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CONSIDERATION ON SUPREMUM IN A MULTI-DIMENSIONAL SPACE AND CONJUGATE DUALITY IN MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT

The first part of this paper is devoted to consideration on the definition of "supremum" in a multi-dimensional Euclidean space. A desirable definition is looked for among several possible alternatives. In the second part conjugate duality in multiobjective optimization is developed. Supremum is defined in the extended multi-dimensional Euclidean space on the basis of consideration in the first part. Some useful concepts such as conjugate maps and subgradients are introduced for vector-valued set-valued maps. Finally a strong duality result for a multiobjective optimization problem is proved under a regularity condition.

CONTENTS

Consideration on Supremum in a Multi-Dimensional Space and Conjugate Duality in Multiobjective Optimization

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1. Introduction

In this paper we develop a conjugate duality theory in multiobjective optimization. Conjugate duality has been fully developed in scalar optimization by Rockafellar [1] and provides a unified framework for the duality theory. The author and Sawaragi extended it to the case of multiobjective optimization by introducing some new concepts such as conjugate maps and subgradients for vector-valued, set-valued maps ([2],[3]). Their results are based on the efficiency (Pareto maximality). Kawasaki refined their results and obtained a reciprocal duality by introducing the concept of "supremum set" in an extended Euclidean space ([4],[5]). His supremum is based on the weak efficiency (weak Pareto maximality) of the closure of a set in an extended sense.

On the other hand, there are some other definitions of "supremum" (Zowe [6], Gross [7], Nieuwenhuis [8], Brumelle [9], Ponstein [10] and so on). In the first part of this paper, we consider the definition of "supremum" in the multi-dimensional Euclidean spaces. As a conclusion, the definition based on the weak efficiency seems to be the most appropriate for our purpose from the mathematical point of view.

In the second part we define the supremum of a set in the extended multi-dimensional Euclidean space on the basis of weak efficiency. We also show that our definition is almost equivalent to that of Kawasaki [4]. Conjugate maps and subgradients are defined for vecot-valued, set-valued maps as extensions of ordinary conjugate functions and subgradients, respectively. Finally, duality results in multiobjective optimization are provided. Since our definition of supremum is almost equivalent to Kawasaki's, the duality results finally obtained are similar to his results in [5]. However, our approach makes the proofs easier and refines some results along with some new properties concerning "supremum".

Part I: Consideration on the Definition of Supremum in R^p

2. Several definitions of maximum and supremum in R^p

In this section we will consider several kinds of definitions of maximum and supremum for sets in the p-dimensional Euclidean space R^p ($p \geqq 1$). Though we deal only with maximum and supremum, analogous results can be also obtained for minimum and infimum.

Let D be a fixed pointed closed convex cone with the nonempty interior in R^p. The reader might imagine the simplest case where D is equal to the nonnegative orthant R^p_+. We use the following three symbols as inequalities: For $y, y' \in R^p$,

$$y \geqq y' \leftrightarrow y - y' \in D$$

$$y \geq y' \leftrightarrow y - y' \in D \setminus \{0\}$$

$$y > y' \leftrightarrow y - y' \in \operatorname{int} D.$$

The relation $\geqq$ is transitive, reflexive and antisymmetric; while $\geq$ and $>$ are transitive and irreflexive. Note that $\geq$ is equivalent to $>$ and $\geqq$, $\not\leq$ and $\not<$ are all equivalent when $p = 1$ and $D = R_+$.

First, we consider the maximum of a set Y in R^p. We must recall the definition in the case $p = 1$:

$$\hat{y} \in Y \subset R \text{ is a maximum of } Y \leftrightarrow \hat{y} \in Y \text{ and } \hat{y} \geqq y \quad \forall y \in Y.$$

Since $\geqq$, $\not\leq$ and $\not<$ are equivalent in this case, we may consider three kinds of extensions.

Definition 2.1 For $Y \subset R^p$,

$$\hat{y} = \max Y \leftrightarrow \hat{y} \in Y \text{ and } \hat{y} \geqq y \quad \forall y \in Y,$$

$$\hat{y} = P\text{-}\max Y \leftrightarrow \hat{y} \in Y \text{ and } \hat{y} \not\leq y \quad \forall y \in Y,$$

$$\hat{y} = WP\text{-}\max Y \leftrightarrow \hat{y} \in Y \text{ and } \hat{y} \not< y \quad \forall y \in Y,$$

Here note that $\max Y$ is a single point, but $P\text{-}\max Y$ and $WP\text{-}\max Y$ are sets. It is clear that

$$\max Y \in P\text{-}\max Y \subset WP\text{-}\max Y \subset Y.$$

We usually call $\max Y$ the greatest element of Y and an arbitrary y in $P-\max Y$ a maximal element (efficient or Pareto maximal point) of Y. An element in $WP-\max Y$ is called a weakly efficient or weakly Pareto maximal point.

Now we turn to the definition of supremum of Y. If $p=1$, the supremum of Y, $\sup Y$, is defined as the smallest upper bound for Y, namely.

$$\hat{y}=\sup Y \leftrightarrow \begin{cases} (i)\ \hat{y} \geqq y \ \ \forall y \in Y; \text{ and} \\ (ii)\ \text{if } y' \geqq y \ \ \forall y \in Y, \text{ then } y' \geq \hat{y}. \end{cases}$$

The second condition can be written in the contrapositive form:

$\{(ii')\}$ if $y' < \hat{y}$, then there exists $y \in Y$ such that $y' < y$.

As in the case of maximum, we can consider the following three kinds of supremum:

Definition 2.2 For $Y \subset R^p$

$$\hat{y}=\sup Y \leftrightarrow \begin{cases} (1-i)\ \ \hat{y} \geqq y \ \ \ \forall y \in Y; \text{ and} \\ (1-ii)\ \ \text{if } y' \geqq y \ \ \forall y \in Y, \text{ then } y' \geqq \hat{y}. \end{cases} \tag{S1}$$

$$\hat{y} \in P-\sup Y \leftrightarrow \begin{cases} (2-i)\ \ \hat{y} \not\leq y \ \ \ \forall y \in Y; \text{ and} \\ (2-ii)\ \ \text{if } y' \leq \hat{y} \text{ then there exists } y \in Y, \text{ such that } y' \leq y. \end{cases} \tag{S2}$$

$$\hat{y} \in WP-\sup Y \leftrightarrow \begin{cases} (3-i)\ \ \hat{y} \not< y \ \ \ \forall y \in Y; \text{ and} \\ (3-ii)\ \ \text{if } y' < \hat{y}, \text{ then there exists } y \in Y, \text{ such that } y' < y. \end{cases} \tag{S3}$$

Remark 2.1 If we use the ordering cone D explicitly, these definitions can be rewritten as follows:

$$\hat{y}=\sup Y \leftrightarrow \begin{cases} (1-i)\ \ \ Y \subset \hat{y}-D; \text{ and} \\ (1-ii)\ \ \text{if } Y \subset y'-D, \text{ then } \hat{y} \in y'-D. \end{cases} \tag{S1}$$

$$\hat{y} \in P-\sup Y \leftrightarrow \begin{cases} (2-i)\ \ \ (\hat{y}+D \setminus \{0\}) \cap Y=\phi; \text{ and} \\ (2-ii)\ \ \hat{y}-D \setminus \{0\} \subset Y-D \setminus \{0\}. \end{cases} \tag{S2}$$

$$\hat{y} \in WP-\sup Y \leftrightarrow \begin{cases} (3-i)\ \ \ (\hat{y}+\operatorname{int} D) \cap Y=\phi; \text{ and} \\ (3-ii)\ \ \hat{y}-\operatorname{int} D \subset Y-\operatorname{int} D. \end{cases} \tag{S3}$$

In the above definitions, $\sup Y$ is a point, while $P-\sup Y$ and $WP-\sup Y$ are sets. It is clear that $P-\sup Y \subset WP-\sup Y$. However $\sup Y$ is not generally contained in $P-\sup Y$. The first definition (S1) was used by Brumelle [9] and many others. The third definition (S3) was adopted by Nieuwenhuis [8].

Noting that

$$\sup Y = \max[cl(Y - R_+)]$$

when $p = 1, D = R_+$ and Y is bounded above, we may obtain other definitions of supremum.

Definition 2.3 Given a set $Y \subset R^p$, we define the following:

$$\sup{}' Y = \max[cl(Y - D)] \qquad \text{(S1')}$$

$$P-\sup{}' Y = P-\max[cl(Y - D)] \qquad \text{(S2')}$$

$$WP-\sup{}' Y = WP-\max[cl(Y - D)] \qquad \text{(S3')}$$

Note that $cl(Y - D) = cl(Y - \text{int}D)$. The definitions (S2') and (S3') are essentially equivalent to those given by Gross [7] and Kawasaki [4], respectively.

3. Some properties of several kinds of supremum

In this section we will study some properties of supremum defined in the previous section. First, we may expect that the definitions with and without the prime "′" are closely related to each other.

Proposition 3.1 *(1) If Y is bounded above,* $\sup Y = \sup{}' Y$. *If Y is not bounded above, neither* $\sup Y$ *nor* $\sup{}' Y$ *exists.*

(2) $WP-\sup Y = WP-\sup{}' Y = [cl(Y - D)] \setminus (Y - \text{int}D)$.

(Proof) (1) Easy. (2) The proof of the fact $WP-\sup Y = [cl(Y - D)] \setminus (Y - \text{int}D)$ can be found in Nieuwenhuis [8] (Theorem I-17). Hence we will show that $WP-\sup{}' Y = [cl(Y - D)] \setminus (Y - \text{int}D)$. First, if $\hat{y} \in WP-\sup{}' Y = WP-\max[cl(Y - D)]$, then it is clear that $\hat{y} \in cl(Y - D)$ and $\hat{y} \notin Y - \text{int}D$. Suppose conversely that $\hat{y} \in [cl(Y - D)] \setminus (Y - \text{int}D)$ but not $\hat{y} \in WP-\max[cl(Y - D)]$. Then there exists $y \in cl(Y - D)$ such that $\hat{y} < y$, i.e. $y \in \hat{y} + \text{int}D$. Hence, for sufficiently small $\varepsilon > 0$, $y + \varepsilon B \subset \hat{y} + \text{int}D$, where B is the unit ball in R^p. Since $y \in cl(Y - D)$, there exists $y' \in Y$ and $d \in D$ such that $y' - d \in y + \varepsilon B$. Therefore, $y' - d \in \hat{y} + \text{int}D$, namely $y' - \hat{y} \in d + \text{int}D \subset \text{int}D$. This implies that $\hat{y} \in Y - \text{int}D$ and so leads to a contradiction. Hence $\hat{y} \in WP-\sup{}' Y$ and the proof is completed.

Thus we need not discriminate between $\sup Y$ (resp. $WP-\sup Y$) and $\sup' Y$ (resp. $WP-\sup' Y$).

Remark 3.1 There is no inclusion relation between $P-\sup Y$ and $P-\sup' Y$. For example, if $Y=\{y\in R^2: y_1<1, y_2<1\}\cup\{y\in R^2: y_1\leqq 0, y_2=1\}$ and if $D=R^2_+$, then

$$P-\sup Y=\{(0,1)\} \text{ and } P-\sup' Y=\{(1,1)\}.$$

The following proposition is obvious.

Proposition 3.2 *$P-\sup Y\subset WP-\sup Y$ and $P-\sup' Y\subset WP-\sup Y$. Moreover we have some relationships between maximum and supremum.*

Proposition 3.3 *(1) If $\sup Y\in Y$, then $\sup Y=\max Y$.*

(2) $P-\max Y=Y\cap P-\sup Y\supset Y\cap P-\sup' Y$.

(3) $WP-\max Y=Y\cap WP-\sup Y$.

(Proof) The proofs are easy.

Remark 3.2 The relation $Y\cap P-\sup' Y=P-\max Y$ does not always hold. In fact, in the example in Remark 3.1,

$$P-\max Y=\{(0,1)\} \text{ and } WP-\sup' Y\cap Y=\phi.$$

Now we shall study some properties of the definitions given above. It seems to be better that the definition of supremum satisfies those properties. The most interesting property is related to the existence of supremum and is called the axiom of continuity of real numbers in the case $p=1$. This axiom asserts that a set which is bounded above has the supremum. Namely, if Y is a nonempty set and if there exists some $\bar{y}\in R$ such that $\bar{y}\geqq y$ for all $y\in Y$ (i.e. such that $Y\subset\bar{y}-D$), then there exists the supremum of Y. The assumption of boundedness can also be written as $Y-D\neq R$ or $Y-\text{int}D\neq R$. As an extension of this axiom, we have the following theorem:

Theorem 3.1 *For a nonempty set* $Y \subset R^p$,

(1) *there exists* $\sup Y$ *if and only if there exists* $\bar{y} \in R^p$ *such that* $\bar{y} \geqq y$ *for all* $y \in Y$;

(2) $P-\sup Y \neq \phi$ *if there exists* $\bar{y} \in R^p$ *such that* $\bar{y} \geqq y$ *for all* $y \in Y$, *and only if* $Y - D \neq R^p$.

(3) $PW-\sup Y \neq \phi$ *if and only if* $Y - D \neq R^p$.

(Proof) (1),(2) Not so difficult. (3) Due to Nieuwenhuis [8] (Theorem I-18).

Remark 3.3 (1) Even when there exists $\bar{y} \in R^p$ such that $\bar{y} \geqq y$ for all $y \in Y$, $P-\sup Y$ may be empty. For example, let $Y = \{y \in R^p : y < 0\}$. Then $P-\sup Y = \phi$, though $Y = -D = 0 - D$.

(2) $P-\sup' Y$ may not be empty when Y is not bounded above. For example, if $Y = \{y \in R^2 : y_1 + y_2 = 0\}$ and $D = R^2_+$, then $P-\sup' Y = Y$ which is not empty but unbounded above.

Another interesting and important property of the supremum is the fact that it divides the whole real line into two parts in the case $p = 1$. This result can be extended to the multi-dimensional case only when we consider the weak Pareto supremum.

Theorem 3.2 *(Nieuwenhuis [8], Lemma I-27) If* $WP-\sup Y \neq \phi$, *then*

$$R^p = (WP-\sup Y) \cup (WP-\sup Y + \operatorname{int} D) \cup (WP-\sup Y - \operatorname{int} D),$$

where the three sets in the right-hand side are disjoint.

Corollary 3.1 *If* $WP-\sup Y \neq \phi$, *then*

$$Y \subset WP-\sup Y - \operatorname{int} D \cup \{0\}.$$

As a counterexample which shows that the theorem and corollary are not valid if we replace $WP-\sup Y$ by $P-\sup Y$ or $P-\sup' Y$, we may consider the example in Remark 3.1.

Taking these results into account, we may conclude that $WP-\sup Y$ is the most appropriate as the definition of supremum of a set Y in R^p from the mathematical point of view. Namely, it satisfies the extensions of the desirable properties of the ordinary supremum in the uni-dimensional Euclidean space. Hence, in the second part of this paper, we will define the supremum of Y essentially by $WP-\sup Y$ and develop the conjugate duality in multiobjective optimization.

Part II: Conjugate Duality in Multiobjective Optimization

4. Definition of Supremum in $\bar{R}^p$

In this section we redefine the supremum of a set not only in R^p but also in the extended Euclidean space $\bar{R}^p$. In the ordinary case ($p=1$), we put $\sup Y = +\infty$ if Y is not bounded above and $\sup Y = -\infty$ if Y is empty. Moreover these two imaginary points $+\infty$ and $-\infty$ are quite useful in optimization theory. Therefore we add $+\infty$ and $-\infty$ to the p-dimensional space R^p and denote the extended space by $\bar{R}^p$. These two points satisfy the following for any $y \in R^p$:

$$-\infty < y < \infty, \quad +\infty + y = +\infty \text{ and } -\infty + y = -\infty.$$

Of course we assume that $-(+\infty) = -\infty$. The sum $+\infty - \infty$ is not considered, since we can avoid it.

Given a set $Y \subset \bar{R}^p$, we define the set $A(Y)$ of all points above Y, and the set $B(Y)$ of all points below Y by

$$A(Y) = \{y \in \bar{R}^p : y > y' \text{ for some } y' \in Y\}$$

and

$$B(Y) = \{y \in \bar{R}^p : y < y' \text{ for some } y' \in Y\}.$$

respectively. Clearly $A(Y) \subset R^p \cup \{+\infty\}$ and $B(Y) \subset R^p \cup \{-\infty\}$.

Definition 4.1 Given a set $Y \subset \bar{R}^p$, a point $\hat{y} \in \bar{R}^p$ is said to be a maximal point of Y if $\hat{y} \in Y$ and $\hat{y} \notin B(Y)$, that is, if $\hat{y} \in Y$ and there is no $y' \in Y$ such that $\hat{y} < y'$. The set of al maximal points of Y is called the maximum of Y and is denoted by $\operatorname{Max} Y$.

Definition 4.2 Given a set $Y \subset \bar{R}^p$, a point $\hat{y} \in \bar{R}^p$ is said to be a supremal point of Y if $\hat{y} \notin B(Y)$ and $B(\hat{y}) \subset B(Y)$[†], that is, if there is no $y \in Y$ such that $\hat{y} < y$ and if the relation $y' < \hat{y}$ implies the existence of some $y \in Y$ such that $y' < y$. The set of all supremal points of Y is called the supremum of Y and is denoted by $\operatorname{Sup} Y$.

Remark 4.1 (1) $\operatorname{Max} \phi = \phi$ and $\operatorname{Sup} \phi = \{-\infty\}$.

[†] $B(\{\hat{y}\})$ is simply denoted by $B(\hat{y})$.

(2) $-\text{Max}(-Y) = \text{Min}\, Y$[†], and $-\text{Sup}(-Y) = \text{Inf}\, Y$[†].

Proposition 4.1 (cf. Proposition 3.3)

$$\text{Max}Y = Y \cap \text{Sup}Y$$

(Proof) If $\hat{y} \in Y \cap \text{Sup}Y$, it is clear that $\hat{y} \in \text{Max}\, Y$. Conversely, if $\hat{y} \in \text{Max}\, Y$, then $\hat{y} \in Y \setminus B(Y)$. Since $\hat{y} \in Y$, $B(\hat{y}) \subset B(Y)$. Hence $\hat{y} \in \text{Sup}Y$.

Proposition 4.2 *(1)* $\text{Sup}\, Y = \{-\infty\}$ *if and only if* $B(Y) = \phi$. *This is the case when and only when* $Y = \phi$ *or* $Y = \{-\infty\}$.

(2) $\text{Sup}\, Y = \{+\infty\}$ *if and only if* $B(Y) = R^p \cup \{-\infty\}$.

(3) *Except the above two cases,* $\text{Sup}\, Y \subset R^p$.

(Proof) (1) It is clear that $B(Y) = \phi$ when and only when $Y = \phi$ or $Y = \{-\infty\}$. It is also obvious that $\text{Sup}\, Y = \{-\infty\}$ if $B(Y) = \phi$. Finally, if $\text{Sup}\, Y = \{-\infty\}$, then $-\infty \notin B(Y)$, which implies that $B(Y) = \phi$.

(2) If $B(Y) = R^p \cup \{-\infty\}$, then $\text{Sup}\, Y \subset \{+\infty\}$. Since $B(+\infty) = R^p \cup \{-\infty\} = B(Y)$, $+\infty \in \text{Sup}\, Y$. Hence $\text{Sup}\, Y = \{+\infty\}$. Suppose conversely that $\text{Sup}\, Y = \{+\infty\}$. Then $+\infty \notin B(Y)$ and $B(+\infty) \subset B(Y)$. Since $B(+\infty) = R^p \cup \{-\infty\}$, $B(Y) = R^p \cup \{-\infty\}$.

(3) Since $-\infty < y < +\infty$ for any $y \in R^p$, $\text{Sup}\, Y \subset R^p$ except the above two special cases.

Now we shall consider a characterization of $\text{Sup}\, Y$ as the maximum of the closure of $B(Y)$ in $\bar{R}^p$ (cf. Proposition 3.1). The above proposition suggests us to define the closure of $B(Y)$ in $\bar{R}^p$ as follows: For $Y \subset \bar{R}^p$, let

$$cl\, B(Y) = \begin{cases} \{-\infty\} & \text{if } B(Y) = \phi \\ \bar{R}^p & \text{if } B(Y) = R^p \cup \{-\infty\} \\ cl\, (B(Y) \cap R^p) \cup \{-\infty\} & \text{otherwise.} \end{cases}$$

Here the symbol "*cl*" in the right-hand side means the usual closure in R^p.

[†] Minimum and infimum can be defined analogously to maximum and supremum, respectively.

Lemma 4.1 $B(cl\,B(Y)) = B(Y)$.

(Proof) If $B(Y) = \phi$ or $B(Y) = R^p \cup \{-\infty\}$, the lemma is obviously true. Hence we consider the remaining general case. The point $-\infty$ is clearly contained in both sets. Thus let $y \in B(Y)$ and $y \neq -\infty$. Then there exists $y' \in Y \cap R^p$ such that $y < y'$. Hence

$$\alpha y + (1-\alpha)y' \in B(Y) \quad \text{for any } \alpha,\ 0 < \alpha < 1.$$

Taking the limit when $\alpha \to 0$, we can prove that $y' \in cl\,B(Y)$ and so $y \in B(cl\,B(Y))$. Conversely suppose that $y \in B(cl\,B(Y))$ and $y \neq -\infty$. Then there exists a sequence $\{y^k\} \subset B(Y) \cap R^p$ such that $y^k \to y'$ and $y < y'$. Therefore, $y < y^k$ for sufficiently large k, and so $y \in B(Y)$.

Proposition 4.3 $\text{Sup}\, Y = [cl\,B(Y)] \setminus B(Y) = \text{Max}\,[cl\,B(Y)]$

(Proof) Since $\text{Max}\,Z = Z \setminus B(Z)$ generally, the right equality follows directly from Lemma 4.1. Hence we prove the left equality, namely that $\text{Sup}\,Y = [cl\,B(Y)] \setminus B(Y)$. Let $\hat{y} \in [cl\,B(Y)] \setminus B(Y)$. It suffices to consider the case where $\hat{y} \in cl\,(B(Y) \cap R^p)$, since the proposition is trivial in the other cases. Then there exists a sequence $\{y^k\} \subset B(Y) \cap R^p$ such that $y^k \to \hat{y}$. For any $y < \hat{y}$, we have $y < y^k$ for sufficiently large k. Hence $B(\hat{y}) \subset B(Y)$ and so $\hat{y} \in \text{Sup}\,Y$. Conversely suppose that $\hat{y} \in \text{Sup}\,Y$. Then $\hat{y} \notin B(Y)$ and $B(\hat{y}) \subset B(Y)$. Since we may assume that $\hat{y} \in R^p$, for an arbitrary fixed $d \in \text{int}\,D$,

$$\hat{y} - \alpha d \in B(\hat{y}) \subset B(Y) \quad \text{for any} \quad \alpha > 0.$$

By taking the limit when $\alpha \to 0$, we can prove that $\hat{y} \in cl\,B(Y)$. Hence $\hat{y} \in [cl\,B(Y)] \setminus B(Y)$. This completes the proof.

Corollary 4.1 $\text{Sup}\,Y = \text{Sup}\,(B(Y)) = \text{Sup}\,(cl\,B(Y))$

(Proof) From Proposition 4.3,

$$\text{Sup}\,Y = \text{Max}\,[cl\,B(Y)],$$

$$\text{Sup}\,(B(Y)) = \text{Max}\,[cl\,B(B(Y))]$$

and

$$\mathrm{Sup}\,(cl\,B(Y)) = \mathrm{Max}\,[cl\,B(cl\,B(Y))].$$

Since $B(B(Y)) = B(Y)$ clearly and $B(cl\,B(Y)) = B(Y)$ from Lemma 4.1, the above three sets coincide.

Lemma 4.2 *If $y \in R^p$ and $d \in \mathrm{int}\,D$, then there exists a positive number α_0 such that $y + \alpha d \in \mathrm{int}\,D$ for all $\alpha \geqq \alpha_0$.*

(Proof) If such α_0 does not exist, we can take a sequence of positive number $\{\alpha_k\}$ such that $\alpha_k \to +\infty$ and $y - \alpha_k d \notin \mathrm{int}D$. Since $\mathrm{int}D$ is a cone, $y / \alpha_k - d \notin \mathrm{int}D$. Noting that $(\mathrm{int}\,D)^c$ is a closed set and taking the limit when $k \to \infty$, we have $-d \notin \mathrm{int}D$. However, this is a contradiction.

Proposition 4.4 $B(Y) = B(\mathrm{Sup}\,Y)$.

(Proof) It is clear that $B(\mathrm{Sup}\,Y) \subset B(Y)$ from the definition of supremum. We shall prove the converse inclusion $B(Y) \subset B(\mathrm{Sup}\,Y)$. if $\mathrm{Sup}\,Y = \{+\infty\}$ or $\{-\infty\}$, the relation is obvious. In the other case, $-\infty$ is contained in both sets. Let $\hat{y} \in B(Y)$ and $\hat{y} \neq -\infty$. Then there exists $y \in Y$ such that $\hat{y} < y$. Take an arbitrary $d \in \mathrm{int}\,D$. Then there exists a positive number α_0 such that $y + \alpha d \notin cl\,B(Y)$ for all $\alpha > \alpha_0$, since otherwise $cl\,B(Y) \supset R^p$ (see Lemma 4.2). Thus we can define a finite nonnegative number $\bar{\alpha}$ by

$$\bar{\alpha} = \sup\,\{\alpha : y + \alpha d \in cl\,B(Y)\}.$$

Then it is clear that $y + \bar{\alpha} d \in \mathrm{Sup}\,Y = \mathrm{Max}\,[cl\,B(Y)]$. Since $\hat{y} < y \leqq y + \bar{\alpha} d$, we have proved that $B(Y) \subset B(\mathrm{Sup}\,Y)$.

Corollary 4.2

$$Y \subset cl\,B(Y) = \mathrm{Sup}\,Y \cup B(Y) = \mathrm{Sup}\,Y \cup B(\mathrm{Sup}\,Y).$$

(Proof) It is clear that $Y \subset cl\,B(Y)$. Since $\mathrm{Sup}\,Y = [cl\,B(Y)] \setminus B(Y)$ from Proposition 4.3 and $B(Y) \subset cl\,B(Y)$,

$$cl\,B(Y) = \mathrm{Sup}\,(Y) \cup B(Y).$$

The last equality directly follows from Proposition 4.4.

Proposition 4.5 (cf. Theorem 3.2)

$$\bar{R}^p = \text{Sup}\, Y \cup A(\text{Sup}\, Y) \cup B(\text{Sup}\, Y)$$

and the above three sets are disjoint. (They may be empty.)

(Proof) It is clear that the three sets are disjoint. Since

$$\text{Sup}\, Y \cup B(\text{Sup}\, Y) = cl\, B(Y)$$

from Corollary 4.2, it suffices to prove that $y \in A(\text{Sup}\, Y)$ if $y \notin cl\, B(Y)$. When $\text{Sup}\, Y = \{-\infty\}$ or $\{+\infty\}$, the above statement is obviously true. So we consider the remaining ordinary case. Since $+\infty \in A(\text{Sup}\, Y)$, we take $y \notin cl\, B(Y)$ with $y \neq +\infty$ and prove that $y \in A(\text{Sup}\, Y)$. Fix an arbitrary $d \in \text{int}\, D$. Since $Y \cap R^p \neq \phi$ in this case, $y - \alpha d \in B(Y)$ for sufficiently large $\alpha > 0$ by Lemma 4.2. Let

$$\bar{\alpha} = \inf\{\alpha > 0 : y - \alpha d \in B(Y)\}$$

and $\bar{y} = y - \bar{\alpha} d$. Showing that $\bar{y} \in \text{Sup}\, Y$ completes the proof. Since $\bar{y} \in cl\, B(Y) = \text{Sup}\, Y \cup B(Y)$, it suffices to show that $\bar{y} \notin B(Y)$. If we suppose to the contrary that $\bar{y} \in B(Y)$, then $y - \alpha d \in B(Y)$ from some α slightly smaller than $\bar{\alpha}$, which contradicts the definition of $\bar{\alpha}$. Therefore $\bar{y} \notin B(Y)$ as was to be proved.

Proposition 4.6 *If $Y_1 \subset Y_2 \subset \bar{R}^p$, then*

$$\text{Sup}\, Y_1 \subset \text{Sup}\, Y_2 \cup B(\text{Sup}\, Y_2)$$

(Proof)

$$Y_1 \subset Y_2 \Longrightarrow B(Y_1) \subset B(Y_2)$$

$$\Longrightarrow cl\, B(Y_1) \subset cl B(Y_2)$$

$$\Longrightarrow \text{Sup}\, Y_1 \subset cl\, B(Y_1) \subset cl\, B(Y_2) = \text{Sup}\, Y_2 \cup B(\text{Sup}\, Y_2).$$

Lemma 4.3 *(1) $B(Y_1 + Y_2) = B(Y_1) + B(Y_2)$ for $Y_1, Y_2 \subset \bar{R}^p$, where it is assumed that the sum $+\infty - \infty$ does not occur.*

(2) $B(\bigcup_{i \in I} Y_i) = \bigcup_{i \in I} B(Y_i)$ for $Y_i \subset \bar{R}^p$ $(i \in I)$.

(Proof) Not so difficult.

Proposition 4.7 *Let F_1 and F_2 be set-valued maps from a space X to $\bar{R}^p$. Here the sum $+\infty-\infty$ is assumed not to occur. Then*

$$\mathrm{Sup}\bigcup_x [F_1(x)+F_2(x)]=\mathrm{Sup}\bigcup_x [F_1(x)+\mathrm{Sup}F_2(x)]$$

(Proof)

$$\mathrm{Sup}\bigcup_x [F_1(x)+F_2(x)]$$

$$=\mathrm{Sup}(B(\bigcup_x [F_1(x)+F_2(x)])) \quad (\textit{Corollary } 4.1)$$

$$=\mathrm{Sup}(\bigcup_x B[F_1(x)+F_2(x)]) \quad (\textit{Lemma } 4.3,(2))$$

$$=\mathrm{Sup}\bigcup_x [B(F_1(x))+B(F_2(x))] \quad (\textit{Lemma } 4.3,(1))$$

$$=\mathrm{Sup}\bigcup_x [B(F_1(x))+B(\mathrm{Sup}F_2(x))] \quad (\textit{Proposition } 4.4)$$

$$=\mathrm{Sup}\bigcup_x B[F_1(x)+\mathrm{Sup}F_2(x)] \quad (\textit{Lemma } 4.3,(1))$$

$$=\mathrm{Sup}B(\bigcup_x [F_1(x)+\mathrm{Sup}F_2(x)]) \quad (\textit{Lemma } 4.3,(2))$$

$$=\mathrm{Sup}\bigcup_x [F_1(x)+\mathrm{Sup}F_2(x)] \quad (\textit{Corollary } 4.1)$$

Corollary 4.3 *If F is a set-valued map from X to $\bar{R}^p$, then*

$$\mathrm{Sup}\bigcup_x F(x)=\mathrm{Sup}\bigcup_x \mathrm{Sup}F(x).$$

(Proof) Take $F_1(x)=F(x)$ and $F_2(x)=\{0\}$ in Proposition 4.7.

Corollary 4.4 *For $Y_1, Y_2\subset\bar{R}^p$,*

$$\mathrm{Sup}(Y_1\cup Y_2)=\mathrm{Sup}(\mathrm{Sup}Y_1\cup\mathrm{Sup}Y_2)$$

(Proof) Take an arbitrary $\bar{x} \in X$, and let $F(\bar{x}) = Y_1$ and $F(x) = Y_2$ for any $x \neq \bar{x}$ in Corollary 4.3.

Corollary 4.5 Sup (Sup Y) = Sup Y *for* $Y \subset \bar{R}^p$.

(Proof) Let $Y_1 = Y_2 = Y$ in Corollary 4.4.

Proposition 4.8 *Given a set* $Y \subset \bar{R}^p$,

$$\mathrm{Sup}(\mathrm{Inf}\, Y) = \mathrm{Inf}\, Y.$$

(Proof) From Corollary 4.2,

$$\mathrm{Inf}\, Y \subset \mathrm{Sup}\,(\mathrm{Inf}\, Y) \cup B(\mathrm{Inf}\, Y).$$

Since $\mathrm{Inf}\, Y \cap B(\mathrm{Inf}\, y) = \phi$, $\mathrm{Inf}\, Y \subset \mathrm{Sup}(\mathrm{Inf}\, Y)$. Conversely, if $\hat{y} \notin \mathrm{Inf}\, Y$, then $\hat{y} \in A(\mathrm{Inf}\, Y) \cup B(\mathrm{Inf}\, Y)$ from Proposition 4.5. If $\hat{y} \in B(\mathrm{Inf}\, Y)$, then $\hat{y} \notin \mathrm{Sup}\,(\mathrm{Inf}\, Y)$. If $\hat{y} \in A(\mathrm{Inf}\, Y)$, then $\hat{y} \notin cl\,(B(\mathrm{Inf}\, Y))$ and so $\hat{y}$ Sup(Inf Y). Therefore $\mathrm{Sup}(\mathrm{Inf}\, Y) \subset \mathrm{Inf}\, Y$. This completes the proof.

The final proposition in this section provides a characterization of the supremum by scalarization under the convexity assumption.

Proposition 4.9

$$\mathrm{Sup}\, Y \supset \bigcup_{\mu \in D^0 \setminus \{0\}} \{ \hat{y} \in cl\, B(Y) : <\mu, \hat{y}> = \sup_{y \in Y} <\mu, y> \}$$

and the converse inclusion is also valid if $cl\, B(Y)$ *is a convex set. Here* D^0 *is a dual (positive polar) cone of* D, *i.e.*

$$D^0 = \{ \mu \in R^p : <\mu, d> \geqq 0 \quad \forall d \in D \}.$$

(Proof) A similar version of this proposition is known well (e.g. [3] Chapter 3) and the proof of this proposition can be easily modified from that of the existent result. Hence it is omitted here.

5. Conjugate maps

In this section we shall define the conjugate map of a set-valued map from a linear topological space X to $\bar{R}^p$. This concept is an extension of that of well-known conjugate functions.

Given a linear topological space X, we consider the space of all linear continuous operators from X to R^p as a dual space of X with respect to R^p. This space is simply denoted by X^* in this paper. Namely, for $x \in X$ and $T \in X^*$, Tx represents an element in R^p. If $X=R^n$, then T is identified with a $p \times n$ matrix. In this section let F be a set-valued map from X to $\bar{R}^p$.

Definition 5.1 A set-valued map F^* from X^* to $\bar{R}^p$ defined by

$$F^*(T) = \mathrm{Sup} \bigcup_{x \in X} [Tx - F(x)] \quad \text{for} \quad T \in X^*$$

is called the conjugate map of F. Moreover, a set-valued map F^{**} from X to $\bar{R}^p$ defined by

$$F^{**}(x) = \mathrm{Sup} \bigcup_{T \in X^*} [Tx - F^*(T)] \quad \text{for} \quad x \in X$$

is called the biconjugate map of F. When f is a function from X to $\bar{R}^p$, its conjugate map and biconjugate map can be defined by identifying it with a set-valued map $x \mapsto \{f(x)\}$.

Proposition 5.1 *Let $\bar{x}$ be a point in X. If we define a set-valued map G from X to $\bar{R}^p$ by $G(x)=F(x+\bar{x})$ for all $x \in X$, then*

(1) $G^*(T)=F^*(T)-T\bar{x}$,

(2) $G^{**}(x)=F^{**}(x+\bar{x})$.

(Proof) (1)

$$\begin{aligned} G^*(T) &= \mathrm{Sup} \bigcup_{x} [Tx - G(x)] \\ &= \mathrm{Sup} \bigcup_{x} [Tx - F(x+\bar{x})] \\ &= \mathrm{Sup} \bigcup_{x'} \{[Tx' - F(x')] - T\bar{x}\} \\ &= \mathrm{Sup} \bigcup_{x'} [Tx' - F(x')] - T\bar{x} \end{aligned}$$

$$= F^*(T) - T\bar{x}.$$

(2)

$$G^{**}(x) = \text{Sup} \bigcup_T [Tx - G^*(T)]$$

$$= \text{Sup} \bigcup_T [Tx - F^*(T) + T\bar{x}] = F^{**}(x + \bar{x}).$$

Proposition 5.2 *Let $\bar{y}$ be a point in R^p. Then*

(1) $(F + \bar{y})^*(T) = F^*(T) - \bar{y}$

(2) $(F + \bar{y})^{**}(x) = F^{**}(x) + \bar{y}$.

(Proof) (1)

$$(F + \bar{y})^*(T) = \text{Sup} \bigcup_x [Tx - F(x) - \bar{y}]$$

$$= \text{Sup} \bigcup_x [Tx - F(x)] - \bar{y}$$

$$= F^*(T) - \bar{y}.$$

(2)

$$(F + \bar{y})^{**}(x) = \text{Sup} \bigcup_x [Tx - F^*(T) + \bar{y}]$$

$$= \text{Sup} \bigcup_x [Tx - F^*(T)] + \bar{y} = F^{**}(x) + \bar{y}.$$

Lemma 5.1 *Let* Inf$F$ *be another set-valued map from* X *to* $\bar{R}^p$ *defined by* $(\text{Inf}\, F)(x) = \text{Inf}\, F(x)$ *for all* $x \in X$. *Then*

$$F^*(T) = (\text{Inf}\, F)^*(T) \text{ and } F^{**}(x) = (\text{Inf}\, F)^{**}(x).$$

(Proof)

$$(\text{Inf}\, F)^*(T) = \text{Sup} \bigcup_x [Tx - (\text{Inf}\, F)(x)]$$

$$= \mathrm{Sup}\bigcup_x \mathrm{Sup}[Tx - F(x)]$$

$$= \mathrm{Sup}\bigcup_x [Tx - F(x)] = F^*(T).$$

Proposition 5.3 (Extension of Fenchel's inequality) *If* $y \in F(x)$ *and* $y' \in F^*(T)$, *then* $y + y' \nleq Tx$. *In other words, for any* $x \in X$ *and any* $T \in X^*$,

$$[F(x) - Tx] \cap B(-F^*(T)) = \phi.$$

(Proof) Since

$$F^*(T) = \mathrm{Sup}\bigcup_x [Tx - F(x)],$$

it is clear from Corollary 4.2 and Proposition 4.5 that

$$[Tx - F(x)] \cap A(F^*(T)) = \phi.$$

This proves the proposition.

Corollary 5.1 *If* $y \in F(0)$ *and* $y' \in -F^*(T)$, *then* $y \nleq y'$.

(Proof) Let $y' = -y'$ and $x = 0$ in Proposition 5.3.

Corollary 5.2 *If* $y \in F(x)$ *and* $y' \in F^{**}(x)$, *then* $y \nleq y'$. *In other words,*

$$F(x) \subset F^{**}(x) \cup A(F^{**}(x)).$$

(Proof) From Proposition 5.3,

$$F(x) \cap B(Tx - F^*(T)) = \phi.$$

However, $B(Tx - F^*(T)) = B(F^{**}(x))$ by Proposition 4.4 and hence

$$F(x) \cap B(F^{**}(x)) = \phi.$$

This implies

$$F(x) \subset F^{**}(x) \cup A(F^{**}(x))$$

from Proposition 4.5.

6. Subgradients

In this section we introduce the concept of subgradients for set-valued maps from a linear topological space to $\bar{R}^p$. The differentiability of a map is closely connected with a relationship between itself and its biconjugate map. In this section F is assumed to be a set-valued map from a linear topological space X to $\bar{R}^p$.

Definition Let $\hat{x} \in X$ and $\hat{y} \in F(\hat{x})$. An element $T \in X^*$ is said to be a subgradient of F at $(\hat{x};\hat{y})$ if

$$T\hat{x} - \hat{y} \in \operatorname{Max} \bigcup_x [Tx - F(x)].$$

The set of all subgradients of F at $(\hat{x};\hat{y})$ is called the subdifferential of F at $(\hat{x};\hat{y})$ and is denoted by $\partial F(\hat{x};\hat{y})$. Moreover we let $\partial F(\hat{x}) = \bigcup_{\hat{y} \in F(\hat{x})} \partial F(\hat{x};\hat{y})$. We can similarly define $\partial f(\hat{x})$ for a function f. When $\partial F(\hat{x};\hat{y}) \neq \phi$ for every $\hat{y} \in F(\hat{x})$, F is said to be subdifferentiable at $\hat{x}$.

The first result is a characterization of a minimal point of a set-valued map.

Proposition 6.1 *A point $\hat{y} \in F(\hat{x})$ is in $\operatorname{Min} \bigcup_x F(x)$ if and only if $0 \in \partial F(\hat{x};\hat{y})$.*

(Proof) Obvious from the definition of the subgradient.

The second result is a relationship between the subgradient and the conjugate map.

Proposition 6.2 *Let $\hat{y} \in F(\hat{x})$ for some $\hat{x} \in X$. Then $T \in \partial F(\hat{x};\hat{y})$ if and only if $T\hat{x} - \hat{y} \in F^*(T)$.*

(Proof) From the definition of the subgradient, $T \in \partial F(\hat{x};\hat{y})$ if and only if

$$T\hat{x} - \hat{y} \in \mathrm{Max} \bigcup_x [Tx - F(x)].$$

The latter condition is equivalent to the following by Proposition 4.1:

$$T\hat{x} - \hat{y} \in F^*(T) \cap \bigcup_x [Tx - F(x)].$$

Hence the proposition is obviously true.

Moreover, the subdifferentiability guarantees a relationship between a map and its biconjugate.

Theorem 6.1 *If F is subdifferentiable at $\hat{x} \in X$, then $F(\hat{x}) \subset F^{**}(\hat{x})$. Moreover, if $F(\hat{x}) = \mathrm{Inf}\, F(\hat{x})$ in addition, then $F(\hat{x}) = F^{**}(\hat{x})$.*

(Proof) In view of Proposition 5.1, it suffices to prove the case $\hat{x} = 0$. First, let $\hat{y} \in F(0)$. Since F is subdifferentiable at 0, there exists $\hat{T} \in X^*$ such that $\hat{y} \in -F^*(\hat{T})$. Then, from Corollary 5.1,

$$\hat{y} \in \mathrm{Max} \bigcup_T [-F^*(T)] \subset \mathrm{Sup} \bigcup_T [-F^*(T)] = F^{**}(0).$$

Thus we have proved that $F(\hat{x}) \subset F^{**}(\hat{x})$. Next we assume that $F(0) = \mathrm{Inf}\, F(0)$ and take an arbitrary $\hat{y} \in F^{**}(0)$. From Proposition 4.5,

$$\bar{R}^p = F(0) \cup A(F(0)) \cup B(F(0)).$$

In view of Corollay 5.2, $\hat{y} \notin A(F(0))$. If we suppose that $\hat{y} \in B(f(0))$, there exists $y' \in F(0)$ such that $\hat{y} < y'$. Then there exists $T' \in X^*$ such that $y' \in -F^*(T')$ since F is assumed to be subdifferentiable at 0. However, this implies that $\hat{y} \in B(-F^*(T'))$ and hence contradicts the assumption $\hat{y} \in F^{**}(0) = \mathrm{Sup} \bigcup_T [-F^*(T)]$. Therefore $\hat{y} \in F(0)$ and we have proved that $F^{**}(\hat{x}) \subset F(\hat{x})$.

Finally we will show that a convexity assumption guarantees the subdifferentiability of a map as in the ordinary case of a scalar-valued function. To this end, we must define the convexity of a set-valued map F by

$$\mathrm{epi}\, F = \{(x, y) \in X \times R^p : y \geq y' \text{ for some } y' \in F(x)\},$$

and say that F is convex when $\mathrm{epi}\, F$ is a convex set in $X \times R^p$. We also define the effective domain of F by

$$\mathrm{dom}\, F = \{x \in X : F(x) \cap R^p \neq \phi\}.$$

Proposition 6.3 *If a set-valued map F from a locally convex linear topological space X to $R^p \cup \{+\infty\}$ is convex, if $\hat{x} \in \operatorname{int}\operatorname{dom} F$ and if $F(\hat{x}) \subset \operatorname{Inf} F(\hat{x})$, then F is subdifferentiable at $\hat{x}$.*

(Proof) Let $\hat{y} \in F(\hat{x})$. Since $F(\hat{x}) \subset \operatorname{Inf} F(\hat{x})$, $(\hat{x},\hat{y})$ is clearly a boundary pont of epi$F$ in $X \times R^p$. Therefore there exists a hyperplane which supports the convex set epi$F$ at $(\hat{x},\hat{y})$. Namely there exists a nonzero vector $(\lambda,\mu) \in X' \times R^p$ such that

$$<\lambda,\hat{x}> + <\mu,\hat{y}> \leqq <\lambda,x> + <\mu,y> \qquad \forall (x,y) \in \operatorname{epi} F,$$

where X' is a usually paired space of X and $<\cdot,\cdot>$ denotes the pairing or the inner product. Since $\hat{x} \in \operatorname{int}\operatorname{dom} F$, μ is not equal to the zero vector. Hence we can take some $T \in X^*$ so that

$$<\mu, Tx> = -<\lambda,x> \qquad \forall x \in X.$$

In fact, there exists a vector $e \in R^p$ such that $<\mu,e> = -1$ since $\mu \neq 0$. So we may define T as $Tx = <\lambda,x>e$. Then

$$<\mu,\hat{y} - T\hat{x}> \leqq <\mu, y - Tx> \quad \text{for any } x \in X \text{ and } y \in F(x).$$

In view of Proposition 4.9, this implies that

$$\hat{y} - T\hat{x} \in \operatorname{Inf} \bigcup_x [F(x) - Tx].$$

namely, that $T \in \partial F(\hat{x};\hat{y})$. Thus F is subdifferentiable at $\hat{x}$.

7. Duality in multiobjective optimization

In this section we shall derive duality results in multiobjective optimization. A domination cone D, which is of course a pointed closed convex cone, is assumed to be given in R^p. Let f be a function from a locally convex linear topological space X to $R^p \cup \{+\infty\}$ and consider a multiobjective optimization problem

(P) minimize $f(x)$

Solving this problem means to find the set

$$\operatorname{Min}(\mathrm{P}) = \operatorname{Min}\{f(x) : x \in X\}$$

or the set

$$\mathrm{Inf}\,(\mathrm{P}) = \mathrm{Inf}\,\{f(x) : x \in X\}.$$

We introduce a perturbation parameter $u \in U$ and imbed the primal problem (P) into a family of multiobjective optimization problems, where U is another locally convex linear topological space. Let φ be a function from $X \times U$ to $R^p \cup \{+\infty\}$ such that

$$\varphi(x,0) = f(x) \qquad \forall x \in X.$$

Then the perturbed problem is the following:

$$(\mathrm{P}_u) \qquad \underset{x}{\text{minimize}}\ \varphi(x,u).$$

Definition 7.1 The set-valued map W from U to $\bar{R}^p$ defined by

$$W(u) = \mathrm{Inf}\,(\mathrm{P}_u) = \mathrm{Inf}\,\{\varphi(x,u) : x \in X\}$$

is called the perturbation map for Problem (P).

Of course, $\mathrm{Inf}\,(\mathrm{P}) = W(0)$.

Now we consider the conjugate map of φ:

$$\varphi^*(T,\Lambda) = \mathrm{Sup}\,\{Tx + \Lambda u - \varphi(x,u) : x \in X, u \in U\} \ \text{ for } T \in X^* \text{ and } \Lambda \in U^*.$$

Then

$$-\varphi^*(0,\Lambda) = -\mathrm{Sup}\,\{\Lambda u - \varphi(x,u) : x \in X, u \in U\}$$

$$= \mathrm{Inf}\,\{\varphi(x,u) - \Lambda u : x \in X, u \in U\}.$$

We define the dual problem to (P) as follows:

$$(\mathrm{D}) \qquad \underset{\Lambda}{\text{maximize}}\ -\varphi^*(0,\Lambda).$$

Since $-\varphi^*(0,\cdot)$ is not a function but a set-valued map from U^* to $\bar{R}^p$, the dual problem is not a usual multiobjective optimization problem. However it can be understood as a problem to obtain the set $\mathrm{Sup} \bigcup_{\Lambda} [-\varphi^*(0,\Lambda)]$, that is,

$$\mathrm{Sup}\,(\mathrm{D}) = \mathrm{Sup} \bigcup_{\Lambda} [-\varphi^*(0,\Lambda)].$$

Remark 7.1 If we would like to make much of the symmetry between the primal and dual problems, we may consider a set-valued map Φ from $X\times U$ to $R^p\cup\{+\infty\}$ such that $\Phi(x,0)=\text{Inf}\{f(x)\}$ for all $x\in X$. Then the primal and dual problems may be written as

(P) $\quad \underset{x}{\text{minimize}}\ \Phi(x,0)$, and

(D) $\quad \underset{\Lambda}{\text{maximize}}\ -\Phi^*(0,\Lambda)$.

The first result we can prove is a weak duality theorem, which states that any feasible value of the primal problem is not below any feasible value of the dual problem.

Proposition 7.1 *For any $x\in X$ and $\Lambda\in U^*$,*

$$\varphi(x,0)\notin B(-\varphi^*(0,\Lambda)).$$

And hence

$$\text{Inf}\,(\text{P})\cap B(\text{Sup}\,(\text{D}))=\phi.$$

(Proof) From Proposition 5.3,

$$\varphi(x,0)-0x-\Lambda 0\notin B(-\varphi^*(0,\Lambda)),$$

that is

$$\varphi(x,0)\notin B(-\varphi^*(0,\Lambda)).$$

This completes the proof of the proposition.

The following relationship between the perturbation map and the dual map is quite important.

Lemma 7.1 $W^*(\Lambda)=\varphi^*(0,\Lambda)$.

(Proof)

$$W^*(\Lambda)=\text{Sup}\bigcup_u[\Lambda u-W(u)]$$

$$=\text{Sup}\bigcup_u[\Lambda u-\text{Inf}\{\varphi(x,u):x\in X\}]$$

$$= \text{Sup} \bigcup_{u} [\Lambda u + \text{Sup} \{ -\varphi(x,u) : x \in X \}]$$

$$= \text{Sup} \bigcup_{u} [\text{Sup} \{ \Lambda u - \varphi(x,u) : x \in X \}]$$

$$= \text{Sup} \bigcup_{u} \{ \Lambda u - \varphi(x,u) : x \in X \} \quad \text{(Corollary 4.3)}$$

$$= \text{Sup} \{ \Lambda u - \varphi(x,u) : x \in X, u \in U \}$$

$$= \varphi^{*}(0, \Lambda).$$

In view of this lemma, we can rewrite Sup(D) as

$$\text{Sup}\,(\text{D}) = \text{Sup} \bigcup_{\Lambda} [-W^{*}(\Lambda)] = W^{**}(0).$$

Since Inf (P) = $W(0)$, the relationship between the primal problem and the dual problem is nothing but the relationship between the values of the perturbation map and its biconjugate map at the nominal point $u = 0$. Hence we may pay attention to the following class of problems.

Definition 7.2 The primal problem (P) is said to be stable if the perturbation map W is subdifferentiable at 0.

We can obtain the strong duality for this class of problems.

Theorem 7.1 *If Problem* (P) *is stable, then*

$$\text{Inf}\,(\text{P}) = \text{Sup}\,(\text{D})$$

(Proof) Obvious from Theorem 6.1.

The following proposition and corollary show that convexity is essentially sufficient for guaranteeing the stability of the primal problem.

Proposition 7.2 *If the function* $\varphi: X \times U \to R^{p} \cup \{+\infty\}$ *is convex, then the perturbation map* W *is a convex set-valued map.*

(Proof) Let

$$Y(u) = \{ \varphi(x,u) : x \in X \} \subset R^p \cup \{+\infty\}$$

for each $u \in U$. Then, by Proposition 4.3,

$$W(u) = \mathrm{Inf}\, Y(u) = [cl\, A(Y(u))] \setminus A(Y(u)).$$

Let $(u^1, y^1), (u^2, y^2) \in \mathrm{epi}\, W$. Then

$$y^i \in W(u^i) + D \subset cl\, A(Y(u^i)) \quad \text{for } i = 1,2.$$

For each α such that $0 \leqq \alpha \leqq 1$,

$$\alpha y^1 + (1-\alpha) y^2 \in \alpha\, cl\, A(Y(u^1)) + (1-\alpha)\, cl\, A(Y(u^2))$$

$$\subset cl\, \{ \alpha A(Y(u^1)) + (1-\alpha) A(Y(u^2)) \}.$$

Since φ is convex, we can easily prove that

$$\alpha A(Y(u^1)) + (1-\alpha) A(Y(u^2)) \subset A(Y(\alpha u^1 + (1-\alpha) u^2)).$$

Therefore

$$\alpha y^1 + (1-\alpha) y^2 \in cl\, A(Y(\alpha u^1 + (1-\alpha) u^2)).$$

which implies that

$$\alpha(u^1, y^1) + (1-\alpha)(u^2, y^2) \in \mathrm{epi}\, W.$$

Hence epi W is a convex set in $U \times R^p$, that is, W is a convex set-valued map.

Corollary 7.1 *If the function φ is convex and if $0 \in \mathrm{int\,dom}\, \varphi(x,u)$ for some x, then Problem* (P) *is stable.*

(Proof) Obvious from Propositions 7.2 and 6.3.

8. Conclusion

In this paper we have considered what kind of definition is appropriate for the supremum of a set in the multi-dimensional Euclidean space. From a mathematical point of view, a definition based on the weak efficiency seems to be the most appropriate, though the oridinary efficiency is better from a practical point of view. Therefore we have defined the supremum of a set in the extended Euclidean space containing two imaginary points $\pm\infty$, on the basis of the weak maximality. This definition satisfies some desirable fundamental properties.

Some useful concepts such as conjugate maps and subgradients have been introduced for vector-valued set-valued maps also on the basis of weak maximality. These concepts have enabled us to develop the conjugate duality in multiobjective optimization. Although the results obtained are quire similar to the earlier works by the author and Sawaragi [2] or Kawasaki [5], our new approach makes the proofs much easier and more understandable.

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