ALGEBRA OF QUADRIFORM NUMBERS

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Introduction

In [1], a set of hypernumbers, called quadriform, were defined in terms of 4 × 4 real matrices which represent a kind of generalization of 2 × 2 matrix representation of complex numbers. The object was to provide for the other two square roots of unity which are always ignored in the complex field and then to be able to take square roots of all numbers in terms of real matrices. Although it was found possible to do so, the resulting set of entities do not form a consistent and usable algebra. This effort and the negative results obtained are summarized in Appendix B.

It was found, however, that the complete set of 2 × 2 real matrices as defined earlier in [1] do lead to a consistent and usable algebra provided one is prepared to recognize certain restrictions on the range of allowable quantities. In this paper, it is this set which is referred to by the name "quadriform". It is not quite true that they form a field but they are not less than a field in the sense of a ring, but rather more than a field, in particular the complex field which is a proper subset of the quadriforms. The limitations on the range of allowable quadriforms are well defined and present no unusual difficulties with the following exception: addition (and subtraction) of two allowable numbers may give an unallowable result. This does not occur within the complex subset and all attributes of the complex field are retained within the subset.

In general, quadriforms are not commutative under multiplication. Although this does not prevent a consistent algebra, it does impose severe limitations on the generalization of complex functions. For example, there is no generalization of the exponential and natural log functions. More precisely, there are several possible generalizations of the exponential function, each of which is the same as e^z for the complex subset, but which do not retain the desired characteristics over the full quadriform set. This is discussed at some length in the last part of the paper.

Definition of Units

In [1], four 2 × 2 real matrices were defined and called r (real unit), i (imaginary unit), m and w, the latter two being the other square roots of r. For consistency in notation, we relabel these as t_n. They are as follows:

$$t_0(=r) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad t_1(=m) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$t_2(=i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad t_3(=w) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

It is readily verified that

$$t_1^2 = t_0 \ , \qquad t_2^2 = -t_0 \ , \qquad t_3^2 = t_0$$

and also that

$$t_1 t_2 = -t_3 \ , \qquad t_1 t_3 = -t_2$$

$$t_2 t_1 = t_3 \ , \qquad t_2 t_3 = -t_1$$

$$t_3 t_1 = t_2 \ , \qquad t_3 t_2 = t_1$$

so that the noncomutative nature appears immediately.

Any quadriform number v is represented as

$$v = (v_0, v_1, v_2, v_3) = \sum_{n=0}^{3} v_n t_n$$

where the v_n are real scalars. Thus in matrix form, any v has the form

$$v = \begin{bmatrix} v_0 + v_1 & -v_2 + v_3 \\ v_2 + v_3 & v_0 - v_1 \end{bmatrix} .$$

The determinant of v is $(v_0^2 - v_1^2) + (v_2^2 - v_3^2)$. For a pure complex number, i.e., $v_1 = v_3 = 0$, this reduces to the square of the absolute value in usual fashion. Hence we define the <u>absolute value</u> of v as

$$|v| = (v_0^2 - v_1^1 + v_2^2 - v_3^2)^{\frac{1}{2}}$$

and say v is <u>allowable</u> if $|v|^2 \geq 0$. Note that we have <u>not</u> used the words modulus or magnitude which will be needed later for other quantities. The three are not the same for quadriforms except over the complex subset.

Square Roots

Since the initial motivation for quadriforms was based on square roots, we will launch into a discussion of them immediately before developing the other attributes. If square roots cannot be calculated in consistent style, then there is not much left to discuss. Fortunately, square roots behave quite well.

Let v be any quadriform and w be its square root, if it exists. Then

$$w^2 = \begin{bmatrix} w_0+w_1 & -w_2+w_3 \\ w_2+w_3 & w_0-w_1 \end{bmatrix}^2 = \begin{bmatrix} (w_0+w_1)^2+(w_3^2-w_2^2) & 2w_0(w_3-w_2) \\ 2w_0(w_2+w_3) & (w_0-w_1)^2+(w_3^2-w_2^2) \end{bmatrix}$$

and we have four simultaneous quadratic equations to satisfy in real numbers. Let

$$w_0 + w_1 = a \qquad -w_2 + w_3 = d$$

$$w_2 + w_3 = c \qquad w_0 - w_1 = b \quad .$$

Then

$$a^2 + cd = v_0 + v_1$$

$$b^2 + cd = v_0 - v_1$$

$$(a + b)c = v_2 + v_3$$

$$(a + b)d = -v_2 + v_3 \quad .$$

Adding and subtracting the first pair and then the second gives the following:

$$a^2 + b^2 + 2cd = 2(w_0^2 + w_1^2 - w_2^2 + w_3^2) = 2v_0$$

$$a^2 - b^2 = 4w_0w_1 = 2v_1$$

$$(a + b)(c + d) = (2w_0)(2w_3) = 2v_3$$

$$(a + b)(c - d) = (2w_0)(2w_2) = 2v_2 \quad .$$

Hence we have the equations:

$$\begin{aligned} w_0^2 + w_1^2 - w_2^2 + w_3^2 &= v_0 \\ 2w_0w_1 &= v_1 \\ 2w_0w_2 &= v_2 \\ 2w_0w_3 &= v_3 \quad . \end{aligned}$$

If v_1 and v_3 are zero, there is always a real solution since we need only set $w_1 = w_3 = 0$ and solve the reduced set as shown in Appendix A. Furthermore, there is no indeterminacy about it. Since $v_1 = v_3 = 0$ defines a subset of numbers equivalent to the complex field, this is hardly surprising but it is encouraging. Note that the determinant is then $v_0^2 + v_2^2 \geq 0$ and the absolute value is well defined in real numbers in standard fashion.

If v_1 or v_3 is not zero, then $w_0 \neq 0$ and we can solve for the others. Thus,

$$w_0^2 + \left(\frac{v_1}{2w_0}\right)^2 - \left(\frac{v_2}{2w_0}\right)^2 + \left(\frac{v_3}{2w_0}\right)^2 = v_0$$

or

$$w_0^4 - v_0w_0^2 + \frac{1}{4}(v_1^2 - v_2^2 + v_3^2) = 0 \quad .$$

This discriminant is

$$(v_0^2 - v_1^2) + (v_2^2 - v_3^2)$$

which is simply the determinant. If this is nonnegative, there is a real solution for w_0^2 but w_0 is real only if

$$w_0^2 = \frac{1}{2}\{v_0 + [(v_0^2 - v_1^2) + (v_2^2 - v_3^2)]^{\frac{1}{2}}\} \geq 0 \quad .$$

We have the following results thus far:

1. If w_0^2 is not a real, nonnegative number, then we regard $\sqrt{v}$ as undefined.
2. If $v_1 = v_3 = 0$, $\sqrt{v}$ is always defined as for complex numbers.
3. If $v_0 \geq 0$, $v_1 = v_2 = v_3 = 0$, then $w_0 = \pm\sqrt{v_0}$ as with real numbers, i.e., $\sqrt{v} = (\pm\sqrt{v_0}, 0, 0, 0)$.

We must yet verify the cases where $v_1 = v_3 = 0$ does not hold. First assume w_0^2 is real and positive. Note that in this case,

$$v_0 = w_0^2 + \frac{1}{4w_0^2}(v_1^2 - v_2^2 + v_3^2) \quad .$$

Substituting back in the original matrix for w^2, we have

<u>For $v_0 + v_1$</u>

$$(w_0 + \frac{v_1}{2w_0})^2 + \frac{v_3^2 - v_2^2}{4w_0^2} = \frac{4w_0^4 + v_1^2 + 4w_0^2 v_1 + v_3^2 - v_2^2}{4w_0^2}$$

$$= v_1 + w_0^2 + \frac{1}{4w_0^2}(v_1^2 - v_2^2 + v_3^2)$$

$$= v_0 + v_1 \quad .$$

<u>For $v_0 - v_1$</u>

$$(w_0 - \frac{v_1}{2w_0})^2 + \frac{v_3^2 - v_2^2}{4w_0^2} = v_0 - v_1 \quad .$$

<u>For $v_2 \pm v_3$</u>

$$2w_0(\frac{v_2}{2w_0} \pm \frac{v_3}{2w_0}) = v_2 \pm v_3 \quad .$$

There are some remaining cases; consider this one: $v_0 = v_1 = 0$, $v_2 = v_3 \neq 0$. In this case the determinant is zero and $w_0^2 = 0$ but $w_0 w_2 = w_0 w_3 \neq 0$. This can only be satisfied if $w_2 = w_3 = \infty$. In matrix form,

$$v = \begin{bmatrix} 0 & 0 \\ v_2 + v_3 & 0 \end{bmatrix}$$

and no finite square root exists, real or complex. Hence we must also outlaw these exceptional cases. However, there are other exceptional cases, for example:

$$v_0 = v_1 \quad , \qquad v_2 = v_3 \quad .$$

In this case, the determinant is also zero but

$$w_0^2 = \frac{1}{2} v_0 \quad .$$

If $v_0 > 0$, then

$$w_0 = \frac{1}{\sqrt{2}}\sqrt{v_0} \quad , \qquad w_n = \frac{v_n}{\sqrt{2}\ \sqrt{v_0}} \quad , \qquad n = 1,2,3 \quad .$$

Since $v_0 = v_1$ and $v_2 = v_3$,

$$w_1 = w_0 \quad , \qquad w_2 = w_3 \quad .$$

In matrix format,

$$v = \begin{bmatrix} 2v_0 & 0 \\ 2v_2 & 0 \end{bmatrix} \quad , \qquad w = \begin{bmatrix} \sqrt{2v_0} & 0 \\ \dfrac{2v_2}{\sqrt{2v_0}} & 0 \end{bmatrix} \quad .$$

These cases will be displayed more fully under singular numbers.

We can now state the following rules:

A. Any quadriform number b is allowable if $|v|$ is real.

B. Any allowable v has square roots if $\frac{1}{2}(v_0 + |v|) > 0$.

The number $v = 0$ is exceptional and is its own square root.

<u>Summarization of Arithmetic Properties</u>

Any v has the form

$$v = \begin{bmatrix} v_0 + v_1 & -v_2 + v_3 \\ v_2 + v_3 & v_0 - v_1 \end{bmatrix}$$

For any v

$$v - v = 0 = \begin{bmatrix} 0 & 0 \\ & \\ 0 & 0 \end{bmatrix}$$

$$0 - v = -v$$

$$t_0 v = v t_0 = v$$

$$(-t_0)v = v(-t_0) = -v \quad .$$

From the theory of determinants:

$$|v|^2 = |-v|^2 \quad .$$

If v is allowable, -v is also.
If u and v are allowable, uv is also.
If u and v are both unallowable or if either absolute value is 0, then uv is allowable.

Multiplication of units is as follows:

t_0	t_1	t_2	t_3
t_1	t_0	$-t_3$	$-t_2$
t_2	t_3	$-t_0$	$-t_1$
t_3	t_2	t_1	t_0

(Read "stub × head") .

$|t_1|^2 = |t_3|^2 = -1$, hence they are unallowable alone.

$$|t_m + t_n|^2 = |t_m|^2 + |t_n|^2 \quad , \qquad m \neq n \quad .$$

If $|v|^2 > 0$, then

$$v^{-1} = \frac{1}{|v|^2}\begin{bmatrix} v_0 - v_1 & +v_2 - v_3 \\ -v_2 - v_3 & v_0 + v_1 \end{bmatrix}$$

$$|v^{-1}| = \frac{1}{|v|}$$

$$vv^{-1} = v^{-1}v = t_0 \quad .$$

If $|v|^2 \neq 0$, v^{-1} always exists but is unallowable if v is. Note that

$$v^{-1} = \frac{1}{|v|^2}(v_0, -v_1, -v_2, -v_3) \quad .$$

Thus reciprocation gives a result of the same form and does not change allowability; the same is true of negation. Addition and subtraction give a result of the same form but may change allowability.

Mulitiplication, and hence multiplication by a reciprocal ("division"), is not, in general, commutative. For any u, v:

$$uv = \left[\begin{array}{c|c} (u_0v_0+u_1v_1-u_2v_2+u_3v_3) & -(u_0v_2-u_1v_3+u_2v_0+u_3v_1) \\ +(u_1v_0+u_0v_1-u_2v_3+u_3v_2) & +(u_0v_3-u_1v_2+u_2v_1+u_3v_0) \\ \hline (u_0v_2-u_1v_3+u_2v_0+u_3v_1) & (u_0v_0+u_1v_1-u_2v_2+u_3v_3) \\ +(u_0v_3-u_1v_2+u_2v_1+u_3v_0) & -(u_1v_0+u_0v_1-u_2v_3+u_3v_2) \end{array}\right] \quad .$$

Thus multiplication gives a result of the same form. Let

$$x = uv \quad , \qquad y = vu \quad .$$

Then, by considering the expressions for uv and also with u and v interchanged, we get the following:

$$x_0 = y_0 \quad \text{in all cases}$$

$$x_1 = y_1 \quad \text{only if } u_2v_3 = u_3v_2$$

$$x_2 = y_2 \quad \text{only if } u_1v_3 = u_3v_1$$

$$x_3 = y_3 \quad \text{only if } u_1v_2 = u_2v_1 \quad .$$

This simply reflects the noncommutativity of the units.

It is clear that addition and subtraction are commutative and associative and that multiplication is right or left distributive over addition.

Any nonzero number with square roots has two allowable square roots. It may also have six unallowable square roots. The unallowable square roots of t_0 are

$$\pm t_1 = (0,\pm 1,0,0)$$

$$\pm t_3 = (0,0,0,\pm 1)$$

$$\pm\frac{1}{\sqrt{2}}(t_1+t_3) = (0,\ \pm\frac{1}{\sqrt{2}},0,\ \pm\frac{1}{\sqrt{2}}) \quad .$$

However, since these do not, in general, commute with other numbers, they cannot be used to form other roots. Thus, suppose

$$\sqrt{v} = w = (w_0,w_1,w_2,w_3) \quad .$$

Then, in general,

$$wt_1 \neq \sqrt{v} \quad , \qquad t_1w \neq \sqrt{v} \quad , \qquad t_1w \neq wt_1$$

even though

$$t_1^2 = t_0 \quad .$$

Instead, we have

$$(wt_1)(t_1w) = v \quad .$$

Conjugate Numbers

Quadriforms have three forms of conjugation. They are achieved by pre- and post-multiplication by a unit other than t_0,

t_1 conjugation

$$t_1vt_1 = (v_0,v_1,-v_2,-v_3)$$

t_2 conjugation

$$t_2vt_2 = (-v_0,v_1,-v_2,v_3)$$

t_3 conjugation

$$t_3vt_3 = (v_0,-v_1-v_2,v_3) \quad .$$

Note that for a complex number, t_2 conjugation gives the negative and either t_1 or t_3 gives what is usually defined as conjugation of a complex number.

Conjugation does not change allowability and the absolute value, if proper, is unchanged.

Reciprocation can be considered a fourth, "complete" form of conjugation since, provided $|v|^2 \neq 0$,

$$|v|^2v^{-1} = (v_0,-v_1,-v_2,-v_3) \quad .$$

This is the only form which commutes with v since, in general,

$$vt_nvt_n \neq t_nvt_nv \quad , \qquad n \neq 0 \quad .$$

Singular Numbers and Divisors of Zero

Any number with $|v|^2 = 0$ has no reciprocal. We call these singular numbers. Apart from 0, there are ten distinct sets* of such numbers. Let x and y represent any nonzero real numbers. Then:

	v_0	v_1	v_2	v_3
S_0:	0	0	x	x
S_1:	0	0	x	-x
S_2:	x	x	0	0
S_3:	x	-x	0	0
S_4:	x	x	y	y
S_5:	x	x	y	-y
S_6:	x	-x	y	y
S_7:	x	-x	y	-y
S_8:	x	0	0	±x
S_9:	0	x	±x	0

These and their squares have the following forms:

$$S_0: \quad \begin{bmatrix} 0 & 0 \\ 2v_2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$S_1: \quad \begin{bmatrix} 0 & -2v_2 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$S_2: \quad \begin{bmatrix} 2v_0 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 4v_0^2 & 0 \\ 0 & 0 \end{bmatrix}$$

*There is an eleventh, general class for which $|v|^2 = 0$ but no pattern is immediately evident.

$$S_3:\ \begin{bmatrix} 0 & 0 \\ 0 & +2v_0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 4v_0^2 \end{bmatrix}$$

$$S_4:\ \begin{bmatrix} 2v_0 & 0 \\ 2v_2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 4v_0^2 & 0 \\ 4v_0v_2 & 0 \end{bmatrix}$$

$$S_5:\ \begin{bmatrix} 2v_0 & -2v_2 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 4v_0^2 & -4v_0v_2 \\ 0 & 0 \end{bmatrix}$$

$$S_6:\ \begin{bmatrix} 0 & 0 \\ 2v_2 & 2v_0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 4v_0v_2 & 4v_0^2 \end{bmatrix}$$

$$S_7:\ \begin{bmatrix} 0 & -2v_2 \\ 0 & 2v_0 \end{bmatrix}^2 = \begin{bmatrix} 0 & -4v_0v_2 \\ 0 & 4v_0^2 \end{bmatrix}.$$

Note that in all above cases, $v^2 = 2v_0v$. Since $v_0 = 0$ for S_0 and S_1, these are square roots of 0. They are allowable, as are all singular numbers.

$$S_8:\ \begin{bmatrix} v_0 & \pm v_0 \\ \pm v_0 & v_0 \end{bmatrix}^2 = \begin{bmatrix} 2v_0^2 & \pm 2v_0^2 \\ \pm 2v_0^2 & 2v_0^2 \end{bmatrix}$$

$$S_9:\ \begin{bmatrix} v_1 & \mp v_1 \\ \pm v_1 & -v_1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For S_8 and S_9 also, $v^2 = 2v_0v$ and, since $v_0 = 0$ for S_9, this is another set of square roots of 0. One might regard S_8 and

S_9 as each constituting two sets; the patterns do not change but signs do.

Singular numbers do not commute, even with themselves except for S_2 and S_3, and, of course, squares. If multiplied by other numbers, they create singular numbers in other sets, depending on whether they multiply on the left or right.

Multiplication of nonsingular numbers never create singular numbers but, clearly, addition and subtraction may, just as with allowable and unallowable numbers. Singular numbers form the boundary between allowable numbers ($|v|$ real) and unallowable numbers ($|v|$ imaginary).

The allowable numbers for which no square roots exist are now identified, namely, the square roots of zero. The other singular numbers have square roots of their own form as is clear from the fact that the squares of such numbers are of their own form. In fact, for such v,

$$\sqrt{v} = \frac{1}{\sqrt{2v_0}} v \quad .$$

Transposition and Exponential Forms

What shall, or can, we mean by e^v where v is a quadriform. First, we would want it to include the usual formulas for complex numbers, that is:

If $z = x + iy \quad , \qquad e^z = w = u + iv \quad .$

Then

$$x = \frac{1}{2} \ln (u^2 + v^2) = \frac{1}{2} \ln |w|^2$$

$$\cos y = \frac{u}{|w|} \quad , \qquad \sin y = \frac{v}{|w|}$$

$$w = e^x(\cos y + i \sin y) \quad .$$

Writing these in quadriform format:

$$z = (z_0, 0, z_2, 0) \quad , \qquad e^z = w = (w_0, 0, w_2, 0)$$

$$z_0 = \frac{1}{2} \ell n \ |w|^2$$

$$\cos z_2 = \frac{w_0}{|w|} \quad , \qquad \sin z_2 = \frac{w_2}{|w|}$$

$$w = e^{z_0}(\cos z_2, 0, \sin z_2, 0) \quad .$$

Two difficulties appear immediately. First, for a general v, we may have $|v|^2 < 0$ so its log is complex. However, this simply means that if v is unallowable, its log is improper. This also is apparent from the fact that $|v|$ is complex, i.e., pure imaginary.

A more serious difficulty is that of noncommutativity. If u, v, z and w are complex numbers with

$$e^u = z \quad , \qquad e^v = w$$

then

$$zw = wz = e^{u+v} = e^{v+u} \quad .$$

But if these are quadriforms, then, in general, $zw \neq wz$ and yet $u + v = v + u$. How can this be arranged?

Let us examine more closely the noncommutativity of multiplication.

Suppose $w = uv$. This can be represented by a row × matrix multiplication as follows:

$$(w_0, w_1, w_2, w_3) = (u_0, u_1, u_2, u_3) \begin{bmatrix} v_0 & v_1 & v_2 & v_3 \\ v_1 & v_0 & -v_3 & -v_2 \\ -v_2 & -v_3 & v_0 & v_1 \\ v_3 & v_2 & v_1 & v_0 \end{bmatrix} \quad .$$

(This pattern is the easiest way to remember multiplication rules in terms of components.) One would like to be able to represent vu by treating u as a column on the right but it doesn't work due to minus signs. Notice that otherwise the matrix is symmetric. Let z = vu and compare the components of w and z:

$$w_0 = u_0v_0 + u_1v_1 - u_2v_2 + u_3v_3$$
$$z_0 = u_0v_0 + u_1v_1 - u_2v_2 + u_3v_3$$

$$w_1 = u_0v_1 + u_1v_0 - u_2v_3 + u_3v_2$$
$$z_1 = u_0v_1 + u_1v_0 + u_2v_3 - u_3v_2$$

$$w_2 = u_0v_1 - u_1v_3 + u_2v_0 + u_3v_1$$
$$z_2 = u_0v_2 + u_1v_3 + u_2v_0 - u_3v_1$$

$$w_3 = u_0v_3 - u_1v_2 + u_2v_1 + u_3v_0$$
$$z_3 = u_0v_3 + u_1v_2 - u_2v_1 + u_3v_0 \quad .$$

Now notice that we can get z from w in the following way:

(1) Change the signs of u_2 and v_2 before multiplying.

(2) Change the sign of z_2 after multiplying.

Changing the signs of the coefficient of t_2 amounts to transposing the matrix. If a prime denotes transposition, then

$$v' = \begin{bmatrix} v_0 + v_1 & -v_2 + v_3 \\ v_2 + v_3 & v_0 - v_1 \end{bmatrix}' = \begin{bmatrix} v_0 + v_1 & v_2 + v_3 \\ -v_2 + v_3 & v_0 - v_1 \end{bmatrix} \quad .$$

Hence

$$(u'v')' = vu \quad .$$

This fundamental relationship must somehow be built into our analytical tools. Note that for pure complex numbers there is no effect. This explains why no fundamental distinction has been found between i and -i in the theory of complex numbers.

Returning to exponential forms, let $\ell_n = \ell n t_n$. Then we should get

$$e^{\ell_0} = (1,0,0,0)$$

$$e^{\ell_1} = (0,1,0,0)$$

$$e^{\ell_2} = (0,0,1,0)$$

$$e^{\ell_3} = (0,0,0,1)$$

and we would like to have $\ell_0 = 0$. Since $|t_1| = |t_3| = i$, these values are improper but we would still hope that, formally, the functions would give the right result. We may expect to find a mixture of real and imaginary numbers for these purely formal purposes.

To be consistent with the complex field, we can define e^v in the following general form. Let

$$F(v), f_0(v), f_1(v), f_2(v), f_3(v)$$

be five complex-valued functions of the four real variables v_0, v_1, v_2, v_3. Then

$$e^v = F(v)(f_0(v), f_1(v), f_2(v), f_3(v)) \quad .$$

If $v_1 = v_3 = 0$, these must reduce to the following:

$$F = e^{v_0}$$

$$f_0 = \cos v_2 \quad , \qquad f_2 = \sin v_2$$

$$f_1 = f_3 = 0$$

and, more generally, if v is allowable, all five functions should be real. Furthermore, each f_n must take on all real values in some interval and F must be an entire function with no zeros. Hence

$$F = e^{g(v)}$$

where g(v) is an entire function. But by the identity theorem for analytic functions, since $g(v) = v_0$ for any v_0, v_2 (with $v_1 = v_3 = 0$), $g(v) \equiv v_0$. Hence $F = e^{v_0}$. We can now turn our attention to the f_n.

Let g_0 and g_2 be two real valued functions of the real variables v_1 and v_3 and set

$$f_0 = (\cos v_2)g_0 \quad , \qquad f_2 = (\sin v_2)g_2 \quad .$$

Then $g_0(0,0) = g_2(0,0) = 1$. Since $\sin v_2$ and $\cos v_2$ are never zero together, it might appear that g_0 and g_2 must have zeros together to allow $w_0 = w_2 = 0$. But, on the contrary, we cannot have both w_0 and w_2 zero when either w_1 or w_3 is nonzero, for then $|w|^2 < 0$. Hence $g_0 g_2 \neq 0$ for any real v_1 and v_3. But since v_1 and v_3 vary independently, g_0, g_2 cannot have zeros for real arguments. This again suggests exponential forms.

There are several other requirements on f_1, f_3, g_0 and g_2 if e^v is to behave in familiar fashion and if it is to be possible to define the inverse function $\ell n\, w$ so that $\ell n(e^v) = v \pm u$, where u contains multiples of π. Two of these conditions are that

$$f_0^2 - f_1^2 + f_2^2 - f_3^2 = 1$$

and

$$(e^v)^2 = e^{2v} \quad ,$$

or, more generally,

$$(e^v)^c = e^{cv} \quad .$$

The writer defined e^v in three different ways which seemed to meet all but the last condition. He then spent two very frustrating weeks trying to calculate $\ell n\ w$ for general w. This was to no avail. Everything worked fine for pure complex numbers and for general w within restricted domains. However, no way could be found to generalize. This did not appear to be a question of analytic continuation. Before continuing, we will examine the special requirement

$$(e^v)^2 = e^{2v}$$

using partially metamathematical arguments.

First note that the general formula for the square of a general quadriform w is

$$w^2 = (w_0^2 + w_1^2 - w_2^2 + w_3^2, 2w_0w_1, 2w_0w_2, 2w_0w_3)$$

$$= (2w_0w_0 - |w|^2, 2w_0w_1, 2w_0w_2, 2w_0w_3) \quad .$$

Suppose $w = e^v$ with $v_0 = 0$ so the scale factor $e^{v_0} = 1$. Then

$$w^2 = 2w_0w - t_0 \quad .$$

This is perfectly consistent with the exponential of a pure imaginary v (or complex v if $v_0 \neq 0$) since

$$(e^{v_2 i})^2 = e^{2v_2 i} = \cos(2v_2) + i \sin(2v_2)$$

$$= (2\cos^2 v_2 - 1) + i(2\cos v_2 \sin v_2)$$

$$= 2\cos v_2(\cos v_2 + i \sin v_2) - 1 \quad .$$

Now consider two different quadriforms, u and v, and let

$$w = e^u \quad , \qquad z = e^v \quad .$$

If u and v are pure complex, we can define the exponential so that

$$wz = zw = e^{u+v} = e^{v+u} \quad .$$

But, for general quadriforms, $wz \neq zw$, even though $u + v = v + u$. Since we cannot distinguish $u + v$ from $v + u$ but must distinguish wz from zw, it is logically impossible to extend the exponential function into the quadriform domain and retain all its characteristics in the general case.

To retain as much consistency as possible, we may look at the geometry of 4-space. Since intuition fails even here, we begin with 3-space by setting $v_3 = 0$ and assuming this makes $w_3 = 0$ (which may not be true). Now if $v_1 = 0$, i.e., v is pure complex, then, by proper definition of e^v, we will get

$$e^v = w = e^{v_0}(\cos v_2, 0, \sin v_2, 0) \quad .$$

Let

$$\bar{w} = e^{-v_0} w \quad .$$

Then

$$|\bar{w}| = 1$$

and $\bar{w}$ lies on the unit circle in the complex plane. But if $v_1 \neq 0$, we must assume $w_1 \neq 0$ since otherwise $\ell n\ w$ requires (or at least admits) a noncomplex value for pure complex w. This seems like more indeterminancy than we can allow. But if $w_1 \neq 0$, then, assuming $|\bar{w}| = 1$ still holds, we have

$$\bar{w}_0^2 + \bar{w}_2^2 = 1 + \bar{w}_1^2 > 1$$

so that $\bar{w}$ does not lie on the unit circle in the complex plane but "above" a larger circle. We define the (complex) <u>modulus</u> of w as

$$/w/ = (|w|^2 + w_1^2 + w_3^2)^{\frac{1}{2}} \doteq (w_0^2 + w_2^2)^{\frac{1}{2}} \geq 0 \quad .$$

Thus, for pure complex numbers, $|w| = /w/$. For ease of reference, we will define two more related quantities. The <u>hypermodulus</u> of w is

$$\backslash w \backslash = (w_1^2 + w_3^2)^{\frac{1}{2}} \geq 0$$

and the <u>magnitude</u> of w is

$$||w|| = (w_0^2 + w_1^2 + w_2^2 + w_3^2)^{\frac{1}{2}} \geq 0 \quad .$$

For purely complex numbers,

$$\backslash w \backslash = 0$$

$$|w| = /w/ = ||w||$$

and, in general,

$$|w|^2 = /w/^2 - \backslash w \backslash^2$$

$$||w||^2 = /w/^2 + \backslash w \backslash^2 \quad .$$

Now suppose $|\bar{w}| = 1$, $\backslash \bar{w} \backslash > 0$, so $/\bar{w}/ > 1$. Then

$$\bar{w}_0 = /\bar{w}/ \cos v_2$$

$$\bar{w}_2 = /\bar{w}/ \sin v_2$$

would give the right complex components. These definitions would be circular, however, unless we know $|\bar{w}| = 1$ and compute $/\bar{w}/$ by

$$/w/^2 = 1 + \bar{w}_1^2 + \bar{w}_3^2 \quad .$$

This, in turn, implies that

$$g_0(v_1,v_3) = g_2(v_1,v_3) = (1 + f_1^2(v) + f_3^2(v))^{\frac{1}{2}} \quad .$$

It turns out that if we set

$$f_1(v) = \sinh v_1 \quad , \qquad f_3(v) = \sinh v_3$$

all the conditions we have found or assumed thus far are satisfied. To simplify notation, we will use the following throughout the sequel:

$$C_1 = \cosh v_1 \quad , \qquad S_1 = \sinh v_1$$

$$c_2 = \cos v_2 \quad , \qquad s_2 = \sin v_2$$

$$C_3 = \cosh v_3 \quad , \qquad S_3 = \sinh v_3 \quad .$$

Now we define the function $E(v)$ as follows:

$$E(v) = e^{v_0}(c_2(1 + S_1^2 + S_3^2)^{\frac{1}{2}}, S_1, s_2(1 + S_1^2 + S_3^2)^{\frac{1}{2}}, S_3) \quad .$$

First note that if $v_1 = v_3 = 0$, then

$$E(v) = e^{v_0}(c_2, 0, s_2, 0) = e^{v}$$

as in the complex domain. For general v,

$$|E(v)| = e^{v_0}[(c_2^2 + s_2^2)(1 + S_1^2 + S_3^2) - S_1^2 - S_3^2] = e^{v_0}$$

so that if $\bar{w} = e^{-v_0}E(v)$, then $|\bar{w}| = 1$. To compare E(2v) with the square of E(v), we can assume without loss of generality that $v_0 = 0$. Then

$$E(2v) = [(c_2^2-s_2^2)(1+4C_1^2S_1^2+4C_3^2S_3^2)^{\frac{1}{2}}, 2C_1S_1, 2c_2s_2(1+4C_1^2S_1^2+4C_3^2S_3^2)^{\frac{1}{2}}, 2C_3S_3]$$

$$(E(v))^2 = [2c_2^2(1+S_1^2+S_3^2)-1, 2c_2(1+S_1^2+S_3^2)^{\frac{1}{2}}S_1, 2c_2s_2(1+S_1^2+S_3^2)^{\frac{1}{2}}, 2c_2(1+S_1^2+S_3^2)^{\frac{1}{2}}S_3] \quad .$$

We first observe that these can be equal only if

$$C_1 = C_3 = c_2(1 + S_1^2 + S_3^2)^{\frac{1}{2}} = c_2(1 + 2S_1^2)^{\frac{1}{2}} = c_2\sqrt{\cosh\ (2v_1)}$$

and

$$(1 + 4C_1^2S_1^2 + 4\ C_3^2S_3^2)^{\frac{1}{2}} = (1 + S_1^2 + S_3^2)^{\frac{1}{2}} \quad .$$

But if $C_1 = C_3$, then the last equation is

$$(1 + 8C_1^2S_1^2)^{\frac{1}{2}} = (1 + 2\ \sinh^2(2v_1))^{\frac{1}{2}} = (1 + 2S_1^2)^{\frac{1}{2}}$$

and

$$\sinh\ (2v_1) = \pm\ \sinh\ v_1$$

which can only hold for $v_1 = 0 = v_3$. Therefore, without examining the first component, we have

$$E(2v) = (E(v))^2 \iff v_1 = v_3 = 0$$

i.e., v is pure complex. Thus the question of noncommutativity is avoided. However, logorithms are not additive for general quadriforms.

If we define the function L(w) so that

$$E(L(w)) = w$$

then there is no problem in computing L(w) for any w for which $|w|^2 > 0$. We can proceed as follows.

Let $|w|^2 > 0$ and $v = L(w)$. Set $v_0 = \frac{1}{2}\ln|w|^2$ and

$$\bar{w}_n = \frac{w_n}{|w|} , \qquad n = 0,1,2,3 \quad .$$

Then $|\bar{w}| = 1$. Set

$$v_1 = \sinh^{-1}\bar{w}_1 , \qquad v_3 = \sinh^{-1}\bar{w}_3$$

and

$$c_2 = \frac{\bar{w}_0}{/\bar{w}/} , \qquad s_2 = \frac{\bar{w}_2}{/\bar{w}/} \quad .$$

Then $c_2^2 + s_2^2 = 1$ and

$$v_2 = \cos^{-1}c_2 = \sin^{-1}s_2$$

which determines both the sign and magnitude of v_2 for

$$-\pi \leq v_2 \leq \pi \quad .$$

It is obvious that $E(v) = w$, and that, if $w_1 = w_3 = 0$, then $L(w) = \ln w$ as in the complex field.

We can now evaluate the ℓ_n introduced earlier:

$$\ell_0 = L(1,0,0,0) = (0,0,0,0) = 0$$

$$\ell_2 = L(0,0,1,0) = (0,0,\tfrac{\pi}{2},0)$$

$$\ell_1 = L(0,1,0,0) = (\tfrac{\pi}{2}i,-\tfrac{\pi}{2}i,0,0)$$

$$\ell_3 = L(0,0,0,1) = (\tfrac{\pi}{2}i,0,0,-\tfrac{\pi}{2}i) \quad .$$

Hence ℓ_1 and ℓ_3 are improper, as predicted, but formally correct. Note that

$$|t_1| = |t_3| = i$$

$$\frac{1}{i} = -i$$

$$\ln i = \frac{\pi}{2}i$$

$$\sinh(-\tfrac{\pi}{2}i) = -(\sin \tfrac{\pi}{2})i = -i$$

so

$$\sinh^{-1}(-i) = -\frac{\pi}{2}i \quad .$$

Also,

$$|\ell_1| = |\ell_3| = -\frac{\pi^2}{4} + \frac{\pi^2}{4} = 0$$

but

$$|\ell_2| = \frac{\pi^2}{4} \quad .$$

Hence both $L(\ell_2)$ and $\sqrt{\ell_2}$ exist and are proper.

Since we have lost additivity of logorithms exactly where we lose commutativity of multiplication (except a few special cases) and have retained other elementary characteristics of

e^z and $\ell n\ z$ for complex z, this seems about the best we can hope for.

The H-Functions

Although E(v) and L(v) have proper values for all allowable arguments - indeed some unallowable ones for E(v) - they are not very interesting. Another set of functions exist which also reduce to e^v and $\ell n\ w$ for pure complex numbers but which have a much stronger infrastructure. This imposes tight limits on the range of validity but, within these limitations, the functions have much more interesting properties. We call these simply H-functions (for hyper-functions).

There are at least two ways to define the H-functions. The "more natural" way leads to an unnecessary asymmetry in indices and in the inverse functions. Hence we adopt the "less natural" way. In either case, the definition is based on a product of factors, analogous to the definition of e^z as

$$e^z = e^{x+iy} = e^x \cdot e^{iy} \quad .$$

We define three factor functions as follows:

$$h_1(v_1) \quad , \quad h_2(v_2) \quad , \quad h_3(v_3)$$

and then

$$H(v) = e^{v_0} \cdot h_1(v_1) \cdot h_2(v_2) \cdot h_3(v_3) \quad .$$

One way to define the h_n (the more natural) is as follows:

$$\bar{h}_1(v_1) = (\cosh v_1, \sinh v_1, 0, 0) = \begin{bmatrix} C_1 + S_1 & 0 \\ 0 & C_1 - S_1 \end{bmatrix}$$

$$h_2(v_2) = (\cos v_2, 0, \sin v_2, 0) = \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix}$$

$$\bar{h}_3(v_3) = (\cosh v_3, 0, 0, \sinh v_3) = \begin{bmatrix} C_3 & S_3 \\ S_3 & C_3 \end{bmatrix} .$$

The absolute value of each of these is unity, so their product is unity. If $v_1 = v_3 = 0$, then $\bar{h}_1(v_1)$ and $\bar{h}_3(v_3)$ reduce to the identity leaving only $h_2(v_2)$, the usual definition of e^v.

It is somewhat more convenient, however, to define h_1 and h_3 as follows:

$$h_1(v_1) = (0, \sinh v_1, \cosh v_1, 0) = \begin{bmatrix} S_1 & -C_1 \\ C_1 & -S_1 \end{bmatrix}$$

$$h_3(v_3) = (0, 0, -\cosh v_3, \sinh v_3) = \begin{bmatrix} 0 & C_3 + S_3 \\ -C_3 + S_3 & 0 \end{bmatrix} .$$

These are both unallowabel alone since

$$|h_1|^2 = |h_3|^2 = -1$$

but

$$|h_1 \cdot h_2 \cdot h_3|^2 = 1 \quad .$$

Furthermore, if $v_1 = v_3 = 0$, then the product is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_2 & c_2 \\ -c_2 & s_2 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} = h_2 \quad .$$

In fact, the product here is simply the negative of t_2 - conjugation of h_2, which returns h_2.

The general product is

$$\begin{bmatrix} S_1 & -C_1 \\ C_1 & -S_1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 & C_3 + S_3 \\ -C_3 + S_3 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} -C_1 s_2 + S_1 c_2 & -C_1 c_2 - S_1 s_2 \\ C_1 c_2 - S_1 s_2 & -C_1 s_2 - S_1 c_2 \end{bmatrix} \begin{bmatrix} 0 & C_3 + S_3 \\ -C_3 + S_3 & 0 \end{bmatrix} \quad .$$

Now define

$$\bar{w}_0 = C_1 c_2 C_3 - S_1 s_2 S_3$$

$$\bar{w}_1 = -C_1 c_2 S_3 + S_1 s_2 C_3$$

$$\bar{w}_2 = C_1 s_2 C_3 - S_1 c_2 S_3$$

$$\bar{w}_3 = -C_1 s_2 S_3 + S_1 c_2 C_3 \quad . \qquad (1)$$

Then it is readily verified that

$$h_1(v_1) \cdot h_2(v_2) \cdot h_3(v_3) = \begin{bmatrix} \bar{w}_0 + \bar{w}_1 & -\bar{w}_2 + \bar{w}_3 \\ \bar{w}_2 + \bar{w}_3 & \bar{w}_0 - \bar{w}_1 \end{bmatrix}$$

$$= (\bar{w}_0, \bar{w}_1, \bar{w}_2, \bar{w}_3) = \bar{w}$$

and we set

$$H(v) = e^{v_0}\bar{w} \quad .$$

Note again that, if $v_1 = v_3 = 0$, then, since

$$C_1 = C_3 = 1 \quad , \qquad S_1 = S_3 = 0$$

the system (1) reduces to

$$\bar{w} = (c_2, 0, s_2, 0)$$

or

$$H(v) = e^{v_0}(c_2, 0, s_2, 0)$$

which is equivalent to Euler's formula

$$e^{v_0 + iv_2} = e^{v_0}(\cos v_2 + i \sin v_2) \quad .$$

Hence, products of H(v) for such v commute. Also,

$$H(0) = (1,0,0,0) \quad .$$

In multiplying the factors h_1, h_2, h_3, we used an arbitrary order. There are clearly six possible orders and they are not equivalent since the factors do not commute if $\|v\| \neq 0$. If we label the above order (123), then the six orders are

(123) , (321) , (213) , (231) , (312) , (321) .

We will investigate the effects of various orders subsequently. Properly, we should subscript H with the order used, viz.:

$$H_{(123)}(v)$$

for the above order. If the subscripting sequence is omitted, (123) is to be assumed.

It might appear that H(v) can take on all possible values but, in fact, it can not. This becomes apparent if one attempts to calculate the inverse function. It is obvious that if we compute $w = H(v)$, then we can compute $\bar{v} = H^{-1}(w)$ where $\bar{v}$ differs from v only by multiples of π in v_2. For, h_1 and h_3 have inverses which can be used to peel off the right and left factors, leaving h_2. However, if one starts from a general w, then $H^{-1}(w)$ may not exist in real numbers. We proceed to show this.

Assume a given w with $|w|^2 > 0$ for which v is to be found such that $H(v) = w$. We call this $H^{-1}(w)$. Begin by setting

$$v_0 = \tfrac{1}{2}\ln|w|^2$$

$$\bar{w}_n = \frac{w_n}{|w|} \quad .$$

Then $|\bar{w}| = 1$. We can ignore the factor $|w|$ and omit the overbars. Then we have $|w| = 1$ and must solve the system of equations

$$C_1c_2C_3 - S_1s_2S_3 = w_0$$

$$-C_1c_2S_3 + S_1s_2C_3 = w_1$$

$$C_1s_2C_3 - S_1c_2S_3 = w_2$$

$$-C_1s_2S_3 + S_1c_2C_3 = w_3$$

$$C_1^2 - S_1^2 = 1$$

$$c_2^2 + s_2^2 = 1$$

$$C_3^2 - S_3^2 = 1 \qquad (2)$$

for C_1, S_1, c_2, s_2, C_3, S_3. It may appear that we have seven equations in six unknowns but the products of the squares of the first four are related by $|w| = 1$, so only six equations are independent, at most.

We will, from time to time, utilize the following readily apparent facts:

For real x,

$$\cosh x \geq 1 \quad , \quad \cosh x \neq \sinh x \quad , \quad \cosh x > |\sinh x|$$

$$1 \leq |\cos x| + |\sin x| \leq \sqrt{2} \quad .$$

If $|\cos x| = |\sin x|$, they each equal $\frac{1}{\sqrt{2}}$.

Now suppose $w_1 = w_3 = 0$. Then

$$C_1c_2S_3 = S_1s_2C_3$$

$$C_1s_2S_3 = S_1c_2C_3 \quad .$$

These are not equal to zero unless $S_1 = S_3 = 0$. In this case, the first and third equations of (2) reduce to

$$c_2 = w_0 \quad , \quad s_2 = w_2$$

which is consistent with $w_0^2 + w_2^2 = 1$. On the other hand, if not both S_1 and S_3 are zero, then both are nonzero. But then neither c_2 nor s_2 can be zero and, by dividing the first pair above,

$$\frac{c_2}{s_2} = \frac{s_2}{c_2}$$

or

$$c_2 = \pm s_2 = \pm\frac{1}{\sqrt{2}} \quad .$$

But then $|C_1S_3| = |C_3S_1|$. It is easy to show from the definition of hyperbolic functions that this requires $C_1 = C_3$, $S_1 = \pm S_3$, i.e., $v_1 = \pm v_3$, the sign depending on the signs of c_2, s_2, w_0 and w_2, all of which must be consistent, including the magnitudes of w_0 and w_2. The first and third equations of (2) become

$$\pm C_1^2 \pm S_1^2 = w_0\sqrt{2}$$

$$\pm C_1^2 \pm S_1^2 = w_2\sqrt{2}$$

where the sign choices are independent. We have the following implications:

$$w_0 > 0 \rightarrow c_2 > 0$$

$$w_0 < 0 \rightarrow c_2 < 0$$

$$w_2 > 0 \rightarrow s_2 > 0$$

$$w_2 < 0 \rightarrow s_2 < 0 \quad .$$

These follow from the dominance of cosh over sinh. Then

$$c_2 > 0 \;, \quad s_2 > 0 \rightarrow w_0 = w_2 > 0$$

$$c_2 > 0 \;, \quad s_2 < 0 \rightarrow w_0 = -w_2 > 0$$

$$c_2 < 0 \;, \quad s_2 > 0 \rightarrow -w_0 = w_2 > 0$$

$$c_2 < 0 \;, \quad s_2 < 0 \rightarrow -w_0 = -w_2 > 0 \;.$$

Suppose $|w_0| \neq |w_2|$. Then we must have $v_1 = v_3 = 0$ and revert to the complex subset. However, we have the following indeterminate possibility:

Indeterminancy 1

If $w_0^2 = w_2^2 = \frac{1}{2}$, $w_1 = w_3 = 0$, then any value for $|v_1| = |v_3|$ satisfies

$$v = H^{-1}(w)$$

with $v_2 = \sin^{-1} w_2 = \cos^{-1} w_0$ which is completely determined within

$$-\pi \leq v_2 \leq \pi \;.$$

The signs of v_1 and v_3 agree or disagree according as the signs of w_0 and w_2 agree or disagree.

Thus the pure complex numbers w lying on either diagonal in the complex plane constitute slits through which any values for v_1 and v_3 may be generated by $H^{-1}(w)$.

Now suppose not both w_1 and w_3 are zero. To facilitate product expansions, we derive the folloing pattern. Let ①, ②, ③ represent three signs of which either one or three are negative. Then

$$(C_1 \textcircled{3} S_1)(c_2 \textcircled{2} s_2)(C_3 \textcircled{1} S_3) =$$

$$(C_1 c_2 \textcircled{23} S_1 s_2 \textcircled{2} C_1 s_2 \textcircled{3} S_1 c_2)(C_3 \textcircled{1} S_3) =$$

$$\left.\begin{array}{r} C_1 c_2 C_3 \textcircled{123} S_1 s_2 S_3 \\ \textcircled{1} C_1 c_2 S_3 \textcircled{23} S_1 s_2 C_3 \\ \textcircled{2} C_1 s_2 C_3 \textcircled{13} S_1 s_2 S_3 \\ \textcircled{12} C_1 s_2 S_3 \textcircled{3} S_1 c_2 C_3 \end{array}\right] = \begin{array}{r} w_1 \\ - \textcircled{1} w_1 \\ + \textcircled{2} w_2 \\ + \textcircled{3} w_3 \end{array} .$$

Hence

$$\begin{array}{ccccccc} \textcircled{3} & \textcircled{2} & \textcircled{1} & = & -\textcircled{1} & \textcircled{2} & \textcircled{3} \\ \hline (C_1 - S_1) & (c_2 + s_2) & (C_3 + S_3) & = & w_0 - w_1 & + w_2 & - w_3 \\ (C_1 + S_1) & (c_2 - s_2) & (C_3 + S_3) & = & w_0 - w_1 & - w_2 & + w_3 \\ (C_1 + S_1) & (c_2 + s_2) & (C_3 - S_3) & = & w_0 + w_1 & + w_2 & + w_3 \\ (C_1 - S_1) & (c_2 - s_2) & (C_3 - S_3) & = & w_0 + w_1 & - w_2 & - w_3 \end{array} \quad . \qquad (3)$$

Multiplying the first and third,

$$(c_2 + s_2)^2 = 1 + 2(w_0 w_2 - w_1 w_3)$$

and the second and fourth,

$$(c_2 - s_2)^2 = 1 - 2(w_0 w_2 - w_1 w_3)$$

Now

$$(c_2 + s_2)^2 = 1 + 2c_2s_2$$

$$(c_2 - s_2)^2 = 1 - 2c_2s_2 \quad .$$

Either product, therefore, requires that

$$c_2s_2 = w_0w_2 - w_1w_3$$

but, for real v_2,

$$-\frac{1}{2} \leq c_2s_2 \leq \frac{1}{2} \quad .$$

Therefore, for a real solution to exist, we must have

<u>Discriminant 1</u>

$$(w_0w_2 - w_1w_3)^2 \leq \frac{1}{4} \quad .$$

Note that this is always satisfied if $w_1 = w_3 = 0$ since then $|w_0| \leq \frac{1}{\sqrt{2}}$, $|w_2| \leq \frac{1}{\sqrt{2}}$. It is also satisfied if $w_0w_2 = w_1w_3 = 0$. This bears further investigation.

Returning to equations (2), and multiplying the first by the third and the second by the fourth, we have:

$$c_2s_2[C_1^2C_3^2 + S_1^2S_3^2 - 2C_1C_3S_1S_3] = w_0w_2$$

$$c_2s_2[C_1^2S_3^2 + S_1^2C_3^2 - 2C_1C_3S_1S_3] = w_1w_3$$

or

$$c_2s_2[C_1C_3 - S_1S_3]^2 = c_2s_2 \cosh^2(v_1 - v_3) = w_0w_2$$

$$c_2s_2[C_1S_3 - S_1C_3]^2 = c_2s_2 \sinh^2(v_1 - v_3) = w_1w_3 \quad . \qquad (4)$$

Hence, if either w_0 or w_2 is zero, then either c_2 or s_2 is zero. From the first and third of (2), it is clear that the pairing is unique:

$$w_0 = 0 \rightarrow c_2 = 0 \quad , \qquad w_2 = 0 \rightarrow s_2 = 0 \quad .$$

But if $c_2 s_2 = 0$, then $w_1 w_3 = 0$. From the second and fourth of (2), we get the following:

$$c_2 = 0 \text{ and } w_1 = 0 \rightarrow S_1 = 0$$

$$c_2 = 0 \text{ and } w_3 = 0 \rightarrow S_3 = 0$$

$$s_2 = 0 \text{ and } w_1 = 0 \rightarrow S_3 = 0$$

$$s_2 = 0 \text{ and } w_3 = 0 \rightarrow S_1 = 0 \quad .$$

Thus, combining:

If $w_0 = 0$ and either w_1 or w_3 is zero, then the corresponding v_1 or v_3 is zero.

If $w_2 = 0$ and either w_1 or w_3 is zero, then the contrary v_1 or v_3 is zero.

We can now obtain solutions for the four cases.

A. $w_0 = w_1 = 0 \quad , \qquad w_3 \neq 0$

The third and fourth equations of (2) become

$$s_2 C_3 = w_2 \quad , \qquad s_2 = \pm 1$$

$$-s_2 S_3 = w_3 \quad .$$

Hence $v_3 = \sinh^{-1}(\pm w_3)$

$|v_2| = \frac{\pi}{2}$ with sign to agree with above.

$v_1 = 0 \quad .$

B. $w_0 = w_3 = 0$, $w_1 \neq 0$

From the second and third equations:

$$S_1 s_2 = w_1 \quad , \qquad s_2 = \pm 1$$

$$-C_1 s_2 = w_2 \quad .$$

Hence

$$v_1 = \sinh^{-1}(\pm w_1)$$

$$|v_2| = \frac{\pi}{2} \text{ with sign to agree with above.}$$

$$v_3 = 0 \quad .$$

C. $w_2 = w_1 = 0$, $w_3 \neq 0$

Using the first and fourth equations:

$$C_1 c_2 = w_0 \quad , \qquad c_2 = \pm 1$$

$$S_1 c_2 = w_3 \quad .$$

Hence

$$v_1 = \sinh^{-1}(\pm w_3)$$

$v_2 = 0$ or π so c_2 sign matches above

$$v_3 = 0 \quad .$$

D. $w_2 = w_3 = 0$, $w_1 \neq 0$

Using the first and second equations:

$$c_2 C_3 = w_0 \quad , \qquad c_2 = \pm 1$$

$$-c_2 S_3 = w_1 \quad .$$

Hence

$$v_3 = \sinh^{-1}(\pm w_1)$$

$$v_2 = 0 \text{ or } \pi \quad \text{so } c_2 \text{ sign matches above}$$

$$v_1 = 0 \quad .$$

Note that the formula

$$\sinh^{-1}x = \log(x + \sqrt{x^2 + 1})$$

is valid for any x.

Let us now consider the cases where only one of four components are zero. It is clear from equations (4) that, if either w_0 or w_2 is zero, then one of w_1 and w_3 is zero. However, if $w_0 w_2 \neq 0$, then we may still have $w_1 w_3 = 0$ if $v_1 = v_3$. But then $S_1 = S_3$ and $C_1 = C_3$. Hence, from the second and fourth equations of (2), $w_1 = -w_3$ and they are zero or nonzero together. Hence we have shown:

<u>Discriminant 2</u>

There is no solution if either w_1 or w_3 is the only zero component of w.

We may now turn our attention to the case where all four components are nonzero. Note that the other case implied above, $w_1 = w_3 = 0$, has already been fully covered. However, we may expect to find some indeterminancy as nonzero w_1 and w_3 approach zero together.

If all four components are nonzero, then from (4) it is clear that

$$c_2 \neq 0 \quad , \qquad s_2 \neq 0 \quad , \qquad v_1 \neq v_3$$

and therefore $\cosh^2(v_1 - v_3) > 1$. Thus

$$0 < |c_2 s_2| < |w_0 w_2| \quad .$$

If we subtract the second equation from the first in (4), we get essentially Discriminant 1. If it is satisfied, we can get two solutions for $c_2 s_2$, the signs depending on the sign of $d = w_0 w_2 - w_1 w_3$. That is,

$$c_2 s_2 = d \quad .$$

Squaring and replacing s_2^2 with $1 - c_2^2$,

$$c_2^2 - c_2^4 = d^2$$

or

$$c_2^4 - c_2^2 + d^2 = 0$$

whence

$$c_2^2 = \frac{1}{2}(1 \pm \sqrt{1 - 4d^2}$$

which gives two solutions unless $d^2 = \frac{1}{4}$. If $d^2 = \frac{1}{4}$, then $c^2 = \pm \frac{1}{\sqrt{2}}$, our old friends the diagonals. If $d^2 < \frac{1}{4}$, then there are two values for c_2^2 and hence four for c_2. However, we must choose signs for c_2 and s_2 which satisfy equations (2).

Note further that as d^2 approaches zero, the two values of c_2^2 approach 1 and 0 and hence s_2^2 approaches 0 and 1. We have the following situation:

As $w_0 w_2$ approaches $w_1 w_3$, d approaches 0.

As d approaches 0, either c_2 or s_2 approaches 0.

If either $c_2 = 0$ or $s_2 = 0$, equations (3) vanish.

Now $w_0 w_2$ can approach $w_1 w_3$ with indefinitely large values of the components, but not with indefinitely small. This follows from the definition of $|w|$ and the assumption that $|w|^2 = 1$. Looking again at (4), $w_0 w_2$ approaching $w_1 w_3$ for large values means $\cosh^2(v_1 - v_3) - \sinh^2(v_1 - v_3)$ approaching zero which only occurs for $v_1 - v_3 = \infty$. Hence for allowable w with nonzero components, d is bounded away from zero unless some of the components approach zero in a way already analyzed.

If $0 < d^2 < \frac{1}{4}$, therefore, we can get two solutions for c_2, s_2 which are fully determinate for principal values of v_2, and such that $c_2 s_2 \neq 0$. Furthermore $|c_2| \neq |s_2|$. Now referring to equations (3), we can divide through by $c_2 + s_2 \neq 0$ or $c_2 - s_2 \neq 0$. We can then eliminate either C_1 and S_1 or C_3 and S_3 by multiplying appropriate pairs. More explicitly,

$$1^{st} \text{ by } 2^{nd} \text{ gives } (C_3 + S_3)^2 = a_3$$

$$3^{rd} \text{ by } 4^{th} \text{ gives } (C_3 - S_3)^2 = b_3 \quad .$$

Both must be nonnegative so that

$$C_3 + S_3 = \sqrt{a_3}$$

$$C_3 - S_3 = \sqrt{b_3} \quad .$$

From these, v_3 can be determined. Then

$$2^{nd} \text{ by } 3^{rd} \text{ gives } (C_1 + S_1)^2 = a_1$$

$$1^{st} \text{ by } 4^{th} \text{ gives } (C_1 - S_1)^2 = b_1$$

and, as above, v_1 can be determined from these.

Now suppose $d^2 = \frac{1}{4}$. Then either $c_2 - s_2 = 0$ or $c_2 + s_2 = 0$. Hence only one pair of equations from (3) can be used. Suppose it is the first and third. Then we can get

$$(C_1 - S_1)(C_3 + S_3) = a$$

$$(C_1 + S_1)(C_3 - S_3) = b \quad .$$

Evidently their product is 1. This is readily verified by expanding a and b and taking their product which gives

$$\frac{|w| + 2d}{(c_2 + s_2)^2} = \frac{1 + 2d}{1 + 2cs} = 1 \quad .$$

However, the equations are satisfied for any values $v_1 = v_3$. If the second and fourth equations must be used, then any values $v_1 = -v_3$ satisfy the equations. Thus the above is consistent with our earlier results.

Interpretation of H-Functions

Discriminant 1 for $H^{-1}(w)$ implies the following:
Given v and having computed $w = H(v)$, then (assuming $|w| = 1$)

$$0 \leq |w_0 w_2 - w_1 w_3| \leq \frac{1}{2} \quad .$$

Discriminant 2, and the other results of the previous section, further imply the following mutually exclusive cases:

(1)	$v_1 = v_3 = 0$,	$v_2 = 0$ or π ,		$\Rightarrow w = (\pm 1,0,0,0)$
(2)	$v_1 = v_3 = 0$,	$v_2 \neq 0, \pm\frac{k\pi}{4}$,		$\Rightarrow w = (w_0,0,w_2,0)$
(3)	Any $v_1 = \pm v_3$,	$v_2 = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$,		$\Rightarrow w = (\frac{1}{\sqrt{2}},0,\pm\frac{1}{\sqrt{2}},0)$
(4)	$v_1 = 0$,	$v_2 = \pm\frac{\pi}{2}$,	$v_3 \neq 0$	$\Rightarrow w = (0,0,w_2,w_3)$
(5)	$v_1 \neq 0$,	$v_2 = \pm\frac{\pi}{2}$,	$v_3 = 0$	$\Rightarrow w = (0,w_1,w_2,0)$
(6)	$v_1 \neq 0$,	$v_2 = 0$ or π ,	$v_3 = 0$	$\Rightarrow w = (w_0,0,0,w_3)$
(7)	$v_1 = 0$,	$v_2 = 0$ or π ,	$v_3 \neq 0$	$\Rightarrow w = (w_0,w_1,0,0)$

(8) $v_1 = v_3 = 0 \quad , \quad v_2 = \pm\frac{\pi}{2} \Rightarrow w = (0,0,\pm 1,0)$

(9) All v_1, v_2, v_3 nonzero $\Rightarrow w = (w_0,w_1,w_2,w_3)$.

The following results are impossible:

$(0,w_1,w_2,w_3)$, $(w_0,0,w_2,w_3)$

$(w_0,w_1,0,w_3)$, $(w_0,w_1,w_2,0)$

$(0,w_1,0,0)$, $(0,0,0,w_3)$, $(0,w_1,0,w_3)$

$(0,0,0,0)$.

These results are more easily seen in a table.

v_1	v_2	v_3	w_0	w_1	w_2	w_3
0	0	0	1	0	0	0
0	$\pm\frac{\pi}{2}$	0	0	0	± 1	0
0	π	0	-1	0	0	0
x	$\pm\frac{\pi}{4}, \frac{3\pi}{4}$	$\pm$x	$\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	0
0	other	0	w_0	0	w_2	0
0	$\pm\frac{\pi}{2}$	x	0	0	w_2	w_3
x	$\pm\frac{\pi}{2}$	0	0	w_1	w_2	0
x	$0, \pi$	0	w_0	0	0	w_3
0	$0, \pi$	x	w_0	w_1	0	0
x	x	x	w_0	w_1	w_2	w_3

Allowing for duplicates, only 8 of the 16 zero-nonzero patterns for w can occur. It is clear that the value of v_2 is pivotal since special values change the pattern of w.

As for v, after allowing for duplicates and doubles, all eight possible patterns are used. Hence we have not omitted any possibilities. The v-pattern x0x is included in xxx as is easily checked from the definition of H(v), equations (1). One can similarly check the patterns 0xx, xx0 for general v_2. In reality, the xxx pattern includes all the others below the dotted line as special cases. Those above the line are pure complex in both v and w, except for the special case of the diagonals in the complex part of w.

If we now include the scale factor e^{v_0}, it becomes clear that H(v) does not take on all values in any spherical domain beyond the complex plane, no matter how small, while at the same time it takes on values larger than those contained in any finite domain, no matter how large. Furthermore, this is true no matter whether one measures with $|w|$, $/w/$ or $||w||$.

Let us again assume $|w| = 1$ and that the discriminants are satisfied. If we set

$$D^2 = (w_0w_2 - w_1w_3)^2 \leq \frac{1}{4}$$

then $|D| \leq \frac{1}{2}$. Let us also assume w does not lie in the complex plane but that it has been generated from an allowable v by $w = H(v)$. Evidently then, under these conditions w does take on all values within the domain defined by $|D| \leq \frac{1}{2}$. What does this domain look like?

Let q be some real number such that

$$0 < q < \frac{1}{2} \quad .$$

Then, staying with positive components to start with,

$$w_0^2 + w_2^2 = 1 + w_1^2 + w_3^2 = /w/^2$$

$$w_0w_2 = w_1w_3 \pm q = k \quad .$$

If we fix attention on w_0 and w_2, holding w_1 and w_3 fixed, then $P = (w_0, w_2)$ must lie on two curves in the complex plane: the circle with radius $/w/$, and the hyperbola $w_0 w_2 = k$. If we move the point P on the circle, then w_1 and w_3 remain fixed but q must change. We can do this until q approaches 0 or $\frac{1}{2}$. If we move P along the hyperbola, then $/w/$ changes and so must w_1 and w_3. But since k remains unchanged, q must again adjust and we can do this only until q approaches 0 or $\frac{1}{2}$.

For convenience, let us call the (w_1, w_3)-plane the <u>transverse plane</u>. Then focusing now on the transverse plane, $Q = (w_1, w_3)$ must also lie on the circle with radius $\backslash w \backslash$ and the hyperbola $w_1 w_3 = K = k \mp q$. If we move Q along either curve, we get the complementary situation to that with P. Hence all components must move together except for the tolerance in q. If one circle gets larger, so must the other; if one hyperbola moves farther from the origin, so must the other. However, these double constraints take effect only when both w_1 and w_3 are nonzero which explains why w cannot have only one zero component. Let us trace a path to see what occurs.

Suppose we start at $v = 0$ so w is unity. If v_0 is varied, w simply traces out the reals, so let v_0 stay at 0. If v_1 is increased, then w_0 and w_3 both increase:

$$w_0 \text{ from } 1 \text{ to } \cosh v_1$$

$$w_3 \text{ from } 0 \text{ to } \sinh v_1 \quad .$$

Thus w moves on an hyperbola in the (w_0, w_3)-plane defined by

$$w_0^2 - w_3^2 = (w_0 - w_3)(w_0 + w_3) = 1 \quad .$$

There is no limit to how large these may become, but w_0 and w_3 approach the same value asymptotically. Suppose we stop when $w_0 = \frac{5}{3}$, $w_3 = \frac{4}{3}$, or v_1 slightly less than 1.1.

Now suppose v_3 starts to increase. Then both w_1 and w_2 decrease to negative values while w_0 and w_3 increase further by a factor of cosh v_3. If v_3 decreases instead, then w_1 and w_2 increase and w_0 and w_3 increase in the same way. Notice that v is still at zero in the complex plane. If v_3 goes to -1.1, approximately, we have:

$$v = (0,1.1,0-1.1)$$

$$w = \left(\frac{25}{9},\frac{20}{9},\frac{16}{9},\frac{20}{9}\right) \quad .$$

Note that $|w|^2 = \frac{625 - 400 + 256 - 400}{81} = 1$, but

$$/w/^2 = \frac{881}{81} = 10.8765432,09876...$$

$$\backslash w\backslash^2 = \frac{800}{81} = 9.8765432,09876...$$

$$||w||^2 = \text{sum} = 20.7530864,19753... \quad .$$

Hence w_0 has moved out to 2.777... on a circle in the complex plane with radius nearly 3.3, while w_2 has become 1.777... . All this occurs without changing the scale factor $e^{v_0} = 1$. Note that v is unallowable but w is allowable.

If we now vary v_2, w varies in a more complicated way which almost defies hand calculation. The value c_2 will decrease from 1 and s_2 will move from 0. It also becomes nearly impossible to visualize the path of w. However, let us move to $v^2 = \frac{\pi}{4}$ so $c_2 = s_2 = \frac{1}{\sqrt{2}}$, holding v_1 and v_3 at approximately +1.1 and -1.1. Then:

$$w_0 = \frac{1}{\sqrt{2}}\left(\frac{25}{9} + \frac{16}{9}\right) = \frac{1}{\sqrt{2}} \cdot \frac{41}{9}$$

$$w_1 = \frac{1}{\sqrt{2}}\left(\frac{20}{9} + \frac{20}{9}\right) = \frac{1}{\sqrt{2}} \cdot \frac{40}{9}$$

$$w_2 = \frac{1}{\sqrt{2}}\left(\frac{25}{9} + \frac{16}{9}\right) = \frac{1}{\sqrt{2}} \cdot \frac{41}{9}$$

$$w_3 = \frac{1}{\sqrt{2}}\left(\frac{20}{9} + \frac{20}{9}\right) = \frac{1}{\sqrt{2}} \cdot \frac{40}{9}$$

$$/w/^2 = 20.7530864,19753...$$

$$\backslash w\backslash^2 = 19.7530864,19753...$$

$$||w||^2 = 39.5061728,39506...$$

$$|w|^2 = 1.0 \quad .$$

Note that $/w/^2$ has taken on the old value of $||w||^2$. Notice also the great difference if $v_3 = +1.1$ instead of -1.1. Then

$$w = \frac{1}{\sqrt{2}}(1,0,1,0)$$

demonstrating the arbitrary values of v_1 and v_3 when w is on a diagonal in the complex plane. We still have v unallowable but can make it allowable by setting $v_0 > |\frac{\pi^2}{16} - 2.42|^{\frac{1}{2}}$ which simply scales w.

<u>E(v) and H(v) for Special Arguments</u>

To illustrate the pseudo-exponentiation of singular numbers and also to compare E(v) with H(v), we show symbolic values for general numbers from the sets S_0 to S_9.

	V	E(v)	H(v)
S_0:	$(0,0,x,x)$	$(c_2R_3,0,s_2R_3,S_3)$	$(c_2C_3,-c_2S_3,s_2C_3,-s_2S_3)$
S_1:	$(0,0,x,-x)$	$(c_2R_3,0,s_2R_3,-S_3)$	$(c_2C_3,-c_2S_3,s_2C_3,s_2S_3)$
S_2:	$(x,x,0,0)$	$e^x(R_1,S_1,0,0)$	$e^x(C_1,0,0,S_1)$
S_3:	$(x,-x,0,0)$	$e^x(R_1,-S_1,0,0)$	$e^x(C_1,0,0,-S_1)$
S_4:	(x,x,y,y)	$e^x(c_2R,S_1,s_2R,S_3)$	$e^x(w_0,w_1,w_2,w_3)$
S_5:	$(x,x,y,-y)$	$e^x(c_2R,S_1,s_2R,-S_3)$	
S_6:	$(x,-x,y,y)$	$e^x(c_2R_1,-S_1,s_2R,S_3)$	
S_7:	$(x,-x,y,-y)$	$e^x(c_2R,-S_1,s_2R,-S_3)$	
S_8:	$(x,0,0,\pm x)$	$e^x(R_3,0,0,\pm S_3)$	$e^x(1,\mp S_3,0,0)$
S_9:	$(0,x,\pm x,0)$	$(c_2R_1,S_1,s_2R_1,0)$	$(C_1c_2,S_1s_2,C_1s_2,S_1c_2)$

<u>Legend</u>

$c_2 = \cos v_2,\ s_2 = \sin v_2$

$C_1 = \cosh v_1,\ C_3 = \cosh v_3$

$S_1 = \sinh |v_1|,\ S_3 = \sinh |v_3|$

(w_0,w_1,w_2,w_3) requires general formulae.

$R_1 = (1 + S_1^2)^{\frac{1}{2}},\ R_3 = (1 + S_3^2)^{\frac{1}{2}}$

$R = (1 + S_1^2 + S_3^2)^{\frac{1}{2}}$

Next we show E(v) and H(v) for the four units.

	v	$E(v)$	$H(v)$
t_0:	(1,0,0,0)	(e,0,0,0)	(e,0,0,0)
t_1:	(0,1,0,0)	$(R_1,S_1,0,0)$	$(C_1,0,0,S_1)$
t_2:	(0,0,1,0)	$(c_2,0,s_2,0)$	$(c_2,0,s_2,0)$
t_3:	(0,0,0,1)	$(R_3,0,0,S_3)$	$(C_3,S_3,0,0)$

It is apparent that the functions are quite different off the complex plane.

H-Function Permutations and Manifolds

We return now to the order of multiplication of the factors h_1, h_2, h_3 of $H(v)$. The order we have been using is (123). We wish to investigate the other five. It is easy to show that, for general values, there are six different results. Let a, b, c be three matrices. Then:

$$(123) = abc$$

$$(321) = ((a'b')''c')' = (a'b'c')'$$

$$(213) = (a'b')'c$$

$$(312) = ((ab)'c')'$$

$$(132) = a(b'c')'$$

$$(231) = (a'(bc)')' \quad .$$

However, the terms in the expressions for w_n do not change, only the signs. Let

$$P_{10} = C_1c_2C_3 \;, \qquad P_{20} = S_1s_2S_3$$

$$P_{11} = C_1c_2S_3 \;, \qquad P_{21} = S_1s_2C_3$$

$$P_{12} = C_1s_2C_3 \;, \qquad P_{22} = S_1c_2S_3$$

$$P_{13} = C_1s_2S_3 \;, \qquad P_{23} = S_1c_2C_3 \;.$$

Then

$$w_n = \pm P_{1n} \pm P_{2n} \;, \qquad n = 0,1,2,3 \;.$$

Let S_{mn} be six signs for P_{mn} (m = 1,2). The pattern for (123) is:

$S_{mn}(123)$:

n \ m	1	2
0	+	−
1	−	+
2	+	−
3	−	+

Now, although there are 256 ways to assign eight signs, there are only three complementary pairs of changes in the factors, h_1, h_2, h_3. Hence we only expect $2^3 = 8$ patterns. Since there are only six orders, two of these eight must be either redundant or impossible.

One can easily find $S_{mn}(312)$ by multiplying the factors above (1) in the opposite order. Only the left column need be generated which gives $w_0 + w_1$ and $w_2 + w_3$, hence all signs in natural state. One finds:

$S_{mn}(312)$:

n \ m	1	2
0	+	−
1	+	−
2	+	+
3	−	−

This is all one can get out of these factors. Next form the product $h_2 \cdot h_1$, putting terms in the same order:

$$\begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} \begin{bmatrix} S_1 & -C_1 \\ C_1 & -S_1 \end{bmatrix} = \begin{bmatrix} -C_1 s_2 + S_1 c_2 & -C_1 c_2 + S_1 s_2 \\ C_1 c_2 + S_1 s_2 & -C_1 s_2 - S_1 c_2 \end{bmatrix} .$$

The result is only to change the sign on the $S_1 s_2$ combination. Hence we can immediately write:

$S_{mn}(213)$:

n \ m	1	2
0	+	+
1	-	-
2	+	-
3	-	+

$S_{mn}(321)$:

n \ m	1	2
0	+	+
1	+	+
2	+	+
3	-	-

Note that the number of minus signs is not constant as one might guess, a priori. Now we must form both $h_1 \cdot h_3$ to get (132) and $h_3 \cdot h_1$ to get (231) since we have already derived (213) and (312) the other way. Hence two combinations are redundant.

$$h_1 \cdot h_3 = \begin{bmatrix} S_1 & -C_1 \\ C_1 & -S_1 \end{bmatrix} \begin{bmatrix} 0 & C_3+S_3 \\ -C_3+S_3 & 0 \end{bmatrix} = \begin{bmatrix} C_1C_3-C_1S_3 & S_1C_3+S_1S_3 \\ S_1C_3-S_1S_3 & C_1C_3+S_1S_3 \end{bmatrix}$$

$$h_3 \cdot h_1 = \begin{bmatrix} C_1C_3+C_1S_3 & -S_1C_3-S_1S_3 \\ -S_1C_3+S_1S_3 & C_1C_3-C_1S_3 \end{bmatrix} .$$

Here all signs change except the C_1C_3 combination. If one multiplies $h_1 \cdot h_3$ by h_2 on the right, the result is:

$S_{mn}(132)$:

n \ m	1	2
0	+	+
1	−	+
2	+	−
3	+	+

One can verify the prior result for (213) by taking the opposite order. Finally, $h_2 \cdot (h_3 \cdot h_1)$ gives:

$S_{mn}(231)$:

n \ m	1	2
0	+	−
1	+	+
2	+	+
3	+	−

Multiplying the other way verifies (312). We can summarize in a combined table,

	(123)		(231)		(312)		(132)		(213)		(321)	
0	+	−	+	−	+	−	+	+	+	+	+	+
1	−	+	+	+	+	−	−	+	−	−	+	+
2	+	−	+	+	+	+	+	−	+	−	+	+
3	−	+	+	−	−	−	+	+	−	+	−	−

(These are probably members of a group but, if so, it is not obvious.)

We are at once faced with the ugly prospect of having to analyze each case separately in all detail. It seems clear that each order generates a different domain of values. Since the signs do not permute in any regular way, there seems no way to generalize the elimination of variables. However, we can make some progress with a different approach.

Consider the combined variables C_1C_3, C_1S_3, S_1C_3, S_1S_3, and write them as variables in linear equations with variable coefficients c_2 and s_2. For example, for (123):

C_1C_3	C_1S_3	S_1C_3	S_1S_3	
c_2			$-s_2$	$= w_0$
	$-c_2$	$+s_2$		$= w_1$
s_2			$-c_2$	$= w_2$
	$-s_2$	$+c_2$		$= w_3$.

The determinant of the matrix is $-(c_2^2 - s_2^2)^2$ which vanishes on the diagonals in the complex plane. Hence we could have predicted some sort of trouble there. Now take the case for (231). We will write only the matrices and omit subscripts during this discussion.

$$(231): \begin{bmatrix} c & & & -s \\ & c & s & \\ s & & & c \\ & s & -c & \end{bmatrix} .$$

This determinant is $(c^2 + s^2)^2 = 1$, which never vanishes. Hence, we can expect less trouble. Let us try the others.

$$(312): \begin{bmatrix} c & & & -s \\ & c & -s & \\ s & & & c \\ & -s & -c & \end{bmatrix}, \quad \text{determinant } (c^2 + s^2)^2 = 1$$

$$(132): \begin{bmatrix} c & & & s \\ & -c & s & \\ s & & & -c \\ & s & c & \end{bmatrix}, \quad \text{determinant } -(c^2 + s^2)^2 = -1$$

$$(213): \begin{bmatrix} c & & & s \\ & -c & -s & \\ s & & & -c \\ & -s & c & \end{bmatrix}, \quad \text{determinant } -(c^2 + S^2)^2 = -1$$

$$(321): \begin{bmatrix} c & & & s \\ & c & s & \\ s & & & c \\ & -s & -c & \end{bmatrix}, \quad \text{determinant } (c^2 - s^2)^2 \quad .$$

Thus the natural forward order and the complete reverse are exactly the ones with vanishing determinants. To see the effect of a negative determinant, let us analyze (213) which would seem the second most natural order. (Complex components first, followed by v_1 and v_3 in order.)

<u>The $H_{(213)}(v)$ Manifold</u>

Assume w in the (213) manifold with $|w|^2 = 1$ so we can ignore v_0. We must find v_1, v_2, v_3 to satisfy the following equations:

$$C_1c_2C_3 + S_1s_2S_3 = w_0$$

$$-C_1c_2S_3 - S_1s_2C_3 = w_1$$

$$C_1s_2C_3 - S_1c_2S_3 = w_2$$

$$-C_1s_2S_3 + S_1c_2C_3 = w_3 \quad .$$

If one squares and adds pairs to get $/w/^2$ and $\backslash w \backslash^2$, all the cross products cancel and one is left with the tautology 1 = 1, as should be. But if one multiplies alternate pairs, and letting $X = C_1C_3S_1S_3$,

$$(C_1^2C_3^2 - S_1^2S_3^2)c_2s_2 + X(s_2^2 - c_2^2) = w_0w_2$$

$$(C_1^2S_3^2 - S_1^2C_3^2)c_2s_2 + X(s_2^2 - c_2^2) = w_1w_3$$

and subtracting

$$(C_1^2 + S_1^2)(C_3^2 - S_3^2)c_2s_2 = w_0w_2 - w_1w_3$$

or

$$(C_1^2 + S_1^2)c_2s_2 = w_0w_2 - w_1w_3 \quad .$$

Now

$$(C_1^2 + S_1^2) \geq 1 \quad , \qquad 0 \leq |c_2s_2| \leq \frac{1}{2} \quad .$$

So now we have an escape for the discriminant, i.e.,

If $(w_0w_2 - w_1w_3)^2 > \frac{1}{4}$,

Then $v_1 \neq 0$.

It appears, therefore, that $H_{(213)}(v)$ takes on all values, and one suspects that $H_{(132)}(v)$ does also. We must leave the task of fully analyzing all the H-functions and their relationships for a later time.

Future Study

This paper has already gone somewhat beyond its title of algebra, although many other tasks remain even there. Before stopping, however, we wish to point out some necessary next steps if quadriforms are to have practical value.

1. Further study of the H-functions, or similar ones, needs to be made to fully understand the surprising effects of noncommutativity.
2. Since we have been unable, and it appears logically impossible, to find a function equivalent to $\ell n\ z$ which is additive, how can general powers and roots be calculated? Indeed, it is not clear, for example, that commutative cube roots exist for the general case.
3. Can we define differentiation in an unambiguous way? It appears that limits may not be independent of path. The inverse problem of integration is more formidable. How do we define a path in 4-space?
4. Under what circumstances do polynomials have zeros? Is there a valid extension to the fundamental theorem of algebra?
5. What happens to Taylor expansions and power series? Will the concepts and tools of analysis carry over?
6. We have already encountered what appear to be group-theoretic problems. Can this theory be brought to bear to simplify the analysis of quadriform functions?
7. Finally, but perhaps the key issue, is there some interpretation of natural phenomena by which quadriforms can be put to good, or even necessary, use. It is somewhat of a puzzle to conceive of a meaningful function of a complex variable which generates quadriforms off the complex plane.

In spite of this rather staggering set of tasks, it is this writer's opinion that enough has already been shown to indicate that there is potentially a very rich theory here that should be developed.

Reference

[1] Orchard-Hays, W. "Hypernumbers: Real or Imaginary?" IIASA RM

Appendix A

Some Elementary Relationships

Lemma 1

If $x^2 + y^2 = a^2$, $a > 0$

then max $x + y$ is given by $x = y = \frac{a}{\sqrt{2}}$.

Proof:

$$y = \sqrt{a^2 - x^2} , \quad \text{assume } x \neq \sqrt{a}$$

$$\frac{d}{dx}\left(x + \sqrt{a^2 - x^2}\right) = 1 - \frac{x}{\sqrt{a^2 - x^2}} = 0$$

$$x^2 = \frac{a}{\sqrt{2}} , \quad y = \frac{a}{\sqrt{2}} \quad (y = -x \text{ gives min} = 0) .$$

Lemma 2

If $x^2 + y^2 = a^2$, $a > 0$

Then max xy is given by $x = y = \frac{a}{\sqrt{2}}$.

Proof:

$$y = \sqrt{a^2 - x^2} , \quad \text{assume } x \neq \sqrt{a}$$

$$\frac{d}{dx}(x\sqrt{a^2 - x^2}) = \sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} = 0$$

$$a^2 - x^2 - x^2 = 0$$

$$x = \frac{a}{\sqrt{2}} , \quad y = \frac{a}{\sqrt{2}} \quad (y = -x \text{ gives min} = -\text{max}) .$$

<u>Lemma 3</u>

If $x + y = a > 0$

then max xy is given by $x = y = \frac{a}{2}$.

Proof:

$y = a - x$

$\frac{d}{dx}(ax - x^2) = a - 2x = 0$

$x = \frac{a}{2}$, $y = \frac{a}{2}$ (min $= -\infty$) .

<u>Lemma 4</u>

The pair of equations

$$x^2 - y^2 = a$$

$$xy = b$$

always has a real solution.

Proof:

1. If $a = b = 0$, $x = y = 0$.

2. If $a = 0$, $b \neq 0$, then

   $|x| = |y| = \sqrt{|b|}$.

   Give x sign of b, $y \geq 0$ or vice versa.

3. If $a \neq 0$, $b = 0$, then:

$$\text{if } a > 0 \ , \quad x = \pm\sqrt{a} \ , \quad y = 0$$

$$\text{if } a < 0 \ , \quad x = 0 \ , \quad y = \pm\sqrt{-a} \ .$$

4. If $ab \neq 0$, let $y = \frac{b}{x}$, then

$$x^2 - \frac{b^2}{x^2} = a$$

$$x^4 - ax^2 - b^2 = 0$$

$$x^2 = \frac{1}{2}(a + \sqrt{a^2 + 4b^2})$$

$$x = \pm\sqrt{x^2} \ .$$

<u>Relations Among Hyperbolic Functions</u>

Let $C_n = \cosh x_n$, $S_n = \sinh x_n$. Then

$$C_n^2 - S_n^2 = 1 = (C_n - S_n)(C_n + S_n)$$

$$C_1C_2 \pm S_1S_2 = \cosh(x_1 \pm x_2)$$

$$S_1C_2 \pm C_2S_1 = \sinh(x_1 \pm x_2)$$

$$\cosh(-x) = \cosh x \ , \quad \sinh(-x) = -\sinh x \ .$$

Appendix B

Background

In [1], a set of hypernumbers were defined as follows:

Let

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{, a real } 2 \times 2 \text{ matrix ,}$$

$$u_0 = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} , \quad u_1 = \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} ,$$

$$u_2 = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} , \quad u_3 = \begin{bmatrix} 0 & r \\ r & 0 \end{bmatrix}$$

four 4×4 real matrices representing hypernumber units. Then a hypernumber v is the sum

$$v = v_0u_0 + v_1u_1 + v_2u_2 + v_3u_3$$

written briefly as

$$v = (v_0, v_1, v_2, v_3)$$

where v_n is a real number, n = 0,1,2,3.

It was stated, somewhat arbitrarily, that the forms

$$v = (v_0, 0, 0, 0)$$

are isomorphic to the reals, and the forms

$$v = (v_0, 0, v_2, 0)$$

to the complex field. Several other attributes of these hypernumbers were also displayed in [1].

Basic Arithmetic Operations

The symbol 0 will be used for the real number zero, the 2×2 zero matrix, and the 4×4 zero matrix. Then for any hypernumber v,

$$v - v = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The operations of negation, addition, subtraction and multiplication are taken to be those of matrix algebra with 4×4 real matrices.

Any number v as defined above has the form

$$v = \begin{bmatrix} v_0+v_1 & 0 & -v_2+v_3 & 0 \\ 0 & v_0+v_1 & 0 & -v_2+v_3 \\ v_2+v_3 & 0 & v_0-v_1 & 0 \\ 0 & v_2+v_3 & 0 & v_0-v_1 \end{bmatrix} . \qquad (1)$$

It is apparent that negation, addition and subtraction give results of the same form. A number v can also be indicated with real numbers a, b, c, d by

$$v = \begin{bmatrix} a & 0 & d & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & c & 0 & b \end{bmatrix} . \qquad (2)$$

Then the product v_1v_2 is

$$\begin{bmatrix} a_1 & 0 & d_1 & 0 \\ 0 & a_1 & 0 & d_1 \\ c_1 & 0 & b_1 & 0 \\ 0 & c_1 & 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 & d_2 & 0 \\ 0 & a_2 & 0 & d_2 \\ c_2 & 0 & b_2 & 0 \\ 0 & c_2 & 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2+d_1c_2 & 0 & a_1d_2+d_1b_2 & 0 \\ 0 & a_1a_2+d_1c_2 & 0 & a_1d_2+d_1b_2 \\ c_1a_2+b_1c_2 & & c_1d_2+b_1b_2 & 0 \\ 0 & c_1a_2+b_1c_2 & 0 & c_1d_2+b_1b_2 \end{bmatrix} .$$

Thus multiplication gives a result on the same form. However,

$$v_2v_1 = \begin{bmatrix} a_1a_1+d_2c_1 & 0 & a_2d_1+d_2b_1 & 0 \\ 0 & a_2a_1+d_2c_1 & 0 & a_2d_1+d_2b_1 \\ c_2a_1+b_2c_1 & 0 & c_2d_1+b_2b_1 & 0 \\ 0 & c_2a_1+b_2c_1 & 0 & c_2d_1+b_2b_1 \end{bmatrix}$$

so that multiplication is not, in general, commutative. If, in (1), $v_1 = v_3 = 0$, then in (2), $b = a$ and $d = -c$. For such numbers, multiplication is commutative. More generally, multiplication is commutative among any of the following four sets:

$$b = a \quad , \quad d = c$$

$$b = a \quad , \quad d = -c$$

$$b = -a \quad , \quad d = c$$

$$b = -a \quad , \quad d = -c$$

but not for mixtures. For example, suppose

$$b_1 = a_1 \quad , \quad d_1 = c_1 \quad , \quad b_2 = a_2 \quad , \quad d_2 = -c_2 \quad .$$

Then

$$a_2a_1 + d_2c_1 = a_2a_1 - c_2c_1$$

$$a_1a_2 + d_1c_2 = a_1a_2 + c_1c_2$$

which are not equal unless $c_1c_2 = 0$.

The determinant of (1) is

$$[(v_0^2 - v_1^2) + (v_2^2 - v_3^2)]^2 \quad . \tag{3}$$

This determinant is nonnegative and it is zero if and only if

$$|v_0| = |v_1| \ \underline{\text{and}}\ |v_2| = |v_3| \quad .$$

For nonzero numbers of the form $v = (v_0, 0, v_2, 0)$, the determinant is never zero. These numbers behave like the complex field except that, if we define the absolute value of v to be the square root of (3), i.e.,

$$|v| = v_0^2 - v_1^2 + v_2^2 - v_3^2$$

so that $|v|$ always exists as a real number, then $v_0^2 + v_2^2$ is the square of the absolute value of the corresponding complex number, and v_0^2 is the square of the absolute value of the corresponding real number.

If (3) is not zero, then (1) has an inverse which we can take as the reciprocal:

$$v^{-1} = \frac{1}{ab - cd}\begin{bmatrix} b & 0 & -d & 0 \\ 0 & b & 0 & -d \\ -c & 0 & a & 0 \\ 0 & -c & 0 & a \end{bmatrix} \quad . \tag{4}$$

Note that

$$ab - cd = (v_0 + v_1)(v_0 - v_1) - (v_2 + v_3)(-v_2 + v_3)$$

$$= (v_0^2 - v_1^2) + (v_2^2 - v_3^2) \quad .$$

Since reciprocation gives again a matrix of the same form, the set is closed under the basic arithmetic operations. However, the equivalent of division, i.e., multiplication by a reciprocal, is, in general, doubly noncommutative. That is, not only is

$$v_1 v_2^{-1} \neq v_2 v_1^{-1}$$

but, in general,

$$v_1 v_2^{-1} \neq v_2^{-1} v_1 \quad .$$

<u>Divisors of Zero</u>

If (3) is zero, v has no reciprocal. Let S be the set of nonzero numbers for which (3) is zero. (Zero is always an exception and hence is not included in S.) Then for $s \in S$, s has one of the four forms:

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \end{bmatrix} , \qquad \begin{bmatrix} 0 & 0 & g & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$$

and their transposes. If s_1 has the first form and s_2 the transpose of the second, then $s_1 s_2 \equiv 0$ but $s_2 s_1$ is not identically zero. For example,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_2 & 0 & f_2 & 0 \\ 0 & g_2 & 0 & f_2 \end{bmatrix} \begin{bmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_1 & 0 & 0 \\ g_1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_2f_1+f_2g_1 & 0 & 0 & 0 \\ 0 & g_2f_1+f_2g_1 & 0 & 0 \end{bmatrix} .$$

However, $(s_2s_1)^2 \equiv 0$, so that there are both nonzero divisiors of zero and nonzero square roots of zero. In both cases, the rank of the matrix is 2 instead of 4. Only 0 has a rank of 0. Note that s_2s_1 has a degenerate form of either factor, with $f = 0$.

Let $s \in S$, v be any number, and w an unknown. Under what conditions are there solutions to the equations

$$sw = v \quad , \qquad ws = v \quad .$$

Taking s as the first form and using form (1) for w,

$$sw = \begin{bmatrix} f(w_0+w_1) & 0 & f(-w_2+w_3) & 0 \\ 0 & f(w_0+w_1) & 0 & f(-w_2+w_3) \\ g(w_0+w_1) & 0 & g(-w_2+w_3) & 0 \\ 0 & g(w_0+w_1) & 0 & g(-w_2+w_3) \end{bmatrix} .$$

If $sw = v$, then

$$f(w_0 + w_1) = (v_0 + v_1) \qquad f(-w_2 + w_3) = (-v_2 + v_3)$$

$$g(w_0 + w_1) = (v_2 + v_3) \qquad g(-w_2 + w_3) = (v_0 - v_1) \quad .$$

By assumption, not both f and g are zero. If $f = 0$, then

$$v_0 = -v_1 \quad , \qquad v_2 = v_3 \quad .$$

If both these are zero, then any values for w_0 and w_2 , with $w_1 = -w_0$ and $w_3 = w_2$, satisfies the equation sw = v(=0). The same is true if $v_0 \neq 0$, $v_2 = 0$. In fact, the value of v_0 is completely immaterial. If $v_2 \neq 0$, then any values for w_0 and w_1 such that

$$w_0 + w_1 = \frac{2v_2}{q}$$

give a solution. Similar conditions hold if $f \neq 0$, $g = 0$.

If f and g are both nonzero, then for some ratios p and q,

$$(v_0 + v_1) = p(v_2 + v_3)$$

$$(v_0 - v_1) = q(-v_2 + v_3) \quad .$$

These are satisfied, for example, by $v_0 = v_3$ and $v_1 = v_2$ with $p = q = 1$. To illustrate, suppose $f = q = 2$, $v_0 = v_3 = 1$, $v_1 = v_2 = 3$. Then

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_0+w_1 & 0 & -w_2+w_3 & 0 \\ 0 & w_0+w_1 & 0 & -w_2+w_3 \\ w_2+w_3 & 0 & w_0-w_1 & 0 \\ 0 & w_2+w_3 & 0 & w_0-w_1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 \\ 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 \end{bmatrix}$$

or

$$2(w_0 + w_1) = 4 \quad , \qquad 2(-w_2 + w_3) = -2 \quad .$$

Clearly, there are C^2 possible assignments of values to satisfy these. (C = cardinality of continuum.)

Hence divisors of zero used as coefficients do not necessarily mean that no solution to an equation exists. There are infinities of right hand sides for which solutions exist and, for each one, either C or C^2 possible solutions. In short, such an equation is completely indeterminate, and hence virtually meaningless.

Square Roots and Complex Forms

The chief motivation for defining these hypernumbers was to complete the set of square roots of unity and their roots. It was shown in [1] that u_1 and u_3 (and their negatives) are both square roots of u_0 and that two others exist as well; $\pm u_2$ are the square roots of $-u_0$. Furthermore, all these have square roots. We consider now the square roots of a general number v.

If w is the square root of v, then

$$|w|^2 = (v_0^2 - v_1^2) + (v_2^2 - v_3^2) \quad .$$

If $|w|$ is to be real then we must have

$$v_0^2 + v_2^2 \geq v_1^2 + v_3^2 \quad .$$

Clearly, there are v for which this is not true, for example,

$$v = (0,v_1,0,0) \quad , \quad v_1 \neq 0$$

or

$$v = \begin{bmatrix} v_1 & 0 & 0 & 0 \\ 0 & v_1 & 0 & 0 \\ 0 & 0 & -v_1 & 0 \\ 0 & 0 & 0 & -v_1 \end{bmatrix} \quad , \quad \text{determinant} = v_1^4 \quad .$$

Suppose $v_1 > 0$. Then, in spite of the above remarks,

$$\sqrt{v} = \begin{bmatrix} \sqrt{v_1} & 0 & 0 & 0 \\ 0 & \sqrt{v_1} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{v_1} \\ 0 & 0 & \sqrt{v_1} & 0 \end{bmatrix} \quad , \quad \text{determinant} = v_1^2 \quad .$$

Hence we seem to be faced with a basic inconsistency. We proceed to straighten this out.

If we use the form (2) for v and take $|v| = ab - cd$, then $|v| = -v_1^2$ which is consistent with the determinant v_1^4. If we attempt to take $|\sqrt{v}| = \sqrt{-v_1^2}$, we would need an imaginary number. However, by defining the "imaginary" unit

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Then

$$\sqrt{v} = \begin{bmatrix} \sqrt{v_1}\, r & \\ 0 & \sqrt{v_1}\, i \end{bmatrix} , \qquad v = \sqrt{v}^2 = \begin{bmatrix} v_1 r & 0 \\ 0 & -v_1 r \end{bmatrix} .$$

We have now to define $|\sqrt{v}|$ so its square is $-v_1^2$. The standard determinant of the 4×4 real matrix for $\sqrt{v}$ is clearly v_1^2, so that either square root does not give the desired result. However, the 2×2 form for $\sqrt{v}$ in terms of r and i is perfectly consistent if we interpret the elements as complex numbers. Clearly this is the only way it can be done, i.e.,

$$|\sqrt{v}| = \sqrt{v_1}\; i \quad .$$

Hence, if we insist on absolute values, we must allow not only negative but imaginary ones. Nevertheless, arithmetic with the 4×4 matrices does not require imaginary or complex numbers. For the present, we will simply abandon the concept of absolute values.

However, we have created another problem. The matrix for $\sqrt{v}$ is not in standard format, as defined and used up to now. It is evident that, by considering other numbers, the unit i can appear in any of the four corners. How many new units in addition to the u_n will be required for all purposes? Before attempting to answer this, let us look at a general square root.

It appears by now that we will require general numbers of the form

$$w = \begin{bmatrix} ar + \alpha i & dr + \delta i \\ cr + \gamma i & br + \beta i \end{bmatrix} .$$

Then

$$w^2 = \begin{bmatrix} (a^2 - \alpha^2 + cd - \gamma\delta)r & (ad - \alpha\delta + bd - \beta\delta)r \\ + (2a\alpha + c\delta + \gamma d)i & + (a\delta + \alpha d + b\delta + \beta d)i \\ (ac - \alpha\gamma + bc - \beta\gamma)r & (b^2 - \beta^2 + cd - \gamma\delta)r \\ + (a\gamma + \alpha c + b\gamma + \beta c)i & + (2b\beta + c\delta + \gamma d)i \end{bmatrix} .$$

Suppose $w^2 = v$ and denote the elements of v by overbars. Then we have eight simultaneous quadratic equations:

$$a^2 - \alpha^2 + cd - \gamma\delta = \bar{a}$$

$$2a\alpha + c\delta + \gamma d = \bar{\alpha}$$

$$b^2 - \beta^2 + cd - \gamma\delta = \bar{b}$$

$$2b\beta + c\delta + \gamma d = \bar{\beta}$$

$$ac - \alpha\gamma + bc - \beta\gamma = \bar{c}$$

$$a\gamma + \alpha c + b\gamma + \beta c = \bar{\gamma}$$

$$ad - \alpha\delta + bd - \beta\delta = \bar{d}$$

$$a\delta + \alpha d + b\delta + \beta d = \bar{\delta} .$$

If our assumption that the form of w is the most general required, then these must always have a solution in real numbers. To prove this, we reorganize the variables somewhat. Let

$$f = a + b \quad , \qquad \varphi = \alpha + \beta \quad .$$

Then the last four equations can be written in tableau form as if they were linear in c, γ, d, δ:

c	γ	d	δ	
f	$-\varphi$			$= \bar{c}$
φ	f			$= \bar{\gamma}$
		f	$-\varphi$	$= \bar{d}$
		φ	f	$= \bar{\delta}$.

These are solvable as two sets of two variables. There is always a solution if $f^2 + \varphi^2 \neq 0$. Assume this condition holds and f and φ are known; also that c, γ, d, δ are computed. Then

$$a^2 - \alpha^2 = \bar{\bar{a}} = \bar{a} - cd + \gamma\delta$$

$$2a\alpha = \bar{\bar{\alpha}} = \bar{\alpha} - c\delta - \gamma d$$

$$b^2 - \beta^2 = \bar{\bar{b}} = \bar{b} - cd + \gamma\delta$$

$$2b\beta = \bar{\bar{\beta}} = \bar{\beta} - c\delta - \gamma d \quad .$$

These are again two sets of equations of the form

$$x^2 - y^2 = p$$

$$2xy = q$$

which always have at least one real solution as shown in Appendix A.

We assumed that not both (a + b) and ($\alpha + \beta$) are zero. Suppose they are. This can happen in two ways (actually four). First suppose all four numbers are zero. Then from the original eight equations, the last four are zero, or $\bar{c} = \bar{\gamma} = \bar{d} = \bar{\delta} = 0$. Furthermore, the first four give $\bar{a} = \bar{b}$, $\bar{\alpha} = \bar{\beta}$. Hence v has the form

$$v = \begin{bmatrix} \bar{a}r + \bar{\alpha}i & 0 \\ 0 & \bar{a}r + \bar{\alpha}i \end{bmatrix} .$$

The implied square root

$$w = \begin{bmatrix} 0 & dr + \delta i \\ cr + \gamma i & 0 \end{bmatrix}$$

is formally correct but is not at all unique. First of all, there is no reason why (c,γ) should differ from (d,δ). If we set them equal, then we have a system of equations as discussed above:

$$c^2 - \gamma^2 = \bar{a}$$

$$2c\gamma = \bar{\alpha}$$

which always has a real solution. However, nothing forces them to be equal. For example, suppose

$$\bar{a} = 4 \quad , \qquad \bar{\alpha} = 0 \quad .$$

Then

$$\begin{bmatrix} 0 & 1r \\ 4r & 0 \end{bmatrix} \begin{bmatrix} 0 & 1r \\ 4r & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2r \\ 2r & 0 \end{bmatrix} \begin{bmatrix} 0 & 2r \\ 2r & 0 \end{bmatrix} = \begin{bmatrix} 4r & 0 \\ 0 & 4r \end{bmatrix} .$$

Besides this nonuniqueness, the form is nonunique:

$$\begin{bmatrix} 2r & \\ & 2r \end{bmatrix}^2 = \begin{bmatrix} 4r & \\ & 4r \end{bmatrix} .$$

When this "proper" form is used, however, the values are unique, except for sign.

Now suppose $a = -b$, $\alpha = -\beta$ (one pair may be zero). Then again the last four of the original eight equations go to zero and the first four give the same form for v as above. However, w has the general form

$$w = \begin{bmatrix} ar + \alpha i & dr + \delta i \\ cr + \gamma i & br + \beta i \end{bmatrix} .$$

But these numbers are really determined by two equations:

$$a^2 - \alpha^2 + cd - \gamma\delta = \bar{a}$$

$$2a\alpha + c\delta + \gamma d = \bar{\alpha}$$

since

$$a^2 = b^2 \quad , \qquad \alpha^2 = \beta^2 \quad , \qquad a\alpha = b\beta \quad .$$

Again there is a high degree of indeterminacy.

What does it mean for not both (a + b) and ($\alpha + \beta$) to be zero? It says that the main diagonal of the root cannot be zero and cannot consist of two quantities which are the negatives of each other. Two roots of u_0 which violate this are u_1 and u_3. It was precisely the inclusion of these units which led to multiple roots of unity and nonzero divisors and roots of zero. But u_2, which is analogous to i, also violate it. What happens if we try to calculate the square root of $-u_0$, as though we didn't know one.

We have the following:

$$a^2 - \alpha^2 + cd - \gamma\delta = -1$$

$$2a\alpha + c\delta + \gamma d = 0$$

$$b^2 - \beta^2 + cd - \gamma\delta = -1$$

$$2b\beta + c\delta + \gamma d = 0$$

$$ac - \alpha\gamma + bc - \beta\gamma = 0$$

$$a\gamma + \alpha c + b\gamma + \beta c = 0$$

$$ad - \alpha\delta + bd - \beta\delta = 0$$

$$a\delta + \alpha d + b\delta + \beta d = 0 \quad .$$

One sees four immediate solutions:

$$\alpha = \pm 1 \quad , \qquad \beta = \pm 1 \quad , \qquad \text{all others zero} \quad .$$

Hence

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}^2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^2 = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}^2 = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}$$

and we don't need u_2 at all. This leads to the suspicion that either the set u_n was an incorrect choice, or the use of i is improper, or both.

Reconsideration of Units and Roots

It is clear by now that the use of u_n plus i allows too much indeterminancy. The u_n were defined so as to complete the formal set of square roots of unity, in the hope that the resulting set of numbers would have useful properties. However, to find roots of these roots, without using the complex value i, we had to invent the 2 × 2 unit i. Even this left absolute values undefinable without using the complex i. Furthermore, use of the unit i creates forms not representable in terms of the u_n. As things stand, we have an incomplete and inconsistent system of numbers, which at the same time allow too many solutions to equations to be meaningful.

It is worth noting in passing that we have shown that the set of 2 × 2 complex matrices do not form a consistent algebra. The preceding calculations for roots can be directly interpreted in terms of such matrices. It is rather surprising that non-singular equations can have infinite sets of solutions in complex numbers, and even in reals, as witness the roots of the identity.

Numerical Examples

An interactive computer program, which effectively creates a quadriform hand-calculator, has been written for the PDP-11. All arithmetic is single precision REAL except cube roots which are double precision.

Four sets of calculations are shown in the following pages:

A. Ten quadriform numbers, $v^{(n)}$

$$n(0,.1,.5,0) \quad , \quad n = 1,\dots,10$$

were generated. Two sets of output, each with lines 1 to 40, are shown, as follows:

Set 1

Lines 1-10: $v^{(1)} \dots v^{(10)}$

11-20: $L(v^{(1)}) \dots L(v^{(10)})$

21-30: $E(^{(1)}) \dots E(v^{(10)})$

31-40: $H(v^{(1)}) \dots H(v^{(10)})$, $H = H_{123}$.

Set 2

Lines 1-40: the various norms of the numbers in Set 1.

B. Same as A for the numbers

$$n(0,.1,-.5,.1) \quad , \quad n = 1,\dots,10 \quad .$$

C. Same as A for the numbers

$$n(.1,-.1,.5,-.1) \quad , \quad n = 1,\dots,10 \quad .$$

D. Ten quadriform numbers, $v^{(n)}$

$$n(2,-1,1,1) \quad , \quad n = 1,\dots,10$$

were generated. Two sets of output, each with lines 1 to 40, are shown, as follows:

Set 1

Lines 1-10: $v^{(1)} \ldots v^{(10)}$

11-20: $(v^{(1)})^2 \ldots (v^{(10)})^2$

21-30: $\sqrt{v^{(1)}} \ldots \sqrt{v^{(10)}}$

31-40: $(v^{(1)})^{-1} \ldots (v^{(10)})^{-1}$.

Set 2

Lines 1-40: the various norms of the numbers in Set 1.

A. Set 1

```
cki 1 10
         1        0.00000        0.10000        0.50000        0.00000
         2        0.00000        0.20000        1.00000        0.00000
         3        0.00000        0.30000        1.50000        0.00000
         4        0.00000        0.40000        2.00000        0.00000
         5        0.00000        0.50000        2.50000        0.00000
         6        0.00000        0.60000        3.00000        0.00000
         7        0.00000        0.70000        3.50000        0.00000
         8        0.00000        0.80000        4.00000        0.00000
         9        0.00000        0.90000        4.50000        0.00000
        10        0.00000        1.00000        5.00000        0.00000
  command :
cki 11 20
        11       -0.71356        0.20273        1.57080        0.00000
        12       -0.02041        0.20273        1.57080        0.00000
        13        0.38505        0.20273        1.57080        0.00000
        14        0.67274        0.20273        1.57080        0.00000
        15        0.89588        0.20273        1.57080        0.00000
        16        1.07820        0.20273        1.57080        0.00000
        17        1.23235        0.20273        1.57080        0.00000
        18        1.36588        0.20273        1.57080        0.00000
        19        1.48367        0.20273        1.57080        0.00000
        20        1.58903        0.20273        1.57080        0.00000
  command :
cki 21 30
        21        0.88197        0.10017        0.48182        0.00000
        22        0.55114        0.20134        0.85836        0.00000
        23        0.07394        0.30452        1.04272        0.00000
        24       -0.44988        0.41075        0.98302        0.00000
        25       -0.90339        0.52110        0.67485        0.00000
        26       -1.17360        0.63665        0.16729        0.00000
        27       -1.17541        0.75858       -0.44029        0.00000
        28       -0.87421        0.88811       -1.01217        0.00000
        29       -0.30209        1.02652       -1.40089        0.00000
        30        0.43771        1.17520       -1.47970        0.00000
  command :
cki 31 40
        31        0.88197        0.04802        0.48182        0.08790
        32        0.55114        0.16942        0.85836        0.10878
        33        0.07394        0.30376        1.04272        0.02154
        34       -0.44988        0.37350        0.98302       -0.17093
        35       -0.90339        0.31186        0.67485       -0.41747
        36       -1.17360        0.08984        0.16729       -0.63028
        37       -1.17541       -0.26610       -0.44029       -0.71038
        38       -0.87421       -0.67212       -1.01217       -0.58050
        39       -0.30209       -1.00345       -1.40089       -0.21639
        40        0.43771       -1.12693       -1.47970        0.33336
  command :
```

A. Set 2

nrm 1 10

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.24000	0.48990	0.50000	0.10000	0.50990
2	0.96000	0.97980	1.00000	0.20000	1.01980
3	2.16000	1.46969	1.50000	0.30000	1.52971
4	3.84000	1.95959	2.00000	0.40000	2.03961
5	6.00000	2.44949	2.50000	0.50000	2.54951
6	8.64000	2.93939	3.00000	0.60000	3.05941
7	11.76000	3.42929	3.50000	0.70000	3.56931
8	15.36000	3.91918	4.00000	0.80000	4.07922
9	19.44000	4.40908	4.50000	0.90000	4.58912
10	24.00000	4.89898	5.00000	1.00000	5.09902

command :
nrm 11 20

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
11	2.93547	1.71332	1.72527	0.20273	1.73714
12	2.42672	1.55779	1.57093	0.20273	1.58396
13	2.57457	1.60455	1.61730	0.20273	1.62996
14	2.87887	1.69672	1.70879	0.20273	1.72078
15	3.22890	1.79691	1.80831	0.20273	1.81964
16	3.58382	1.89442	1.90523	0.20273	1.91599
17	3.94499	1.98620	1.99652	0.20273	2.00679
18	4.29194	2.07170	2.08160	0.20273	2.09144
19	4.62757	2.15118	2.16071	0.20273	2.17020
20	4.95131	2.22515	2.23437	0.20273	2.24355

command :
nrm 21 30

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
21	1.00000	1.00000	1.00500	0.10017	1.00998
22	1.00000	1.00000	1.02007	0.20134	1.03975
23	1.00000	1.00000	1.04534	0.30452	1.08879
24	1.00000	1.00000	1.08107	0.41075	1.15648
25	1.00000	1.00000	1.12763	0.52110	1.24221
26	1.00000	1.00000	1.18547	0.63665	1.34561
27	1.00000	1.00000	1.25517	0.75858	1.46659
28	1.00000	1.00000	1.33744	0.88811	1.60545
29	1.00000	1.00000	1.43309	1.02652	1.76280
30	1.00000	1.00000	1.54308	1.17520	1.93964

command :
nrm 31 40

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
31	1.00000	1.00000	1.00500	0.10017	1.00998
32	1.00000	1.00000	1.02007	0.20134	1.03975
33	1.00000	1.00000	1.04534	0.30452	1.08879
34	1.00000	1.00000	1.08107	0.41075	1.15648
35	1.00000	1.00000	1.12763	0.52110	1.24221
36	1.00000	1.00000	1.18547	0.63665	1.34561
37	1.00000	1.00000	1.25517	0.75858	1.46659
38	1.00000	1.00000	1.33744	0.88811	1.60545
39	1.00000	1.00000	1.43309	1.02652	1.76280
40	1.00000	1.00000	1.54308	1.17520	1.93964

command :
end
@

B. Set 1

```
command :
command :
command :
cki 1 10
         1      0.00000      0.10000     -0.50000      0.10000
         2      0.00000      0.20000     -1.00000      0.20000
         3      0.00000      0.30000     -1.50000      0.30000
         4      0.00000      0.40000     -2.00000      0.40000
         5      0.00000      0.50000     -2.50000      0.50000
         6      0.00000      0.60000     -3.00000      0.60000
         7      0.00000      0.70000     -3.50000      0.70000
         8      0.00000      0.80000     -4.00000      0.80000
         9      0.00000      0.90000     -4.50000      0.90000
        10      0.00000      1.00000     -5.00000      1.00000
command :
cki 11 20
        11     -0.73484      0.20703     -1.57080      0.20703
        12     -0.04169      0.20703     -1.57080      0.20703
        13      0.36377      0.20703     -1.57080      0.20703
        14      0.65146      0.20703     -1.57080      0.20703
        15      0.87460      0.20703     -1.57080      0.20703
        16      1.05692      0.20703     -1.57080      0.20703
        17      1.21107      0.20703     -1.57080      0.20703
        18      1.34460      0.20703     -1.57080      0.20703
        19      1.46239      0.20703     -1.57080      0.20703
        20      1.56775      0.20703     -1.57080      0.20703
command :
cki 21 30
        21      0.88634      0.10017     -0.48421      0.10017
        22      0.56178      0.20134     -0.87492      0.20134
        23      0.07702      0.30452     -1.08606      0.30452
        24     -0.48126      0.41075     -1.05158      0.41075
        25     -0.99519      0.52110     -0.74343      0.52110
        26     -1.33214      0.63665     -0.16989      0.63665
        27     -1.37340      0.75858      0.51446      0.75858
        28     -1.04939      0.88811      1.21501      0.88811
        29     -0.37159      1.02652      1.72319      1.02652
        30      0.55020      1.17520      1.65997      1.17520
command :
cki 31 40
        31      0.89120     -0.13661     -0.49304      0.13661
        32      0.59631     -0.28378     -0.89748      0.28378
        33      0.16980     -0.34005     -1.09655      0.34005
        34     -0.33294     -0.21898     -0.99250      0.21898
        35     -0.85618      0.11909     -0.54344     -0.11909
        36     -1.33406      0.64067      0.20295     -0.64067
        37     -1.67720      1.22565      1.09152     -1.22565
        38     -1.76611      1.67531      1.86927     -1.67531
        39     -1.46298      1.74813      2.22971     -1.74813
        40     -0.64894      1.22454      1.89153     -1.22454
command :
```

B. Set 2

nrm 1 10

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.24000	0.48990	0.50000	0.10000	0.50990
2	0.96000	0.97980	1.00000	0.20000	1.01980
3	2.16000	1.46969	1.50000	0.30000	1.52971
4	3.84000	1.95959	2.00000	0.40000	2.03961
5	6.00000	2.44949	2.50000	0.50000	2.54951
6	8.64000	2.93939	3.00000	0.60000	3.05941
7	11.76000	3.42929	3.50000	0.70000	3.56931
8	15.36000	3.91918	4.00000	0.80000	4.07922
9	19.44000	4.40908	4.50000	0.90000	4.58912
10	24.00000	4.89898	5.00000	1.00000	5.09902

command :

nrm 11 20

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
11	2.93547	1.71332	1.72527	0.20273	1.73714
12	2.42672	1.55779	1.57093	0.20273	1.58396
13	2.57457	1.60455	1.61730	0.20273	1.62996
14	2.87887	1.69672	1.70879	0.20273	1.72078
15	3.22890	1.79691	1.80831	0.20273	1.81964
16	3.58882	1.89442	1.90523	0.20273	1.91599
17	3.94499	1.98620	1.99652	0.20273	2.00679
18	4.29194	2.07170	2.08160	0.20273	2.09144
19	4.62757	2.15118	2.16071	0.20273	2.17020
20	4.95131	2.22515	2.23437	0.20273	2.24355

command :

nrm 21 30

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
21	1.00000	1.00000	1.00500	0.10017	1.00998
22	1.00000	1.00000	1.02007	0.20134	1.03975
23	1.00000	1.00000	1.04534	0.30452	1.08879
24	1.00000	1.00000	1.08107	0.41075	1.15648
25	1.00000	1.00000	1.12763	0.52110	1.24221
26	1.00000	1.00000	1.18547	0.63665	1.34561
27	1.00000	1.00000	1.25517	0.75858	1.46659
28	1.00000	1.00000	1.33744	0.88811	1.60545
29	1.00000	1.00000	1.43309	1.02652	1.76280
30	1.00000	1.00000	1.54308	1.17520	1.93964

command :

nrm 31 40

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
31	1.00000	1.00000	1.00500	0.10017	1.00998
32	1.00000	1.00000	1.02007	0.20134	1.03975
33	1.00000	1.00000	1.04534	0.30452	1.08879
34	1.00000	1.00000	1.08107	0.41075	1.15648
35	1.00000	1.00000	1.12763	0.52110	1.24221
36	1.00000	1.00000	1.18547	0.63665	1.34561
37	1.00000	1.00000	1.25517	0.75858	1.46659
38	1.00000	1.00000	1.33744	0.88811	1.60545
39	1.00000	1.00000	1.43309	1.02652	1.76280
40	1.00000	1.00000	1.54308	1.17520	1.93964

command :

end

@

C. Set 1

```
  command :
  command :
  command :
  command :
  command :
cki 1 10
        1      0.10000     -0.10000      0.50000     -0.10000
        2      0.20000     -0.20000      1.00000     -0.20000
        3      0.30000     -0.30000      1.50000     -0.30000
        4      0.40000     -0.40000      2.00000     -0.40000
        5      0.50000     -0.50000      2.50000     -0.50000
        6      0.60000     -0.60000      3.00000     -0.60000
        7      0.70000     -0.70000      3.50000     -0.70000
        8      0.80000     -0.80000      4.00000     -0.80000
        9      0.90000     -0.90000      4.50000     -0.90000
       10      1.00000     -1.00000      5.00000     -1.00000
  command :
cki 11 20
       11     -0.71356     -0.20273      1.37340     -0.20273
       12     -0.02041     -0.20273      1.37340     -0.20273
       13      0.38505     -0.20273      1.37340     -0.20273
       14      0.67274     -0.20273      1.37340     -0.20273
       15      0.89588     -0.20273      1.37340     -0.20273
       16      1.07820     -0.20273      1.37340     -0.20273
       17      1.23235     -0.20273      1.37340     -0.20273
       18      1.36588     -0.20273      1.37340     -0.20273
       19      1.48367     -0.20273      1.37340     -0.20273
       20      1.58903     -0.20273      1.37340     -0.20273
  command :
cki 21 30
       21      0.97956     -0.11070      0.53514     -0.11070
       22      0.63616     -0.24591      1.06863     -0.24591
       23      0.10396     -0.41106      1.46603     -0.41106
       24     -0.71796     -0.61277      1.56877     -0.61277
       25     -1.64079     -0.85914      1.22570     -0.85914
       26     -2.42732     -1.16006      0.34601     -1.16006
       27     -2.76569     -1.52760     -1.03599     -1.52760
       28     -2.33546     -1.97652     -2.70405     -1.97652
       29     -0.91397     -2.52432     -4.23837     -2.52482
       30      1.49560     -3.19453     -5.05591     -3.19453
  command :
cki 31 40
       31      0.97429      0.04430      0.52543     -0.04430
       32      0.64502     -0.07555      1.04269      0.07555
       33     -0.02052     -0.39822      1.46249      0.39822
       34     -0.95443     -0.87804      1.69012      0.87804
       35     -1.94746     -1.35593      1.61331      1.35593
       36     -2.63927     -1.55552      1.09253      1.55552
       37     -2.56448     -1.12297     -0.02771      1.12297
       38     -1.27363      0.27270     -1.86538     -0.27270
       39      1.46873      2.77427     -4.39154     -2.77427
       40      5.43601      6.12523     -7.27156     -6.12523
  command :
```

C. Set 2

nrm 1 10

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	0.24000	0.48990	0.50990	0.14142	0.52915
2	0.96000	0.97980	1.01980	0.28284	1.05830
3	2.16000	1.46969	1.52971	0.42426	1.58745
4	3.84000	1.95959	2.03961	0.56569	2.11660
5	6.00000	2.44949	2.54951	0.70711	2.64575
6	8.64000	2.93939	3.05941	0.84853	3.17490
7	11.76000	3.42929	3.56931	0.98995	3.70405
8	15.36000	3.91918	4.07922	1.13137	4.23320
9	19.44000	4.40908	4.58912	1.27279	4.76235
10	24.00000	4.89898	5.09902	1.41421	5.29150

command :

nrm 11 20

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
11	2.31319	1.52092	1.54771	0.28671	1.57404
12	1.80445	1.34330	1.37355	0.28671	1.40316
13	1.95230	1.39725	1.42636	0.28671	1.45489
14	2.25660	1.50220	1.52931	0.28671	1.55596
15	2.60663	1.61451	1.63977	0.28671	1.66464
16	2.96655	1.72237	1.74607	0.28671	1.76945
17	3.32272	1.82283	1.84524	0.28671	1.86738
18	3.66967	1.91564	1.93697	0.28671	1.95808
19	4.00529	2.00132	2.02176	0.28671	2.04198
20	4.32903	2.08063	2.10029	0.28671	2.11977

command :

nrm 21 30

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
21	1.22140	1.10517	1.11620	0.15656	1.12713
22	1.49182	1.22140	1.26995	0.34777	1.31671
23	1.82212	1.34986	1.46971	0.58133	1.58051
24	2.22554	1.49182	1.72526	0.86659	1.93067
25	2.71828	1.64872	2.04805	1.21501	2.38134
26	3.32012	1.82212	2.45185	1.64057	2.95009
27	4.05520	2.01375	2.95336	2.16035	3.65916
28	4.95303	2.22554	3.57299	2.79522	4.53646
29	6.04965	2.45960	4.33580	3.57064	5.61681
30	7.38906	2.71828	5.27248	4.51775	6.94328

command :

nrm 31 40

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
31	1.22140	1.10517	1.10694	0.06265	1.10872
32	1.49182	1.22140	1.22607	0.10684	1.23071
33	1.82212	1.34986	1.46263	0.56317	1.56731
34	2.22554	1.49182	1.94099	1.24174	2.30420
35	2.71828	1.64872	2.52891	1.91758	3.17372
36	3.32012	1.82212	2.85646	2.19983	3.60536
37	4.05520	2.01375	2.56463	1.58812	3.01653
38	4.95304	2.22554	2.25871	0.38565	2.29140
39	6.04964	2.45960	4.63064	3.92341	6.06926
40	7.38910	2.71829	9.07887	8.66238	12.54841

command :

end

D. Set 1

cki 1 10				
1	2.00000	-1.00000	1.00000	1.00000
2	4.00000	-2.00000	2.00000	2.00000
3	6.00000	-3.00000	3.00000	3.00000
4	8.00000	-4.00000	4.00000	4.00000
5	10.00000	-5.00000	5.00000	5.00000
6	12.00000	-6.00000	6.00000	6.00000
7	14.00000	-7.00000	7.00000	7.00000
8	16.00000	-8.00000	8.00000	8.00000
9	18.00000	-9.00000	9.00000	9.00000
10	20.00000	-10.00000	10.00000	10.00000
command :				
cki 11 20				
11	5.00000	-4.00000	4.00000	4.00000
12	20.00000	-16.00000	16.00000	16.00000
13	45.00000	-36.00000	36.00000	36.00000
14	80.00000	-64.00000	64.00000	64.00000
15	125.00000	-100.00000	100.00000	100.00000
16	180.00000	-144.00000	144.00000	144.00000
17	245.00000	-196.00000	196.00000	196.00000
18	320.00000	-256.00000	256.00000	256.00000
19	405.00000	-324.00000	324.00000	324.00000
20	500.00000	-400.00000	400.00000	400.00000
command :				
cki 21 30				
21	1.36603	-0.36603	0.36603	0.36603
22	1.93185	-0.51764	0.51764	0.51764
23	2.36603	-0.63397	0.63397	0.63397
24	2.73205	-0.73205	0.73205	0.73205
25	3.05453	-0.81846	0.81846	0.81846
26	3.34607	-0.89658	0.89658	0.89658
27	3.61416	-0.96841	0.96841	0.96841
28	3.86370	-1.03528	1.03528	1.03528
29	4.09808	-1.09808	1.09808	1.09808
30	4.31975	-1.15747	1.15747	1.15747
command :				
cki 31 40				
31	0.66667	0.33333	-0.33333	-0.33333
32	0.33333	0.16667	-0.16667	-0.16667
33	0.22222	0.11111	-0.11111	-0.11111
34	0.16667	0.08333	-0.08333	-0.08333
35	0.13333	0.06667	-0.06667	-0.06667
36	0.11111	0.05556	-0.05556	-0.05556
37	0.09524	0.04762	-0.04762	-0.04762
38	0.08333	0.04167	-0.04167	-0.04167
39	0.07407	0.03704	-0.03704	-0.03704
40	0.06667	0.03333	-0.03333	-0.03333
command :				

D. Set 2

nrm 1 10

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
1	3.00000	1.73205	2.23607	1.41421	2.64575
2	12.00000	3.46410	4.47214	2.82843	5.29150
3	27.00000	5.19615	6.70820	4.24264	7.93725
4	48.00000	6.92820	8.94427	5.65685	10.58301
5	75.00000	8.66025	11.18034	7.07107	13.22876
6	108.00000	10.39231	13.41641	8.48528	15.87451
7	147.00000	12.12436	15.65248	9.89950	18.52026
8	192.00000	13.85641	17.88854	11.31371	21.16601
9	243.00000	15.58846	20.12461	12.72792	23.81176
10	300.00000	17.32051	22.36068	14.14214	26.45751

command :
nrm 11 20

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
11	9.00000	3.00000	6.40312	5.65685	8.54400
12	144.00000	12.00000	25.61250	22.62742	34.17602
13	729.00000	27.00000	57.62812	50.91169	76.89603
14	2304.00000	48.00000	102.44999	90.50967	136.70407
15	5625.00000	75.00000	160.07811	141.42136	213.60010
16	11664.00000	108.00000	230.51248	203.64676	307.58414
17	21609.00000	147.00000	313.75308	277.18585	418.65619
18	36864.00000	192.00000	409.79996	362.03867	546.81628
19	59049.00000	243.00000	518.65308	458.20520	692.06433
20	90000.00000	300.00000	640.31244	565.68542	854.40039

command :
nrm 21 30

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
21	1.73205	1.31607	1.41421	0.51764	1.50597
22	3.46410	1.86121	2.00000	0.73205	2.12977
23	5.19615	2.27951	2.44949	0.89658	2.60842
24	6.92820	2.63215	2.82843	1.03528	3.01194
25	8.66025	2.94283	3.16228	1.15747	3.36745
26	10.39231	3.22371	3.46410	1.26795	3.68886
27	12.12436	3.48200	3.74166	1.36954	3.98443
28	13.85641	3.72242	4.00000	1.46410	4.25953
29	15.58846	3.94822	4.24264	1.55291	4.51791
30	17.32051	4.16179	4.47214	1.63692	4.76230

command :
nrm 31 40

n	sq(abs.val)	abs.value	modulus	hypermodulus	magnitude
31	0.33333	0.57735	0.74536	0.47140	0.88192
32	0.08333	0.28868	0.37268	0.23570	0.44096
33	0.03704	0.19245	0.24845	0.15713	0.29397
34	0.02083	0.14434	0.18634	0.11785	0.22048
35	0.01333	0.11547	0.14907	0.09428	0.17638
36	0.00926	0.09623	0.12423	0.07857	0.14699
37	0.00680	0.08248	0.10648	0.06734	0.12599
38	0.00521	0.07217	0.09317	0.05893	0.11024
39	0.00412	0.06415	0.08282	0.05238	0.09799
40	0.00333	0.05774	0.07454	0.04714	0.08819

command :
end
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