RR-75-5

A PREDATOR-PREY MODEL FOR DISCRETE-TIME COMMERCIAL FISHERIES

M. Gatto

S. Rinaldi

C. Walters

March 1975

A Predator-Prey Model for Discrete-Time Commercial Fisheries[1]

M. Gatto,[2] S. Rinaldi,[2] and C. Walters[3]

Abstract

A very simple discrete-time predator (boats) - prey (fish) model for the description of the dynamic behavior of a fishery is presented. The stability properties of the system are analyzed in some detail and the sensitivity of the equilibrium with respect to the catchability coefficient, the length of the fishing season and the investment coefficient of the fleet is analyzed. Finally, a simple procedure is presented and used for estimating the characteristic parameters of the fleet of a few fisheries. The agreement between the data and the predicted results is quite satisfactory when considering the crudeness of the model.

1. Introduction

In the literature on commercial fisheries, the dynamics of fish populations is often described by means of a set of differential (difference) equations in which variables such as effort and dimensions of the fleet enter as constant parameters or as driving variables. However, in the real world, economic variables are not fully controllable and are strongly influenced by the dynamics of the fish population itself. A fleet is normally sensitive (at least over long periods of time) to catches in recent years, or in other words, to investment (Smith [11]; Fullenbaum, Carlson,

---

[1]This work has been supported by the International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria.

[2]Centro di Teoria dei Sistemi, C.N.R., Milano, Italy.

[3]Institute of Animal Resource Ecology, U.B.C., Canada.

Bell, and Smith [5]; Wang [12]). Thus it should be, in general, more appropriate to consider the dimension of the fleet (e.g. number of boats) as a state variable rather than as a parameter or as a control variable. Modern modelling techniques and system theory make it possible to add such dimensions without losing the analytical tractability that is considered a virture of classical fishery dynamics models.

The structure of a general model which is consistent with this suggestion is shown in Fig. 1. The driving forces acting on each subsystem are constant in time only if the fishery is not controlled by a supervisory agency and if the surrounding environment of the fishery does not vary in time (no trends in the economy, no improvements in fishing technology, no deterioration of the habitat,...). This limit case of behavior of the system will be called "natural evolution" of the fishery in order to distinguish it from cases of "controlled evolution" obtained when decision makers fix over time the values of some of the driving forces (e.g. number of spawners to be released from hatcheries, length of fishing season, taxes, number of licenses, subsidies,...). A controlled evolution is usually obtained through a feedback as shown in Fig. 2, where the controller receives information about the state of the system and consequently makes a decision. To analyze and compare the controlled evolution of a fishery corresponding to different feedback policies, it is first necessary to have a model for the description of the natural evolution of the fishery and to know how basic properties of that model (e.g. equilibrium and its stability) are influenced by parameter values.

The aim of this paper is to present a very simple discrete-time model of the kind described in Fig. 1 (see Sect.2), and then prove the existence of an asymptotically stable equilibrium for its natural evolution (see Sect. 3) and discuss the sensitivity of this equilibrium with respect to those parameters which are potential driving variables of a controlled evolution (see Sect. 4). Finally, a very simple scheme for the estimation of the parameters of the model is given in Sect. 5.

The model presented in this paper is very crude because both the fish population dynamics and the evolution of the fleet are described by means of a first order difference equation. Thus, the fishery turns out to be considered as a classical predator (boats) - prey (fish) system. It must be noted that this paper does not represent the first attempt to describe a fishery as a predator-prey system. Commercial fisheries have already been described as continuous-time predator-prey systems (e.g. Smith [11], Fullenbaum, Carlson, Bell, and Smith [5], Wang [12]). The continuous time description is, in general, more elegant but can give rise to serious disadvantages when the model is used for designing the best control policy: continuous-time models require that the decision maker is operating continuously in time, while in almost all commercial fisheries decision makers are operating in discrete time (e.g. once per year). Moreover, in some special fisheries (e.g. Pacific salmon) the discrete-time description is definitely necessary because of the short, pulsed character of fishery effort. Finally, the particular type of data available for commercial fisheries makes it possible to estimate the parameters of discrete models only.

2. The Model

Let B_t, N_t and C_t be, respectively, the number of boats, the number of fish and the total catch in year t. Then, the model is specified by two difference equations for the dynamic behavior of boats and fish and by an equation giving the catch C_t as a function of B_t and N_t. The particular equations used in the remainder of this paper are as follows:

$$B_{t+1} = sB_t + i\,\frac{C_t}{B_t} \quad , \tag{1a}$$

$$N_{t+1} = (N_t - C_t)\ \exp\left[a\left(1 - \frac{N_t - C_t}{N_E}\right)\right] \quad , \tag{1b}$$

$$C_t = N_t\left[1 - \exp\ (-cB_tT)\right] \quad . \tag{1c}$$

In the first equation (fleet dynamics) s and i are "survival" and "investment" coefficients of the fleet; therefore $0 < s < 1$ and $i > 0$.

The second equation is the well-known Ricker model where $(N_t - C_t)$ is the number of spawners in year t, N_E is the natural equilibrium of the fishery and e^a is the growth factor $(0 \leq a \leq 2)$.

The last equation is the commonly used "catch equation" and simply states that the catch C_t is proportional to the recruitment N_t and is an increasing and bounded function of the fishing rate cB_tT (c is the usual catchability coefficient and B_tT is the effort = number of boats x length of the fishing season). The three pairs of parameters $(s,i),(a,N_E),(c,T)$ appearing in Eq. (1) are assumed for the foregoing discussion to be constant in time.

By substituting the catch expression into the first two equations one obtains the description of the dynamics of the fishery in the form

$$B_{t+1} = f_B(B_t,N_t) \quad , \tag{2a}$$

$$N_{t+1} = f_N(B_t,N_t) \quad , \tag{2b}$$

where the functions f_B and f_N are given by

$$f_B(B_t,N_t) = sB_t + i \frac{N_t}{B_t} \left[1 - \exp\ (-cB_tT)\right] \quad , \tag{3a}$$

$$f_N(B_t,N_t) = N_t \exp \left[a - cB_tT - a \frac{N_t}{N_E} \cdot \exp\ (-cB_tT)\right] \quad , \tag{3b}$$

so that the natural evolution of the fishery is nothing but a trajectory in the state space of the system described by Eqs. (2-3).

Some comments on the assumptions underlying Eq. (1) are now needed in order to bound the validity of the model.

The weakest point of the model is certainly the description of the dynamics of the fleet. There are in fact different reasons why Eq. (1a) might not be considered satisfactory. First, there may be a considerable time lag between investment decisions and actual appearance of boats in the fleet. Second, Eq. (1a) does not take into account the age structure of the fleet which could be of some importance, especially in the case of a sudden change in fishing technology (note that, by definition, this cannot occur during the natural evolution of the system). Third, the investment $I_t = iC_t/B_t$ is assumed to be linearly related to the catch per boat while a more realistic assumption should be that the investment is an increasing and strictly convex function of the catch per boat; however, this assumption would seriously increase the difficulty of the discussion below. Fourth, and probably most important, is that in real fisheries the investment I_t does not depend only upon the catch per boat of the previous year, but also upon all the prior history of the fishery. This could be taken into account by assuming that I_t is a weighted sum of the catches per boat in the past, i.e.

$$I_t = \sum_{k=o}^{t} i^{t-k+1} \frac{C_k}{B_k} \quad , \tag{4}$$

so that

$$I_{t+1} = iI_t + i \frac{C_{t+1}}{B_{t+1}} \quad .$$

Thus, under this assumption the fishery would be described by a third order model of the kind

$$B_{t+1} = f_B(B_t, I_t) \quad ,$$

$$I_{t+1} = f_I(B_t, N_t, I_t) \quad ,$$

$$N_{t+1} = f_N(B_t, N_t) \quad ,$$

and the dynamic behavior of such a model would certainly be smoother than the one predicted by Eq. (2), because of the "filtering" effect introduced by Eq. (4). Finally, in many fisheries the number of boats present every year is subject to apparently random fluctuations due to the mobility of the boats and the competition among fisheries. Thus, the dynamics of the fishery can be described only very roughly by Eq. (1a). As an alternative, one could use a stochastic description of the kind

$$B_{t+1} = sB_t + i \frac{C_t}{B_t} + \Delta_t \tag{5}$$

with a fairly high variance of the noise Δ_t (in Sect. 5, the stochastic process Δ_t will be assumed to be normally distributed).

For the dynamics of the fish population, the situation is not as fuzzy because the limits of validity of the Ricker model (1b) have been well studied (e.g. Cushing and Harris [2]). The most important phenomena that are missing in this model are the effects of the age structure of the population, a time delay in the stock-recruitment relation and the stochasticity induced by random fluctuations of the quality of the habitat. The first two criticisms could in principle be overcome by using a higher order model, while the third requires a detailed description of the influence that some suitable environmental indicators have on the life cycle of the fish, a very difficult problem indeed. A synthetic way of solving this problem consists of multiplying the stock-recruitment function by a random factor α_t, i.e.

$$N_{t+1} = \alpha_t (N_t - C_t) \exp \left[a \left(1 - \frac{N_t - C_t}{N_E} \right) \right] , \qquad (6)$$

where α_t can be interpreted as a measure of the probability of survival in year t. Since the number of causes of death in the life cycle of a fish is very high and since these causes can be considered essentially as independent of each other, it follows that the stochastic process α_t can be reasonably assumed to be lognormal.

Finally, the catch equation is open to considerable criticism (Paloheimo and Dickie [10]), since it does not take schooling and nonrandom boat searching into account. To add some realism, a stochastic term can be included to give

$$C_t = N_t \left[1 - \exp \left(-\beta_t c B_t T \right) \right] , \qquad (7)$$

where β_t is again a lognormal stochastic process because it arises as a product of several essentially independent efficiency factors such as weather.

In the next two sections the deterministic behavior ($\Delta_t = 0$, $\alpha_t = 1$, $\beta_t = 1$) of the fishery is analyzed. In Sect. 5, Eqs. (5-7) and the assumptions of the stochastic processes Δ_t, α_t and β_t are used to devise a satisfactory scheme for the estimation of the parameters.

3. Stability Properties

The purpose of this section is to find the equilibrium states of the model, discuss their stability and, in general, study the properties of the natural evolution of the fishery.

By definition, the equilibrium states are the solutions of Eq. (2) with $B_t = B_{t+1} = \bar{B}$ and $N_t = N_{t+1} = \bar{N}$, i.e.

$$\bar{B} = s\bar{B} + i\,\frac{\bar{N}}{\bar{B}}\,[1 - \exp\,(-c\bar{B}T)] \quad , \qquad (8a)$$

$$\bar{N} = \bar{N}\,\exp\left[a - c\bar{B}T - a\,\frac{N}{N_E}\,\exp\,(-c\bar{B}T)\right] \quad . \qquad (8b)$$

A trivial solution of this system of equations is given by the origin of the state space, $(\bar{B},\bar{N}) = (0,0)$. Since $\bar{B} = 0$ if and only if $\bar{N} = 0$, it is possible to assume $\bar{B} \neq 0$ and $\bar{N} \neq 0$ in Eqs. (8) and solve them with respect to $\bar{N}$:

$$\bar{N} = \frac{1 - s}{i}\,B^2 \Big/ [1 - \exp\,(-c\bar{B}T)] = v(\bar{B}) \quad , \qquad (9a)$$

$$\bar{N} = N_E\left(1 - \frac{c\bar{B}T}{a}\right)\exp\,c\bar{B}T = h(\bar{B}) \quad . \qquad (9b)$$

The shapes of the two isoclines $v(\bar{B})$ and $h(\bar{B})$ given by Eqs. (9) appear in Fig. 3; these isoclines demonstrate that there always exists one and only one equilibrium state $(\bar{B},\bar{N})$ with $\bar{B} \neq 0$ and $\bar{N} \neq 0$, which is called the productive equilibrium state from now on.

Let us now linearize the system around its two equilibrium states in order to study their stability properties. The linearized system is

$$\begin{bmatrix} \Delta B_{t+1} \\ \\ \Delta N_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{df_B}{dB} & \frac{df_B}{dN} \\ \\ \frac{df_N}{dB} & \frac{df_N}{dN} \end{bmatrix} , \begin{bmatrix} \Delta B_t \\ \\ \Delta N_t \end{bmatrix} = F \begin{bmatrix} \Delta B_t \\ \\ \Delta N_t \end{bmatrix} \qquad (10)$$

where ΔB_t and ΔN_t are the variations with respect to a steady state and the matrix F is evaluated at the equilibrium.

In the case of the origin the matrix F turns out to be given by

$$F = \begin{bmatrix} s & icT \\ 0 & \exp(a) \end{bmatrix} ,$$

so that the eigenvalues are s and exp (a). The former is smaller than one, while the latter is greater than one, and this implies the origin in an unstable equilibrium state. More precisely, the origin is a saddle point, the eigenvectors being the B axis and the vector

$$\begin{bmatrix} 1 \\ icT \quad \exp(a) - s \end{bmatrix} ,$$

and the trajectories in the neighborhood of the origin are shown in Fig. 4 where successive states are joined by a straight line.

Working out the derivatives indicated in Eq. (10) and using Eq. (9) it is possible to prove that the matrix F evaluated at the productive equilibrium $(\bar{B},\bar{N})$ is given by

$$F = \begin{bmatrix} icT\,\frac{\bar{N}}{\bar{B}} - c\bar{B}T + sc\bar{B}T + 2s - 1 & (1 - s)\,\frac{\bar{B}}{\bar{N}} \\ -c\bar{N}T(1 - a + c\bar{B}T) & 1 - a + c\bar{B}T \end{bmatrix} .$$

Since $(\bar{B},\bar{N})$ is not available in closed form, explicit computation of the eigenvalues is impossible. Nevertheless, the discussion of the stability of the equilibrium can be performed in an indirect way recalling that the eigenvalues of a 2 x 2 matrix lie within the unit circle when the following two inequalities are satisfied

$$|\Pi| < 1 \quad , \tag{11a}$$

$$|\Sigma| < 1 + \Pi \quad , \tag{11b}$$

where Π and Σ are, respectively, the product and the sum of the eigenvalues. Since Π and Σ are the determinant and the trace of the matrix F, it is possible to show that under the assumption

$$c\bar{B}T < 1 \quad ,$$

which is satisfied in most commercial fisheries, conditions (11a) and (11b) are verified, i.e. the productive equilibrium is always asymptotically stable. A proof of this statement can be found in Appendix 1.

Though the analysis so far performed is a stability analysis in the small, there is no evidence for the productive equilibrium state not being stable in the large. This assertion is essentially validated by the existence of a region of attraction R containing $(\bar{B},\bar{N})$, i.e. a region satisfying the following two properties:

a) any trajectory starting from a point in R is contained in R (R is an invariant set),

b) any trajectory starting from a point outside of R reaches R in a finite number of transitions.

A proof of the existence of such a region can be found in Appendix 2.

Finally, simulation of the model shows that, depending upon the values of the parameters, monotonic or oscillatory transients can be obtained. In Fig. 5 an example corresponding to the exploitation of a virgin fishery $(B_O = O, N_O = N_E)$ is shown. Two transients are plotted for two different values of parameter cT: trajectory A is obtained in the case of poor technology and/or short length of fishing season $(cT = 1.5 \times 10^{-3})$, while trajectory B is obtained in the opposite case $(c = 3.5 \times 10^{-3})$. It is worthwhile noticing that in case A there is no oscillatory behavior, while in case B there are periods of temporary overinvestment followed by periods of overexploitation of the fish population, a fact which has been observed in commercial fisheries.

4. Sensitivity of the Productive Equilibrium

As pointed out in the previous section, the productive equilibrium $(\bar{B},\bar{N})$ cannot be given a closed form expression. Nevertheless, the sensitivity of this steady state with respect to some parameters can be determined in a qualitative way.

With this aim, it is convenient to study first how the isoclines v(B) and h(B) are influenced by the parameters. It is interesting to notice (see Fig. 6) that curve v(B) does not depend separately on s and i, but on $\frac{1-s}{i}$, i.e. on the ratio between mortality and investment, and that it approaches, for large values of B^2, a limit parabola independent of c and T. On the other hand, curve h(B) does not depend (see Fig. 7) upon s and i, but only upon cT, a, and N_E. By intersecting h(B) with v(B), it is easy to understand how the equilibrium point varies with $\frac{1-s}{i}$ and cT: these variations are shown in Fig. 8.

The following general conclusions can be drawn:

a) If $a < 1$, the population $\bar{N}$ is decreasing with cT and increasing with $\frac{1-s}{i}$. If $a > 1$, then the statement above is still valid for large values of cT and low values of $\frac{1-s}{i}$. In simple terms, if the fishery is characterized by a low reproduction rate then the size of the stock at the equilibrium is decreasing with the catchability coefficient, with the length of the fishing season, and with the survival and investment coefficient of the fleet. If, on the contrary, the fishery is characterized by a high reproduction rate, then the stock size is a dome-shaped function of the same parameters.

b) The number of boats $\bar{B}$ is decreasing with $\frac{1-s}{i}$ while it is first increasing and then decreasing with cT. In other words, greater values of the survival and investment coefficients imply larger sizes of the fleet, while too large values of the catchability coefficient and of the length of the fishing season give rise to a small equilibrium fleet size.

As for the equilibrium catch $\bar{C}$, observe that Eq. (1a) yields

$$\bar{C} = \frac{1 - s}{i} \bar{B}^2 \quad , \tag{12}$$

which is the limit parabola shown in Fig. 6. With this in mind, it is easy to realize that the catch $\bar{C}$ is a dome-shaped function of $\frac{1 - s}{i}$ and cT. An important index for the fishery is the equilibrium catch per boat $\bar{J}$ which (see Eq. (12)) turns out to be given by

$$\bar{J} = \frac{1 - s}{i} \bar{B} \quad . \tag{13}$$

The following two simple but important properties of this index can be proved to be valid:

c) The catch per boat is increasing with the ratio $\frac{1 - s}{i}$.

d) The catch per boat is first increasing and then decreasing with cT.

To study how $\bar{J}$ varies with $\frac{1 - s}{i}$, it is sufficient to plot the curves of constant catch per boat given by

$$\frac{N}{B} \left[1 - \exp\left(-BcT\right)\right] = \text{const.}$$

and intersect them with the curve of Fig. 8b, which is the locus of the equilibrium states obtained for different values of $\frac{1 - s}{i}$ (see Fig. 9). It is easy to verify that, since a < 2, the curves of constant catch per boat intersect the equilibria locus only once; therefore $\bar{J}$ is an increasing function of $\frac{1 - s}{i}$.

To prove property d) it is sufficient to remark that in view of Eq. (13), $\bar{J}$ has the same dependence upon cT as the number of boats, i.e. it is first increasing and then decreasing with cT (see Fig. 10). Therefore, there exists a length of the fishing season which maximizes the catch per boat.

Property d) is of particular interest because it points out the possibility for a fishery to be in the equilibrium state B of Fig. 10. A suitable change of the length of the fishing season will then generate a transient from state A to state B, the latter being characterized by the same number of boats and the same catch per boat but by a greater number of fish and by a shorter length of the fishing season, a definite advantage in the management of the fishery. The transient from state A to state B is characterized by a remarkable initial disinvestment which, nevertheless, could be compensated for by temporarily providing subsidies to the fishery.

5. Parameter Estimation

A procedure for the estimation of the parameters of the model is outlined below. The method consists in working out separately the least squares estimation of the parameters of the three components of the fishery.

Suppose that the variables B_t, C_t, N_t and T_t (note that the length of the fishing season is now allowed to be varying in time) have been measured for a certain number of years ($t = 1,2,\ldots,n$) during which there has been no evidence of relatively important changes in the economy (s and i are constant), in technology (c is constant) and in the quality of the environment (a and N_E are constant). Then, consider first the catch function in the form given by Eq. (7); from this expression one obtains

$$\log c = \frac{1}{n} \sum_{t=1}^{n} \log \left(\frac{1}{B_t T_t} \log \frac{N_t}{N_t - C_t} \right) - \frac{1}{n} \sum_{t=1}^{n} \log \beta_t \quad , \tag{14}$$

in which the term $\frac{1}{n} \sum_{t=1}^{n} \log \beta_t$ goes to zero as n approaches infinity because it is an estimate of the mean value of a normally distributed random variable which is known to have zero mean value (recall the assumptions on β_t). Thus

$$\log \hat{c} = \log \sqrt[n]{\prod_{t=1}^{n} \frac{1}{B_t T_t} \log \frac{N_t}{N_t - C_t}} \tag{15}$$

is an unbiased estimate of log c and the variance of this estimate is proportional to the variance of the noise and decreases with n as $\frac{1}{n}$. Moreover, this estimate is the one which minimizes the expected value of the square of the difference between log c given by Eq. (14) and all its possible estimates.

As far as the estimation of the parameters s and i is concerned, it is very simple to prove (e.g. Lee [7]) that if the noise Δ_t in Eq. (5) is a normally distributed independent noise with zero mean value, then the least squares estimate is unbiased, consistent, and given by

$$\begin{bmatrix} \hat{s} \\ \hat{i} \end{bmatrix} = (P'P)^{-1} \; P'p \quad , \tag{16}$$

where the matrix P and the vector p are given by

$$P = \begin{bmatrix} B_1 & C_1/B_1 \\ B_2 & C_2/B_2 \\ \vdots & \vdots \\ B_{n-1} & C_{n-1}/B_{n-1} \end{bmatrix}, \quad p = \begin{bmatrix} B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix}, \tag{17}$$

and P' denotes the transpose of P.

Finally, the estimation of parameters a and N_E can also be carried out by means of a linear expression of the kind (16) as pointed out in the literature (Dahlberg [3]). In fact, from Eq. (6) one obtains

$$a + (C_t - N_t)\frac{a}{N_E} = \log \frac{N_{t+1}}{N_t - C_t} - \log \alpha_t ,$$

and $\log \alpha_t$ has the same properties as Δ_t in Eq. (5). Thus, in this case

$$\begin{bmatrix} \hat{a} \\ \frac{\hat{a}}{\hat{N}_E} \end{bmatrix} = (Q'Q)^{-1} Q'q , \tag{18}$$

where

$$Q = \begin{bmatrix} 1 & C_1 - N_1 \\ 1 & C_2 - N_2 \\ \vdots & \vdots \\ 1 & C_{n-1} - N_{n-1} \end{bmatrix}, \quad q = \begin{bmatrix} \log N_2/N_1 - C_1 \\ \log N_3/N_2 - C_2 \\ \vdots \\ \log N_n/N_{n-1} - C_{n-1} \end{bmatrix}. \tag{19}$$

In conclusion, the estimation of the parameters of the fishery can be carried out separately for the three subsystems shown in Fig. 1 by means of Eqs. (15-19). Thus, through this procedure one can separately evaluate the validity of Eqs. (1a), (1b) and (1c) and therefore deduce which parts of the model are satisfactory and, eventually, which are not. Moreover, this scheme requires only simple subproblems to be solved, a definite advantage from a computational point of view (for example, in this case two 2 x 2 matrices must be inverted instead of a 4 x 4 matrix). In this respect, it is important to note that if the number of fish N_t is unknown (which is usually the case) the scheme outlined above cannot be used. However, the estimation of the parameters can still be carried out by introducing Eq. (1c) into Eq. (1b) in such a way that N_t and N_{t+1} are eliminated. Thus, a new difference equation is obtained that can be used to estimate the three parameters a, N_E and c. The disadvantages introduced by the lack of information on N_t are that the estimation procedure is no longer linear and that a problem of dimension three must be solved instead of two subproblems of dimension two and one.

Since there is already a large body of literature on estimation of catchability coefficients and parameters of the Ricker model, further examples are unnecessary. Fig. 12 demonstrates the effort model fit for five fisheries; two kinds of predictions are shown:

1) one year forecasts (predicted values based on observed values from previous year),

2) simulation forecasts (predicted values based on simulated values from previous year).

The one year forecasts are reasonably good in most cases: at least the qualitative direction of change is usually predicted correctly. On the other hand, the simulation forecasts usually lead to large cumulative errors after a few years. These errors suggest some major weaknesses of the simple effort model:

1) investment time lags may delay effort growth (example: fin whales, 1950-1960),

2) effort changes may reflect mobility to other fishing areas (example: halibut and cod),

3) sudden large effort pulses may occur without apparent simple explanation (examples: Peru anchovy, California sardine).

Thus it appears inadvisable to use the simple effort model except for qualitative, short run forecasts.

6. Conclusion

The model outlined in this paper is obviously too crude for practical, quantitative application. Our intent has been to suggest an approach to development of wider perspectives on problems of fishery dynamics, in hope of identifying new management strategies which take the dynamics of fishing, as well as fish, into account. The qualitative conclusions in Sect. 4 may be reasonable guidelines for the design of such strategies. Probably the greatest weakness of our simple analysis is failure to take alternative fishing locations and species into account; with modern, flexible fishing gear it may be economical to deplete some stocks (zero productive equilibrium) while subsisting on or profiting from others.

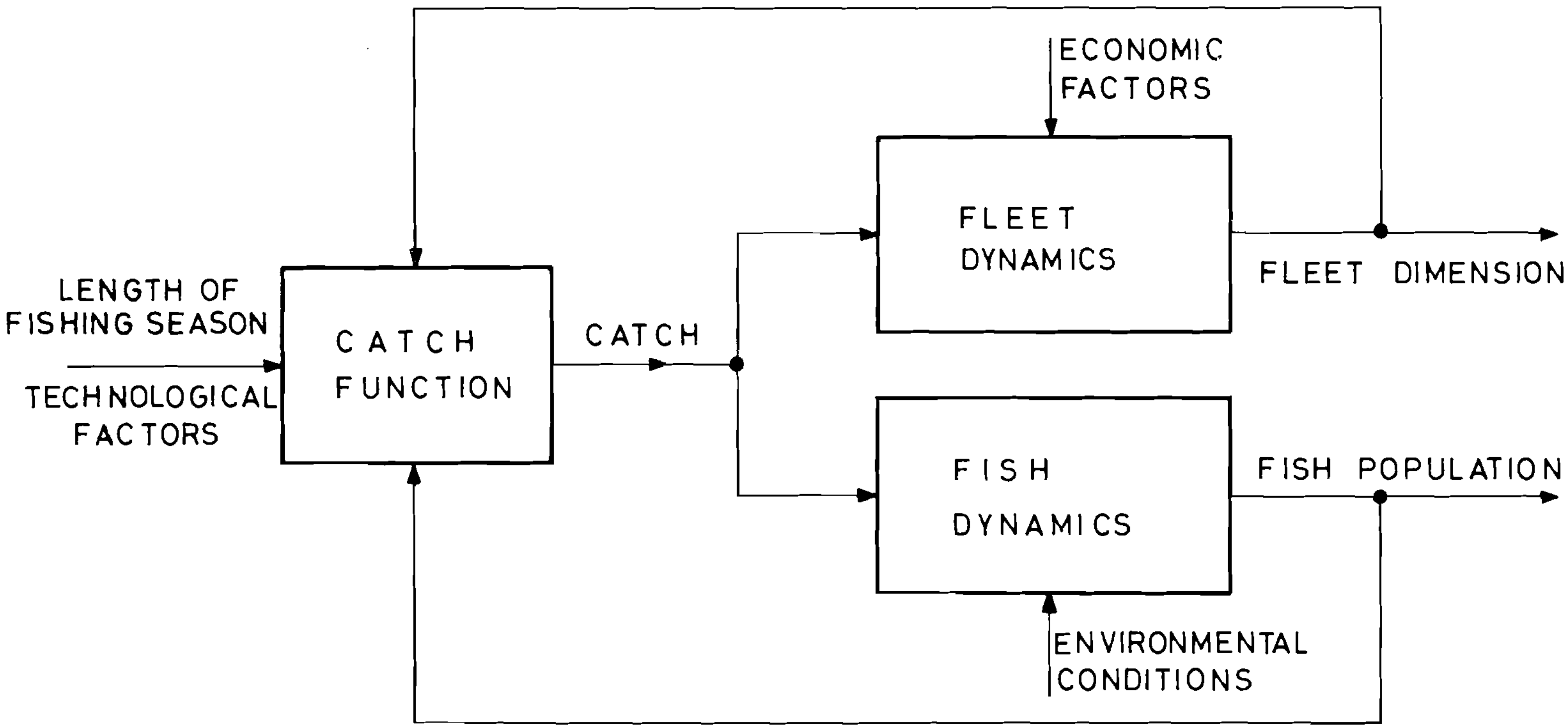


FIGURE 1. GENERAL STRUCTURE OF A FISHERY MODEL.

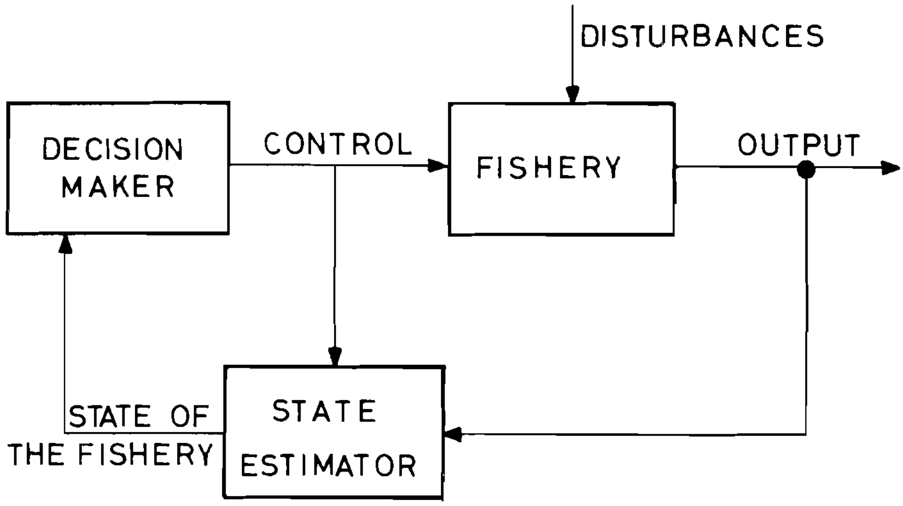


FIGURE 2. CONTROLLED EVOLUTION OF A FISHERY
(CONTROL = LENGTH OF FISHING SEASON, TAXES, SUBSIDIES, ...;
OUTPUT = SAMPLES OF CATCH, NUMBER OF BOATS,
SAMPLES OF RECRUITMENT, ...;
DISTURBANCES = TRENDS IN THE ECONOMY, DETERIORATION OF
THE HABITAT, CHANGE IN TECHNOLOGY, ...).

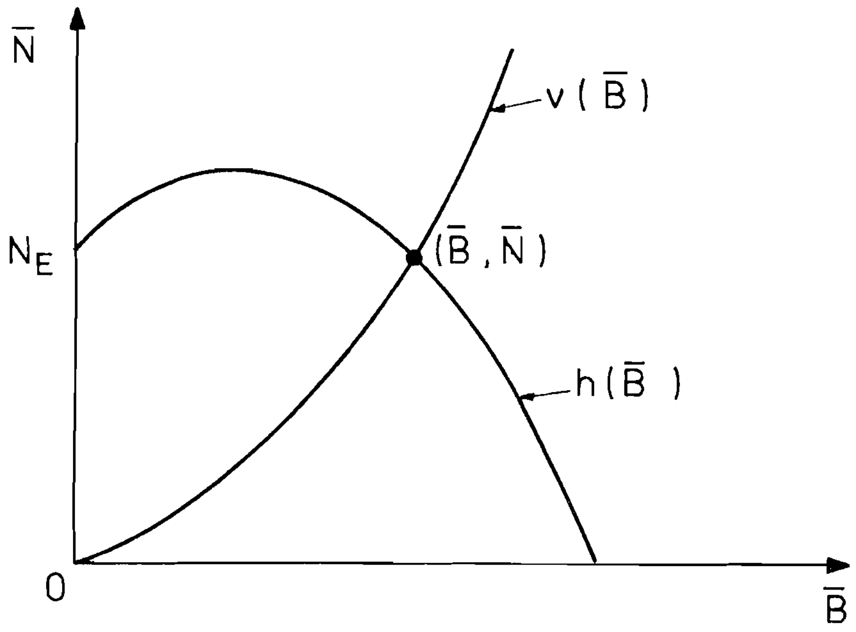


FIGURE 3. THE ISOCLINES $v(\bar{B})$ AND $h(\bar{B})$.

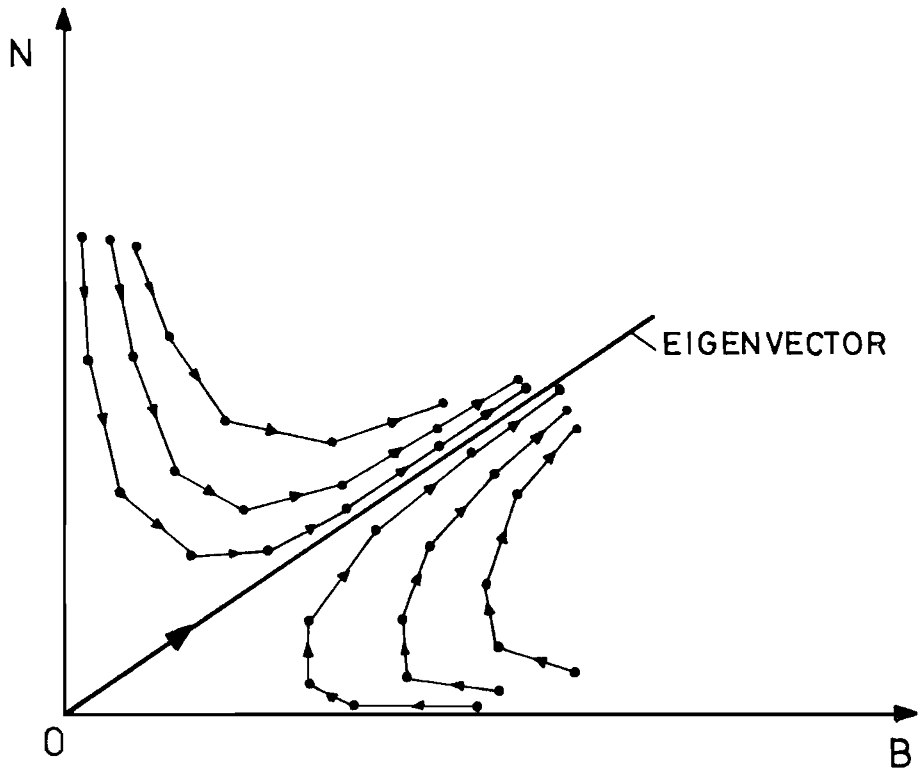


FIGURE 4. THE ORIGIN IS A SADDLE POINT.

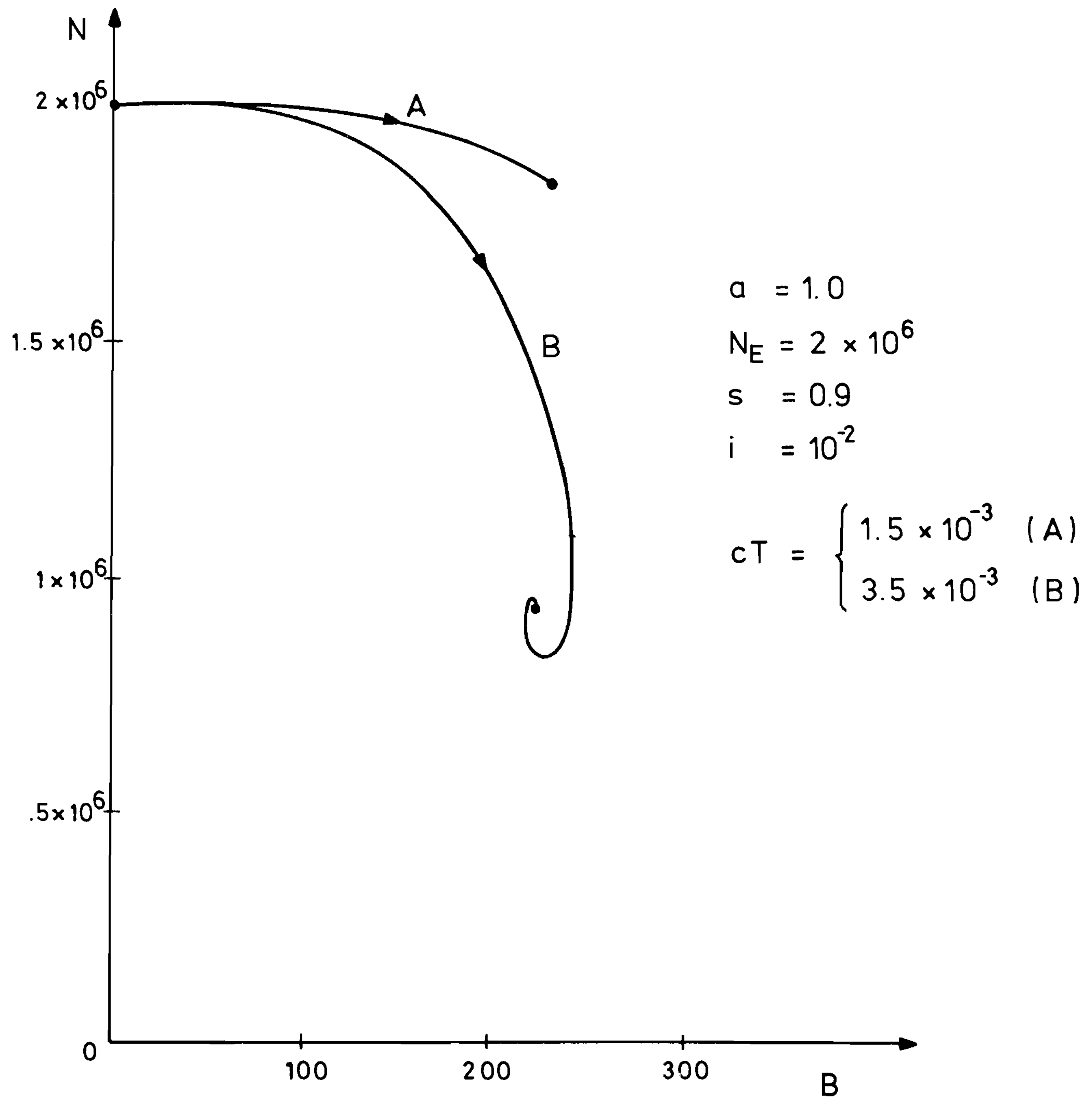


FIGURE 5. NATURAL EVOLUTIONS OF A VIRGIN FISHERY.

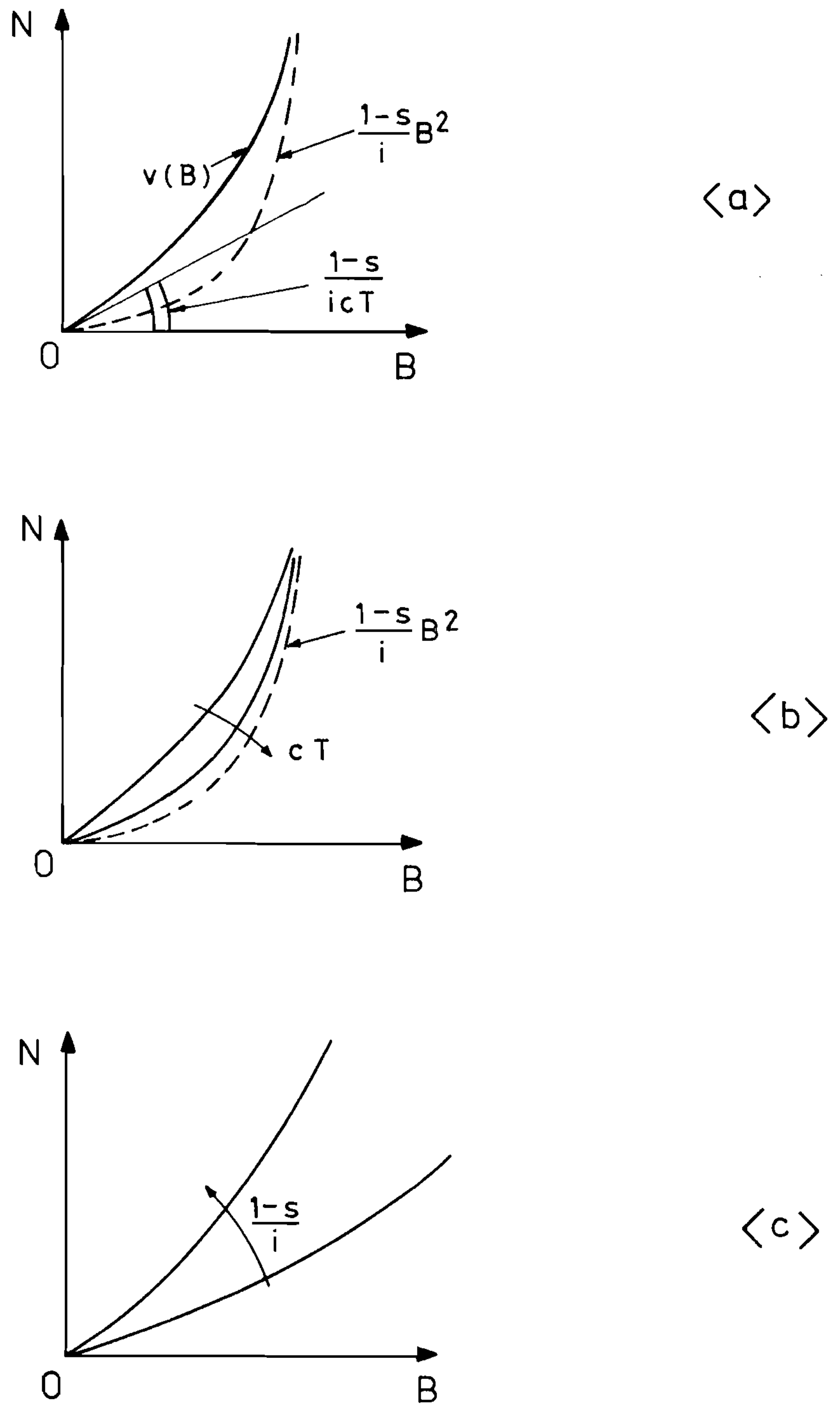


FIGURE 6. THE INFLUENCE OF THE PARAMETERS ON THE ISOCLINE $v(B)$ ⟨a⟩ THE CURVE $v(B)$, ⟨b⟩ THE INFLUENCE OF cT, ⟨c⟩ THE INFLUENCE $\frac{1-s}{i}$.

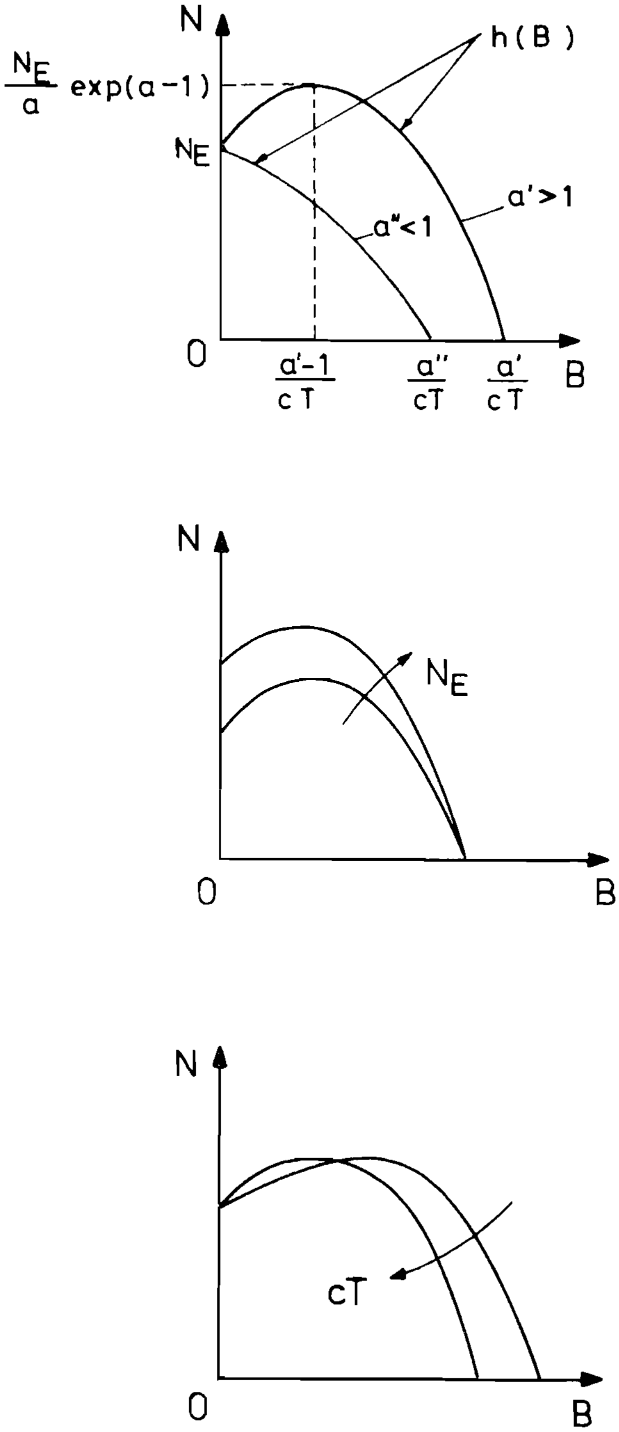


FIGURE 7. THE INFLUENCE OF THE PARAMETERS ON THE ISOCLINE h(B)
<a> THE CURVE h(B), <b> THE INFLUENCE OF N_E,
<c> THE INFLUENCE OF cT.

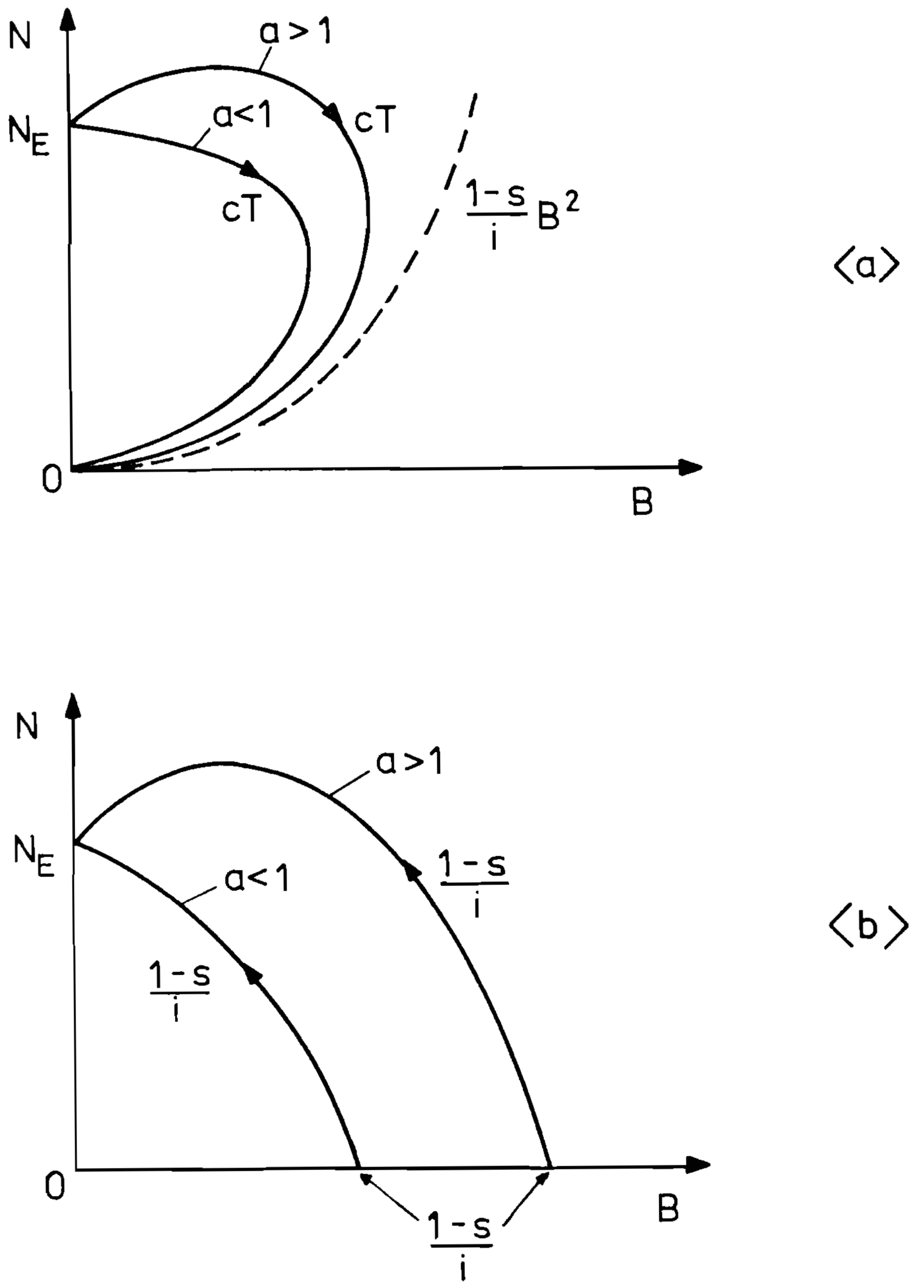


FIGURE 8. VARIATIONS OF THE PRODUCTIVE EQUILIBRIUM
⟨a⟩ WITH RESPECT TO cT
⟨b⟩ WITH RESPECT TO $\frac{1-s}{i}$.

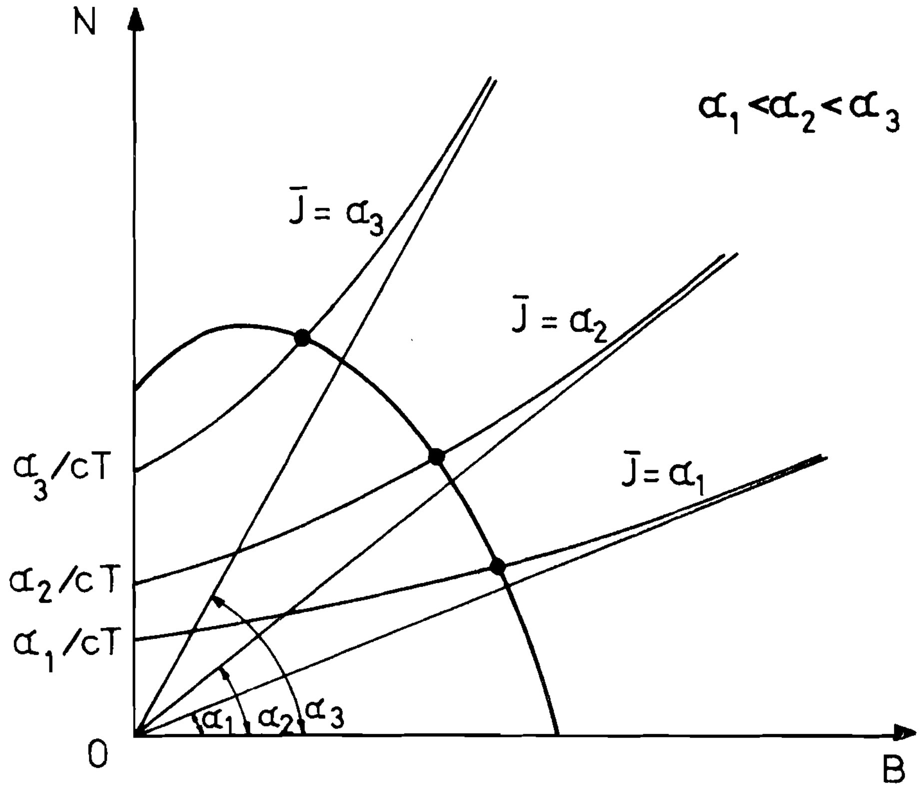


FIGURE 9. THE CATCH PER BOAT AS A FUNCTION OF $\frac{1-s}{i}$.

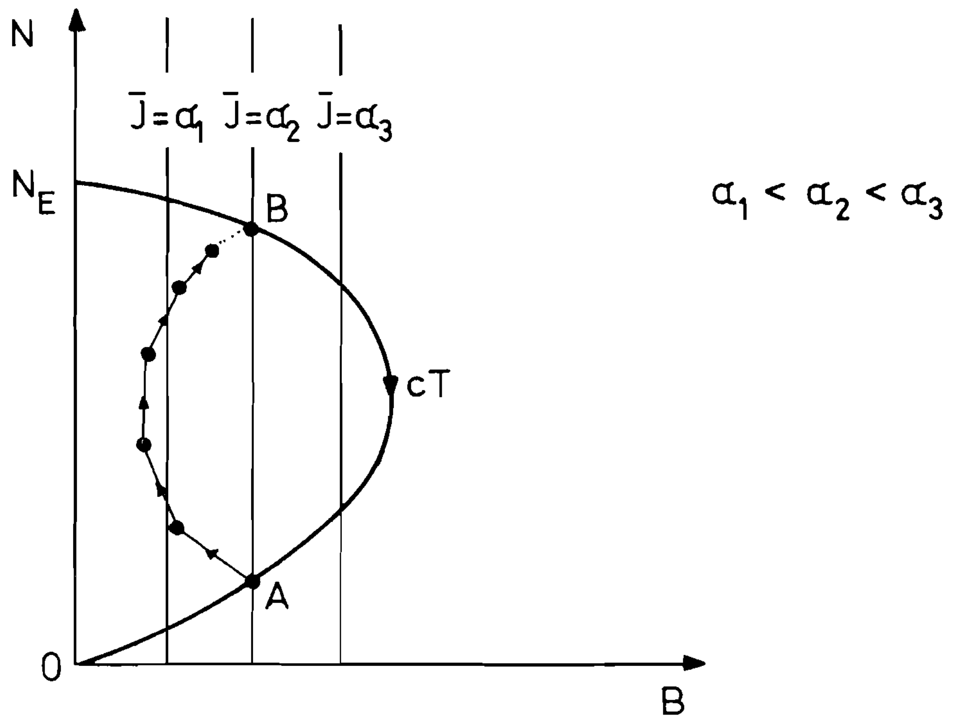


FIGURE 10. EVOLUTION OF THE FISHERY FROM PRODUCTIVE EQUILIBRIUM A TO PRODUCTIVE EQUILIBRIUM B.

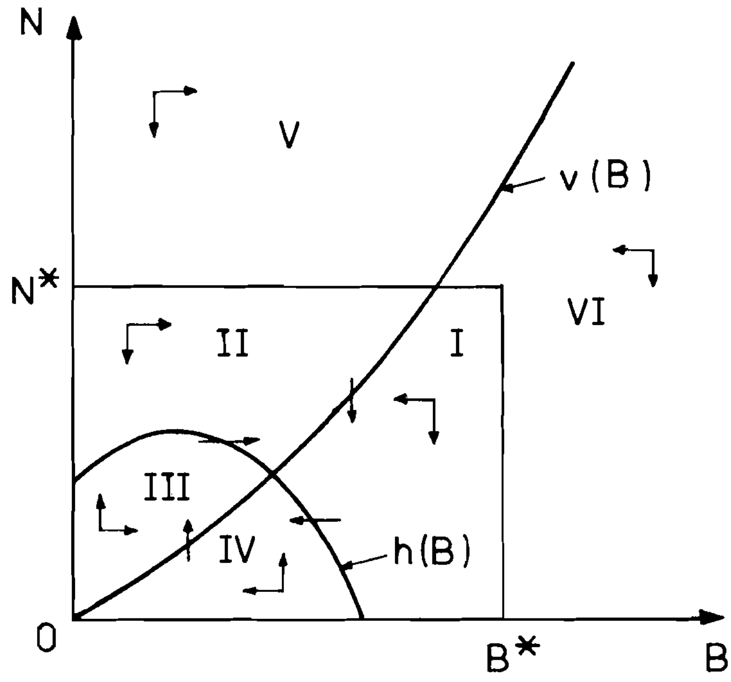
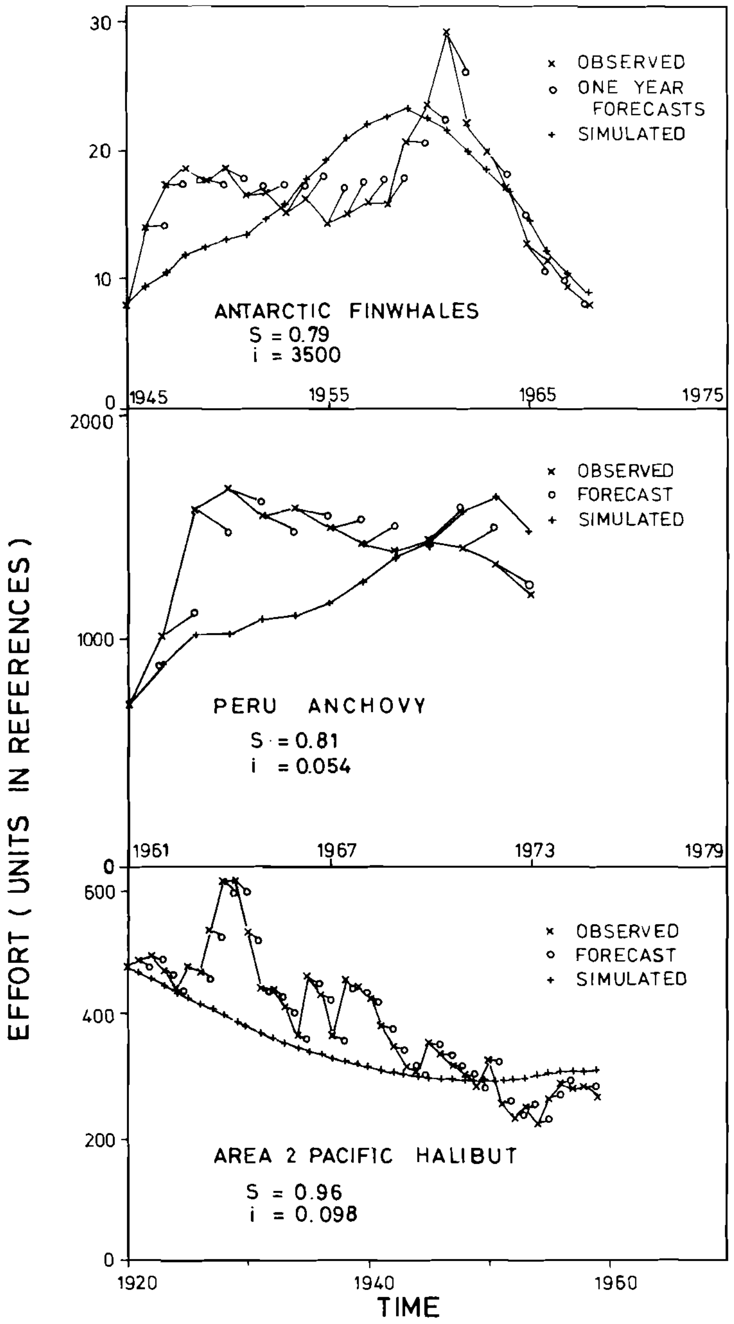


FIGURE 11. THE REGION OF ATTRACTION.



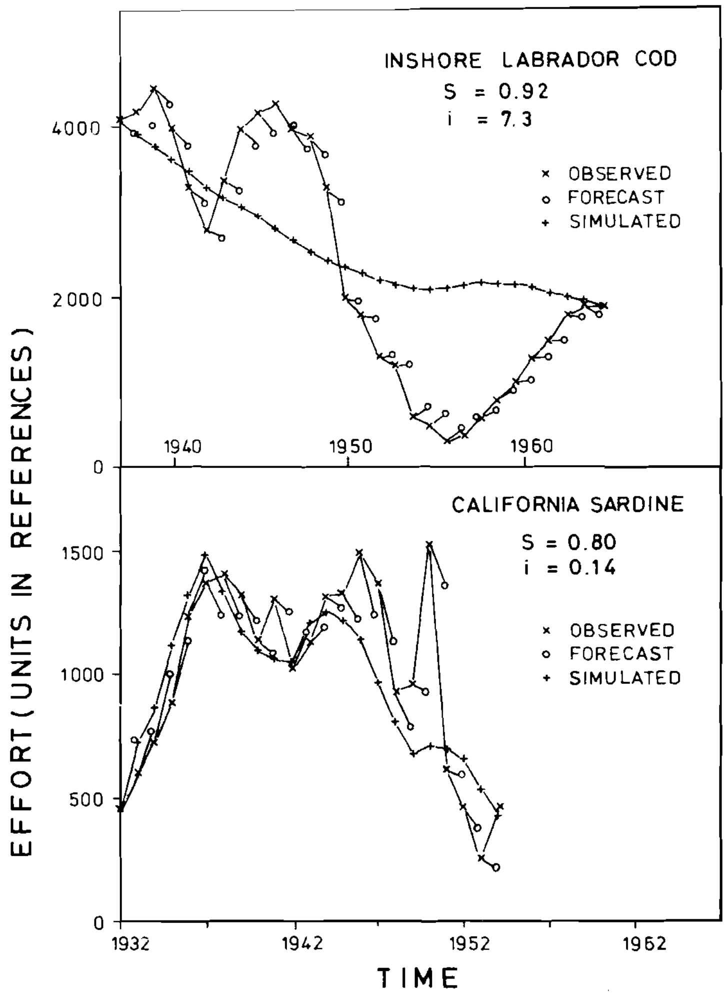


FIGURE 12. EXAMPLE OF PARAMETER IDENTIFICATION FOR THE FLEET DYNAMICS EQUATION. "FORECAST" VALUES ARE PREDICTED FROM PREVIOUS OBSERVED VALUES; "SIMULATED" VALUES ARE PREDICTED FROM PREVIOUS PREDICTED VALUES. DATA SOURCES WERE: WHALES-INTL.WHALING COMMISSION [6]; ANCHOVY-ANCHOVETA EXPERT PANEL [4]; HALIBUT-CHAPMAN ET AL [1]; COD-MAY [8]; SARDINE-MURPHY [9].

APPENDIX 1

Let λ_1, λ_2 be the eigenvalues of the system obtained by linearization around the productive equilibrium $(\bar{B},\bar{N})$. Moreover, let

$$\Sigma = \lambda_1 + \lambda_2 \quad , \qquad \Pi = \lambda_1 \lambda_2 \quad ,$$

and suppose

$$0 < a < 2 \quad , \qquad 0 < s < 1 \quad , \qquad \bar{B}cT \leq 1 \quad .$$

The aim of this appendix is to prove that

a) $|\Pi| < 1$,

b) $|\Sigma| < 1 + \Pi$.

Proof of a)

First of all recall that Π is the determinant of the matrix F, i.e.

$$\Pi = (1 - a + \bar{B}cT)(icT \frac{\bar{N}}{\bar{B}} + 2s - 1) \quad ;$$

since $\bar{B}cT < a$ (easy to check),

$$-1 < 1 - a < 1 - a + \bar{B}cT < 1 \quad .$$

Therefore, it is sufficient to prove that

$$-1 < icT \frac{\bar{N}}{\bar{B}} + 2s - 1 < 1 \quad ,$$

or, replacing $\bar{N}$ with $v(\bar{B})$ given by Eq. (9a),

$$-1 < (1 - s)\frac{\bar{B}cT}{1 - \exp(-\bar{B}cT)} + 2s - 1 < 1 \quad . \qquad \text{(A1)}$$

Notice that $\dfrac{\bar{B}cT}{1 - \exp(-\bar{B}cT)}$ is an increasing function of $\bar{B}cT$; hence, since $0 \leq \bar{B}cT \leq 1$, its minimum value is 1 (for $\bar{B}cT = 0$) and its maximum value is $\dfrac{1}{1 - \exp(-1)}$ (for $\bar{B}cT = 1$). Thus, the first inequality in (A1) is proved. As for the second one, note that

$$(1 - s)\frac{\bar{B}cT}{1 - \exp(-\bar{B}cT)} + 2s - 1 < \frac{1 - s}{1 - \exp(-1)}$$

$$+ 2s - s = \frac{(1 - 2\exp(-1))s + \exp(-1)}{1 - \exp(-1)} \quad .$$

But since $0 < s < 1$, it follows that

$$(1 - 2\exp(-1))s + \exp(-1) < 1 - \exp(-1) \quad ,$$

which implies the second inequality in (A1).

Proof of b)

Remember that Σ is the trace of F, i.e.

$$\Sigma = 2s - a + \bar{B}cT\left(s + i\frac{\bar{N}}{\bar{B}^2}\right) \quad .$$

Let us first prove that

$$-1 - \Pi < \Sigma \quad .$$

In fact

$$-1 - \Pi - \Sigma = -1 - icT \frac{\bar{N}}{\bar{B}} - 2s + 1 + aicT \frac{\bar{N}}{\bar{B}}$$

$$+ 2sa - a - ic^2T^2\bar{N} - 2s\bar{B}cT$$

$$+ \bar{B}cT - 2s + a - s\bar{B}cT - icT \frac{\bar{N}}{\bar{B}} \quad ,$$

or substituting $\bar{N}$ with $v(\bar{B})$,

$$1 + \Pi + \Sigma = 2s(2 - a) - \bar{B}cT(1 - 3s)$$

$$+ (2 - a)(1 - s) \cdot \frac{\bar{B}cT}{1 - \exp(\bar{B}cT)}$$

$$+ \frac{(1 - s)(\bar{B}cT)^2}{1 - \exp(-\bar{B}cT)} \quad .$$

If $3s - 1 > 0$, of course $1 + \Pi + \Sigma > 0$; otherwise, notice that

$$\frac{(1 - s)(\bar{B}cT)^2}{1 - \exp(-\bar{B}cT)} > (1 - 3s)\ \bar{B}cT \quad ,$$

so that $1 + \Pi + \Sigma > 0$.

Now, it must be proved that $\Sigma < 1 + \Pi$. After some cumbersome computations, one obtains

$$\Sigma - 1 - \Pi = (1 - s)\left(\bar{B}cT\left[\frac{a - \bar{B}cT}{1 - \exp(-\bar{B}cT)} + 1\right] - 2a\right) \quad , \tag{A2}$$

and, since $s < 1$, the second term of the right-hand side of Eq. (A2) must be proved to be negative. Now, since

$$\frac{a - \bar{B}cT}{1 - \exp(-\bar{B}cT)} < \frac{a - \bar{B}cT}{\bar{B}cT - \frac{(\bar{B}cT)^2}{2}} \quad ,$$

it turns out that

$$\bar{B}cT\left[\frac{a - \bar{B}cT}{1 - \exp\,(-\bar{B}ct)} + 1\right] - 2a < \frac{2a - (\bar{B}cT)^2}{2 - \bar{B}cT} - 2a$$

$$= \frac{-2a(1 - \bar{B}cT) - (\bar{B}cT)^2}{2 - \bar{B}cT} \quad ,$$

and the last expression, in view of the assumption $\bar{B}cT < 1$, is negative.

APPENDIX 2

In this appendix the region R given by

$$0 \leq N \leq \frac{N_E}{a} \exp\,(2a - 1) = N^*$$

$$0 \leq B \leq s\sqrt{\frac{i\,N_E \exp\,(2a - 1)}{a(1 - s)}} + icT\,\frac{N_E}{a} \exp\,(2a - 1) = B^*$$

is proved to be a region of attraction.

To achieve this purpose it is necessary to prove that

a) any trajectory starting from a point in R is contained in R,

b) any trajectory starting from the outside of R reaches R in a finite number of transitions.

Proof of a)

First of all, notice that if $N_t \geq 0$, $B_t \geq 0$, then $N_{t+1} \geq 0$, $B_{t+1} \geq 0$ (this follows trivially from Eqs. (2)-(3)). Therefore, a) is proved once it is proved that $N_t \leq N^*$ and $B_t \leq B^*$ imply $N_{t+1} \leq N^*$ and $B_{t+1} \leq B^*$. An inspection of Fig. 11 (where the arrows show the direction of the transitions) suggests that the last statement is proved if

i) (N_t, B_t) belonging to regions II or III implies $B_{t+1} \leq B^*$, and

ii) (N_t, B_t) belonging to regions III or IV implies $N_{t+1} \leq N^*$.

In order to prove i) notice that (N_t, B_t) belonging to region II or III is equivalent to

$$\frac{1 - s}{i}\;\frac{B_t^2}{1 - \exp\,(-cB_tT)} < N_t \leq N^* \quad , \quad B_t > 0 \quad .$$

From equation

$$B_{t+1} = sB_t + iN_t \left(\frac{1 - \exp(-cB_tT)}{B_t} \right)$$

it follows that

$$B_{t+1} \leq sB_t + icTN_t \quad .$$

But

$$B_t^2 < \frac{i}{1-s} (1 - \exp(-cB_tT))N_t \leq \frac{i}{1-s} N_t \quad ,$$

i.e.

$$B_t < \sqrt{\frac{i}{1-s} N_t} \quad .$$

Then

$$B_{t+1} < s \sqrt{\frac{i}{1-s} N_t} + icTN_t$$

and, since $N_t \leq N^*$, it follows that $B_{t+1} < B^*$. To prove ii), recall that

$$N_{t+1} = N_t \exp \left[a - cB_tT - a \frac{N_t}{N_E} \exp(-cB_tT) \right] \quad .$$

Since $N_t \geq 0$ and $B_t \geq 0$ it turns out that

$$N_{t+1} \leq \exp(a) N_t \quad .$$

On the other hand, if (N_t, B_t) belongs to regions III or IV, then $N_t \leq \frac{N_E}{a} \exp(a-1)$ (see Fig. 11). Therefore, it follows that $N_{t+1} \leq N^*$.

Proof of b)

Consider Fig. 11 and notice that in regions V and VI there is no equilibrium state and no cycle, since every transition starting from there is characterized by a decrease of N. Therefore, a trajectory starting from outside of R will reach, after a finite number of transitions, a point (B_t, N_t) such that $N_t < N^*$. If (B_t, N_t) belongs to R, property (b) is proved; otherwise it must belong to region VI, and therefore, after a suitable number of transitions, will be: $B_t < B^*$, i.e. $(B_t, N_t) \varepsilon R$.

References

[1] Chapman, D.G., Myre, R.J. and Southwood, G.M. "Utilization of Pacific Halibut Stocks: Estimation of MSY, 1960." Rep. Internat. Hal. Comm., No. 31, 1962.

[2] Cushing, D.H. and Harris, J.G.K. "Stock and Recruitment and the Problem of Density Dependence." In Fish Stocks and Recruitment. B.B. Parrish, P.A. Larkin and W.E. Ricker, Eds. Denmark, Conseil Int. pour l'Exploration de la Mer, 1973.

[3] Dahlberg, M.L. "Stock and Recruitment Relationships and Optimum Escapements of Sockeye Salmon Stocks of the Chignik Lakes, Alaska." In Fish Stocks and Recruitment. B.B. Parrish, P.A. Larkin and W.E. Ricker, Eds. Denmark, Conseil Int. pour l'Exploration de la Mer, 1973.

[4] Expert Panel on Stock Assessment, 1974. "Report of the Second Session of the Panel of Experts on Stock Assessment on Peruvian Anchoveta." Bol. Inst. Mar. Peru-Callo, 2, 10 (1974), 605-719.

[5] Fullenbaum, R., Carlson, E., Bell, F. and Smith, V.L. "Economics of Production from Natural Resources: Comment and Reply." Amer. Econ. Rev., 61 (1971), 483-491.

[6] International Whaling Commission. "Twentieth Report." 1969.

[7] Lee, R.C.K. Optimal Estimation, Identification and Control. Cambridge, Mass., M.I.T. Press, 1964.

[8] May, A.W. "Effect of Offshore Fishing on the Inshore Labrador Cod Fishery." ICNAF Research Bul., 4 (1967), 67-74.

[9] Murphy, G.I. "Population Biology of the Pacific Sardine (Sardings caervlea)." Proc. Calif. Acad. Sci., 4, 34 (1966), 1-84.

[10] Paloheimo, J.E. and Dickie, L.M. "Abundance and Fishing Success." Denmark, Conseil Int. pour l'Exploration de la Mer, 155 (1964), 152-163.

[11] Smith, V.L. "Economics of Production from Natural Resources." Amer. Econ. Rev., 58 (1968), 409-431.

[12] Wang, A. "Economics of Production from Fishery Resources." Ph.D. Dissertation, Cincinnati, Ohio, University of Cincinnati, 1974.