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SOME SYSTEM APPROACHES TO WATER RESOURCES PROBLEMS
II. STATISTICAL EQUILIBRIUM OF PROCESSES IN DAM STORAGE

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1. Suppose we are dealing with a reservoir with a capacity R designed to meet water demands over a long period of time.

One realizes that the water demands W_t during the time period $(t, t + \Delta)$ are connected with general climatological processes in the region concerned. Suppose these processes can be described by a multidimensional vector (or more abstract) function

$$\omega = \omega_t \quad , \qquad -\infty < t < \infty \tag{1}$$

of the time parameter t.

Because of the obvious annual cycle in a climate evaluation, it seems reasonable to assume that water demands during time period t are the same as during time period t_o if $u = t - t_o$ is a period of integer years and the corresponding historical records $\{\omega' = w'_s, s \leq t\}$ and $\{\omega = \omega_s, s \leq t_o\}$ differ from each other only by a shift in time, i.e. $\omega'_s = \omega_{s-u}$:

$$W_t(\omega_s, s \leq t) = W_{t_o}(\omega_{s+u}, s \leq t_o) \quad , \quad u = t - t_o \quad . \tag{2}$$

A similar property seems natural for the reservoir inflow. Say that during a period $(t, t + \Delta)$ the reservoir receives an amount of water

$$\chi_t = \int_t^{t+\Delta} x_s \, ds \quad ,$$

where the inflow velocity x_t depends on the climatological processes (1) in such a way that

$$x_t(\omega_s, s \leq t) = x_{t_o}(\omega_{s+u}, s \leq t_o) \tag{3}$$

if $u = t - t_o$ is a period of integer years. One can assume, for example, that W_t and x_t are components of Equation (1); in this case the relation (3) becomes trivial.

Let us consider a sequence of intervals $(t, t + \Delta)$ where Δ is a time unit (year, month, etc.) and according to $\Delta = 1$, the parameter t runs integer numbers. A discrete time operation usually corresponds to the following description: the release of an amount of water is $Z_t = W_t$ if there is enough water in the reservoir. Otherwise the release is $Z_t = V_t$, where V_t is the total amount of water available during the time period t. Furthermore, if there is too much water and the reservoir cannot contain it all, the amount $Z_t = V_t - R$ overflows. Thus, we have the so-called "Z-shaped" policy

$$Z_t = \begin{cases} V_t, & V_t \leq W_t \\ W_t, & W_t \leq V_t \leq W_t + R \\ V_t - R, & W_t + R \leq V_t \end{cases}, \tag{4}$$

where the operation parameter W_t has to be chosen on the basis of the previous historical record ω_s, $s \leq t - 1$, in order to meet the actual water demands (see Figure 1).

The rest of the water at the end of time period t which is allowed to be used for operation of the reservoir during the next time period is

$$Y_t = \begin{cases} 0 , & V_t \leq W_t \\ V_t - W_t , & W_t \leq V_t \leq W_t + R , \\ R , & W_t + R \leq V_t \end{cases} \tag{5}$$

(see Figure 2), and the total amount of water available at the next step will be

$$V_{t+1} = Y_t + X_{t+1} , \tag{6}$$

if X_{t+1} is the new inflow.

The continuous analogue of the reservoir model (4)-(6) may be described as follows.

If the useful reservoir volume Y_t is not zero, then in accordance with a general climate condition, a release velocity (a discharge) $z_t = w_t$ is a function of the corresponding historical record $\omega = \omega_s$, $s \leq t$. In the case of a water spill ($Y_t = R$), the release velocity must not be less than the inflow velocity x_t and moreover, $Z_t = 0$ if $Y_t = 0$. Thus

$$Z_t = \begin{cases} 0 , & Y_t = 0 \\ w_t(\omega_s \; ; \; s \leq t) , & 0 < Y_t < R , \\ \max(x_t, w_t) , & Y_t = R \end{cases} \tag{7}$$

where the "demand velocity" function w_t is assumed to be of the type described above (see (2)).

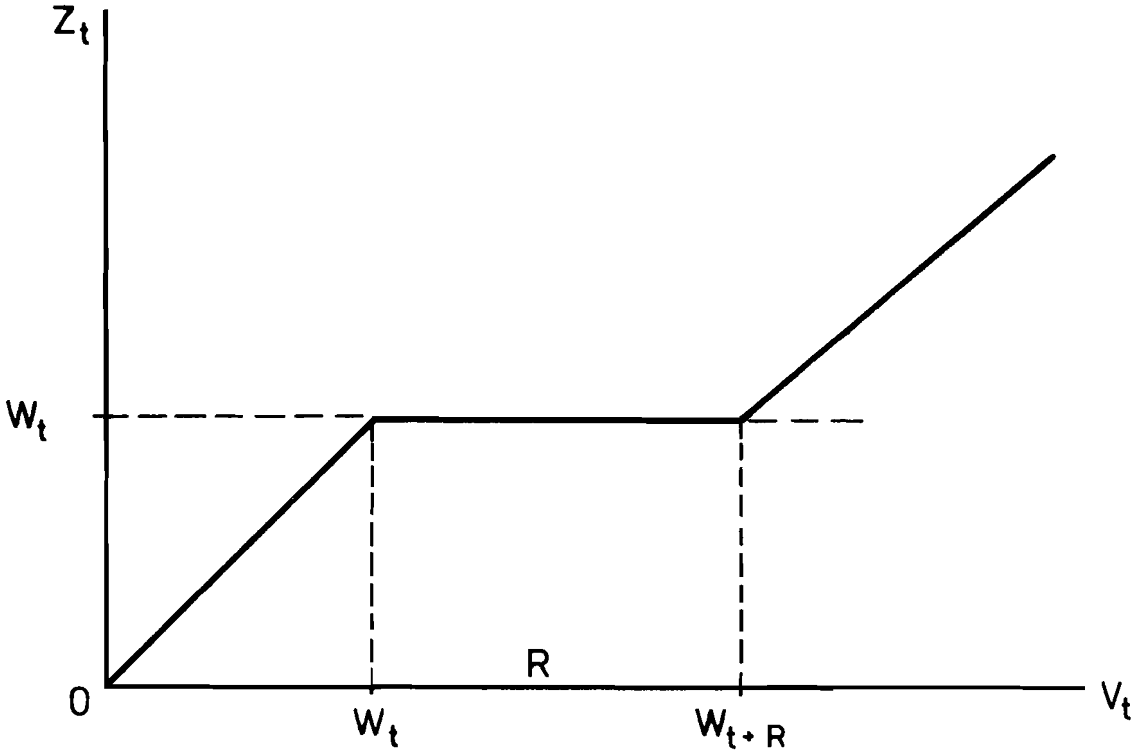

FIGURE 1.

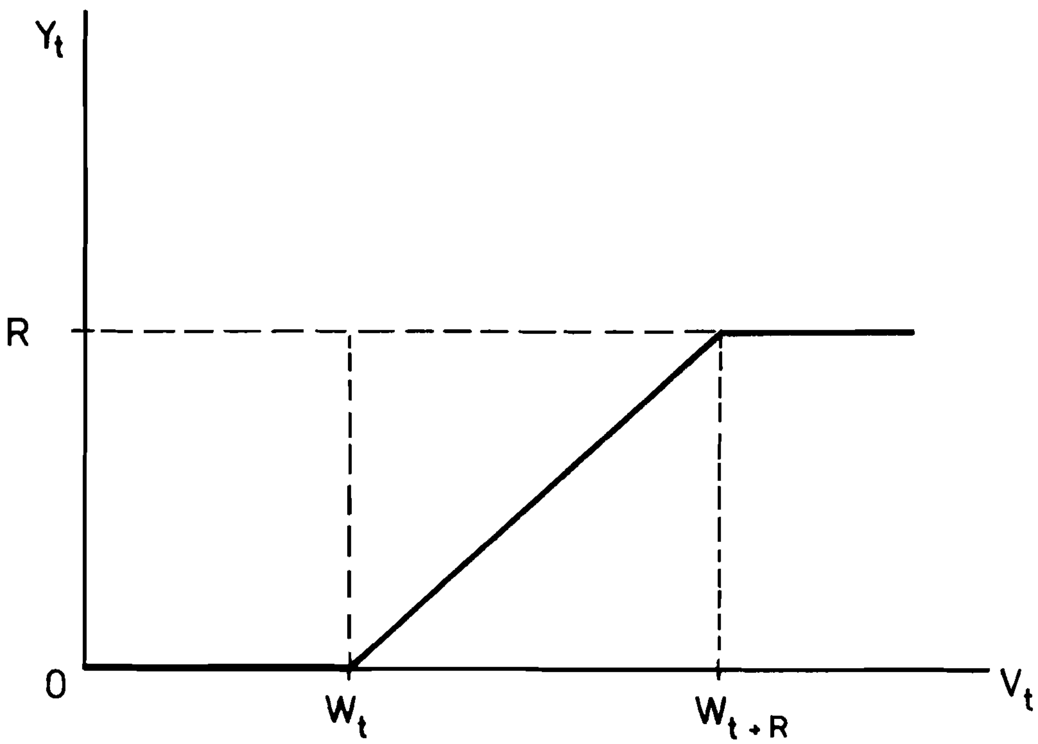

FIGURE 2.

Suppose the reservoir system evaluation in time is considered a random process on a probability space (Ω, ϑ, P) of all possible elementary outcomes $\omega \in \Omega$ where ϑ is an ensemble of possible events A that may occur with probabilities

$$P(A) \quad , \quad A \in \vartheta \quad .$$

One can assume that each elementary outcome $\omega \in \Omega$ is described by the entire historical record

$$\omega = \omega_t \quad , \quad -\infty < t < \infty$$

of the climatological processes used above (see (1) and (3)).

As is known, most practical applications of any stochastic reservoir model are based on the assumption that random processes in the reservoir system will eventually reach a so-called statistical equilibrium; this means that during a long term operation the probability of an annual event A becomes independent of a particular year and initial reservoir conditions, and moreover, the frequency of event A during a series of years N is approximately equal to the corresponding probability P(A):

$$\frac{\nu_n(A)}{n} \approx P(A) \quad , \tag{8}$$

where $\nu_n(A)$ is a number of years in which event A occurs.

It will be shown that a statistical equilibrium phenomenon occurs under a general assumption concerning climatological random processes[1].

Let A be an event which may or may not occur, depending upon the behaviour of the processes

$$\omega = \omega_t \quad , \qquad t > t_o \quad ,$$

and let

$$P\left(A \middle|_{\omega = \omega_s,\ s \le t}\right)$$

be a probability of A under a fixed previous historical record $\omega = \omega_s$, $s \le t_o$, up to time t_o. One can realize that a similar historical record $\omega' = \omega_{s-u}$, $s \le t_o + u$ might occur at the other time period $t_o + u$, and that event S_uA may occur which is similar to A but shifted in time u (see Figure 3). We assume that if historical records $\omega = \omega_s$, $s \le t_o$, $\omega' = \omega_{s-u}$, $s \le t_o + u$, and events A, S_uA differ from each other only by a time shift over a number of years, then the probabilities of such events are the same:

$$P\left(A \middle|_{\omega = \omega_s,\ s \le t_o}\right) = P\left(S_uA \middle|_{\omega' = \omega_{s-u},\ s \le t_o + u}\right) \qquad (9)$$

where u is a period of integer years.

One supposes that Equation (9) may conform to reality because of the obvious annual cycle in climate processes which may be considered homogeneous over a long period of time.

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[1] This problem was posed by M. Fiering in a personal communication.

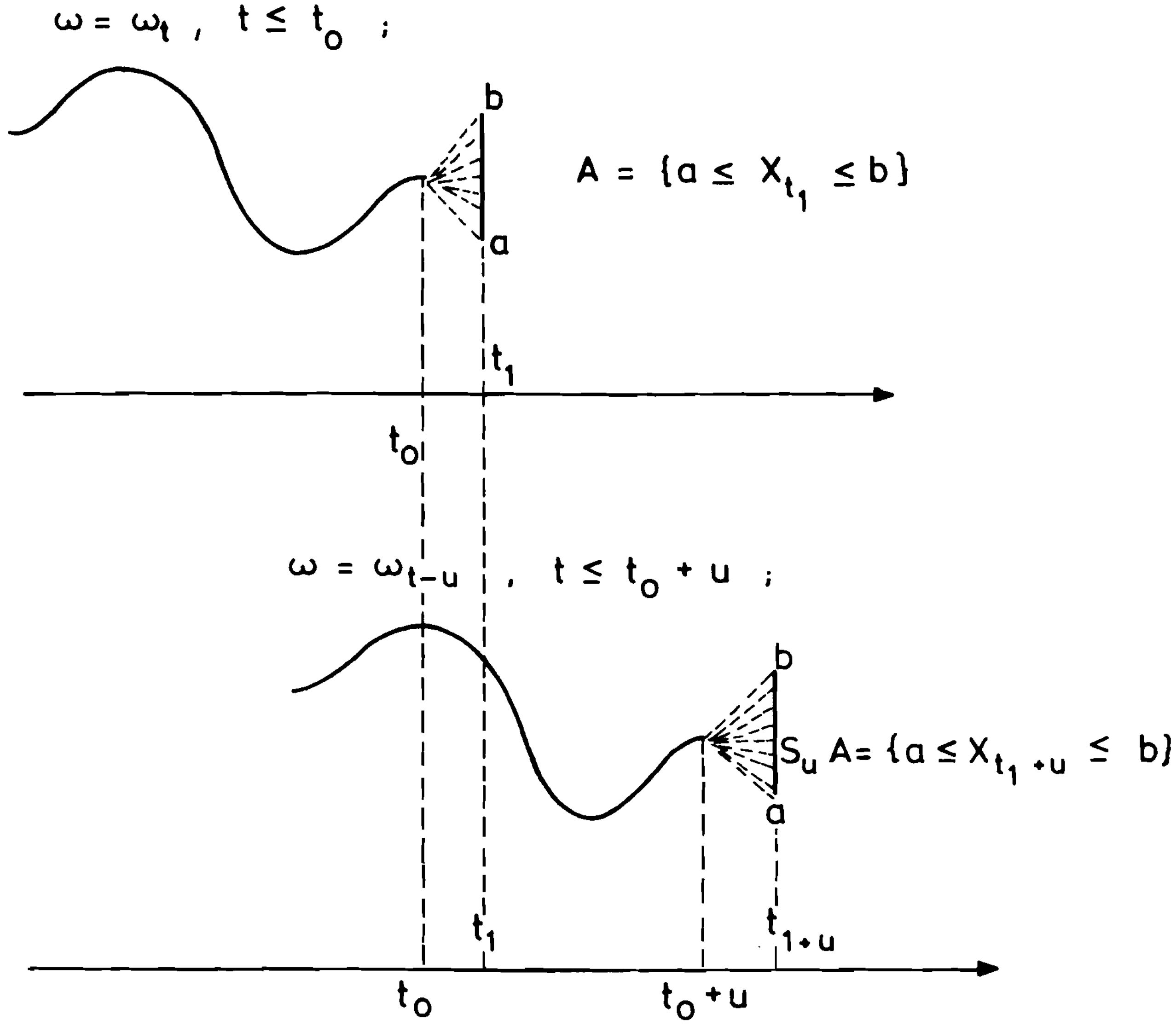

FIGURE 3.

Note that for the year unit of time, Equation (9) holds true in the case of the stationary or homogeneous Markov process ω_t, $-\infty < t < \infty$.

Let a general process which satisfies Equation (9) (with respect to the shift transformations S_u) be called homogeneous. It is also assumed that a non trivial event A cannot be predicted more precisely from an infinite past history; if one can find out by any historical record ω_s, $s \leq t_o$ ($t_o \to -\infty$) the event A certainly did or did not occur, then such an event is trivial, i.e.

$$P(A) = 1 \text{ or } 0 \quad . \tag{10}$$

This is the so-called regularity condition (well known in the theory of stationary processes) which holds true particularly for homogeneous ergodic Markov processes (see, for example, [1]).

Our purpose is to show that generally, in a regular homogeneous case, the process (V_t, Y_t, Z_t) in the reservoir system considered will reach statistical equilibrium, i.e. the Large Numbers Low (8) is valid.

It is worthwhile to note that an error in the approximate Equation (8) strongly depends on the time required for the process to reach the state of statistical equilibrium.

Example. Let the inflow X_t, $t \geq 0$ be a sequence of independent identically distributed random variables which may take values 0,1 with corresponding probabilities p, 1 - p

(Bernoulli scheme). Suppose that

$$W_t \equiv R = 1$$

and an initial amount of water in the reservoir is $Y_o = 1$. Then after a random series of events

$$X = 1,\dots,X_\tau = 1$$

certainly occurs, a zero inflow $X_{\tau+1} = 0$, and $V_t \equiv X_t$, $Y_t \equiv 0$, $Z_t = X_t$, for all $t > \tau$ (see Figure 4).

One can say that the process (V_t, Y_t, Z_t) reaches statistical equilibrium after time τ. The corresponding stationary probability distribution is such that, for example,

$$P\ \{Y_t = 0\} = 1 \quad .$$

Therefore, a frequency of the event $A = \{Y_t = 0\}$ by Equation (8) should be equal to 1. However, if the probability p of the zero inflow X_t is very small, then a long series of non-zero inflows $X_o = 1,\dots,X_\tau = 1$ is very likely:

$$P\ (\tau > n) = (1 - p)^n \quad ,$$

and for a series of years n, $n \leq \tau$, one has $\nu_n(A) = 0$. So the corresponding frequency of the event A will be equal to 0.

Concerning a method of the statistical equilibrium proof, it should be mentioned that in a case of homogeneous Markov processes the standard ergodic theorem on the convergence

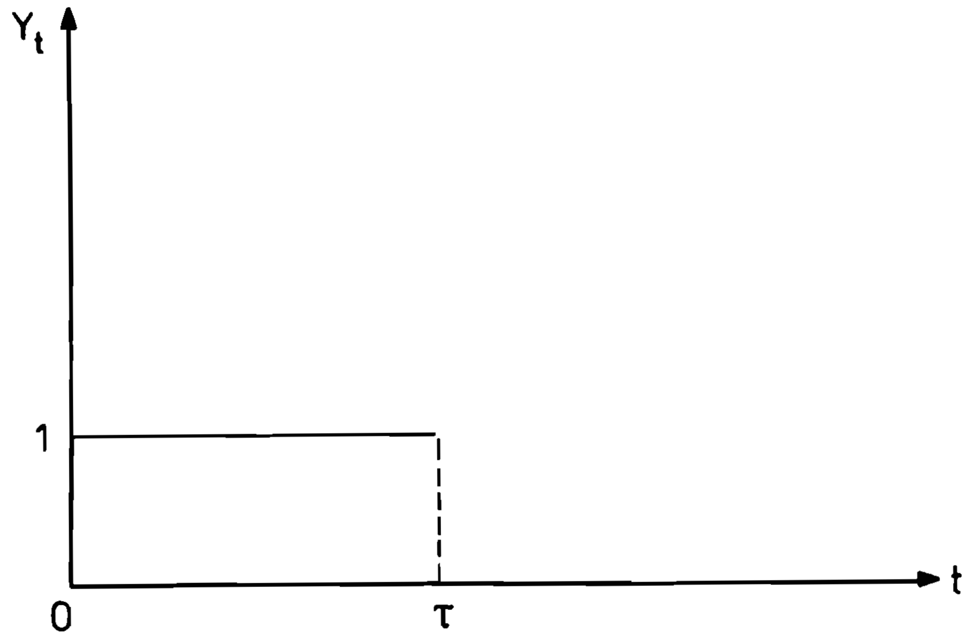

FIGURE 4.

to the stationary probability distribution can be applied (see for example, [2]-[5]). This method is quite different and may be called a method of imbedded stationary processes.

2. Let us consider a standard probability space (Ω, ϑ, P) of abstract functions

$$\omega = \omega_t \quad , \qquad -\infty < t < \infty \quad ,$$

which is invariant with respect to shift transformations

$$S_u \omega = \omega_{t-u} \quad , \qquad -\infty < t < \infty \quad , \tag{11}$$

where

$u = n\Delta$,

n = runs integer numbers ,

Δ = a time unit .

Suppose that $\omega = \omega_t$, $-\infty < t < \infty$, as the random process of the probability space (Ω, ϑ, P) is regular and homogeneous, i.e. Equations (9) and (10) hold true. Remember that the shifted event $S_u A$ which was used in Equation (9) is

$$S_u A = \{\omega : S_{-u}\, \omega \in A\} \quad , \quad A \in \vartheta \quad .$$

The regularity condition (10) means that

$$\bigcap_t \vartheta_{-\infty}^{t} = \emptyset \quad , \tag{12}$$

where $\vartheta_{-\infty}^{t}$ denotes a 6-algebra of events generated by $\omega = \omega_s, s \leq t$, and $\emptyset$ is a trivial 6-algebra of events $A \in \vartheta$ with probability 1 or 0.

Suppose that in addition, the following property holds:

$$P(\Omega^* \cap S_u\Omega^*) > 0 \quad , \tag{13}$$

if a set $\Omega^* \subseteq \Omega$ contains almost all of the elements $\omega \in \Omega$. Then

$$P(S_uA) = P(A) \quad , \quad A \in \vartheta \quad . \tag{14}$$

One can verify this using the well-known limit relation

$$\lim_{t\to-\infty} P\left(A\Big|_{\omega = \omega_s,\ s \leq t}\right) = P(A) \tag{15}$$

for almost all $\omega \in \Omega$ (see, for example, [1]); i.e. for every fixed $A \in \vartheta$ there is a set $\Omega^* \subseteq \Omega$ such that the relation (15) holds true for all $\omega \in \Omega$. The similar limit relation concerned with a shift event S_uA,

$$\lim_{t\to-\infty} P\left(S_uA\Big|_{\omega' = S_{-u}\omega_s,\ s \leq t - u}\right) = P(S_uA)$$

fulfils almost all $\omega' = S_{-u}\,\omega \subseteq \Omega^*$ particulary for almost all $\omega \in \Omega^* \cap S_u\Omega^*$ where, according to Equation (13), one has $P(\Omega^* \cap S_u\Omega^*) > 0$. But the process $\omega = \omega_t$ is homogeneous, i.e.

$$P\left(S_uA\Big|_{\omega' = S_{-u}\omega_s,\ s \le t-u}\right) = P\left(A\Big|_{\omega = \omega_s,\ s \le t}\right)$$

and obviously has to be

$$P(S_uA) = P(A) \quad .$$

Thus, one can say that <u>the regular homogeneous process is stationary with respect to the corresponding shift transformations</u>.

<u>Example</u> (non-stationary homogeneous process). Let Ω be a space of all functions

$$\omega : \omega_t = t + \Theta \quad , \quad -\infty < t < \infty \quad ,$$

where Θ is a real parameter, $-\infty < \Theta < \infty$. Let $Q(d\Theta)$ be a probability distribution on the real line and $P(d\omega)$ be the corresponding probability on Ω. Obviously in this case,

$$P\left(A\Big|_{\omega = \omega_s,\ s \le t}\right) = \begin{cases} 1 & , \ \omega \in A \\ 0 & , \ \omega \in A \end{cases}$$

and

$$P\left(S_uA\Big|_{\omega' = S_u\omega_s,\ s \le t+u}\right) = P\left(S_uA\Big|_{\omega = \omega_s,\ s \le t}\right) \quad ,$$

because

$$\omega' = S_u\omega \quad S_u A$$

if and only if $\omega \in A$. The probability distribution $Q(d\Theta)$, $-\infty < \Theta < \infty$, is not uniform and it is easy to verify that the homogeneous random process $\omega = \omega_t$, $-\infty < t < \infty$ is non-stationary. Here the condition (10) does not hold true. Besides, if

$$Q(\Theta = \Theta_o) = 1$$

for some fixed value Θ_o, and Ω^* is the set containing the single function $\omega_t = t + \Theta_o$, $-\infty < t < \infty$ of the probability 1, then

$$P(\Omega^* \cap S_u\Omega^*) = 0 \quad , \qquad u \neq 0 \quad .$$

Thus, Equation (13) also is not valid.

Note that a general stationary process $\omega = \omega_t$, $-\infty < t < \infty$ is homogeneous. Indeed, if

$$\phi(\omega) = P\left(A \middle|_{\omega = \omega_s,\ s \leq t}\right)$$

is a conditional probability of $A \in \vartheta$ with respect to the 6-algebra $\vartheta^t_{-\infty}$, then for any $B \in \vartheta^t_{-\infty}$ we have

$$P(AB) = P(S_u A \cdot S_u B) = \int_B \phi(\omega)P(d\omega)$$
$$= \int_{S_u B} \phi(S_{-u}\omega')P(d\omega') \quad ,$$

because in the stationary case the probability measure $P(d\omega)$ is invariant under the shift transformations

$$\omega \to \omega' = S_u \omega \quad , \qquad u \in \Omega \quad .$$

The set of all events $S_u B$, $B \in \vartheta^{t}_{-\infty}$, coincides with the 6-algebra $\vartheta^{t+u}_{-\infty}$ so that $\psi(\omega') = \phi(S_{-u}\omega')$ is a conditional probability of $S_u A$ with respect to $\vartheta^{t+u}_{-\infty}$:

$$\psi(w') = P\left(S_u A \Big|_{\omega' = \omega_s ,\ s \le t + u}\right)$$

and

$$P\left(A \Big|_{\omega = \omega_s ,\ s \le t}\right) = \phi(\omega) = \psi(S_u \omega)$$

$$\cdot P\left(S_u A \Big|_{\omega' = S_{us} ,\ s \le t + u}\right) \ .$$

Thus the original abstract process (1) and any random process of type (2) or (3) are regular and stationary with respect to the shift transformations S_u.

Remember that for such processes the following phenomenon occurs for any period u of integer years. With the probability 1,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{n=1}^{N} \delta(S_{un} A) = P(A)$$

where $\delta(B)$ is an indicator of an event B ($\delta(B) = 1$ if B occurs otherwise $\delta(B) = 0$). Moreover,

$$P(A) = 1 \text{ or } 0 \tag{16}$$

if an event A is invariant under the shift transformation S_u, i.e. $S_u A \subseteq A$ (see, for example, [1]).

3. Let us consider the discrete time model (4)-(6).

Note that even for the very simple inflow X_t the corresponding random process Y_t may be quite complex.

Example. Let X_t, $t > 0$, be the sequence of independent Bernoulli variables: $X_t = 0$ or R with probabilities p and 1 - p.

Let the water demand function W_t be a constant

$$W_t = W \quad .$$

Then the process Y_t is determined by the homogeneous Markov chain of Equation (4) with the infinite number of states

$$Y_{t+1} = \max\ (0, Y_t - W) \text{ or } \min\ (R, Y_t + R - W)$$

with probabilities p and 1 - p. One can verify that there is an ergodic class of states which can be achieved from points 0 or R, and other states are non-essential. All ergodic states may be described as

$$Y = mR - nW \quad ,$$

where m,n are integer numbers such that

$$\frac{m-1}{n} \leq \frac{W}{R} \leq \frac{m}{n} \quad .$$

Thus, in the case of an irrational value W/R there is an <u>infinite</u> number of ergodic states.

Generally if the process (X_t,W_t) forms a lag-(m,n) Markov chain, that is if

$$(X_t,\ldots,X_{t-m};\ W_t,\ldots,W_{t-n})$$

is a simple Markov chain, then the process

$$(Y_t,X_t,W_t)$$

should be of a similar type. Well-known ergodic theorems can be applied in order to establish the existence of statistical equilibrium (in other words, the existence of a limit stationary distribution--see, for example, [1]).

The process (X_t,W_t) considered is of a much more general type. Of course, it is actually observed from an initial time moment $t_o = 0$ but one can realize that (X_t,W_t), $t > 0$, is part of a regular process

$$(X_t,W_t) \quad , \qquad -\infty < t < \infty \quad ,$$

which is homogeneous with respect to the time shift transformations S_u with period u of integer years, as was described in Section 1.

Remember that the regular homogeneous process (see Section 2) (X_t,W_t), $-\infty < t < \infty$, is stationary with respect to S_u.

Obviously, if the inflow X_t and the water demand W_t precisely coincide:

$$X_t = W_t \quad , \quad t > 0$$

then there is no statistical equilibrium because any variation of initial reservoir conditions absolutely changes the process Y_t: $Y_t = Y_o$, $t > 0$.

A similar phenomenon occurs if, for example, the current water demands are formed in such a way that W_t is equal to the inflow X_{t-1} during the previous period of time and if this inflow is less than the useful reservoir capacity R, say

$$X_t < \tfrac{1}{4}R \quad .$$

That is, under the initial volume Y_o,

$$\tfrac{1}{4}R < Y_o < \tfrac{3}{4}R \quad ,$$

the process Y_t, $t > t_o$, as one can easily verify, will be of the form

$$Y_t = Y_o + X_t - X_o \quad .$$

Obviously, the probability distribution of the variables Y_t even for a distant future time t depends strongly on the initial reservoir condition Y_o.

One can construct more complex examples of <u>nonergodic</u> reservoir processes by considering a relationship between X_t and W_t of the following type:

$$X_t - W_t = \xi_t - \xi_{t-1} \quad , \tag{17}$$

where ξ_t is a stationary process.

Let us assume that the relationship of the type (17) is not valid. Roughly speaking, this means that the differences

$$X_t - W_t \quad , \quad t > 0$$

accumulate in such a way that their sums

$$\eta_t = \sum_{s=1}^{t} (X_s - W_s) \quad , \quad t \to \infty$$

become unbounded, i.e. for every R, there is a number n for which

$$|\eta_n| \geq R$$

with non-zero probability

$$P(|\eta_n| \geq R) > 0 \quad . \tag{18}$$

This condition holds true, for example, for discrete (integer) type variables if, for any historical record of reservoir processes up to the current time t, one can expect with non-zero probability that the inflow X_{t+1} during the next time period will exceed the corresponding water demands W_{t+1}. In this case, even the initially empty reservoir will be full ($\eta_n \geq R$) with non-zero probability at the end of the period n, n = R.

Now one can be sure that sooner or later the random process

$$\eta_t = \sum_{s=1}^{t} (X_s - W_s) \quad , \quad t \geq 1$$

will exceed the upper or lower boundaries R or -R. That is, the event

$$A = \{ |\eta_n| \geq R \quad \text{for some } n \geq 1 \}$$

is invariant with respect to time-shift transformations S_u, $u > 0$:

$$S_u A = \{ |\eta_n| \geq R \text{ for some } n \geq u + 1 \} \subseteq \underline{A} \quad ,$$

and (see (16) P(A) = 1 because of Equation (18).

Moreover, $P(S_u A) = 1$ for any u, i.e. there is certainly a finite time τ at which a sequence

$$\eta_{u,t} = \sum_{s=u}^{t} (X_s - W_s) \quad , \quad t \geq u \quad ,$$

is out of the interval (-R,R). Let $\tau = \tau(u)$ be the first such moment (see Figure 5).

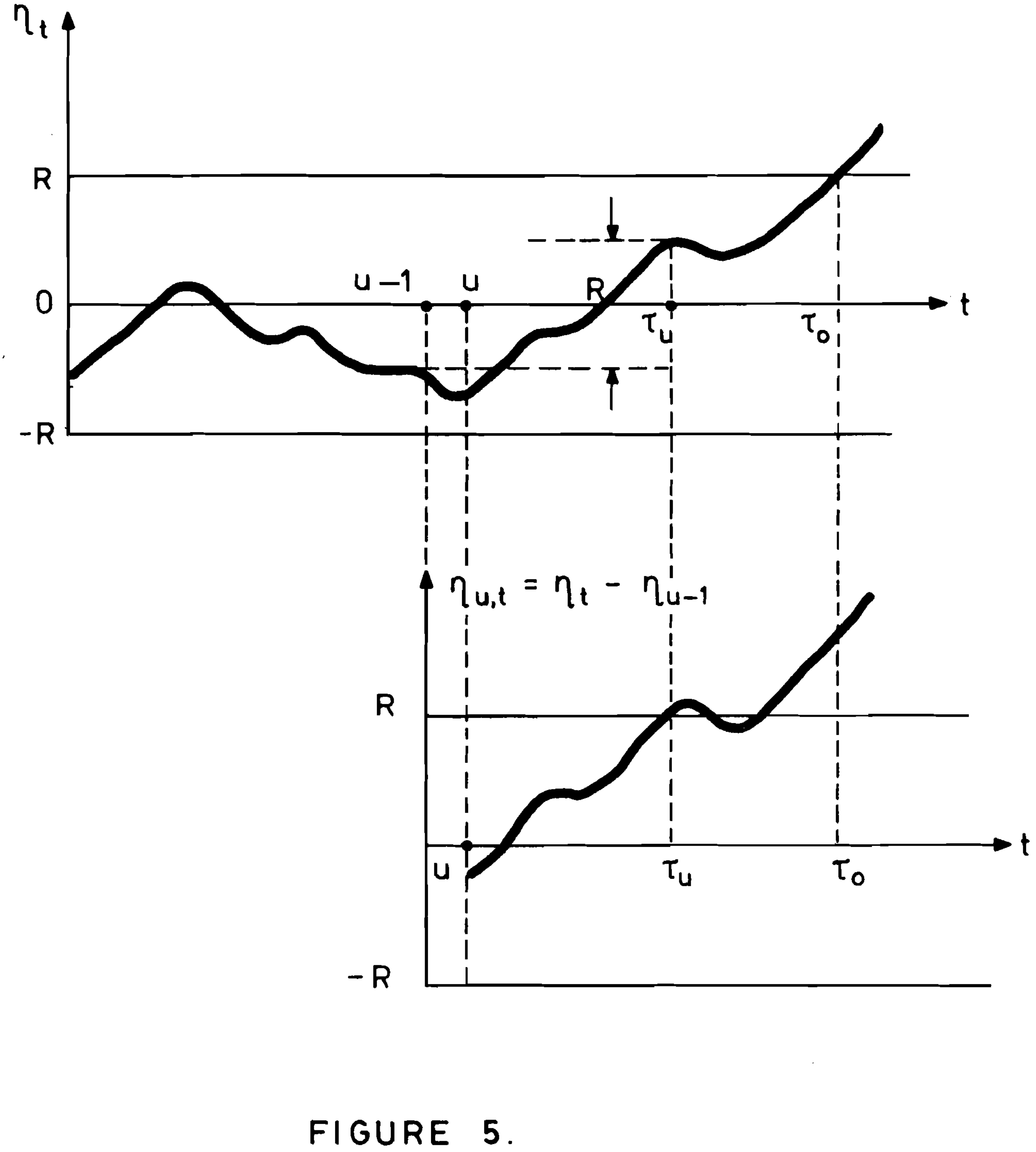

FIGURE 5.

Let it be shown that

$$Y_\tau = \begin{cases} R \ , & \eta_{u,\tau} \geq R \\ 0 \ , & \eta_{u,\tau} \leq R \end{cases} \quad . \tag{19}$$

According to Equation (5) this is obvious for $\tau = u$:

$$Y_u = \begin{cases} R \ , & X_u \geq W_u + R \\ 0 \ , & R+X_u \leq W_u \end{cases} \quad ,$$

and, for any $\tau = n$, follows from a relationship

$$\eta_{u,t} \leq Y_t \leq R + \eta_{u,t} \ , \quad u \leq t < \tau \tag{20}$$

(this will be proved), because if $\eta_{u,n} \leq - R$, then

$$Y_{n-1} + \left(X_n - W_n\right) \leq R + \eta_{u,n} \leq 0 \quad ,$$

$$Y_{n-1} + X_n \leq W_n \quad , \ Y_n = 0 \quad ;$$

and if $\eta_{u,n} \geq R$ then

$$Y_{n-1} + \left(X_n - W_n\right) \geq \eta_{u,n} \geq R \quad ,$$

$$Y_{n-1} + X_n \geq W_n + R \quad , \ Y_n = R \quad .$$

Now let the inequalities (20) be established. In the case where $\tau > u$ we have

$$-R < X_u - W_u < R \quad ,$$

and obviously

$$Y_u \geq \left(X_u - W_u \right) , \qquad \text{if} \quad Y_{u-1} + X_u - W_u < R$$

$$Y_u = R > \left(X_u - W_u \right) , \qquad \text{if} \quad Y_{u-1} + X_u \geq W_u + R$$

$$Y_u < R + \left(X_u - W_u \right) , \qquad \text{if} \quad Y_{u-1} + X_u > W_u$$

$$Y_u = 0 < R + \left(X_u - W_u \right) , \quad \text{if} \quad Y_{u-1} + X_u \leq W_u$$

so that the inequalities (20) are valid for $t = u$. Suppose they are valid for $\tau > n + 1$ and all $t \leq n$. Because

$$Y_{n+1} = Y_n + \left(X_{n+1} - W_{n+1} \right)$$

if

$$W_{n+1} < Y_n + X_{n+1} < W_{n+1} + R ,$$

in this case,

$$\eta_{u,n} + \left(X_{n+1} - W_{n+1} \right) = \eta_{u,n+1} \leq Y_n + \left(X_{n+1} - W_{n+1} \right)$$

$$= Y_{n+1} \leq R + \eta_{u,n} + \left(X_{n+1} - W_{n+1} \right)$$

$$= R + \eta_{u,n+1} .$$

Otherwise

$$Y_{n+1} = 0 < R + \eta_{u,n+1} , \qquad \text{if } Y_n + X_{n+1} \leq W_{n+1} ,$$

$$Y_{n+1} = R > \eta_{u,n+1} , \qquad \text{if } Y_n + X_{n+1} \geq W_{n+1} + R ,$$

so that the inequalities (20) are valid for all t, $u \leq t < \tau$.

The inflow-demand process (X_t, W_t), $-\infty < t < \infty$, will be associated with a stationary process

$$(V_t^*, Y_t^*, Z_t^*) \quad , \quad -\infty < t < \infty \tag{21}$$

which precisely coincides with the reservoir process defined by (4)-(6)

$$(V_t, Y_t, Z_t) \quad , \quad t > 0$$

for all $t \geq \tau$ whatever the history was up time τ (particularly for every initial reservoir condition Y_o) where

$$\tau = \min_{u>0} \tau(u) \quad . \tag{22}$$

The real reservoir system operates (or will operate if not yet designed) from some moment t_o--say $t_o = 0$; and we consider the inflow-demand process (X_t, W_t) during the previous time period as a part of the general climatological processes ω_t, $-\infty < t < \infty$, in the region concerned (see Section 1). Moreover one can assume without any loss of generality that

$$\omega_t = (X_t, W_t) \quad , \quad -\infty < t < \infty \ .$$

Let the random variable $\tau(u) = \tau(\omega, u)$ be defined as the first moment when the random sequence

$$\eta(u,t) = \sum_{s=u}^{t} (X_s - W_s) \quad , \quad t \geq u$$

is out of the interval $(-R, R)$, and let

$$\tau_t(\omega) = \min_{u \geq t} \tau(\omega, u) \ . \tag{23}$$

Obviously

$$\tau_{t_1} \leq \tau_{t_2} \quad , \qquad t_1 \leq t_2 \quad ,$$

and

$$\tau_{t_2}(\omega) = \tau_{t_1}\left(S_\Delta \, \omega + \Delta, \; \Delta = t_2 - t_1 \right)$$

where S_u is the time-shift transformation (see (11)) .

Let us set

$$\Omega_s^t = \{\omega : \tau_s \leq t\} \; , \; \Omega^t = \bigcup_{s \leq t} \Omega_s$$

and

$$\Omega^* = \bigcap_t \; \Omega^t$$

(remember we are considering the integer time parameter).
Because of the relationship

$$\tau_{s+u}(\omega) - u = \tau_s\left(S_{-u} \, \omega\right) \quad , \; - \infty < u < \infty \quad , \tag{24}$$

we have

$$S_u \Omega_s^t = \Omega_{s+u}^{t+u} \subseteq \Omega_{s+u}^t \quad , \quad S_u \Omega^t \subseteq \Omega^t \quad , \quad u \leq 0 \; ,$$

where according to Equation (18)

$$P(\Omega_s^t) > 0 \text{ for } t - s \geq n$$

so the invariant event Ω^t certainly occurs , as does the event Ω^*

$$P(\Omega^*) = 1 \quad .$$

Therefore, for every time moment t there is a random s, $s \leq t$, such that $\tau_s \leq t$.

For any t let the corresponding s, $\tau_s \leq t$ (which is not unique)be taken and define a random variable Y_t^* as in (5):

$$Y_u^* = \begin{cases} 0 , & V_u^* < W_u \\ V_u^* - W_u , & W_u \leq V_u^* \leq W_u + R \\ R , & W_u + R < V_u^* \end{cases} \tag{25}$$

for all u, $\tau_s < u \leq t$, with the initial condition

$$Y_{\tau_s}^* = \begin{cases} R , & \eta_{s,\tau_s} \geq R \\ 0 , & \eta_{s,\tau_s} \leq -R \end{cases} ,$$

where

$$V_u^* = Y_{u-1}^* + X_u , \quad \tau_s < u \leq t . \tag{26}$$

Also let

$$Z_t^* = V_t^* - Y_t^* \tag{27}$$

be defined according to Equation (4) .

Apparently, because of the relationship (19), the variables

$$V_t^* = V_t^* (\omega) , \quad Y_t^* = Y_t^* (\omega) , \quad Z_t^* = Z_t^* (\omega)$$

do not depend on our choice of the corresponding moments

$$s = s(\omega) , \quad \tau_s = \tau_s(\omega) \leq t ,$$

and particularly

$$V_t^*(\omega) = V_t(\omega) \quad , \; Y_t^*(\omega) = Y_t(\omega) \quad , \quad Z_t^*(\omega) = Z_t(\omega), \quad (28)$$

in the case

$$s(\omega) = 0 \quad , \quad \tau_0(\omega) \leq t \quad ,$$

where the variables V_t, Y_t, Z_t have been defined above (see (4)-(6)).

One can verify from (25)-(27) that because of the relationships

$$\tau_{t_2}(\omega) = \tau_{t_1}(S_{-u}\omega) + u$$

$$\eta_{t_2,\tau_{t_1}}(\omega) = \eta_{t_1,\tau_{t_1}}(S_{-u}) \quad , \quad u = t_2 - t_1 \quad ,$$

we have

$$\begin{aligned} V_{t_2}^*(\omega) &= V_{t_1}^*(S_{-u}\omega) \quad , \\ Y_{t_2}^*(\omega) &= Y_{t_1}^*(S_{-u}\omega) \quad , \\ Z_{t_2}^*(\omega) &= Z_{t_1}^*(S_{-u}\omega) \quad , \quad u = t_2 - t_1 \quad . \end{aligned} \quad (29)$$

Thus the random process

$$(V_t^*, Y_t^*, Z_t^*) \quad , \quad -\infty < t < \infty \quad ,$$

as well as the original inflow-demand process

$$\omega_t = (X_t, W_t) \quad , \quad -\infty < t < \infty \quad ,$$

is <u>regular</u>, <u>homogeneous</u> and <u>stationary</u> with respect to the time shift transformations S_u (with a period u of integer years).

As previously mentioned, (see (28)), <u>the reservoir process</u> $\xi_t = (V_t, Y_t, Z_t)$, coincides with $\xi_t^* = (V_t^*, Y_t^*, Z_t^*)$, for all $t \geq \tau_o$.

Let us consider

$$\xi_t = (V_t, Y_t, Z_t) \quad , \quad t \geq 0$$

and its probability distribution $P = P(B)$ on the space of all possible trajectories:

$$P(B) = P\{\xi \in B\} \quad ,$$

where $\xi = \xi_t$, $t \geq 0$, is a trajectory of our random process. Let $P_u = P_u(B)$ be a shifted probability distribution, namely

$$P_u(B) = P\{S_u \; \xi \in B\} \quad ,$$

where $S_u \; \xi_t = \xi_{t+u}$, $t \geq 0$, is a trajectory shifted in time $u \geq 0$.

Let us consider also the stationary process

$$\xi_t^* = (V_t^*, Y_t^*, Z_t^*) \quad , \quad t \geq 0$$

and its probability distribution $P^* = P^*(B)$ which is invariant under shift transformations:

$$P_u^*(B) = P\{S_u \; \xi^* \in B\} = P\{\xi^* \in B\} = P^*(B)$$

(if u is a period of integer years).

One can verify that

$$\sup_B |P_u(B) - P_u^*(B)| \leq 2\ P\{\tau_o > u\}$$

where τ_o is the time moment determined by Equation (23).

Indeed,

$$\begin{aligned}
P_u(B) &= PS_u\xi \in B \\
&= P\left\{S_u\xi \in B \Big|_{\tau_o < u}\right\} \{\tau_o < u\} + P\left\{S_u\xi \in B \Big|_{\tau_o < u}\right\} , \\
&= P_u^*(B) = P\{S_u\xi^* \in B\} = P\left\{S_u\xi^* \in B \Big|_{\tau_o < u}\right\} \\
&\cdot P\{\tau_o < u\} + P\left\{S_u\xi^* \in B \Big|_{\tau_o < u\ P\{\tau_o > u\}}\right\}
\end{aligned}$$

and because under the condition $\tau_o \leq u$ we have

$$\xi_{t+u} = \xi_{t+u}^* , \qquad t \geq 0 ,$$

$$P\left\{S_u\xi \in B \Big|_{\tau_o \leq u}\right\} = P\left\{S_u\xi^* \in B \Big|_{\tau_o \leq u}\right\} ;$$

and thus

$$\begin{aligned}
\left|P_u(B) - P_u^*(B)\right| &= \left|P_u\left\{S_u\xi \in B \Big|_{\tau_o > u}\right\}\right. \\
&\left. - P\left\{S_u\xi^* \in B \Big|_{\tau_o > u}\right\}\right| \cdots P\{\tau_o > u\} < 2\ P\{\tau_o > u\} .
\end{aligned}$$

As for any probability distribution failure,

$$\lim_{u\to\infty} P\{\tau_o > u\} = 0 \quad .$$

Moreover, in the most interesting cases we have an exponential rate of convergence

$$P\{\tau_o > u\} \leq Ce^{-Du}$$

for some positive constants C,D. For example, if apart from Equation (18) we assume that

$$P\left\{|\eta_n| \geq R\Big|_{\omega_s, s \geq 0}\right\} > p > 0 \quad , \tag{29}$$

then for an integer years period t

$$P\left\{\tau_o > t + n\Big|_{\tau_o > t}\right\} \leq 1 - p$$

(remember our process is homogeneous), and

$$P\{\tau_o > t + n\} = P\left\{\tau_o > t + n\Big|_{\tau_o > t}\right\} P\{\tau_o > t\}$$

$$\leq (1 - p)\ P\{\tau_o > t\}$$

so finally we have

$$P\{\tau_o > t\} \leq (1 - p)^{\frac{t}{n} - 1} \quad . \tag{30}$$

Concerning the stationary distribution P^*, the following remark may not be useless in the case when the reservoir process $\xi_t = (V_t, Y_t, Z_t)$ is considered as a component of some Markov process with transition probabilities Q. Namely the proper probabilities P^* can be uniquely determined from the corresponding equation of the type

$$P^* Q = P^* \quad .$$

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