TWO APPROACHES TO MULTIOBJECTIVE PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

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ABSTRACT

Two approaches to the analysis of multiobjective programming problems are presented based on a systematic extension of the traditional formulation of the problem to obtain a formulation applicable for processing information in the form of fuzzy sets. Solutions are based on trade-offs among achieving greater possible degree of nondominance and greater possible degree of feasibility.

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S.A. Orlovski*

1. Introduction

Multiobjective (MO) programming problems with fuzzy information were extensively analyzed and many papers have been published displaying a variety of formulations and approaches to their analysis (see for instance, Zimmerman, 1978; Yager, 1978; Takeda and Nishida, 1980; Hannan, 1981; Lohandjula, 1982; Feng, 1983; Backley, 1983; Tong, 1982). Most of the approaches to fuzzy MO problems are based on the staightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function.

Here we present two approaches based on a systematic extension of the traditional formulation of MO problems with fuzzy parameters to obtain a formulation applicable for processing information in the form of fuzzy sets. The paper is based on results described in Orlovski, 1978, 1980, 1981, 1983, 1984.

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Two aspects of a fuzzy MO problem are of major importance. The first is that whereas in a traditional problem every objective function represents a linear ordering of alternatives, in a fuzzy MO problem we have only fuzzy preference relations between alternatives. Due to this fact the concept of domination requires further definition and we can only speak about determining alternatives with various degrees of nondominance. The second aspect lies in that in a fuzzy MO problem alternatives can be chosen only on the basis of trade-offs among two generally conflicting objectives: achieving greater possible degree of nondominance and greater possible degree of feasibility. Both aspects are considered in this paper.

2. Problem formulation

We assume here that alternatives from a given set X are pairwisely compared with each other using n objective functions $J_i(x, \bar{q})$, $i=1,...,n$ in such a way that greater values of each of these functions are considered to be more preferable. Each of these functions contains a vector of parameters $\bar{q}$ the values of which are known fuzzily and described by means of the membership function $\mu: Q \to [0,1]$. Speaking informally, the problem lies in determining feasible alternatives giving greater possible values of the objective functions.

In a traditional MO problem, when the values of parameters $\bar{q}$ are specified unambiguously, the rational choices are Pareto-maximal alternatives which can be determined using well-known computational techniques. Here, with fuzzily specified values of parameters $\bar{q}$, we can have only fuzzy evaluations of the corresponding objective functions and, therefore, should additionally more precisely define the meaning of a rational choice.

First, in our reasoning we assume that the set feasible alternatives is nonfuzzy and coincides with set X, and consider the fuzzy set of feasible alternatives later for both the approaches.

3. First approach

3.1. Reformulation of the problem

This approach is based on the consideration of levels for the values of the objective functions which can be achieved to a higher possible degree. More formally, we understand our problem in the following way:

$$(J_1^o, \ldots, J_n^o, \alpha) \rightarrow \overline{\max}_{x, J_i^o, \alpha} \tag{1a}$$

$$degree\{J_1(x,\tilde{q}) \geq J_1^o, \ldots, J_n^o(x,\tilde{q}) \geq J_n^o\} \geq \alpha \tag{2a}$$

$$x \in X$$

Maximization in (1a) is understood, of course, in the Pareto sense and to indicate this we use the symbol $\overline{\max}$. Constraint (2a) reflects the fact that with fuzzy values of functions J_i we can only consider satisfying the inequalities $J_i(x,\tilde{q}) \geq J_i^o$ to a certain degree. An essential point in this formulation is that the multiobjective choice in this case should be based not on the trade-offs among the values of the objective functions, which are fuzzy due to the fuzzy nature of parameters $\tilde{q}$, but among the lower estimates of these values obtainable to a certain degree α. This formulation also implies that when deciding upon the trade-offs among the lower estimates of the objectives the decision-maker should consider the possibility degree α of these estimates.

In the following subsection we demonstrate that the above formulation can be reduced to a traditional form of a MO problem.

3.2. Analysis of the problem

For conveniency we shall consider functions $J_i(x,\tilde{q})$, $i=1,\ldots,n$ as components of a vector function $\bar{J}(x,\tilde{q})$ with values from the real vector space R^n. If we denote by $\bar{J}^o$ the vector of levels $(J_1^o, \ldots, J_n^o)$ then problem (1a)-(2a) can be written in the form:

$$(\bar{J}^o, \alpha) \rightarrow \overline{\max}_{x, \bar{J}^o, \alpha} \tag{1b}$$

$$degree\{\bar{J}(x,\bar{q})\geq\bar{J}^o\} \geq \alpha \tag{2b}$$

$$x \in X$$

To formulate constraints (2b) more explicitly, we can directly use the extension principle (Zadeh, 1973) to obtain:

$$degree\{\bar{J}(x,\bar{q})\geq\bar{J}^o\} = \sup_{\bar{q}:\bar{J}(x,\bar{q})\geq\bar{J}^o} \mu(\bar{q}), \tag{3}$$

which, in fact, represents the extention of the "greater or equal" relation from the vector space R^n onto the class of fuzzy vectors - values of the objective vector-function with fuzzy parameters $\bar{q}$.

Finally, using (3) we can formulate problem (1b)-(2b) as follows:

$$(\bar{J}^o,\alpha) \rightarrow \overline{\max}_{x,\bar{J}^o,\alpha}$$

$$\sup_{\bar{q}:\bar{J}(x,\bar{q})\geq\bar{J}^o} \mu(\bar{q}) \geq \alpha \tag{4}$$

$$x \in X$$

If some tuple $(x^s,\bar{J}^{os},\alpha^s)$ is a solution to this problem then the tuple $(J_1^{os}, \ldots, J_n^{os},\alpha^s)$ is Pareto optimal which means that any other alternative x providing for better values of some of the components of $(\bar{J}^o,\alpha)$ gives worse values of some of the other components of this tuple.

Now it is a simple excersize to verify that for continuous in $\bar{q}$ functions $J_i(x,\bar{q})$ (for any $x \in X$), $i=1,\ldots,n$ and $\mu(\bar{q})$ problem (4) can equivalently be formulated as follows:

$$(\bar{J}^o,\alpha) \rightarrow \overline{\max}_{x,\bar{q},\bar{J}^o,\alpha}$$

$$\mu(\bar{q}) \geq \alpha$$

$$\bar{J}(x,\bar{q}) \geq \bar{J}^o$$

$$x \in X,\ \bar{q} \in Q,$$

and finally, in the form:

$$(\bar{J}(x,\bar{q}),\alpha) \rightarrow \overline{\max}_{x,\bar{q},\alpha} \tag{5}$$

$$\mu(\bar{q}) \geq \alpha$$

$$x \in X,\ \bar{q} \in Q.$$

4. Second approach

4.1. Reformulation of the problem

This approach is based on the extention of the natural order on the real line of values of the objective functions onto the class of fuzzy subsets of this line. In this way we obtain preference relations which can be used for comparing with each other fuzzy values of the objective functions for various alternatives. Then, using these relations, we define a fuzzy strict preference relation on the set of alternatives and determine the corresponding fuzzy subset of nondominated alternatives.

As before, we consider n objective functions $J_i(x,\bar{q})$, with $\bar{q}$ being a fuzzily-valued vector of parameters described by membership function $\mu(\bar{q})$. Using the extension principle the corresponding fuzzy values of these functions can be obtained in the following form:

$$\varphi_i(x,r) = \sup_{\bar{q}:J_i(x,\bar{q})=r} \mu(\bar{q}),\ i=1,\ldots,n. \tag{7}$$

Now we can obtain the following fuzzy nonstrict preference relations induced on the set of alternatives by φ_i:

$$\eta_i(x_1,x_2) = \sup_{z \geq y} \min\{\varphi_i(x_1,z),\varphi_i(x_2,y)\},\ i=1,\ldots,n.$$

The next step is to define a way of comparing alternatives with each other using all these n preference relations. To do that we define strict dominance relation on X in the following way. Let $\eta_i^s(x_1,x_2)$ be the fuzzy strict preference relation corresponding to $\eta_i(x_1,x_2)$, and defined as follows (see Orlovski, 1978):

$$\eta_i^s(x_1,x_2) = \begin{cases} \beta=\eta_i(x_1,x_2)-\eta_i(x_2,x_1), \text{ if } \beta>0 \\ 0,\ \textit{otherwise} \end{cases}.$$

Then we say that the degree $\eta^s(x_1,x_2)$ to which alternative x_1 is strictly prefered to alternative x_2 is as follows:

$$\eta^s(x_1,x_2) = \min_i \eta_i^s(x_1,x_2).$$

In a nonfuzzy formulation this would mean that x_1 is strictly preferable to x_2 iff it is strictly better than x_2 with respect to every objective function. The respective nondominated alternatives are commonly reffered to as semiefficient or weakly effective.

Having defined η^s we can describe the corresponding fuzzy subset η^{ND} of nondominated alternatives in the form (Orlovski, 1978):

$$\eta^{ND}(x) = 1 - \sup_{y \in X} \eta^s(y,x) =$$

$$= 1 - \sup_{y \in X} \min_i \eta_i^s(y,x),$$

and using the above formulation of η_i^s, we have:

$$\eta^{ND}(x) = 1 - \sup_{y \in X} \min_i [\eta_i(y,x) - \eta_i(x,y)]. \tag{8}$$

The value $\eta^{ND}(x)$ is the nondominance degree of the respective alternative. If $\eta^{ND}(x) \geq \alpha$ then alternative x may be strictly dominated by some other alternative to a degree smaller than $1-\alpha$.

4.2. Determining alternatives nondominated to a prespecified degree

Now we consider the problem of determining alternatives satisfying:

$$\eta^{ND}(x) \geq \alpha, \tag{9}$$

where α is the desired degree of nondominance.

Let us formulate the following nonfuzzy multiobjective problem:

$$\bar{r} = (r_1,\ldots,r_n) \to \overline{\max_{\bar{r},x}} \tag{10}$$

$$\varphi_i(x,r_i) \geq \alpha, \quad i=1,\ldots,n,$$

$$x \in X, \quad \bar{r} \in R^n.$$

The following theorem states that under some conditions any solution x to problem (10) satisfies (9).

Theorem. *If for any of the functions $\varphi_i(\cdot,\cdot)$, $i=1,\dots,n$ and any $x\in X$ there exist $r_i\in R^1$, $i=1,\dots,n$ such that $\varphi_i(x,r_i)\geq\alpha$, then for any solution to problem (10) we have $\eta^{ND}(x)\geq\alpha$.*

Proof. Let $(x^o,\bar{r}^o)$ be a solution to problem (10). Then, as follows from (8) prove the theorem it suffices to show that

$$\sup_y \min_i [\eta_i(y,x^o)-\eta_i(x^o,y)] \leq 1-\alpha.$$

Assume the contrary, i.e. that $y'\in X$ and $\varepsilon>0$ can be found such that

$$\min_i [\eta_i(y',x^o)-\eta_i(x^o,y')] > 1-\alpha+\varepsilon,$$

or

$$\eta_i(y',x^o)-\eta_i(x^o,y') > 1-\alpha+\varepsilon,\ i=1,\dots,n. \tag{11}$$

Using (7) we can write (11) in the form:

$$\sup_{r_i\geq z_i} \min\{\varphi_i(y',r_i),\varphi_i(x^o,z_i)\} - \sup_{r_i\geq z_i} \min\{\varphi_i(x^o,r_i),\varphi_i(y',z_i)\} >$$

$$> 1-\alpha+\varepsilon \quad i=1,\dots,n. \tag{11a}$$

Let us choose z'_i, $i=1,\dots,n$ such that $\varphi_i(y',z'_i)\geq\alpha$ for all $i=1,\dots,n$ (the existence of z'_i, $i=1,\dots,n$ follows from the assumptions about functions φ_i). Since $(x^o,\bar{r}^o)$ is a solution to problem (10), we have that $r^o_{i_o}\geq z'_i$ for at least one $i=i_o$ among $i=1,\dots,n$. Thus we have

$$\varphi_{i_o}(x^o,r^o_{i_o})\geq\alpha,\ \ \varphi_{i_o}(y',z'_{i_o})\geq\alpha,\ \ r^o_{i_o}\geq z'_{i_o}.$$

Therefore, we have

$$\sup_{r_{i_o}\geq z_{i_o}} \min\{\varphi_{i_o}(x^o,r_{i_o}),\varphi_{i_o}(y',z_{i_o})\} \geq \alpha.$$

Hence, the inequality with index i_o in (11a) does not hold, since its first additive term does not exceed 1. This contradiction proves the Theorem.

Using (6) we can now write problem (11) in the following form:

$$\bar{r} = (r_1, \ldots, r_n) \to \overline{\max_{\bar{r}, x}}$$

$$\sup_{\bar{q} : J_i(x, \bar{q}) = r_i} \mu(\bar{q}) \geq \alpha, \; i = 1, \ldots, n,$$

$$x \in X,$$

or equivalently:

$$\bar{J}(x, \bar{q}) \to \overline{\max_{\bar{q}, x}}$$

$$\mu(\bar{q}) \geq \alpha, \tag{12}$$

$$x \in X, \; \bar{q} \in Q.$$

As can be seen this formulation is quite the same as the corresponding MO formulation (5) for the first approach (see Sect. 3.2) in the case of a fixed α. Therefore, both the approaches are equivalent to each other in the sense that both may lead to choices of the same alternatives.

5. Fuzzy set of feasible alternatives

Let us now additionally assume that the set of feasible alternatives is described by the following system of inequalities:

$$\bar{\psi}(x, \bar{p}) \leq 0, \tag{13}$$

with $\bar{\psi}$ being a given real vector-valued function, and $\bar{p}$ being a vector of parameters with the membership function $\nu: P \to [0,1]$ describing fuzzily its possile values.

To use this type of information we first determine an explicit description of the corresponding fuzzy subset of feasible alternatives in the form of a membership function $w(x)$. If we introduce the notation

$$P(x) = \{\bar{p} \mid \bar{p} \in P, \bar{\psi}(x, \bar{p}) \leq 0\}.$$

Then using the extension principle we can write this membership function in the form:

$$w(x) = \sup_{\bar{p} \in P(x)} \nu(\bar{p}).$$

The value $w(x)$ of this function is understood as the feasibility degree of the corresponding alternative, and these values should also be taken into account when making choices of alternatives.

Alternatives in this case should be evaluated by two generally conflicting factors: their degree of nondominance $\eta^{ND}(x)$ (in the second approach) and their degree of feasibility $w(x)$. Let α be the desired degree of nondominance and β be the desired level of feasibility. Then an alternative having a degree of nondominance not smaller than α and feasible to a degree not smaller than β should satisfy the following inequalities:

$$\eta^{ND}(x) \geq \alpha, \quad w(x) \geq \beta.$$

Therefore, with the fuzzy set of feasible alternatives formulation (12) will have the following additional constraints:

$$\sup_{\bar{p} \in P(x)} \nu(\bar{p}),$$

$$P(x) = \{\bar{p} \mid \bar{p} \in P, \ \bar{\psi}(x, \bar{p}) \leq 0\}.$$

And it can easily be verified that with this type of constraints problem (12) can be written as follows:

$$\bar{J}(x, \bar{q}) \to \overline{\max_{\bar{q}, \bar{p}, x}}$$

$$\bar{\psi}(x, \bar{p}) \leq 0,$$

$$\mu(\bar{q}) \geq \alpha, \ \nu(\bar{p}) \geq \beta,$$

$$x \in X, \ \bar{q} \in Q, \ \bar{p} \in P.$$

By varying the values of α and β we can determine alternatives with various trade-offs among the degrees of nondominance and feasibility.

6. Concluding remarks

Two approaches to MO problems with fuzzy parameters are suggested in this paper. Both are based on the systematic use of the extension principle as the means of processing fuzzy information about parameters. The rationality of choice is based on trade-offs among degrees of feasibility and nondominance. It is shown that rational alternatives in both the approaches can be determined by solving similar MO problems in a traditional form.

The use of fuzzy sets for describing information about real systems is a relatively new area and much further work is needed in order to find practically effective methods allowing to combine the fuzziness of human judgements with the powerful logics and tools of mathematical analysis. Successful development in this direction may help overcome one of the essential obstacles to the application of the mathematical modeling to the analyses of real systems, namely, the existing gap between the language used for mathematical models and the language used by potential users of those models.

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