# SYSTEM SIMILARITIES AND NATURAL LAWS*

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ABSTRACT

The problem of the existence of natural "laws" in the social and behavioral sicences as opposed to empirical relations is considered. It is shown that this question is intimately tied-up with the question of when two systems are equivalent. Tools from the mathematical theory of singularities of smooth mappings are employed to formalize the equivalence issue and to provide an operational mathematical basis for investigating the existence of laws of nature. The theory developed is applied to a spectrum of situations arising in water resource analysis, forestry management and urban development.

# System Similarities and the Existence of Natural Laws

by

John Casti

## 1. Systems and Laws of Nature

One of the main pillars upon which the edifice of modern theoretical physics rests is the existence of "laws" of nature, which specify fixed relationships that must obtain between the variables describing particular physical processes. Laws such as the conservation of mass-energy, conservation of charge, conservation of spin and so forth, provide the foothold upon which mathematical models of a dazzling degree of fidelity are based and, in general, seem to account for much of what we can legitimately say we "know" about relationships in the physical world, at least from the modeling point of view.

In an earlier paper [1], we have argued that natural laws, as that term is understood in physics, do not exist in the social and behavioral sciences and this "lawless" state accounts for much of the difficulty in trying to mimic the techniques of physics for modeling human affairs. Our contention was that in order to qualify as a law of nature, a relationship between variables must possess the properties of independence, invariance and analyticity. Roughly speaking, the first condition means that the relationship must not depend upon the particular physical system in which the related quantities are used (e.g.

we cannot have one version of Henry's law for inductors in a TV set and another for the same inductor in a car stereo). The second requirement simply means that the expression of the law must not depend upon the coordinate system chosen to express it, i.e. the law is a coordinate-free relation between variables. Finally, analyticity implies that local space-time information is sufficient to impose a global rigidity upon the system and it is not necessary to explicitly account for what is happening "at infinity" in the expression of the law.

In this paper we propose a framework suitable for mathematically formalizing the above concepts and to examine the issue of the existence of law outside the physical sciences. As indicated below, the formalization of the notion of a natural law involves the idea of system similarity and is closely connected with the concept of a bifurcation of one system description (or model) from another. During the course of investigating the existence of laws, it will turn out that many of the results from the theory of singularities of smooth mappings play an important role in providing the necessary mathematical underpinnings to make our ideas operational. For this reason, the bare essentials of this theory are sketched in a later section of the paper. Finally, the general ideas are employed to study problems in water basin characteristics, forest growth relationships and urban population migration as illustrations of the difference between a natural law and an empirical relationship.

## 2. System Descriptions and Models

We begin with a natural system $\Sigma$ which, for our purposes, is assumed to be described by some set of real-valued observables $\{f_i\}$ defined on an abstract set of states X, i.e.

$$f_i : X \to R \quad , \qquad i = 1,2,\ldots .$$

For purposes of modeling $\Sigma$, we extract a finite subset of these observables $f = \{f_1,\ldots,f_n\}$, say, and write an abstraction of $\Sigma$ as

$$f : X \to Y \quad ,$$

where $Y \subset R^n$. A mathematical model of $\Sigma$ is then a translation or, following the terminology of Rosen [2], an encoding of the abstraction of $\Sigma$ into some formal mathematical system M, and a subsequent retranslation or decoding of the theorems of M back into properties (relations) of the observables f. The basic situation is depicted in Figure 1. The essence of the modeling relation lies in making effective choices of the formal system M

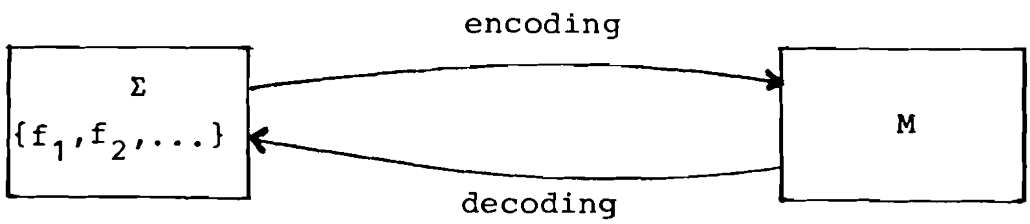


Figure 1. Modeling Relation

and the encoding/decoding operations. For our purposes in this paper, it will not be necessary to pass from an abstraction of $\Sigma$ to its formal model M, but the above diagram should always be kept in mind as a reminder of the intrinsic duality between the real-world of $\Sigma$ and its description in terms of observables, and the abstract world of M and its formal mathematical structures.

Now let $\{f_1,\ldots,f_n\}$ be some abstraction of $\Sigma$, and assume that there exist m relations $\{\Phi_i\}$ linking the observables f, i.e. we have real-valued functions

$$\Phi_i(f_1, f_2,\ldots,f_n) = 0 \quad , \qquad i = 1,2,\ldots,m \quad .$$

The relations $\{\Phi_i\}$ are termed the <u>equations of state</u> for $\Sigma$. Since each $f_i$ is a real-valued function, we can more compactly represent the equations of state as

$$\Phi : R^n \to R^m$$

Clearly, the <u>structure</u> of $\Sigma$ is contained in $\Phi$, while f represents only the measurement process. Thus in seeking natural laws for $\Sigma$, we shall focus attention upon $\Phi$ and, by abuse of terminology, call $\Phi$ a <u>description</u> of $\Sigma$. Our first item of business is to consider the question: when are two descriptions of $\Sigma$, $\Phi$ and $\hat{\Phi}$ <u>equivalent</u>.

Let us imagine that instead of using the observables f and the description $\Phi$, we choose an alternate set of observables $\hat{f} = \{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\}$ and a description $\hat{\Phi} = \{\hat{\Phi}_1, \ldots, \hat{\Phi}_m\}$ of $\Sigma$. i.e.

$$\hat{\Phi} : R^n \to R^m \quad .$$

Now we ask for conditions under which the two descriptions $\Phi$ and $\hat{\Phi}$ are equivalent. Diagrammatically, we seek maps g and h such that the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\Phi} & R^m \\ g \downarrow & & \downarrow h \\ R^n & \xrightarrow[\hat{\Phi}]{} & R^m \end{array}$$

commutes. Thus, we count the descriptions $\Phi$, $\hat{\Phi}$ equivalent if by coordinate changes in the domain and range, we can make $\Phi$ "look like" $\hat{\Phi}$, and conversly. If no such g and h exist, then the descriptions $\Phi$ and $\hat{\Phi}$ convey essentially different, or inequivalent, information about $\Sigma$ and there is a gain in knowledge about $\Sigma$ by employing both descriptions; otherwise, the use of both $\Phi$ and $\hat{\Phi}$ is redundant.

Algebraically, the condition for equivalence is

$$\Phi \circ h = g \circ \hat{\Phi} \quad .$$

As an elementary illustration, consider the case n = 2, m = 1 with

$$\Phi(f_1,f_2) = f_2, \qquad \hat{\Phi}(\hat{f}_1,\hat{f}_2) = \hat{f}_1^2 + \hat{f}_2 \quad .$$

The transformation g defined as

$$\hat{f}_1 = f_1,$$
$$\hat{f}_2 = - f_1^2 + f_2$$

with h = identity, transforms $\Phi$ into $\hat{\Phi}$ as the calculation

$$\hat{\Phi}(\hat{f}_1,\hat{f}_2) = \hat{f}_1^2 + \hat{f}_2 = f_2 = \Phi(f_1,f_2)$$

demonstrates. Thus, the two descriptions $\Phi$ and $\hat{\Phi}$ are equivalent.

The concept of <u>bifurcation</u> of descriptions arises precisely when two descriptions are <u>not</u> equivalent. In this case, there is essential information contained in one description that cannot be obtained from the other by shifting to a new view of $\Sigma$ via transformations g and h. In this case, we say that the description $\Phi$ <u>bifurcates</u> from $\hat{\Phi}$.

In general, in order to provide specific testable conditions under which $\Phi \sim \hat{\Phi}$, we must be more specific about the mathematical properties of the maps $\Phi$, $\hat{\Phi}$ and the admissible class of coordinate tranformations. For a variety of mathematical reasons, it is convenient to require that the functions $\Phi$, $\hat{\Phi}$ be <u>smooth</u>, i.e. $C^\infty$, with the coordinate transformations g, h being diffeomorphisms. Happily, this purely mathematical requirements coincides nicely with our earlier condition that any natural law should possess the property of analyticity. In what follows, we shall express results for smooth $\Phi$, $\hat{\Phi}$ and, hence, for analytic (or real-analytic) $\Phi$, $\hat{\Phi}$. If there is an open neighborhood* U of $\Phi$ in the space of smooth functions such that $\Phi$ is equivalent to each $\hat{\Phi} \in U$, then we call $\Phi$ stable. Thus, the unstable smooth functions represent bifurcation "points" in $C^{\infty}$.

Sometimes instead of fixed functions $\Phi$, $\hat{\Phi}$, we wish to consider parametrized families of functions where now

$$\Phi = \Phi_{\alpha}(f) \quad , \qquad \hat{\Phi}_{\hat{\alpha}} = \hat{\Phi}(\hat{f}) \quad ,$$

with $\alpha$, $\hat{\alpha} \in R^k$ being vectors of parameters. In this case we have the diagram

$$\begin{array}{ccc} R^n \times R^k & \xrightarrow{\Phi_{\alpha}(f)} & R^m \\ g \downarrow & & \downarrow h \\ R^n \times R^k & \xrightarrow[\hat{\Phi}_{\hat{\alpha}}(\hat{f})]{} & R^m \end{array}$$

where now the coordinate change g is such that it acts on the product space $R^n \times R^k$ in the usual fashion. In other words, the families $\Phi_{\alpha}(f)$, $\hat{\Phi}_{\hat{\alpha}}(\hat{f})$ are equivalent if there exist diffeomorphisms $g = (g_1, g_2)$ and h such that

$$g_1 : f \to \hat{f}$$

$$g_2 : \alpha \to \hat{\alpha}$$

$$h : \Phi \to \hat{\Phi}$$

making the above diagram commute.

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*In the so-called Whitney topology on the space of smooth maps. See [2-5] for a detailed definition.

The central questions that now arise are:

A) (Determinacy Problem). Given $\Phi$, $\hat{\Phi}$ (or $\Phi_\alpha$, $\hat{\Phi}_{\hat{\alpha}}$), how can we tell whether or not they are equivalent or, what is the same thing;

A') Given $\Phi$ (or $\Phi_\alpha$), what are all $\hat{\Phi}$ ($\hat{\Phi}_{\hat{\alpha}}$) that are equivalent to it?

B) (Classification Problem). In the equivalence class of $\Phi$, what is "simplest" or canonical representative of that class?

C) (Unfolding Problem). If $\Phi$ is unstable, what is the minimal-parameter family, $\Phi_\alpha$ that we can imbed $\Phi$ within, so that $\{\Phi_\alpha\}$ is stable as a family of maps?

The theory of singularities os smooth mappings has been created specifically to answer these questions.

## 3. Singularity Theory

Here we briefly review elementary aspects of the theory of stable mappings and singularity theory, primarily to give the flavor of the type of results obtainable from the full machinery of singularity theory. Since a full account of the theory is far beyond the scope of this paper, the reader is urged to consult [2-5] for more detailed information.

Let us consider a smooth function $\phi : M \to N$, when M, N are smooth manifolds. Let

$$J\phi(x_0) = \left[\frac{\partial \phi_i}{\partial x_j}\right](x_0)$$

be the Jacobian matrix of $\phi$ at $x_0 \in M$. We assume that $J\phi(x_0)$ is of maximal rank, i.e. min {dim M, dim N} = rank $J\phi(x_0)$. Then we call $\phi$ an immersion at $x_0$ if dim M $\leq$ dim N, a submersion at $x_0$ if dim M $\geq$ dim N and a local diffeomorphism at $x_0$ if dim M = dim N and $\phi$ is a bijective immersion at $x_0$.

The map $\phi : R^n \to R^m$ is called stable at a point $x_0$ if there is a neighborhood U of $x_0$ such that for any neighborhood $\bar{U}$ of $x_0$

contained in U, and for any perturbation $\bar{\phi}$ of $\phi$, there is a point $\bar{x}_0 \in \bar{U}$ and local diffeomorphisms g and h such that the diagram

$$\begin{array}{ccc} (R^n, x_0) & \xrightarrow{\ \phi\ } & (R^m, \phi(x_0)) \\ g \downarrow & & \downarrow h \\ (R^n, \bar{x}_0) & \xrightarrow[\ \bar{\phi}\ ]{} & (R^m, \bar{\phi}(\bar{x}_0)) \end{array}$$

commutes.

The importance of immersions and submersions resides in the following *global* stability results of Mather.

*Theorem 1. Let N be a compact subset of* $R^n$ *and let* $\phi : N \to R^m$ *be a one-to-one immersion. Then* $\phi$ *is stable. Furthermore, if* $m \geq 2 \dim N + 1$*, then* $\phi$ *is a one-to-one immersion if and only if* $\phi$ *is stable.*

*Theorem 2. Let* $\phi : R^n \to R^m$ *be a submersion. Then* $\phi$ *is stable.*

These results are important because they are *global* conclusions from *local* conditions.

Locally, we can use the Implicit Function Theorem to obtain similar conclusions.

*Theorem 3. Let* $\phi : N \to R^m$*, where* $N \subset R^n$ *and let* $\phi$ *be an immersion at* $x_0$*. Then* $\phi$ *is locally stable at* $x_0$ *and there exists a coordinate change* $h : R^m \to R^m$ *such that* $\phi$ *takes the form*

$$h \circ \phi(x_1,\ldots,x_n) = (x_1,\ldots,x_n, 0,\ldots,0) .$$

*Theorem 4. Let* $\phi : R^n \to R^m$ *be a smooth submersion at* $x_0 \in U \subset R^n$*. Then* $\phi$ *is locally stable at* $x_0$ *and there is a coordinate change* $g : R^n \to R^n$ *such that*

$$\phi \circ g(x_1,x_2,\ldots,x_n) = (x_{n-m+1},\ldots,x_n) .$$

For immersions and submersions, the above results dispose of the questions of stability and "simple" representatives for equivalence classes. However, in general we cannot expect maps to be either immersions or submersions. In this event $J\phi(x_0)$ will not have maximal rank and we are naturally led to the idea of a singular point of $\phi$. A point $x_0$ is called a singular point for $\phi$ if rank $J\phi(x_0) < \min\{\dim M, \dim N\}$. Otherwise, $x_0$ is a regular point. We study the stability of $\phi$ near a singular point.

The simplest case, and the one that serves as the model to motivate more general situations, is when $\phi$ is a function, i.e. $m = 1$. The classical situation studied by Morse is when $x_0$ is a singular point and the Hessian of $\phi$ at $x_0$ is nonsingular, in which case $x_0$ is called a non-degenerate critical point. If all the singular points of $\phi$ are non-degenerate critical points, we call $\phi$ a Morse function. The basic stability result for Morse functions is Morse's Lemma.

Theorem 5. $\phi : R^n \to R$ is stable if and only if $\phi$ is a Morse function and the critical values of $\phi$ are distinct; (i.e. if $x_0$, $x_0^*$ are distinct non-degenerate critical points, then $\phi(x_0) \neq \phi(x_0^*)$.

Furthermore, the Morse functions form an open, dense set in $C^\infty(R^n, R)$.

Finally, in the neighborhood of any non-degenerate critical point, there exists an integer $k$, $0 \leq k \leq n$, and a coordinate change $g : R^n \to R^n$, such that

$$\phi \circ g(x_1, x_2, \ldots, x_n) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2 .$$

So, by Morse's Lemma any smooth function can be locally approximated by a Morse function which, in turn, can be made to look like a non-degenerate quadratic form in the neighborhood of a non-degenerate critical point. Furthermore, the only stable smooth functions are of this type. We note in passing that this result forms the mathematical basis for why the so-called "laws" of classical physics all turn out to be expressed as quadratic forms.

While Morse's Lemma, in effect, closes out the stability problem for functions, the following result of Whitney for maps of $R^2 \to R^2$ served as the starting point for what is now the theory of singularities.

Theorem 6. Let M be a compact subset of $R^2$ and let $\phi: M \to R^2$, i.e. $\phi(x,y) \to (u(x,y), v(x,y))$, $\phi$ smooth.

(1) Then $\phi$ is stable at $(x_0,y_0) \in M$ if and only if near $x_0$ $\phi$ is equivalent to one of the three mappings:

(a) $u = x, \quad v = y$, (regular point)

(b) $u = x^2, \quad v = y$, (fold point)

(c) $u = xy-x^3, \quad v = y$, (cusp point).

(2) The stable maps $\phi: M \to R^2$ form an everywhere dense set in $C^\infty(R^2,R^2)$.

(3) $\phi$ is globally stable if and only if

(a) $\phi$ is stable at each point of M

and

(b) The images of folds intersect only pairwise and at non-zero angles and the images of folds do not intersect images of cusps.

Stability and density conditions for general maps $\phi: R^n \to R^m$ require introduction of technical concepts beyond the scope of this paper for their precise statement. However, we can indicate the nature of these conditions in slightly imprecise, but straightforward language. The basic idea, due to Mather, is to try to provide conditions under which the stability of $\phi$ at a point is equivalent to the general stability of $\phi$, and to determine conditions on a finite set of derivatives of $\phi$ that imply point stability.

Let us assume that $\phi(0) = 0$ and let $C_0^\infty(R^n)$ denote functions on $R^n \to R$ that are smooth at 0. Then we have the following characterization of stability of $\phi$ due to Arnol'd: if for every function $\psi \in C_0^\infty(R^n)$ there exists an nxm matrix H of $C_0^\infty(R^n)$ functions and an mxm matrix K of $C_0^\infty(R^m)$ function such that

$$\psi I_m = (J\phi)H + K \circ \phi + 0(|x|^{m+1}), \quad (*)$$

then $\phi$ is stable, where $I_m$ = mxm identity. In other words, if the equation (*) is solvable by matrix functions H and K up to order m = dim $R^m$ for every smooth $\psi \in C_0^\infty (R^n)$, then $\phi$ is stable, and conversly.

The importance of Arnol'd's result is that it provides a necessary and sufficient test for stability of $\phi$ involving only the properties of $\phi$ and its first derivatives at 0, i.e. local properties of $\phi$ are sufficient to provide a global result.

As an example, consider the case when m = 1, n = 2, $\phi(x_1,x_2) = 1/3 \ (x_1^3 + x_2^3)$. Then $J\phi = (x_1^2,x_2^2)$ and Arnol'd's condition for stability of $\phi$ is that for every $\psi \in C_0^\infty(R^2)$, we must have

$$\psi = h_1x_1^2 + h_2x_2^2 + 1/3k(x_1^2 + x_2^2) + 0(|x|^2)$$

for $h_1,h_2,k \in C_0^\infty (R)$. It is clear that this function is not stable, since linear functions $\psi$ cannot satisfy the condition. Of course, this result could have been obtained from Theorem 5, using the fact that the only critical point of $\phi$ is the origin, which is a degenerate critical point.

The failure of $\phi$ to be stable leads to the question of "unfolding", i.e. the existence of a parametrized family of maps containing $\phi$, such that the family is stable in the sense discussed in Section 2. In order to deal with this question, it is necessary to consider the idea of the codimension of $\phi$. For simplicity, we consider only the case when $\phi$ is a function (m = 1), although the general idea can be extended to maps with additional technical effort.

The Jacobian ideal $\Delta(\phi)$ of $\phi$ is the set of all smooth functions expressible as

$$g_1 \frac{\partial \phi}{\partial x_1} + g_2 \frac{\partial \phi}{\partial x_2} + \ldots + g_n \frac{\partial \phi}{\partial x_n} ,$$

where $g_i$ are arbitrary smooth functions. Let

$$m_n = \{\phi \in C_0^\infty(R^n) : \phi(0) = 0\} .$$

Then the codimension of $\phi$ is defined as

$$\text{cod } (\phi) = \dim_R m_n/\Delta(\phi) .$$

Roughly speaking, the codimensions measures the number of independent directions in $C_0^\infty(R^n)$ "missing" from $\Delta(\phi)$. As an example, consider the function used earlier

$$\phi(x_1,x_2) = 1/3(x_1^3 + x_2^3) .$$

Then

$$\frac{\partial \phi}{\partial x_1} = x_1^2 , \qquad \frac{\partial \phi}{\partial x_2} = x_2^2$$

and we see that a basis for $m_n/\Delta(\phi)$ is given by the elements $\{x_1,x_2,x_1x_2\}$. (By convention, the constants generated by 1 are not considered in the calculations). So, the cod $\phi = 3$. On the other hand, the function $\phi(x_1,x_2) = x_1^2\, x_2$ has cod $\phi = \infty$.

One of the deepest and most far-reaching theorems of singularity theory is the result that an unstable function $\phi$ can be stabilized by imbedding it into a k-parameter stable family of functions and that the smallest k that will work is k = cod $\phi$. Furthermore, the parameters can be made to appear linearly and the new functions that must be added to $\phi$ to define the family are exactly these comprising a basis for $m_n/\Delta(\phi)$. Such a stable family is called a universal unfolding of $\phi$. So, for the unstable function $\phi = 1/3(x_1^3 + x_2^3)$, a universal unfolding is given by

$$1/3(x_1^3 + x_2^3) + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_1 x_2 \quad ,$$

$\alpha_i \in R$.

Another way of viewing the unfolding concept, and one that is somewhat more directly related to our goals in this paper, is as follows. Imagine we have the function $\phi(x)$ and we perturb it by a smooth perturbation p(x) to obtain a new function $\hat{\phi}(x)$. Assume that cod $(\phi) = c$ and that the functions $u_1(x),\ldots,u_c(x)$ form a basis for $m_n/\Delta(\phi)$. Then we can write

$$\hat{\phi}(x) = \phi(x) + p(x) = \phi(x) + \sum_{i=1}^{c} \alpha_i u_i(x) + z(x)$$

The basic unfolding theorem then guarantees that in a suitable coordinate system $z(x) \equiv 0$. That is, $\hat{\phi}$ is equivalent to $\phi$ modulo the unfolding terms. All of the "directions" in the perturbation p(x) can be removed by a suitable coordinate change with the exceptions of those directions represented by the $\{u_i(x)\}$. Thus, the universal unfolding of $\phi$ represents the entire family of functions that are equivalent to $\phi$.

For special classes of maps (submersions, immersions, Morse functions, etc.), we have seen earlier that coordinate changes can be found such that a map of the given class can be made equivalent to a certain "simple" canonical form. Now we examine the general classification problem, i.e. for given n,m, determine the number of equivalence classes of smooth maps of $R^n \to R^m$ and determine a "simple" representative, or canonical form, for each class.

A related question that arises is whether or not the stable maps form a dense set in the space of maps from $R^n \to R^m$, i.e. whether any map can be approximated arbitrarily closely by a stable map. We have already seen that if $m = 1$, then the Morse functions are stable and form such a dense set. The answer to the density question can be shown to depend upon the values of $(n,m)$. The basic result is

*Theorem 7. Let $q = m - n$. Then the stable smooth maps of $R^n \to R^m$ are dense if*

(1) $q \geq 4$, $m < 7q + 8$;

(2) $q = 0,1,2,3$, $m < 7q + 9$;

(3) $q = -1$, $m < 8$;

(4) $q = -2$, $m < 6$;

(5) $q \leq -3$, $m < 7$ .

*Corollary. Stable smooth maps of $R^n \to R^m$ are always dense if $n \leq 7$ or $m \leq 5$.*

General classification results exist in the literature. As an illustration of the type of results available, Table 1 shows the classification of all stable maps $R^n \to R^m$ for $n, m \leq 4$.

Table 1. Canonical Stable Maps $R^n \to R^m$, $m, n \leq 4$

| SPACES | CANONICAL STABLE MAPS |
|---|---|
| $R \to R$ | $x \to x$ |
| | $x \to x^2$ |
| $R \to R^2$ | $x \to (x,0)$ |
| $R \to R^3$ | $x \to (x,0,0)$ |
| $R \to R^4$ | $x \to (x,0,0,0)$ |
| $R^2 \to R$ | $(x,y) \to x$ |
| | $(x,y) \to \pm x^2 \pm y^2$ |

Table 1 contd.

| | |
|---|---|
| $R^2 \to R^2$ | $(x,y) \to (x,y)$ |
| | $(x,y) \to (x,y^2)$ |
| | $(x,y) \to (x,y^3 + xy)$ |
| $R^2 \to R^3$ | $(x,y) \to (x,y,0)$ |
| | $(x,y) \to (x,xy,y^2)$ |
| $R^2 \to R^4$ | $(x,y) \to (x,y,0,0)$ |
| $R^3 \to R$ | $(x,y,z) \to x$ |
| | $(x,y,z) \to \pm x^2 \pm y^2 \pm z^2$ |
| $R^3 \to R^2$ | $(x,y,z) \to (x,y)$ |
| | $(x,y,z) \to (x,\pm y^2 \pm z^2)$ |
| $R^3 \to R^3$ | $(x,y,z) \to (x,y,z)$ |
| | $(x,y,z) \to (x,y,z^2)$ |
| | $(x,y,z) \to (x,y,z^3+xz)$ |
| | $(x,y,z) \to (x,y,z^4+xz+yz^2)$ |
| $R^3 \to R^4$ | $(x,y,z) \to (x,y,z,0)$ |
| | $(x,y,z) \to (x,y,z,z^2)$ |
| $R^4 \to R$ | $(x,y,z,t) \to x$ |
| | $(x,y,z,t) \to \pm x^2 \pm y^2 \pm z^2 \pm t^2$ |
| $R^4 \to R^2$ | $(x,y,z,t) \to (x,y)$ |
| | $(x,y,z,t) \to (x,\pm y^2 \pm z^2 \pm t^2)$ |
| | $(x,y,z,t) \to (x,\pm y^2 \pm z^2 + t^3+xt)$ |
| $R^4 \to R^3$ | $(x,y,z,t) \to (x,y,z)$ |
| | $(x,y,z,t) \to (x,y,\pm z^2 \pm t^2)$ |
| | $(x,y,z,t) \to (x,y,\pm z^2+t^3+xt)$ |
| | $(x,y,z,t) \to (x,y,\pm z^2+t^4+xt+yt^2)$ |
| $R^4 \to R^4$ | $(x,y,z,t) \to (x,y,z,t)$ |
| | $(x,y,z,t) \to (x,y,z,t^2)$ |
| | $(x,y,z,t) \to (x,y,z,t^3+xt)$ |
| | $(x,y,z,t) \to (x,y,z,t^4+xt+yt^2)$ |
| | $(x,y,z,t) \to (x,y,z,t^5+xt+yt^2+zt^3)$ |
| | $(x,y,z,t) \to (x,y,zt,z^2 \pm t^2+xz+yt)$ |

In order to classify unstable functions, we need to introduce one final concept, the idea of the corank of a function. Let $\phi : R^n \to R$ be a smooth function having a degenerate critical point at the origin, i.e. grad $\phi$ (0) = det $H\phi$ (0) = 0, where

$H\phi$ is the Hessian matrix of $\phi$. Then the integer $r = n\text{-rank } H\phi(0)$ is called the corank of f.

The importance of the corank is that it can be shown that if corank $\phi = r$, then $\phi$ is equivalent to a function

$$g(x_1,\ldots,x_r) \pm x_{r+1}^2 \pm x_{r+2}^2 \pm \ldots \pm x_n^2 \quad ,$$

where g is $O(|x|^3)$. Furthermore, it can be shown that the classification of $\phi$ depends only upon the similar classification for g. In Table 2 we display the classification of all unstable functions having corank $\leq 2$, codim $\leq 5$, together with their universal unfoldings. (Note that Table 2 omits the irrelevant quadratic terms above). For an account of how these classification theorems

Table 2. The elementary catastrophy of codimension $\leq$. When the $\pm$ sign occurs, germs with sign (+) are called standard, (-) are called dual.

| NAME | FUNCTION | UNIVERSAL UNFOLDING | CORANK | CODIMENSION |
|---|---|---|---|---|
| fold | $x^3$ | $x^3+ax$ | 1 | 1 |
| cusp | $\pm x^4$ | $\pm\ x^4+ax^2+bx$ | 1 | 2 |
| swallowtail | $x^5$ | $x^5+ax^3+bx^2+cx$ | 1 | 3 |
| butterfly | $\pm x^6$ | $\pm\ x^6+ax^4+bx^3+cx^2+dx$ | 1 | 4 |
| wigwam | $x^7$ | $x^7+ax^5+bx^4+cx^3+dx^2+ex$ | 1 | 5 |
| elliptic umbilic | $x^3-xy^2$ | $x^3-xy^2+ax^2+bx+cy$ | 2 | 3 |
| hyperbolic umbilic | $x^3+xy^2$ | $x^3+xy^2+ax^2+bx+cy$ | 2 | 3 |
| parabolic umbilic | $\pm(x^2y+y^4)$ | $\pm(x^2y+y^4)+ax^2+by^2+cx+dy$ | 2 | 4 |
| second elliptic umbilic | $x^5-xy^2$ | $x^5-xy^2+ay^3+bx^2+cy^2+dx+ey$ | 2 | 5 |
| second hyperbolic umbilic | $x^5+xy^2$ | $x^5-xy^2+ay^3+bx^2+cy^2+dx+ey$ | 2 | 5 |
| symbolic umbilic | $\pm(x^3+y^4)$ | $\pm(x^3+x^4)+axy^2+by^2+cxy+dx+ey$ | 2 | 5 |

are established, see references [6-7]. Now we return to the question of natural law and system descriptions.

## 4. Descriptions, Laws and Similarities

Armed with the foregoing tools of singularity theory, we can now rephase our earlier questions surrounding natural laws in much more specific and testable terms. For simplicity, let us assume that we have a single equation of state

$$\Phi(f_1, f_2, \ldots, f_n) = k \quad ,$$

linking the observables $\{f_i\}$ of a description of $\Sigma$. To satisfy our requirement of "analyticity" for a natural law, assume that the function $\Phi$ is analytic, i.e. $\Phi$ equals the sum of its Taylor series. The requirement of "independence" is not a mathematical condition, so we assume it is also satisfied for the $\Phi$ and the physical situaion under consideration. Our interest focuses upon the final requirement for a natural law, "invariance".

If we were to interpret the invariance criterion in the strictest sense, then it would follow that the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\Phi} & R \\ g \downarrow & & \downarrow h \\ R^n & \xrightarrow[\Phi]{} & R \end{array}$$

would be commutative for *all* g and h, i.e.

$$\Phi = h^{-1} \circ \Phi \circ g \quad .$$

Of course, in general this condition is too strict and can only be satisfied in the trivial case $h = g =$ identity. Thus, the real problem associated with natural laws is an *inverse problem*: *given a relation $\Phi$, determine a subgroup G of* diff $(R^n \times R^m)$ *such that*

$\Phi$ is invariant under G. We then say $\Phi$ is a "law" of nature relative to G. This notion of a natural law also shows explicitly that a law is not an absolute in the sense that some "laws" are stronger than others: if a law $\Phi_1$ is invariant under a group $G_1$ and $\Phi_2$ is a law invariant under a group $G_2$ with $G_1$ a subgroup of $G_2$, then $\Phi_2$ is a stronger law than $\Phi_1$. Ideally, our task is to find the largest subgroup of diff $(R^n \times R^m)$ which leaves a given relation $\Phi$ invariant. The physical utility of the law will then hinge upon our ability to interpret the group G within the context of the given system.

Generally speaking, it is extremely difficult to determine such a subgroup of diff $(R^n \times R^m)$ for a given relation $\Phi$. However, one important special case where it can be carried out explicitely is when $\Phi$ is a linear operator, i.e. an mxn matrix, and the coordinate transformations g and h are linear. In this case, we must have

$$\Phi = H^{-1} \Phi G \quad ,$$

where $\Phi$, H and G are the matrix representations of $\Phi$, h and g in some bases in $R^n$ and $R^m$. For simplicity, take the case when $n = m$ and $G = H$. Then the problem is to find the largest subgroup of GL(n) such that

$$H\Phi = \Phi H \quad ,$$

in other words, to characterize all nonsingular matrices H that commute with $\Phi$. This is a classical problem of Frobenius and is treated extensively in, for example, in [8-9]. The simplest result in this direction is when the characteristic polynomial of $\Phi$ coincides with its minimal polynomial, in which case H must be a polynomial in $\Phi$. This means that the set $\{I, \Phi, \Phi^2, \dots, \Phi^{n-1}\}$

generates the maximal subgroup of GL(n) commuting with $\Phi$.

While the case of $\Phi$ linear is the simplest possible situation, it represents an extremely important case in the sense that virtually all of the standard "laws" of classical physics fit into this mold since they are based upon linear relationships of one type or another (e.g. Newton's 2nd Law, Ohm's Law, Maxwell's equations, etc.).

Unfortunately, for nonlinear descriptions $\Phi$ no such body of results exists for determining interesting subgroups of diff $(R^n \times R^m)$ upon which to base a natural law, and we must lower our sights and consider only system similarities of the type discussed in the last section. For pratical purposes, such an approach is a generalization of the idea of law, as a natural law represents only the special case of a "self-similar" description. In the following section, we consider a few examples of the enployment of system similarity concepts in a variety of settings in natural resource and human systems.

## 5. Examples of System Similarities

A. Water Basin Similarities - in a recent paper [10], it has been argued that the probability density functions of the peak and time-to-peak of the instantaneous unit hydrograph can be characterized as specific functions of the rainfall characteristic and the geomorphological features of a river basin. We wish to examine these results to see if it is possible to identify those basins which are similar in the sense of having the same instantaneous unit hydrograph distribution function, modulo a coordinate change.

Following [10], let us consider the instantaneous unit hydrograph, which is the probability density function for the time arrival of a randomly chosen drop of rainfall at the absorption state. The main characteristics of the hydrograph are its peak and time-to-peak, which we denote by $q_p$ and $t_p$, respectively. It has been argued in [10] that the probability density functions for $q_p$ and $t_p$ are given by

$$f(q_p) = 3.534\ \Pi\, q_p^{3/2} \exp\left(-1.412\ \Pi\, q_p^{5/2}\right),$$

$$f(t_p) = 0.656\ t_p^{-7/2} \exp\left(-0.262\ \Pi\, t_p^{-5/2}\right),$$

where

$$\Pi = L_\Omega^{5/2} / (\bar{i}_R A_\Omega R_L \alpha_\Omega^{3/2}) .$$

Here, the quantities comprising $\Pi$ relate to the water basin's geomorphological structure, and to the storm intensity and local climate. We shall not elaborate upon these physical quantities, but focus our attention solely upon the relation for $f(q_p)$, which we shall rewrite in more streamlined fashon as

$$\Phi_c(x) = A\,c\,x^{3/2} \exp\left(-B\,c\,x^{5/2}\right) ,$$

where A and B are regarded as fixed constants, and c is a parameter. In this case, our diagram is

$$\begin{array}{ccc} R & \xrightarrow{\ \phi_c\ } & R \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ R & \xrightarrow[\ \phi_{c'}\ ]{} & R \end{array}$$

and the condition for it to commute is that there exist functions g and h such that

$$h \circ \Phi_c = \Phi_{c'} \circ g$$

or, more explicitly,

$$h(A c x^{3/2} e^{-Bcx^{5/2}}) = A c' x'^{3/2} e^{-Bc'x'^{5/2}} \quad .$$

The coordinate change g is determined if we can find x' as a function of x,c,c'. The simplest way to accomplish this is to let h = identity and solve for x' in the above relation. With h = identity, it is easy to see that $g = \Phi_{c'}^{-1} \circ \Phi_c$, which implies that g is uniquely determined for all c' such that $\Phi_{c'}$ is invertible. This means that $\frac{d\Phi_{c'}}{dx}(x) \neq 0$. Checking this condition yields

$$\frac{d}{dx} \Phi_{c'}(x) = 1/2\ A c' x^{1/2} e^{-B c' x^{5/2}} [3-5 B c' x^{5/2}],$$

so that $\frac{d\Phi_{c'}(x)}{dx} = 0$ for all (c',x) such that

$$3 - 5 B c' x^{5/2} \neq 0 \quad .$$

Reinterpreting this result for the water basin problem, we see that with regard to the distribution function for the peak of the instantaneous unit hydrograph, two basins described by parameters $\Pi$ and $\Pi'$ are equivalent if

$$3 - 5 B \Pi' q_p^{5/2} \neq 0 \quad .$$

That is, we can always find $q_p'$ as a function of $q_p$, $\Pi$, $\Pi'$ as long as the above inequality holds. A similar analysis can be carried out for the time-to-peak function involving $t_p$.

While the preceding analysis is correct, it suffers from the defect that one of our coordinate changes h has been preselected to be the identity. This enabled us to easily derive the foregoing global result for a subgroup of diff (R x R). Let us now proceed in another fashion to obtain local results for all of diff (R x R) and then piece the local result together to obtain a global picture.

An alternate way of looking at the above situation is to employ Theorem 5 on Morse functions. If we return to the function $\Phi_c(x)$, we find that the critical points of $\Phi_c(x)$ are $x^* = 0$ and the positive roots of the equation $x^{*5/2} = 3/(5Bc)$. In terms of the unit hydrograph, these values correspond to the modes of the distribution of the peak flows. To determine whether or not these are degenerate critical points, we calculate $\frac{d^2\Phi_c(x)}{dx^2}$. This yields

$$\frac{d^2}{dx^2}\Phi_c(x) = 1/4\ A c\, e^{-Bc'x^{5/2}}[5x^2(1-5Bc^2) + x^{-1/2}] \quad .$$

Since $\frac{d^2}{dx}\Phi_c(0) = \infty$, $x^* = 0$ is a non-degenerate critical point. Similarly, if we set $\alpha = 3/(5Bc)$ and examine the critical points $x^{*5/2} = \alpha$, we find that $x^*$ is non-degenerate if

$$15\,B\,c\,\alpha^{3/5} \neq 1/2\ (1/\alpha)^{2/5} \quad ,$$

a condition satisfied by all $\alpha > 0$. Thus, for all fixed $c \neq 0$, the critical points of $\Phi_c(x)$ are non-degenerate, so by Theorem 5 we can assert that there exists a coordinate change $x \to g(x)$, such that

$$\Phi_c(x) \to \pm\, x^2 \quad .$$

However, physically the parameter c is replaced by the parameter $\Pi$ which is always positive. Hence, we conclude that <u>locally</u> near the singular points $x^* = 0$, $(3/5\,B\,c)^{-5/2}$, the distribution function $f(q_p) \cong \pm\, q_p^2$, where the sign is determined by $\frac{d^2}{dx^2}\Phi(x^*)$, which in this problem is always positive. So, near a critical point $f(q_p)$ always looks like $q_p^2$, while away from a critical point, the Implicit Function Theorem insures that $f(q_p)$ looks like the linear function $q_p$.

The above results can be summarized by saying that insofar as the distribution of the peaks of the instantaneous unit hydrograph adequately describe a river basin all basins are equivalent: up to a coordinate change in the measurement of the peak, the distribution of the peaks of one basin can be transformed into the peaks of another. Thus, the passage from one basin to another via a change of $\Pi \to \Pi'$ can always be undone by a corresponding change in the measurement scale.

B. Forestry Yield Models - An extremely interesting application of the ideas presented in this paper arises in the modeling of timber yields in a forest as a function of tree diameter and height. If we let

D = tree diameter at breast height (in cm),
H = tree height from breast height (in m),
V = total tree volume, exclusive of bark, stump to tip (in $m^3$),

then the following relationship is empirically derived in [11] relating these quantities:

$$V = 0.0073 + 0.000040\ D^2H$$

$$\doteq \alpha + \beta\ D^2H \quad .$$

We inquire as to what extent this empirical relationship constitutes a natural law for timber yield from a given tree.

For our purposes, the parameter $\alpha$ plays no role so we set $\alpha = 0$. Similarly, our results do not depend upon $\beta$, other than that $\beta$ be non-zero. So, we set $\beta = 1$ and consider the function $\Phi(x,y) = xy^2$.

Computing $\frac{\partial \Phi}{\partial x} = y^2$, $\frac{\partial \Phi}{\partial y} = 2xy$, it is easy to verify that cod $\Phi = +\infty$. Thus, $\Phi$ is as degenerate as a smooth function can possibly be, in the sense that it takes an infinite number of unfolding parameters to imbed $\Phi$ into a stable family of models. In particular, an arbitrarily small perturbation of $\Phi$ will produce a function $\hat{\Phi}$ that is not equivalent to $\Phi$ in our sense. In other words, $\Phi$ is unstable in the strongest possible sense.

Interpretation of this result in the forestry context suggests that the above empirical rule for timber yield is about as far away from being a natural law as any relationship could possibly be. In fact, the high sensitivity of the relationship strongly suggests that extreme caution be employed before utilizing this relationship in any real forestry management situation. Of course, whether or not the type of instability we focus upon here actually matters in the use of the relationship depends upon what purposes are served by the relationship. But mathematically, the empirical rule $V = \alpha + \beta D^2H$ is highly questionable as a basis for any policy-making due to its almost pathological sensitivity to perturbations.

C. Urban Spatial Structure - in simple models of the evolution of urban structures an important role is often played by the flow of money from residents of one region into shops in another region. If we let the zones of the region be labeled $i = 1,2,\ldots,K$ and define

$s_{ij}$ = flow of cash from region i to region j,

$e_i$ = per capita expenditure on shopping goods by residents of zone i,

$P_i$ = population of zone i,

$W_i$ = size of the "center" represented by zone i,

$c_{ij}$ = cost of travel from zone i to zone j,

the the standard aggregate model for $S_{ij}$ is [12]

$$s_{ij} = \frac{e_i P_i W_j^{\alpha} \exp(-\beta c_{ij})}{\sum_{k=1}^{K} W_k^{\alpha} \exp(-\beta c_{ik})}, \qquad i,j = 1,2,\ldots,K,$$

where $\alpha$ and $\beta$ are parameters representing consumer economies and "ease" of travel, respectively. Here we shall investigate the map

$$S : R^K \rightarrow R^{K^2} ,$$

$$W = (W_1, W_2, \ldots, W_K) \rightarrow (s_{11}, s_{12}, \ldots, s_{KK}) \quad .$$

To see things most clearly, we consider the case of two regions, i.e. $K = 2$. Since $K^2 > K$, our first approach is to test $S$ for being an <u>immersion</u> and, if so, invoke Theorem 1 to conclude that $S$ is stable. To check whether $S$ is an immersion, we need to calculate the Jacobian of $S$,

$$[J\,S\,(W)]_{ij} = [\frac{\partial s_{ik}}{\partial W_j}] \quad , \quad i,j = 1,2; \quad k = 1,2 \quad .$$

Since we are interested primarily in the parameters $\alpha, \beta$, we set $e_i = p_i = 1$ for this analysis.

After some algebra, we find

$$J\,S\,(W) = \alpha \cdot \begin{bmatrix} \frac{W_1^{\alpha-1} W_2^{\alpha}}{D_1^2} e^{-\beta(c_{11}+c_{12})} & \frac{-W_1^{\alpha} W_2^{\alpha-1}}{D_1^2} e^{-\beta(c_{11}+c_{12})} \\ \frac{-W_1^{\alpha-1} W_2^{\alpha}}{D_1^2} e^{-\beta(c_{11}+c_{12})} & \frac{W_1^{\alpha} W_2^{\alpha-1}}{D_1^2} e^{-\beta(c_{11}+c_{12})} \\ \frac{W_1^{\alpha-1} W_2^{\alpha}}{D_2^2} e^{-\beta(c_{21}+c_{22})} & \frac{-W_1^{\alpha} W_2^{\alpha-1}}{D_2^2} e^{-\beta(c_{21}+c_{22})} \\ \frac{-W_1^{\alpha-1} W_2^{\alpha}}{D_2^2} e^{-\beta(c_{21}+c_{22})} & \frac{W_1^{\alpha} W_2^{\alpha-1}}{D_2^2} e^{-\beta(c_{21}+c_{22})} \end{bmatrix}$$

It is easy to see that rank $J = 1$ for all W and $\alpha$ and $\beta$, i.e. J is not of maximal rank and, hence, S is not an immersion and, furthermore, every point $W = (W_1, W_2)$ is a singular point of S. Thus, S is an unstable map.

The instability of S means that there exist arbitrarily small perturbations of S, call them S', such that S' is not equivalent to S. S and S' contain inherently different information about the urban structure. By our earlier conditions for a natural law, the instability of S also rules out S as a candidate for being a law of urban structural behavior. It is purely a relationship connecting the "attractiveness" of urban centers as measured by W, with the flow of funds among these centers.

Although S is unstable, Theorem 7 tells us that stable maps of $R^2 \to R^4$ are dense. Thus, while there are maps close to S that are not equivalent to it, there are also other maps equally close that are stable, i.e. S can be arbitrarily closely approximated by a stable map.

## 6. System Equivalence and Complexity

There has been a considerable amount of systems literature devoted to the idea of characterizing the complexity of a system. Most of the complexity measures proposed have tacitly assumed that complexity is a property intrinsic to the given system $\Sigma$, i.e. it is independent of the interaction of $\Sigma$ with any other system S. We take the position that complexity is a contingent property of $\Sigma$ and that it is meaningless to speak of "the" complexity of $\Sigma$ without specifying the system S with which $\Sigma$ is in interaction; a system cannot perceive its own level of complexity. This level is only established by placing $\Sigma$ into interaction with another system.

Following the work of Rosen [13], we shall adopt the view that the complexity $\Sigma$ is equal to the number of non-equivalent descriptions that S can form for $\Sigma$. In other words, if S can provide only a small number of non-equivalent description of $\Sigma$, then relative to S, $\Sigma$ will have a low degree of complexity. Conversly, if $\Sigma$ can display many non-equivalent modes of interaction with S, then S will perceive $\Sigma$ as being very complex.

Relative to the mathematical development described above, the operational realization of this complexity measure is rather straightforward. Given a family $\{\Phi_\alpha\}$, $\alpha \in A$, A an index set, of descriptions of a system $\Sigma$, we define

complexity of $\Sigma$ = the number of equivalence classes which arise from $\Phi_\alpha$, $\alpha \in A$.

Here, each value of $\alpha \in A$ corresponds to a different description of $\Sigma$, and we measure the complexity of $\Sigma$ in terms of the <u>bifurcation points, $\alpha^* \in A$</u>. Each such point $\alpha^*$ represents a description of $\Sigma$ that is not equivalent to descriptions "near" $\alpha^*$, i.e. for some $\alpha$ arbitrarily close to $\alpha^*$, $\Phi_\alpha$ and $\Phi_{\alpha^*}$ are in different equivalence classes. With this definition of complexity, the problem of finite classification of maps $\Phi_\alpha$ takes on more direct system-theoretic significance. In the event $\Phi_\alpha$ is a <u>function</u>, we know from the Thom-Arnol'd theory that there are a finite number of equivalance classes (under $C^\infty$ coordinate changes) only if cod $\Phi_\alpha \leq 6$. The system-theoretic interpretation of this result is still not clear, and even more elusive is the meaning of the "moduli" that enter into the classification for cod $\Phi_\alpha > 6$. These items will form the basis for future investigations.

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