according to perceived response by customers. At the next level, the developers, again acting as oligopolists at each center, make their short-run rent decisions and longer-run capacity expansion decisions depending on their perceptions of response by the retailers. A future challenge is to include the retailer decisions and developer decisions in a formal probabilistic framework (as already done for the customer decisions), with observations on system behavior and appropriate transmission of information between the different levels implicitly describing the perceptions of each group about the others' possible actions. To achieve this purpose, further developments will need to be made in the scope of use of information theory. In the meantime, the models developed above can be fully tested, to determine if it is really necessary in practice to introduce such increased complexity.

REFERENCES

Berge, C. (1957) Theorie Generale des Jeux a n Personne. *Memoriale des Sciences Mathematiques*, Fasc. 138.

Harris, B., and A.G. Wilson (1978) Equilibrium values and dynamics of attractiveness terms in production-constrained spatial interaction models, *Environment and Planning A*, 10, 371-388.

Intriligator, M.D. (1971) *Mathematical Optimization and Economic Theory*, Prentice-Hall, New Jersey.

Johansson, B. (1978) *Contributions to Sequential Analysis of Oligopolistic Competition*, Memorandum No. 73, National-ekonomiska Institutionen, University of Gothenburg.

Lakshmanan, T.R. and W.G. Hansen (1965) A retail market potential model, *Journal of the American Institute of Planners*, 31, 134-143.

Leonardi, G. (1981) A unifying framework for public facility location problems. *Environment and Planning A*, 13, a. Part 1:1001-1028, b. Part 2:1085-1108.

Rijk, F.A., and A.C.F. Vorst (1983) On the uniqueness and existence of equilibrium points in an urban retail model. *Environment and Planning A*, 15, 475-482.

Roy, J.R., and B. Johansson (1981) On planning and forecasting the location of retail and service activity. Revised version of paper presented at *International Conference on Structural Economic Planning in Time and Space*, University of Umeå, Sweden.

Roy, J.R., and P.F. Lesse (1983a) Planning models for non-cooperative situations: A two-player game approach, *Regional Science and Urban Economics*, 13, 205-221.

Roy, J.R., and P.F. Lesse (1983b) On nested recursive entropy models for multistage processes, Submitted to *Environment and Planning A*.

Roy, J.R. (1983) Estimation of singly-constrained nested spatial interaction models, *Environment and Planning B*, 10, 2 (in press).

Shubik, M. (1959) *Strategy and Market Structure: Competition, Oligopoly and the Theory of Games*, Wiley, New York.

Theil, H. (1972) *Statistical Decomposition Analysis*, North-Holland, Amsterdam/London.