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MATHEMATICAL PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

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ABSTRACT

An approach to the analysis of mathematical programming problems is presented that is based on a systematic extension of the traditional formulation of the problem to obtain a formulation applicable for processing information in the form of fuzzy sets. Solutions are based on trade-offs among achieving greater possible degree of nondominance and greater possible degree of feasibility.

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S.A. Orlovski*

1. Introduction

Mathematical programming (MP) problems form a subclass of decision-making problems in which preferences between alternatives are described by means of an objective function defined on the set of alternatives in such a way that greater (or smaller) values of this function correspond to more preferable alternatives. Values of the objective function describe effects from choices of one or other alternative. In economic problems, for an example, these values may reflect profits obtained using various means of production; in water management problems they may have the meaning of electric power production for various water yelds from a reservoir, etc. The set of feasible alternatives in MP problems is described by means of equations and/or inequalities representing relevant relationships between variables. In any case the results of the analysis using given formulation of the MP problem depend largely upon how adequately various factors of the real system or a

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process are reflected in the description of the objective function and of the constraints.

Descriptions of the objective function and of the constraints in a MP problem include parameters. For an example, in problems of rational water, land and other natural resources allocation such parameters can represent crop productivities, requirements for irrigation per unit area of land for different crops, economic parameters like prices for various types of production, labour requirements, etc. The nature of those parameters depends, of course, on the detailization accepted for the model representation, and their values are considered as data which should exogenously be supplied for the analysis.

Clearly, the values of such parameters depend on multiple factors not included into the formulation of the problem. For the above example these factors may include nutrient contents of the soil, soil treatment practices, solar activity, the state of the external market, and many others. If trying to make the model more representative of the real system we include the corresponding complex dependences into it, then the model may become cumbersome and analytically unacceptable. Moreover, it can happen that such attempts to increase "the precision" of the model will be of no practical value due to the impossibility to measure, or to measure to a sufficient accuracy the values of newly introduced parameters. On the other hand, the model with some fixed values of its parameters may still be too crude, since these values are often chosen in a quite arbitrary way.

An intermediate and flexible approach may be based on the introduction into the model the means of a more adequate representation of experts' understanding of the nature of the parameters in the form of fuzzy sets of their possible values. The resultant model, although not taking into account many details of the real system in question is a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. On this way we obtain a new type of MP problems containing fuzzy parameters. And treating such problems requires the application of fuzzy-set-theoretic tools in a logically consistent manner.

MP and related problems with fuzzy information were extensively analyzed and many papers have been published displaying a variety of formulations and approaches to their analysis (see for instance, Dubois and Prade, 1978; Negoita and Sulariu, 1976; Zimmerman, 1976; Luhandjula, 1982; Ostaziewich, 1980; Orlovsky, 1977, 1980). Most of the approaches to fuzzy MP problems are based on the staightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function. This approach although mentioned by Bellman and Zadeh, 1970 in their underlying paper is in fact some special case of the methodology suggested there.

Here we present a different approach based on a systematic extension of the traditional formulation of MP problems to obtain a formulation applicable for the processing information in the form of fuzzy sets. This approach is based on the results described in Orlovski, 1978, 1980 and is outlined in Orlovski, 1981, 1983.

According to this approach two aspects of a fuzzy MP problem are of major importance. The first is that whereas in a traditional MP problem the objective function represents a complete ordering of alternatives, in a fuzzy MP problem we have only a fuzzy preference relation between alternatives. Due to this fact the concept of maximization becomes undefined and we can only speak about determining alternatives with various degrees of nondominance. The second aspect lies in that in a fuzzy MP problem alternatives can be chosen only on the basis of trade-offs among two generally conflicting objectives: achieving greater possible degree of nondominance and greater possible degree of feasibility. Both aspects are considered in this paper.

2. Problem formulation

We consider the following traditional formulation of a mathematical programming (MP) problem:

$$f(x, a_1, \ldots, a_q) \to \max_x \tag{1}$$

$$\psi_j(x, b_{ij}, \ldots, b_{pj}) \le 0, \; j=1,\ldots,n \tag{2}$$

$$x \in X, \; a_i \in R^1, \; b_{ij} \in R^1$$

where $(a_1, \ldots, a_q)=\bar{a}$ and $|b_{ij}|_{p\times n}=B$ are respectively vector and matrix of exogenous parameters.

In the ordinary case when values of parameters a_i and b_{ij} are given as numbers the meanings of the inequality signs in the constraints (2) and of the maximization of the performance function (1) are well understood and are based on our ability of comparing numbers and saying for any two of them which is greater or at least not smaller than the other. But here we consider the case when values of the parameters are described fuzzily by their respective membership functions which we denote by $\chi_i(a_i)$, $i=1,\ldots,q$, and $\nu_{ij}(b_{ij})$, $i=1,\ldots,p$; $j=1,\ldots,n$. In this case for any given alternative values of functions f and ψ_j can also be described only fuzzily and the formulation (1)-(2) of our problem becomes mathematically meaningless and requires further clarification.

To obtain a mathematically precise formulation of the problem in our case we should first define how we compare alternatives with each other using fuzzy values of the performance function in terms "*greater or equal*". More formally, we should extend the natural ordering in the number line onto the class of fuzzy numbers or fuzzy subsets of this line. Second, we should define the set of those alternatives which in a certain sense "*satisfy*" constraints (2) with fuzzy values of parameters b_{ij}.

2.1. Fuzzy objective function

Let us consider first the performance function f and formulate it explicitly as a fuzzy performance function $\varphi(x,r)$, $x \in X$, $r \in R^1$, i.e. a function that gives a fuzzy value for any alternative. To achieve that we can apply what in the fuzzy sets area is traditionally referred to as the extension principle. Let a_i^o, $i=1,\ldots,q$ be some values of the parameters; their membership degrees are given by $\chi_i(a_i^o)$, $i=1,\ldots,q$. Denote by φ^o the minimum of these values, i.e.

$$\varphi^o = \min_{i=1,\ldots,q} \chi_i(a_i^o).$$

If $r^o = f(x, a_1^o, \dots, a_q^o)$ is the corresponding value of f for some alternative $x \in X$ then it is natural to accept that this value belongs to the fuzzy value of the performance function for x to a degree not smaller than φ^o. Using this reasoning we can write the desired fuzzy performance function in the following form:

$$\varphi(x, r) = \sup_{\bar{a} \in Q(x,r)} \chi(\bar{a}), \; x \in X, \; r \in R^1, \tag{3}$$

with

$$Q(x, r) = \{\bar{a} = (a_1, \dots, a_q) \mid f(x, a_1, \dots, a_q) = r\} \tag{4}$$

and

$$\chi(\bar{a}) = \min_{i=1,\dots,q} \chi_i(a_i).$$

For any fixed $x' \in X$ $\varphi(x', r)$ is the membership function of the corresponding fuzzy evaluation (effectiveness) of alternative x'.

Consider two alternatives $x_1, x_2 \in X$ and the respective fuzzy values of the performance function $\varphi(x_1, r)$, $\varphi(x_2, r)$. Clearly, if $\varphi(x_1, r)$ is not worse than $\varphi(x_2, r)$ to a certain degree than we are justified to consider x_1 be not worse that x_2 to the same degree. We define this type of relation between fuzzy values $\varphi(x_1, r)$, $\varphi(x_2, r)$ (and therefore, between x_1, x_2) using the extension principle in the following way. A degree η of $x_1 \succsim x_2$ (*"not worse than"*) is given by:

$$\eta(x_1, x_2) = \sup_{\substack{y, z \in R^1 \\ y \geq z}} \min\{\varphi(x_1, y), \varphi(x_2, z)\}, \; x_1, x_2 \in X \tag{5}$$

Only after having defined this relation (pairwise comparisons between alternatives) we can speak about choosing those alternatives from set X^* which in some sense if not the best are not dominated by other alternatives. Clearly, the fuzziness of relation $\eta(x_1, x_2)$ allows us to speak about alternatives which are nondominated only to a certain

* For the moment we put aside constraints (2) of the original formulation of the problem.

degree. In other words, alternatives can differ in their degrees of nondominance. Using results from Orlovski, 1976, 1983, we define a degree η^{ND} of nondominance of alternative $x \in X$ as follows:

$$\eta^{ND}(x) = 1 - \sup_{y \in X}[\eta(y,x) - \eta(x,y)], \tag{6}$$

where $\eta(\cdot,\cdot)$ is the relation defined by relationships (3)-(5). Function $\eta^{ND}(x)$ is a description of the fuzzy set of nondominated alternatives in set X with fuzzy binary relation $\eta(\cdot,\cdot)$. Using (5) and (6) we obtain the following expression for this function:

$$\eta^{ND}(x) = 1 - \sup_{x' \in X}\{ \sup_{\substack{y,z \in R^1 \\ y \geq z}} \min\{\varphi(x',y),\varphi(x,z)\} - \tag{7}$$

$$- \sup_{\substack{y,z \in R^1 \\ y \geq z}} \min\{\varphi(x,y),\varphi(x',z)\}\}.$$

Referring now to the original description of the MP problem, we can say that in the fuzzy set context (i.e. with given fuzzy values of parameters $a_1,\ldots,a_q$) we understand the "maximization" problem as that of determining the fuzzy set of nondominated alternatives η^{ND}.

In concrete problems, however, the determination of a complete explicit description $\eta^{ND}(x)$ of the fuzzy set of ND alternatives may be difficult and/or not necessary. (This situation is somewhat similar to that in multiobjective optimization when the determination of a complete explicit description of the set of Pareto optimal alternatives is often not considered.) More realistic and practically important would be to have a procedure that allows for the determination of nondominated alternatives with some prespecified properties. In our case it would be useful to have the means of determining alternatives having degrees of nondominance not smaller than some desired level α. Formally, this means the determination of (some) alternatives x satisfying

$$\eta^{ND}(x) \geq \alpha. \tag{8}$$

As is shown in Orlovski, 1981 if the fuzzy values $\chi_i(a_i)$ of parameters a_i are such that $\chi_i(a_i) \geq \alpha$ for some a_i, $i=1,...,q$ then any solution to the following MP problem:

$$r \to \max_{x,r} \tag{9}$$

$$\varphi(x,r) \geq \alpha, \quad x \in X$$

satisfies condition (8), i.e. is nondominated to a degree not smaller than α.

It can further be shown (see Orlovski, 1981) that for continuous functions $\chi_i(a_i)$, $i=1,...,q$ and $f(x,a_1, \ldots, a_q)$ problem (9) is equivalent to the following MP problem:

$$f(x,a_1,...,a_q) \to \max_{x,a_1,...,a_q}$$

$$\chi_i(a_i) \geq \alpha, \quad i=1,...,q, \tag{10}$$

$$x \in X$$

2.2. Fuzzy set of feasible alternatives

Let us now turn our attention to constraints (2) of the original formulation of the problem. As has been mentioned earlier in this paper the question here is to define how we understand the feasibility of alternatives with respect to these constraints. Clearly, with only fuzzy description of values of parameters b_{ij} in functions ψ_j some alternatives can be more feasible than others. In other words, they can differ in their degree of feasibility and we can only consider a fuzzy set of feasible alternatives. Our purpose here is to obtain an explicit description of this fuzzy set by means of a membership function (which we shall denote by $\mu^C(x)$) and we shall reason in the following way.

Consider one of the constraints j in (2) and let b_{ij}^o be some values of the respective parameters. Their membership degrees in the respective fuzzy sets are $\nu_{ij}(b_{ij}^o)$. Denote by μ_j^o the minimum of these degrees, i.e.

$$\mu_j^o = \min_{i=1,\ldots,p} \nu_{ij}(b_{ij}^o).$$

If some alternative $x \in X$ satisfies the inequality

$$\psi_j(x, b_{1j}^o, \ldots, b_{pj}^o) \leq 0,$$

then we can naturally accept that this alternative satisfies constraint j to a degree not smaller than μ_j^o, i.e. we may consider that $\mu_j^C(x) \geq \mu_j^o$. For convenience, we introduce the following notations:

$$\nu_j(\bar{b}_j) = \min_{i=1,\ldots,p} \nu_{ij}(b_{ij}),\ \bar{b}_j = (b_{1j}, \ldots, b_{pj}),$$

$$P_j(x) = \{\bar{b}_j \mid \psi_j(x, b_{1j}, \ldots, b_{pj}) \leq 0,\}$$

Using the above reasoning we can write the membership function of the fuzzy set of alternatives satisfying constraint j in the following form:

$$\mu_j^C(x) = \sup_{\bar{b}_j \in P_j(x)} \nu_j(\bar{b}_j). \tag{11}$$

To each alternative $x \in X$ this function assigns a degree to which this alternative satisfies constraint j.

We can obtain the same function in a more formal way using the extension principle first to extend the definition of function ψ_j for fuzzy values of parameters b_{ij}, and then to extend the ordering ($\leq$) on the number line to fuzzy numbers:

$$\Psi_j(x, r) = \bar{\psi}_j(x, \nu_{1j}(b_{1j}), \ldots, \nu_{pj}(b_{pj})) = \sup_{r = \psi_j(x, b_{1j}, \ldots, b_{pj})} \min_{i=1,\ldots,p} \nu_{ij}(b_{ij}).$$

For any $x \in X$ function $\Psi_j(x, r)$ is the corresponding fuzzy value (fuzzy number) of function $\bar{\psi}_j$ for given fuzzy values $\nu_{ij}(b_{ij})$ of parameters b_{ij}.

Next we consider number 0 as a fuzzy subset of the number line with the following membership function:

$$\lambda_0(r) = \begin{cases} 1, \text{ for } r = 0, \\ 0, \text{ otherwise}, \end{cases}$$

and define for any fixed $x \in X$ a degree to which fuzzy number $\Psi_j(x,r)$ is not greater than $\lambda_0(r)$. Using the extension principle we obtain:

$$\mu_j^C(x) = \{degree\ of\ \Psi_j(x,r) \le \lambda_0(r)\} =$$

$$= \sup_{\substack{y,z \in R^1 \\ y \le z}} \min\{\Psi_j(x,y), \lambda_0(z)\} = \sup_{\substack{y \in R^1 \\ y \le 0}} \Psi_j(x,y).$$

Using the above relationship for $\Psi_j(x,r)$ we have

$$\mu_j^C(x) = \sup_{\substack{y \in R^1 \\ y \le 0}} \sup_{y = \psi_j(x, b_{1j}, \ldots, b_{pj})} \min_{i=1,\ldots,p} \nu_{ij}(b_{ij}) = \sup_{\bar{b}_j \in P_j(x)} \nu_j(\bar{b}_j)$$

($\bar{b}_j$, $P_j(x)$, $\nu_j(\bar{b}_j)$ defined as before) which coincides with equation (11).

Now for a fixed $x \in X$ we have degrees $\mu_j^C(x)$, $j=1,\ldots,n$ to which x satisfies the respective constraints and it is natural to accept that x simultaneously satisfies all of them to a degree:

$$\mu^C(x) = \min_{j=1,\ldots,n} \mu_j^C(x).$$

or using (11):

$$\mu^C(x) = \min_{j=1,\ldots,n} \sup_{\bar{b}_j \in P_j(x)} \nu_j(\bar{b}_j).$$

Apparently, here we accept that the fuzzy set of feasible alternatives is the intersection of the fuzzy sets of alternatives satisfying the respective constraints.

3. Compromise between nondominance and feasibility

Now when we have introduced explicit descriptions of the fuzzy relation $\eta(\cdot,\cdot)$ allowing for the comparison of alternatives with each other and of the membership function of the fuzzy set of feasible alternatives $\mu^C(x)$ we can consider rational choices of alternatives on the basis of this information. Apparently, in making these choices we have two generally conflicting objectives: we would like to choose an alternative having greater possible degree of nondominance α ("maximization"), and at the same time having greater possible degree of feasibility β. If we fix some desired levels of α (nondominance) and β (feasibility) then using the above notation and results we have that any alternative determined as a solution to the following problem:

$$f(x, a_1, \ldots, a_q) \to \max_{x, \bar{a}}$$

$$\chi_i(a_i) \geq \alpha \tag{12}$$

$$\min_{j=1,\ldots,n} \sup_{\bar{b}_j \in P_j(x)} \nu_j(\bar{b}_j) \geq \beta$$

has a degree of nondominance not smaller than α and is feasible (satisfies the constraints) to a degree not smaller than β. As can be verified, if functions $\nu_{ij}(b_{ij})$ and ψ_j are continuous with respect to b_{ij} this problem is equivalent to the following:

$$f(x, a_1, \ldots, a_q) \to \max_{x, a_i, b_{ij}}$$

$$\chi_i(a_i) \geq \alpha, \ i=1,\ldots,q,$$

$$\nu_{ij}(b_{ij}) \geq \beta, \ i=1,\ldots,p; j=1,\ldots,n, \tag{13}$$

$$\psi_j(x, b_{1j}, \ldots, b_{pj}) \leq 0, \ j=1,\ldots,n,$$

$$x \in X.$$

If problem (13) has a solution for $\alpha=\beta=1$ then such solution is an alternative that is unfuzzily (to a degree 1) nondominated (i.e. no other alternative is better to a positive degree) and at the same time is unfuzzily ($\beta=1$) feasible. If no solution to (13) exists for $\alpha=\beta=1$ then the analyst or the decision-maker (DM) should sacrifice either the degree of nondominance or the degree of feasibility (or both) and attempt to determine less "ideal" alternatives (with smaller α and/or β) which agree with his tolerances with regard to α and β. For any alternative $x^0 \in X$ determined in this way $\varphi(x^0, r)$ is the corresponding fuzzy value of the objective function and $\mu^C(x^0)$ is its feasibility degree.

Clearly, "the most rational" solutions would be those corresponding to Pareto optimal pairs (α,β) which in this case can be defined as follows. A pair (α^o,β^o) is Pareto optimal for problem (13) if for any other pair (α,β) such that $\alpha>\alpha^o$, $\beta\geq\beta^o$ or $\alpha\geq\alpha^o$, $\beta>\beta^o$ problem (13) has no solution.

In principle, Pareto optimal α,β can be determined by iterating values of these levels and solving problem (13) at each iteration step. More realistic, however, would be to solve this problem for some increasing values of α,β until these values together with the fuzzy value $\varphi(x,r)$ of the performance function for the solution to problem (13) satisfy the DM.

Remarks:

1. In formulation (12)-(13) of the problem it is assumed that all constraints are of equal importance to the DM and this fact is reflected by assigning the same minimum desired level of feasibility β to all of them. However, if we would like to take into account differences in the importance of constraints we can specify different desired levels of feasibility β_j for different constraints j. In that case the respective constraints will be of the form (for (12)):

$$\sup_{\bar{b}_j \in P_j(x)} \nu_j(\bar{b}_j) \geq \beta_j, \quad j=1,\ldots,n,$$

and (for (13)):

$$\nu_{ij}(b_{ij}) \geq \beta_j, \quad i=1,\ldots,p; \; j=1,\ldots,n.$$

This would mean that we treat all the constraints separately and do not define the fuzzy set of feasible alternatives $\mu^C(x)$ as just the intersection of the respective fuzzy sets $\mu_j^C(x)$.

2. If fuzzy sets $\chi_i(a_i)$ and $\nu_{ij}(b_{ij})$ in our problem are described in a triangular form (that is extensively discussed in the current literature on fuzzy sets) then they can analytically be described as follows (for χ_i as an example):

$$\chi_i(a_i) = \max\{0;\min\{l_i^1(a_i),l_i^2(a_i)\}\},$$

with $l_i^1(a_i)$ and $l_i^2(a_i)$ being two linear functions having slopes of opposite signs. In this case, as can easily be seen, condition $\chi_i(a_i) \geq \alpha$ with $0 \leq \alpha \leq 1$ in formulation (13) is equivalent to the following two linear constraints:

$$l_i^1(a_i) \geq \alpha, \ l_i^2(a_i) \geq \alpha.$$

4. Concluding remarks

We outlined here an approach to processing information in the form of fuzzy sets in problems of choice formulated in the mathematical programming form. The use of fuzzy sets for describing information about real systems is a relatively new area and much further work is needed in order to find practically sound methods allowing to combine the fuzziness of human judgements with the powerfull logic and tools of mathematical analysis. Successful development in this direction may help overcome one of the essential obstacles to the application of mathematical modeling to the analyses of real systems, namely, the existing gap between the language used for mathematical models and the language used by potential users of those models.

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