NOT FOR QUOTATION
WITHOUT THE PERMISSION
OF THE AUTHOR

UTILITY FUNCTIONS FOR INFINITE-PERIOD PLANNING

Charles M. Harvey

November 1986
WP-86-82

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

Foreword

This paper presents a systematic discussion of decision analysis models for attitudes toward risk when the effects of a public policy choice extend into the distant or unbounded future. Several issues of social risk attitudes are identified and discussed. Conditions on preferences are presented by which value judgments concerning these issues can be included in a formal model for a public policy evaluation.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

UTILITY FUNCTIONS FOR INFINITE-PERIOD PLANNING*

Charles M. Harvey

1. Introduction

There are a variety of methods for treating the uncertainty of future outcomes in a multiperiod planning model. Cost-benefit studies often: (1) add a "risk premium" to the discount rate (e.g., Sugden and Williams 1978, p. 60), (2) report a probability distribution of net present values (e.g., Mishan 1976, p. 372), (3) use the net present value function as a utility function, or (4) subtract a multiple of the variance of the net return from the expected net return (Markowitz 1959). These methods do not address preference issues regarding an institution's or society's attitude toward risk.

Decision analysis models that do address these issues have been developed by a number of authors. Fishburn (1965a), (1970) has introduced conditions on preferences that imply a utility function having an additive form. Pollak (1967) has introduced preference conditions that imply a utility function having an additive form or a log-additive form. Meyer (1970), (1972) has shown that weakened assumptions of mutual utility independence imply that the utility function has an additive form, a positive multiplicative form, or a negative multiplicative form (see Keeney 1968, 1974 for similar results in a multiattribute setting). Richard (1975) has introduced a condition of strict multivariate risk aversion that together with mutual utility independence implies a negative multiplicative form. Preference conditions concerning utility dependence, e.g., the dependence of the decision maker's risk attitude for one period on the outcomes in other periods, have been introduced by Bell (1977), Fishburn (1965b), and Meyer (1977). Related work includes that of Barrager (1980), Bell (1974), Bodily (1981), and Fishburn and Rubinstein (1982).

This paper is a systematic study of preference issues regarding attitudes toward risk in multiperiod planning. A number of decision analysis models are presented that when appropriate can be used to formulate social preferences in a risk management study or to formulate corporate preferences in a long range planning study by a private form.

*This research is supported in part by the National Science Foundation under Grant no. SES 84-10665.

The models in this paper are concerned with infinite-period outcomes, i.e., with an unbounded planning horizon. Corresponding results for a bounded planning horizon can be deduced by assuming that the amounts in the periods beyond the horizon are a specified, standard amount.

A general type of planning model is presented in Section 2. Its primary characteristics are that tradeoffs between different periods satisfy conditions of preferential independence and that risk attitudes satisfy conditions of expected utility. Then, in Section 3, four preference issues are identified that can be included in such a planning model (and six preference issues are identified that cannot be included).

Sections 4-8 are concerned with special conditions on social or corporate preferences, and with the implications of these preference conditions for the form of a utility function in a general type of planning model. Sections 4 and 5 discuss attitudes toward multiperiod risk; Section 6 discusses concern for intertemporal equity; and Section 7 discusses attitudes toward present risk. Then, in Section 8, a number of planning models are presented which use a combination of preference conditions regarding these issues.

The planning models discussed in this paper are intended to be prescriptive rather than descriptive or normative; that is, they are intended to clarify the implications from value judgments concerning relatively simple choices to preferences between the relatively complex actual alternatives. The utility functions discussed are intended to be an inobtrusive technical means by which the analyst can evaluate these implications.

A utility function can be determined by the following steps: (1) identify which preference issues are important, (2) verify preference conditions related to these issues, and (3) for each preference issue that is deemed important, make a specific preference assessment. Such a procedure has several advantages: it is possible to choose where to simplify the planning model, the number of value judgments needed is minimized, and a sensitivity analysis of the effects of the value judgments is possible.

2. Expected-Utility Planning Models

This section presents an expected-utility model for infinite-period planning that is based on the additive-value model presented in Harvey (1986a). The reader is referred to that paper for a more detailed discussion of the definitions and results concerning value tradeoffs that are needed for the present discussion of risk attitudes. Moreover, the reader is referred to Appendix A of this paper for much of the technical material (definitions and proofs) used in this section.

Suppose that a unit of time is chosen, e.g., a year, and that the future is divided into a sequence of periods $(t-1, t]$, $t = 1,2,\ldots$, of unit duration. The outcomes during a period, $t = 1,2,\ldots$, are to be described by a single, readily interpretable variable x_t that may involve the pricing out of nonmonetary attributes or the summing of the costs, risks, and benefits that accrue to the individuals in a specific group. The consequences over the future are described by sequences $(x_1, x_2, \ldots)$ of the single-period amounts x_t, $t = 1,2,\ldots$. These consequences will be called *infinite-period consequences*. Uncertain events having a finite number of infinite-period consequences as possible outcomes will be called *infinite-period lotteries*.

Definition 1. A model having the following structure will be called a *planning model*: (a) x^* denotes a specified amount of the variables x_t to be used for normalization purposes, (b) I denotes a common interval on which the variables x_t are defined, (c) C denotes the set of all infinite-period consequences having amounts x_t in I, (d) L denotes the set of all infinite-period lotteries having consequences in C, and (e) $\succsim$ denotes a preference relation defined on the set L of lotteries.

The amount $x_t = x^*$ will be called a *standard amount*. For technical reasons, it will be assumed that the interval I has one of the forms (x^-, x^+), $[x^*, x^+)$, and $(x^-, x^*]$ where each of the amounts x^*, x^-, x^+ can be either finite or infinite. Here, the standard amount x^* will be called *intermediate*, *least preferred*, and *most preferred* respectively. If x^* is a least preferred or most preferred amount in I, then L is to be restricted to those lotteries not having the extreme consequence $(x^*, x^*, \ldots)$ as a possible consequence.

In terms of the above notation, an ordinal-value model is defined in Harvey (1986a) by the preference conditions (A)-(E) restated in Appendix A. The primary result is as follows.

Lemma 1. The conditions (A)-(E) on tradeoffs are satisfied if and only if preferences among the infinite-period consequences in the set C^* (defined in Appendix A) are represented by an infinite-series value function

$$V(x_1,x_2,\ldots) = v(x_1) + a_2 v(x_2) + \cdots \tag{1}$$

such that: $V(x_1,x_2,\ldots)$ converges if and only if $(x_1,x_2,\ldots)$ is in the set C^*; the *timing weights* a_t, $t = 1,2,\ldots$, are positive; and the *tradeoffs function* v is continuous, strictly increasing, and normalized so that $v(x^*) = 0$.

In this paper, the preference conditions (A)-(E) are augmented by two conditions as follows:

(F) For any consequence $(x_1,x_2,\ldots)$ in C^*, there exists an indifferent consequence $(x_1', x^*_2, x^*_3,\ldots)$ in C^* for some amount x_1' in I.

(G) The preference relation $\succsim$ satisfies the conditions for expected utility for the set L^* of lotteries in L having consequences in C^*. (See, e.g., Herstein and Milnor 1953). Moreover, any lottery in L^* has an indifferent consequence in C^*. (This condition on the existence of certainty equivalents is included to ensure that the utility function is continuous.)

Theorem 1. A planning model satisfies the preference conditions (A)-(G) if and only if the preference relation $\succsim$ restricted to the set L^* of infinite-period lotteries is represented by a utility function having the forms:

$$U(x_1,x_2,\ldots) = f(\, v(x_1) + a_2 v(x_2) + \cdots) \tag{2}$$

$$U(x_1,x_2,\ldots) = u \circ v^{-1}(\, v(x_1) + a_2 v(x_2) + \cdots) \tag{3}$$

$$U(x_1,x_2,\ldots) = f(f^{-1}\circ u(x_1) + a_2 f^{-1}\circ u(x_2) + \cdots) \ . \tag{4}$$

Here, the value function $V(x_1,x_2,\ldots) = v(x_1) + a_2 v(x_2) + \cdots$ is as described in Lemma 1; the *multiperiod risk function* f is a continuous, strictly increasing function defined on the range of V; and the ***first-period utility function*** u is a standard conditional utility function for the first period, i.e., a utility function on consequences of the form $(x_1, x_2^*, x_3^*, \ldots)$.

The purpose of Theorem 1 is to provide the general forms (2)-(4) of utility functions for modeling the preference issues (i)-(iv) discussed in the next section. The timing weights a_t model the issue (i) of the importance of the future; the tradeoffs function v models the issue (ii) of intertemporal tradeoffs; the first-period utility function u models the issue (iii) of risk in the present; and the multiperiod risk function f models the issue (iv) of multiperiod risk.

3. Preference Issues

This section first identifies a number of issues concerning individual and social preferences that can be modeled by a utility function of the general forms (2)-(4) described in Theorem 1. It then identifies other issues that are excluded by the assumptions of Theorem 1. References are provided to more detailed discussions of each issue.

(i) *The importance of future periods.* This issue is concerned with the dependence of tradeoffs between two future periods s and t on the futurity of the periods s and t.

In most cost-benefit and risk analysis studies, preferences regarding this issue are modeled by present value discounting, that is, by assuming that the timing weights a_t form a geometric sequence, $1, a, a^2, \ldots$, for some constant $0 < a < 1$. Timing weights of this form are implied by several preference conditions, e.g., the stationarity condition in Koopmans (1960), (1972) and the pairwise invariance condition in Keeney and Raiffa (1976, p. 480) which states that tradeoffs between two periods s and t depend only on the (absolute) difference $t - s$ between the periods. More general timing weights are considered, for example, in Williams and Nassar (1966) and Weibull (1986).

Another preference condition, called relative timing preferences, is introduced in Harvey (1986a). This condition states that tradeoffs between two periods s and t depend only on the relative difference t / s between the periods. It implies that the timing weights a_t form an arithmetic sequence, $1,(1/2)^r,(1/3)^r,\ldots,$ for some constant $r > 0$. The term ***relative value discounting*** is used for a model of this type.

Relative value discounting models accord more importance to the distant future, and hence to future generations, than do present value discounting models. This distinction between the two types of models occurs because any arithmetic sequence $(1/t)^r$, $r > 0$, decays more slowly than any geometric sequence a^{t-1}, $0 < a < 1$.

(ii) *Concern for intertemporal equity.* This issue has two equivalent forms. The first form is a concern for equity in the amounts x_t in different periods; typically, there is an aversion toward large differences in the amounts x_t. The second form is a dependence of the tradeoffs between two periods s and t on the base amount, e.g., asset position or social wealth, in one of the periods; for example, there typically is a willingness to incur greater costs in a period s for a benefit in a period t if the base costs in period s are low than if the base costs in period s are already high.

This issue is not included in most cost-benefit studies. Preferences are modeled by assuming that the tradeoffs function v is linear, i.e., $v(x_t) = x_t$. This form of v is implied by an attitude of neutrality toward intertemporal inequity or, equivalently, by the condition that tradeoffs between two periods are independent of the base amount in one of the periods. Preference conditions that do model aversion toward intertemporal inequity and that imply a parametric form for the tradeoffs function v are introduced in Harvey (1986a).

(iii) *Attitude toward risk in the present.* Suppose that (x_t) denotes an infinite-period consequence having the amount x_t in period t and standard amounts x^* in every other period. Then, this issue can be described as a concern for risk in lotteries having consequences of the form (x_1), i.e., lotteries such that all of the uncertainty is in the first period, $t = 1$.

This issue and that of (iv) below are two components of the overall issue of risk in infinite-period planning. Most cost-benefit studies either do not include the risk issues (iii), (iv) or model the overall issue of risk by one of the methods described in the introduction.

An attitude of neutrality toward risk in the present implies that the first-period utility function u in linear, i.e., $u(x_1) = x_1$. A number of authors have introduced preference conditions that model an attitude of aversion toward risk in the present and that imply a parametric form for the function u. References are provided in Sections 4 and 5.

(iv) *Attitude toward multiperiod risk.* Suppose that (x_s, x_t) denotes an infinite-period consequence having the amounts x_s and x_t in two periods s,t and having standard amounts x^* in every other period. One form of the multiperiod risk issue is a concern for joint probabilities as well as marginal probabilities in assessing preferences between infinite-period lotteries. For example, suppose that $<(x_s, x_t), (x_s', x_t')>$ denotes a lottery having the equally likely consequences (x_s, x_t), (x_s', x_t') and $<(x_s, x_t'), (x_s', x_t)>$ denotes a lottery having the equally likely consequences (x_s, x_t'),(x_s', x_t). Even though these lotteries have equal marginal distributions for the period s and for the period t, if $x_s < x_s'$ and $x_t < x_t'$, then the second lottery may be preferred to the first because of an aversion to "catastrophe" if the consequence (x_s, x_t) occurs. References for this issue are provided in Sections 4 and 5.

An attitude of neutrality toward multiperiod risk (for example, indifference between any two lotteries described above) implies that the multiperiod risk function f is linear, i.e., $f(V) = V$. For such a model, the infinite-series value function $V(x_1, x_2, \ldots)$ in Theorem 1 is also a utility function.

In contrast to the issues (i)-(iv) discussed above, the following issues (v)-(x) are excluded by the preference conditions in Theorem 1 and hence cannot be represented by the models to be discussed in this paper.

(v) Dependence of the tradeoffs between two periods on the amounts in the intervening periods. This issue is excluded by condition (C) in Appendix A. Such preferential complementarity is discussed, for example, in Gorman (1968), and models are proposed in Bordley (1985) and Meyer (1977).

(vi) Dependence of the tradeoffs within a period on the base amounts in that period. For a multiattribute decision problem, there may be, for example, a dependence of the pricing-out amounts for the non-monetary attributes on the base amount of the monetary attribute. For a group decision problem, there may be a concern for equity in the consequences to the affected individuals. These issues are excluded by the requirement that the outcomes in a period t can be described by a single, readily interpretable variable, i.e., that multiple attributes can be reduced to a single attribute by "willingness-to-pay" methods, and group consequences can be aggregated by a "sum-of-benefits" method. Multiattribute tradeoffs dependence is discussed, for example, in Harvey (1985a) and Kirkwood and Sarin (1980); interpersonal equity in consequences is discussed, for example, in Atkinson (1970), Barrager (1980), Fleming (1952), and Harvey (1985b).

(vii) Dependence of tradeoffs comparisons on the period. This issue is excluded by condition (E) in Appendix A, which requires that for any two periods s, t and any two pairs of amounts $x^1 < x^2$ and $x^3 < x^4$, if there is a willingness to tradeoff more to increase x_s from x^1 to x^2 than to increase x_s from x^3 to x^4, then there is a willingness to tradeoff more to increase x_t from x^1 to x^2 than to increase x_t from x^3 to x^4. A general planning model that does not assume condition (E) is presented in Harvey (1986a); however, there are no known preference conditions on dependence of tradeoffs comparisons that could be used to structure such a model.

(viii) Attitude toward intertemporal inequity in risks. This issue must be distinguished from the issue (ii) of concern for intertemporal equity in amounts. The present issue can be described as follows. Consider two pairs of amounts $x_s < x_s'$ and $x_t < x_t'$ such that $(x_s) \sim (x_t)$ and $(x_s') \sim (x_t')$. Then, the consequences (x_s, x_t') and (x_s', x_t) are indifferent and have the same inequity in amounts. By the substitution principle, it would follow that the lottery $<(x_s, x_t'), (x_s', x_t)>$ is indifferent to either of the consequences (x_s, x_t'), (x_s', x_t). However, the lottery might be preferred to, for example, the consequence (x_s, x_t') since the lottery presents equal risk in both periods whereas the consequence (x_s, x_t') is unfair to the period s (and thus to the affected individuals in period s).

Such preferences are excluded by the expected utility conditions (G). Inequity in risk for group decision problems is discussed, for example, in Broome (1982), Harvey (1985c), Keeney (1980), Keeney and Winkler (1985), Kirkwood (1979), and Sarin (1985).

(ix) Regret, Allais paradoxes, and framing effects. These issues are excluded by condition (G). Discussions of these issues may be found, for example, in Allais and Hogen (1979), Bell (1983), (1985), Kahneman and Tversky (1979), and Schoemaker (1982). Models that do not assume condition (G) are presented in Chew (1983), Chew and MacCrimmon (1979), Fishburn (1982), (1983), and Machina (1982).

(x) Time resolution of uncertainty. This group of issues is discussed by Mayer (in Keeney and Raiffa 1976, Chapter 9) where several issues are identified, including: (1) changes in personal or social preferences over time, and (2) the time at which uncertainty is resolved.

The preferences of a future society are included in a utility function of Theorem 1 only in that the preferences of current society depend in part on a concern for such future preferences; the planning models in this paper do not include a probability distribution of future preferences. A separate issue is whether the social preferences represented in a specified utility function change with time. For a present value discounting model, the tradeoffs between the years 2050 and 2051 are the same whether the model represents the preferences of current society or the preferences of society in the year 2050. For a relative value discounting model, however, there is more indifference between the years 2050 and 2051 if the model represents current preferences than if the model represents preferences in the year 2050.

The time at which uncertainty is resolved can be included in a planning model by the usual backward induction of decision-tree analysis. However, the issue of anxiety caused by prolonged uncertainty is not included in the planning models in this paper. This issue of "anxiety along the way" is discussed in the work of Meyer cited above.

4. Absolute Multiperiod Risk

This section and the next discuss preference conditions regarding the issue of multiperiod risk. General conditions are introduced in Sections 4.1 and 5.1 to define the attitudes of neutrality, aversion, and proneness toward multiperiod risk, and special conditions are introduced in Section 4.2 and 5.2.

Each general or special preference condition is shown to be equivalent to a condition on the form of the multiperiod risk function f in Theorem 1. In this sense, the function f can be regarded as that part of a utility function $U(x_1,x_2,\ldots)$ which encodes the multiperiod risk attitude for preferences among infinite-period lotteries. Table 1 below lists all of the special conditions on multiattribute risk and the corresponding special forms of the utility function U. The properties within each of parts (I)-(IV) are equivalent.

Table 1: Conditions on Multiperiod Risk and Forms of $U(x_1,x_2,\ldots)$

(I)	(II)
(Absolute) Risk Neutrality (a)	Relative Risk Neutrality (a)
(Absolute) Risk Neutrality (b)	Relative Risk Neutrality (b)
U linear in v	U logarithmic in v
U additive in u	U log-exp in u
(III)	**(IV)**
Absolute Risk Constancy	Relative Risk Constancy
Utility Independence	Timing Independence
U linear or exponential in v	U logarithmic or power in v
U additive or multiplicative in u	U log-exp or power-root in u

Both single-period risk attitudes and multiperiod risk attitudes can be defined by specifying a period t and considering preferences between those lotteries having consequences (x_t) in which the amounts in the periods other than period t are standard amounts x^*. For such lotteries, there is no uncertainty except for the outcome in period t.

For a single-period risk attitude, the consequences (x_t) are measured directly in terms of the amounts x_t in period t. In particular, the tradeoffs between x_t and the amounts in other periods are not considered. For a multiperiod risk attitude, however, the consequences (x_t) are measured in terms of the amounts $v(x_t)$ by considering the tradeoffs between the amounts x_t in period t and the amounts in other periods; then, changes in the period t will be referred to as *changes in value*.

4.1 Absolute multiperiod risk neutrality and aversion

The multiperiod risk attitudes of neutrality, aversion, and proneness may be defined by a number of equivalent conditions concerning changes in value. Two such conditions are specified in this subsection. The following concepts will be needed for the first of these conditions.

Definition 2. Pairs of amounts x_t, x_t' and y_t, y_t' in a period t will be said to have *equal (absolute) changes in value* provided that

$$(x_t, z') \sim (x_t', z) \text{ and } (y_t, z') \sim (y_t', z)$$

for a common pair of amounts z, z' in another period. In particular, an amount $\bar{x}_t$ will be called the *(absolute) tradeoffs midvalue* of two amounts $x_t < x_t'$ provided that the pairs of amounts $x_t, \bar{x}_t$ and $\bar{x}_t, x_t'$ have equal changes in value.

As a type of example, suppose that preferences for intertemporal equity are not included in the model (as is typically the case for cost-benefit models). Then, two pairs of amounts x_t, x_t' and y_t, y_t' have equal changes in value provided that $x_t' = x_t + h$, $y_t' = y_t + h$ for the same change h, and the tradeoffs midvalue of two amounts $x_t < x_t'$ is the ordinary average of x_t and x_t'.

Lemma 2. For a planning model of type (A)-(F), any two amounts $x_t < x_t'$ in any period t have a unique tradeoffs midvalue $\bar{x}_t$. Any two pairs of amounts x, x' and y, y' in the interval I have equal changes in value when they are in a period t if and only if they have equal changes in value when they are in any other period. Thus, an amount $\bar{x}$ is the tradeoffs midvalue of a pair of amounts $x < x'$ when $x, \bar{x}, x'$ are in a period t if and only if $\bar{x}$ is the tradeoffs midvalue of $x < x'$ when $x, \bar{x}, x'$ are in any other period.

Suppose that a single period t has been chosen, and the tradeoffs midvalue $\bar{x}_t$ of a pair of amounts $x_t < x_t'$ has been assessed. Multiperiod risk attitudes can be defined as in parts (a) of Theorem 2 below by comparing the "average" consequence $(\bar{x}_t)$ with the "risky" lottery $<(x_t),(x_t')>$. For example, *multiperiod risk aversion* can be defined as the condition that $(\bar{x}_t)$ is preferred to $<(x_t),(x_t')>$. In applications, it may often be convenient to choose the period t as the first period, $t = 1$.

These definitions are similar to the definitions in Dyer and Sarin (1982) comparing strength of preference midvalues and risk midvalues (i.e., certainty equivalents). The difference to be emphasized is that the above definitions do not require a precise assessment of the certainty equivalent of $<(x_t),(x_t')>$; instead, $(\bar{x}_t)$ and $<(x_t),(x_t')>$ are compared directly.

Multiperiod risk attitudes can also be defined by considering consequences of the form (x_s, x_t) for two periods s and t. Suppose that the periods s and t have been chosen, and two pairs of amounts x_s, x_s' and x_t, x_t' have been assessed such that

$$x_s < x_s',\ x_t < x_t' \text{ and } (x_s) \sim (x_t),\ (x_s') \sim (x_t') \quad . \tag{5}$$

These indifference relations imply that the two infinite-period consequences (x_s, x_t') and (x_s', x_t) are indifferent.

As one type of example, if s and t are chosen to be adjacent periods in the distant future and the future is valued in accord with the conditions of relative timing preferences (so that $a_s / a_t \approx 1$), then x_t and x_t' can be assessed as approximately $x_t = x_s$ and $x_t' = x_s'$.

Multiperiod risk attitudes can be defined as in parts (b) of Theorem 2 by comparing the "average" consequences (x_s, x_t'), (x_s', x_t) with the "risky" lottery $<(x_s, x_t),(x_s', x_t')>$. For example, *multiperiod risk aversion* can be defined as the condition that the consequences (x_s, x_t'), (x_s', x_t) are preferred to the lottery $<(x_s, x_t),(x_s', x_t')>$.

These definitions are similar to the definitions in Richard (1975) of multivariate risk neutral, strictly multivariate risk averse, and strictly multivariate risk seeking. The difference is that in Richard's definitions the consequences (x_s, x_t'), (x_s', x_t) are not assessed to be indifferent, and the lottery $<(x_s, x_t'), (x_s', x_t)>$ is compared with $<(x_s, x_t), (x_s', x_t')>$. In the terminology of Farquhar (1984), the comparisons in Richard's definitions are evaluated by a paired-gamble method while the comparisons in the above definitions are evaluated by a standard-gamble method.

Theorem 2. For a planning model of type (A)-(G), the properties within each of the following parts (I)-(III) are equivalent.

I. *Multiperiod risk neutrality:*

(a) There exists a period t such that for any amounts $x_t < x_t'$ the consequence $(\bar{x}_t)$ is indifferent to the lottery $<(x_t), (x_t')>$.

(b) There exist two periods s and t such that for any amounts as in (5) the consequences $(x_s, x_t') \sim (x_s', x_t)$ are indifferent to the lottery $<(x_s, x_t), (x_s', x_t')>$.

(c) Preferences are represented by a utility function (2) in which the multiperiod risk function f is linear, that is,

$$U(x_1, x_2, \ldots) = v(x_1) + a_2 v(x_2) + \cdots$$

where the tradeoffs function v is to be assessed. Here, v is normalized so that $v(x^*) = 0$.

(d) Preferences are represented by a utility function (4) of the form

$$U(x_1, x_2, \ldots) = u(x_1) + a_2 u(x_2) + \cdots$$

where the first-period utility function u is to be assessed. Here, u is normalized so that $u(x^*) = 0$.

II. *Multiperiod risk aversion:*

(a) There exists a period t such that for any amounts $x_t < x_t'$ the consequence $(\bar{x}_t)$ is preferred to the lottery $<(x_t),(x_t')>$.

(b) There exist two periods s and t such that for any amounts as in (5) the consequences $(x_s, x_t') \sim (x_s', x_t)$ are preferred to the lottery $<(x_s, x_t),(x_s', x_t')>$.

(c) Preferences are represented by a utility function (2) or (4) in which multiperiod risk function f is strictly concave.

III. *Multiperiod risk proneness:*

Properties analogous to II. (a)-(c) can also be specified.

The salient feature of the two sets (a), (b) of conditions on multiperiod risk attitudes discussed in this subsection is that they require tradeoffs assessments. Part of the assessment task is thereby shifted from risk assessments to tradeoffs assessments.

4.2 Absolute multiperiod risk constancy

This section discusses two equivalent conditions on multiperiod risk attitudes that correspond to the condition on single-period risk attitudes of constant risk aversion (Arrow 1971, Pfanzagl 1959, and Pratt 1964).

Definition 3. Preferences will be said to be ***absolute multiperiod risk constant*** for a period t provided that for any three pairs of amounts $w_t < w_t'$, $x_t < x_t'$, and $y_t < y_t'$ that have equal absolute changes in value, if (x_t) is the certainty equivalent of $<(w_t), (y_t)>$, then (x_t') is the certainty equivalent of $<(w_t'), (y_t')>$.

One procedure for verifying this condition is to select amounts $w_t < y_t$ and assess the certainty equivalent (x_t) of $<(w_t),(y_t)>$. Then, consider an amount $x_t' > x_t$, assess two amounts w_t', y_t' such that the pairs $w_t < w_t'$, $x_t < x_t'$, and $y_t < y_t'$ have equal absolute changes in value, and ask whether (x_t') is the certainty equivalent of $<(w_t'),(y_t')>$.

A second condition equivalent to that of absolute multiperiod risk constancy is the condition of utility independence introduced by Pollak (1967) for multiperiod models and by Keeney (1968), (1974) for multiattribute models. Consider those lotteries having consequences of the form (x_t, z) where z is an amount in a fixed period s other than period t. Period t is said to be *utility independent* of the period s provided that $(x_t, z) \succsim \langle (w_t, z), (y_t, z) \rangle$ implies that $(x_t, z') \succsim \langle (w_t, z'), (y_t, z') \rangle$ for any amount z' in period s, that is, the attitude toward risk for period t is independent of the fixed amount z in the period s.

For a finite-period model or a multiattribute model, the equivalence of properties similar to (b) and (c) below is discussed by Meyer and Pratt (in Keeney and Raiffa 1976, p. 330) and in Pollak (1967), and the equivalence of properties similar to (b) and (d) below is discussed in Keeney (1968), (1974) and Meyer (1970).

Theorem 3. For a planning model of type (A)-(G), the following properties are equivalent:

(a) There exists a period t for which preferences are absolute multiperiod risk constant.

(b) There exist periods s and t that are utility independent.

(c) Preferences are represented by a utility function (2) in which the function f is linear or exponential, that is,

$$U(x_1, x_2, \ldots) = \begin{cases} \exp(r(v(x_1) + a_2 v(x_2) + \cdots)), & r > 0 \\ v(x_1) + a_2 v(x_2) + \cdots, & r = 0 \\ -\exp(r(v(x_1) + a_2 v(x_2) + \cdots)), & r < 0 \end{cases}$$

where the tradeoffs function v and the constant r are to be assessed. Here, v is normalized so that $v(x^*) = 0$.

(d) Preferences are represented by a utility function (4) that is additive or multiplicative, that is

$$U(x_1, x_2, \ldots) = \begin{cases} \Pi_{t=1}^{\infty}(a(u(x_t) - 1) + 1)^{a_t} \\ \Sigma_{t=1}^{\infty} a_t u(x_t) \\ -\Pi_{t=1}^{\infty}(-a(u(x_t) + 1) + 1)^{a_t} \end{cases}$$

where the first-period utility function u and the constant $a > 0$ are to be assessed. Here, u is normalized so that: $u(x^*) = 1$ in the first case, $u(x^*) = 0$ in the second case, and $u(x^*) = -1$ in the third case. Moreover, $a(u(x) - 1) + 1 > 0$ for all x in I in the first case, and $a(u(x) + 1) - 1 < 0$ for all x in I in the third case.

Note: As is well known, a utility function in parts (c), (d) is multiperiod risk averse in the third cases and is multiperiod risk prone in the first cases.

5. Relative Multiperiod Risk

This section discusses preference conditions on relative (or proportional) changes in one or more periods. These preference conditions are analogous to the preference conditions discussed in the preceding section on absolute changes. Preference conditions on relative changes appear to be more realistic than preference conditions on absolute changes for many problems, in particular, for those problems in which the degree of risk aversion for one period decreases as the amounts in the other periods become more favorable.

Consider a planning model of type (A)-(G) having a least preferred standard amount x^*. Relative changes will be measured with respect to x^* as the amount having "zero value." For example, an amount x_t^1 will be said to have "one-half the value" of an amount x_t^2 provided that x_t^1 is the tradeoffs midvalue of x^*_t and x_t^2.

Typically, x^* will be less than any amount of x_t that has an appreciable chance of occurring. As an illustration, suppose that x_t describes the amount of consumption of a non-renewable resource (e.g., helium) during period t. Then, x^* might be chosen as the outcome of no consumption or of a specified minimal level of consumption of the resource being studied.

5.1 Relative multiperiod risk neutrality and aversion

This subsection discusses risk attitudes involving relative changes in value that are analogous to the risk attitudes of neutrality, aversion, and proneness discussed in Section 4.1.

Relative changes in value can be defined by means of the concept of a "standard sequence." Krantz et al. (1971) contains an extensive discussion of this idea in a more general setting. An increasing sequence of amounts

$$x_t^0 < x_t^1 < \cdots < x_t^n$$

in a period t will be called a ***standard sequence*** provided that the pairs x_t^{k-1}, x_t^k, $k = 1,\ldots,n$, have equal absolute changes in value (Definition 2).

For a planning model of type (A)-(F), a sequence of amounts is a standard sequence if and only if $v(x_t^k) - v(x_t^{k-1})$ is constant for $k = 1,\ldots,n$. Thus, a sequence of amounts is a standard sequence in a period t if and only if it is a standard sequence in any other period.

Definition 4. Pairs of amounts x_t, x_t' and y_t, y_t' not equal to x^* will be said to have ***equal relative changes in value*** provided that for any two standard sequences

$$x_t^0 < x_t^1 < \cdots < x_t^n \text{ and } y_t^0 < y_t^1 < \cdots < y_t^n$$

with the initial amounts $x_t^0 = y_t^0 = x^*$, if $x_t = x_t^m$ and $y_t = y_t^m$ for some $m < n$, then $x_t' < x_t^n$ if and only if $y_t' < y_t^n$. In particular, an amount $\hat{x}_t$ will be called the ***relative tradeoffs midvalue*** of two amounts $x_t < x_t'$ provided that the pairs of amounts $x_t, \hat{x}_t$ and $\hat{x}_t, x_t'$ have equal relative changes in value.

Lemma 3. For a planning model of type (A)-(F), two pairs of amounts x_t, x_t' and y_t, y_t' have equal relative changes in value if and only if $v(x_t') = pv(x_t)$ and $v(y_t') = pv(y_t)$ for the same constant $p > 0$. The relative tradeoffs midvalue of two amounts $x_t < x_t'$ is the amount $\hat{x}_t$ such that $v(\hat{x}_t) = pv(x_t)$ and $v(x_t') = pv(\hat{x}_t)$ for the same $p > 0$. The statements of Lemma 2 are true with the adjective "relative" added to the terms "equal changes in value" and "tradeoffs midvalue."

Suppose that a period t has been chosen and the relative tradeoffs midvalue $\hat{x}_t$ of two amounts $x_t < x_t'$ has been assessed. Relative multiperiod risk attitude can be defined as in parts (a) of Theorem 4 below by comparing the "average" consequence $(\hat{x}_t)$ with the lottery $<(x_t),(x_t')>$. The situation is analogous to that for absolute multiperiod risk attitudes. In particular, the discussion in Section 4.1 on standard-gamble methods of verification is equally pertinent in the present context.

The condition (a) below of relative multiperiod risk neutrality corresponds to the single-period risk attitude of logarithmic utility (Harvey 1981, 1986b).

Relative multiperiod risk attitudes can also be defined by considering the same amount in several different periods. For technical reasons, we will consider not only actual periods t but also imaginary periods t having arbitrary timing weights, $0 < a_t < 1$. With the non-restrictive assumption that the sequence of actual timing weights, $a_t, t = 1,2,...$, decreases to zero as t increases, these imaginary periods correspond to actual periods for a different choice of the unit of time.

For a period $t > 1$ (actual or imaginary), a period m where $1 < m < t$ will be called the *temporal midvalue* of the first period and period t provided that the tradeoffs between periods 1 and m coincide with the tradeoffs between periods m and t (Harvey 1986a). For example, in a present value discounting model the temporal midvalue is the arithmetic mean, i.e., that period m such that $m = 1 + h$ and $t = m + h$ for the same increase h. In a relative value discounting model the temporal midvalue is the geometric mean, i.e., that period m such that $m = k \cdot 1$ and $t = k \cdot m$ for the same proportional increase k.

Consider an amount x in one of the three periods $1 < m < t$ where m is the temporal midvalue of 1 and t. Relative multiperiod risk attitudes can be defined as in parts (b) of Theorem 4 by comparing the "average" consequence (x_m) with the "risky" lottery $<(x_1),(x_t)>$ where $x = x_1 = x_m = x_t$.

Theorem 4. For a planning model of type (A)-(G) having a least preferred standard amount, the properties within each of the following parts (I)-(III) are equivalent.

I. *Relative multiperiod risk neutrality:*

(a) There exists a period t such that for any amounts $x_t < x_t'$ not equal to x^* the consequence $(\hat{x}_t)$ is indifferent to the lottery $<(x_t),(x_t')>$.

(b) For any period $t > 1$ (actual or imaginary) and any amount x not equal to x^*, the consequence (x_m) is indifferent to the lottery $<(x_1),(x_t)>$ where m is the temporal midvalue of 1 and t, and $x = x_1 = x_m = x_t$.

(c) Preferences are represented by a utility function (2) in which the function f is logarithmic, that is,

$$U(x_1,x_2,\ldots) = \log(v(x_1) + a_2 v(x_2) + \cdots)$$

where the tradeoffs function v is to be assessed. Here, v is normalized so that $v(x^*) = 0$.

(d) Preferences are represented by a utility function (4) of the form

$$U(x_1,x_2,\ldots) = \log(e^{cu(x_1)} + a_2 e^{cu(x_2)} + \cdots)$$

where the first-period utility function u and the constant $c > 0$ are to be assessed. Here, $u(x^*) = -\infty$ and $u(x^+) = +\infty$.

II. *Relative multiperiod risk aversion:*

(a) There exists a period t such that for any amounts $x_t < x_t'$ not equal to x^* the consequence $(\hat{x}_t)$ is preferred to the lottery $<(x_t),(x_t')>$.

(b) For any period $t > 1$ (actual or imaginary) and any amount x not equal to x^*, the consequence (x_m) is preferred to the lottery $<(x_1),(x_t)>$ where m is the temporal midvalue of 1 and t, and $x = x_1 = x_m = x_t$.

(c) Preferences are represented by a utility function (2) or (4) such that the composite function $f \circ \exp$ is strictly concave.

III. *Relative multiperiod risk proneness:*

Properties analogous to II. (a)-(c) can also be specified.

5.2 Relative multiperiod risk constancy

This subsection discusses two equivalent conditions on multiperiod risk attitudes that correspond to the single-period risk attitude of linear risk aversion (Harvey 1981, 1986b) and constant proportional risk aversion (Pratt 1964).

Definition 5. Preferences will be said to be ***relative multiperiod risk constant*** for a period t provided that for any three pairs of amounts $w_t < w_t'$, $x_t < x_t'$, and $y_t < y_t'$ that have equal relative changes in value, if (x_t) is the certainty equivalent of $<(w_t),(y_t)>$, then (x_t') is the certainty equivalent of $<(w_t'),(y_t')>$.

This preference condition is similar to the multiattribute risk condition of proportional utility dependence defined in Harvey (1984). It may be verified by means of the procedure that is described in Section 4.2 for the condition of absolute multiperiod risk constancy; the only modification is that here the pairs $w_t < w_t'$, $x_t < x_t'$ and $y_t < y_t'$ have equal relative changes in value rather than equal absolute changes in value.

A second condition equivalent to that of relative multiperiod risk constancy can be defined as follows. For each period, $t = 1,2,\ldots$, the preference relation $\succsim$ on infinite-period lotteries induces a preference relation $\succsim_t$ on those lotteries having consequences of the type (x_t). For the discussion of utility independence in Section 4.2, the issue was whether the preference relation $\succsim_t$ is the same as a preference relation on lotteries having consequences (x_t, z) where $z \neq x^*$ is a fixed amount in another period. Now, consider the issue of whether the preference relation $\succsim_t$ is the same as the preference relation $\succsim_s$ for another period s. It will be said that the periods are ***timing independent*** provided that for any two periods s and t (actual or imaginary) the preference relations $\succsim_t$ and $\succsim_s$ are the same.

For a multiattribute model, the equivalence of properties similar to (a) and (d) below is discussed in Harvey (1984); for a group decision model, the equivalence of properties similar to (b) and (d) is discussed in Harvey (1985c).

Theorem 5. For a planning model of type (A)-(G) having a least preferred standard amount, the following properties are equivalent:

(a) There exists a period t for which preferences are relative multiperiod risk constant.

(b) The periods are timing independent.

(c) Preferences are represented by a utility function (2) in which the function f is logarithmic or power, that is,

$$U(x_1,x_2,\ldots = \begin{cases} (v(x_1) + a_2 v(x_2) + \cdots)^r, & r > 0 \\ \log(v(x_1) + a_2 v(x_2) + \cdots), & r = 0 \\ -(v(x_1) + a_2 v(x_2) + \cdots)^r, & r < 0 \end{cases}$$

where the tradeoffs function v and the constant r are to be assessed. Here, v is normalized so that $v(x^*) = 0$.

(d) Preferences are represented by a utility function (4) that is log-exponential or power-root,that is,

$$U(x_1,x_2,\ldots) = \begin{cases} (u(x_1)^{1/r} + a_2 u(x_2)^{1/r} + \cdots)^r, & r > 0 \\ \log(e^{cu(x1)} + a_2 e^{cu(x2)} + \cdots), & r = 0 \\ -((-u(x_1))^{1/r} + a_2(-u(x_2))^{1/r} + \cdots)^r, & r < 0 \end{cases}$$

where the first-period utility function u and the constants $c > 0$ and r are to be assessed. Here, u is normalized so that $u(x^*) = 0$ in the first case, and $u(x^+) = 0$ in the third case.

Note: It follows from Theorem 2 and 4 that a utility function in parts (c), (d) is multiperiod risk averse if $r < 1$ and is relative multiperiod risk averse if $r < 0$.

6. Multiperiod Tradeoffs

This section describes four preference conditions on multiperiod tradeoffs. Each condition is equivalent to a special form for the tradeoffs function v and hence to special forms for the utility functions (2) and (3) that contain v. The special forms for (2) are immediate in terms of v whereas those for (3) require the calculation of v^{-1}. These results follow by combining results in Harvey (1985a), (1986a), and hence are only summarized here. The terminology is chosen to be consistent with that in Sections 4, 5 and that in Harvey (1986b).

Consider a pair of base amounts x_s and x_t in two periods s, t where x_s is fixed and the effects of changes in x_t are to be examined. For a change w_s in period s, the corresponding change w_t in period t such that

$$(x_s, x_t + w_t) \sim (x_s + w_s, x_t)$$

is called the *tradeoffs amount* for the change w_s, and is denoted by $w_t = f(w_s; s, t, x_t)$.

If w_t does not depend on the base amount x_t, that is, $w_t = w_t'$ for any $w_t' = f(w_s; s, t, x_t')$, then there is said to be ***absolute tradeoffs independence***. If $w_t - w_t'$ depends only on the difference $x_t - x_t'$ and on the tradeoffs amount w_t, then there is said to be ***absolute tradeoffs constancy***.

Preferences are in accord with the condition of absolute tradeoffs independence if and only if there is a tradeoffs function of the form

$$v(x_t) = x_t - x^*$$

where x^* is finite. Then, the utility function (3) has the special form

$$U(x_1, x_2, \ldots) = u\left((x_1 - x^*) + \Sigma_{t=2}^{\infty} a_t (x_t - x^*) \right) \quad . \tag{6}$$

Preferences are in accord with the condition of absolute tradeoffs constancy if and only if there is a tradeoffs function of the form

$$v(x_t) = \begin{cases} \exp(qx_t) - \exp(qx^*), & q > 0 \\ x_t - x^*, & q = 0 \\ -\exp(qx_t) + \exp(qx^*), & q < 0 \end{cases}$$

where x^* can be $-\infty$ if $q > 0$ and can be $+\infty$ if $q < 0$. Then, the utility function (3) is (6) if $q = 0$ or if $q \neq 0$ is of the special form

$$U(x_1, x_2, \ldots) = u\left(\frac{1}{q} \log[\ \exp(qx_1) + \Sigma_{t=2}^{\infty} a_t (\exp(qx_t) - \exp(qx^*))\ ] \right) \quad . \tag{7}$$

It is also possible to consider relative changes in the amounts x_t, or more generally relative changes in amounts $y_t = x_t + c > 0$ where c is a constant to be specified or evaluated. The amounts y_t can be interpreted either as sums, i.e.,

$x_t + x^0$ where $x^0 = c$ is an "asset position," or as differences, i.e., $x_t - x^0$ where $x^0 = -c$ is a "point of ruin."

Suppose that for a base amount $y_t = x_t + c$ in period t and a change w_s in period s there is a proportion p_t such that

$$(x_s, x_t + p_t y_t) \sim (x_s + w_s, x_t) \quad .$$

Then, p_t is called the ***tradeoffs proportion*** for the change w_s, and is denoted by $p_t = g(w_s; s, t, x_t)$.

If p_t does not depend on the base amount x_t, that is, $p_t = p_t'$ for any $p_t' = g(w_{s;}\, s, t, x_t')$, then there is said to be ***relative tradeoffs independence***. If $p_t - p_t'$ depends only on the ratio y_t / y_t' and on the tradeoffs proportion p_t, then there is said to be ***relative tradeoffs constancy***.

Preferences are in accord with the condition of relative tradeoffs independence if and only if there is a tradeoffs function of the form

$$v(x_t) = \log((x_t + c)/(x^* + c))$$

where $-c < x^* < +\infty$. Then, the utility function (3) has the special form

$$U(x_1, x_2, \ldots) = u\left((x_1 + c)\Pi_{t=2}^{\infty} \left[\frac{x_t + c}{x^* + c}\right]^{a_t} - c \right) \quad . \tag{8}$$

Preferences are in accord with the condition of relative tradeoffs constancy if and only if there is a tradeoffs function of the form

$$v(x_t) = \begin{cases} (x_t + c)^q - (x^* + c)^q \,, & q > 0 \\ \log((x_t + c)/(x^* + c)) \,, & q = 0 \\ -(x_t + c)^q + (x^* + c)^q \,, & q < 0 \end{cases}$$

where x^* can be $-c$ if $q > 0$ and can be $+\infty$ if $q < 0$. Then, the utility function (3) is (8) if $q = 0$ or if $q \neq 0$ is of the special form

$$U(x_1, x_2, \ldots) = u\left(\left[(x_1 + c)^q + \Sigma_{t=2}^{\infty} a_t ((x_t + c)^q - (x^* + c)^q) \right]^{1/q} - c \right) \quad . \tag{9}$$

Each of the above conditions on multiperiod tradeoffs is equivalent to a condition on attitude toward intertemporal equity. For example, there is absolute tradeoffs independence if and only if there is an attitude of ***intertemporal inequity neutrality***, that is, any two indifferent consequences (x_s', x_t) and (x_s, x_t') are indifferent to the "equable" consequence $(\frac{1}{2}(x_s + x_s'), \frac{1}{2}(x_t + x_t'))$.

7. First-Period Risk

This section briefly describes four preference conditions on risk attitudes for consequences (x_1) having an amount x_1 in the first period and standard amounts x^* in the other periods. These conditions are restatements (with new terminology) of the conditions on single-attribute risk attitudes that are referenced in Section 4.2 and 5.2. Each condition is equivalent to a special form for the first-period utility function u, and hence to special forms for the utility functions (3) and (4).

For two amounts x_1 and x_1' in the first period, let $\hat{x}_1$ denote the ***risk midpoint*** of x_1 and x_1' defined by $(\hat{x}_1) \sim \langle (x_1),(x_1') \rangle$. If for any amounts x_1 and x_1' the risk midpoint $\hat{x}_1$ of x_1, x_1' is the average of x_1 and x_1', then there is said to be ***first-period risk neutrality***. If for any (absolute) change h, the risk midpoint of $x_1 + h$ and $x_1' + h$ is $\hat{x}_1 + h$, then there is said to be ***absolute first-period risk constancy***.

There is first-period risk neutrality if and only if $u(x_1) = x_1$ is a first-period utility function. There is first-period risk constancy if and only if there is a first-period utility function of the form

$$u(x_1) = \begin{cases} \exp(rx_1), & r > 0 \\ x_1, & r = 0 \\ -\exp(rx_1), & r < 0 \end{cases}$$

for some amount of the parameter r.

Conditions on first-period risk attitudes can also be defined by considering proportional changes in the amounts $y_1 = x_1 + c > 0$ where c is a constant to be specified or evaluated. If for any amounts x_1 and x_1' the risk midpoint of x_1 and

x_1' is that amount $\hat{x}_1$ such that $\hat{x}_1 + c = k(x_1 + c)$ and $x_1' + c = k(\hat{x}_1 + c)$ for the same proportion k, then there is said to be ***relative first-period risk neutrality.*** If for any relative change k the risk midpoint of $x_1 + k(x_1 + c)$ and $x_1' + k(x_1' + c)$ is $\hat{x}_1 + k(\hat{x}_1 + c)$, then there is said to be ***relative first-period risk constancy.***

There is relative first-period risk neutrality if and only if $u(x_1) = \log(x_1 + c)$ is a first-period utility function. There is relative first-period risk constancy if and only if there is a first-period utility function of the form

$$u(x_1) = \begin{cases} (x_1 + c)^r, & r > 0 \\ \log(x_1 + c), & r = 0 \\ -(x_1 + c)^r, & r < 0 \end{cases}$$

for some amount of the parameter r.

8. Examples of Planning Models

This section specifies a number of planning models that include the following preference issues: (i) the importance of the future, (ii) intertemporal equity, (iii) first-period risk, and (iv) multiperiod risk. Preference conditions from Sections 4-7 are assumed concerning the issues (ii)-(iv). For each of the resulting models, either the preference condition of present value discounting or that of relative discounting can be used. Thus, it is not necessary to discuss here the issue (i) of the importance of the future, even though this issue is included in the models.

One planning model is that in which there is absolute tradeoffs independence, first-period risk neutrality, and multiperiod risk neutrality. Such preferences are represented by the utility function

$$U(x_1, x_2, \ldots) = x_1 + a_2 x_2 + a_3 x_3 + \cdots \tag{10}$$

that is often used in cost-benefit studies and risk analysis studies.

This model may be regarded either as excluding all three of the preference issues (ii)-(iv) or as formulating these issues in the simplest possible manner. From either viewpoint, the model is as simple as possible.

The next simplest models would be those that assume a preference condition other than the above for only one of the issues (ii)-(iv). However, such models are inconsistent according to the following result.

Theorem 6. For a planning model of type (A)-(G), any two of the preference conditions: absolute tradeoffs independence, first-period risk neutrality, and multiperiod risk neutrality, imply the third.

Thus, any model that assumes another condition for one of the issues (ii)-(iv) must assume another condition for at least one other of these issues. First, consider those models which include two of the issues (ii)-(iv) in the sense that only one of the conditions in Theorem 6 is assumed.

Theorem 7. For a planning model of type (A)-(G):

(a) If there is multiperiod risk neutrality, then the attitude toward intertemporal equity coincides with the attitude toward first-period risk, that is, $v = u$.

(b) If there is absolute tradeoffs independence, then the attitude toward multiperiod risk coincides with the attitude toward first-period risk, that is, $f = u$.

(c) If there is first-period risk neutrality, then the attitudes of intertemporal inequity aversion and multiperiod risk aversion are inconsistent.

As an example of part (a), consider a planning model having an intermediate standard amount. The preference conditions of: (1) multiperiod risk neutrality and (2) either relative tradeoffs independence or relative first-period risk neutrality, imply that preferences are represented by the utility function

$$U(x_1,x_2,...) = \log\frac{x_1+c}{x^*+c} + a_2\log\frac{x_2+2}{x^*+c} + \cdots \quad . \tag{11}$$

As an example of part (b), consider a planning model having a least preferred standard amount. The preference conditions of: (1) absolute tradeoffs independence and (2) either relative multiperiod risk neutrality or relative first-period risk neutrality with respect to $y_t = x_t - x^*$ imply that preferences are represented by the utility function

$$U(x_1,x_2,...) = \log(\,(x_1 - x^*) + a_2(x_2 - x^*) + \cdots\,) \quad . \tag{12}$$

Next consider those planning models which include all three of the preference issues (ii)-(iv) in the sense that none of the three preference conditions in Theorem 6 are assumed.

As an example of such a planning model having an intermediate standard amount, consider the following conditions:

(1) There is absolute multiperiod risk constancy (or utility independence), and there is multiperiod risk aversion.

(2) There is relative tradeoffs independence with respect to $y_t = x_t + c$.

(3) There is relative first-period risk constancy with respect to $y_t = x_t + c$, and there is relative first-period risk aversion.

Theorem 8. Any two of the above conditions (1)-(3) imply the third. Preferences satisfy the conditions (1)-(3) if and only if preferences are represented for some $r < 0$ by a utility function of the form

$$U(x_1, x_2, \ldots) = -\exp(r[\log\frac{x_1+c}{x^*+c} + a_2\log\frac{x_2+c}{x^*+c} + \cdots]) \quad . \tag{13}$$

Note: Here, the attitude toward risk in a period t is represented by the utility function $u_t(x_t) = -(x_t + c)^{ra_t}, r < 0$. Thus, the degree of risk aversion is less for periods in the more distant future.

As an example of a planning model having a least preferred standard amount such that all three of the issues (ii)-(iv) are included, consider the following conditions:

(1) There is relative multiperiod risk neutrality.

(2) There is relative tradeoffs constancy with respect to $y_t = x_t - x^*$.

(3) There is relative first-period risk neutrality.

Theorem 9. Any two of the above conditions (1)-(3) imply the third. Preferences satisfy the conditions (1)-(3) if and only if preferences are represented for some $q > 0$ by a utility function of the form

$$U(x_1, x_2, \ldots) = \log((x_1 - x^*)^q + a_2(x_2 - x^*)^q + \cdots) \quad . \tag{14}$$

Note: Here, the periods are timing independent (Section 5.2), and $u(x_t) = \log(x_t - x^*)$ is a utility function for any period t conditional on $x_s = x^*$ for all $s \neq t$. The degree of risk aversion in a period t is less for fixed amounts in the other periods that are more preferred. There is an attitude of intertemporal inequity aversion if and only if $0 < q < 1$.

Appendix A: General Planning Models

This appendix lists the preference conditions for the expected-utility planning model in Theorem 1 and provides a proof of that result. The definition of the set C^* and the conditions (A)-(E) are as in Harvey (1986a). Conditions (F), (G) are added in the present paper.

The preference relation $\succsim$ defined on the set L of infinite-period lotteries is assumed to be a partial order, i.e., reflexive and transitive, and to be strictly increasing in that larger amounts x_t in each period, $t = 1,2,\ldots$, are preferred.

The following definition specifies that subset of infinite-period consequences on which tradeoffs between the periods are considered.

Definition A1. The set C^* consists of those infinite-period consequences $c = (x_1, x_2, \ldots)$ in C such that:

(a) For any $x_1' > x_1$, there exists an N such that

$$c_n^+ = (x_1', x_2, \ldots, x_n, x^*_{n+1}, x^*_{n+2}, \ldots) \succsim c \quad \text{for all } n \geqq N \ .$$

(b) For any $x_1' < x_1$, there exists an N such that

$$c_n^- = (x_1', x_2, \ldots, x_n, x^*_{n+1}, x^*_{n+2}, \ldots) \succsim c \text{ for all } n \geqq N \ .$$

The tradeoffs conditions (A)-(E) below are used to establish the value function (1) for infinite-period consequences in the set C^*.

(A) The preference relation $\succsim$ is a complete ordering on the set C^*.

(B) The preference relation $\succsim$ is continuous on C^* in each variable.

(C) Each pair of adjacent variables x_t, x_{t+1}, $t = 1,2,\ldots$, is preferentially independent on C^* of the other variables.

(D) Each consequence $c = (x_1, x_2, \ldots)$ in C that satisfies the following conditions is in the set C^*.

(a) For any $x'_1 > x_1$, there exists an N such that

$$c_m^+ \gtrsim c_n = (x_1, \ldots, x_n, x^*_{n+1}, x^*_{n+2}, \ldots) \text{ for all } m, n \geqq N \quad .$$

(b) For any $x'_1 < x_1$, there exists an N such that

$$c_m^- \gtrsim c_n = (x_1, \ldots, x_n, x^*_{n+1}, x^*_{n+2}, \ldots) \text{ for all } m, n \geqq N \quad .$$

(E) Any two periods s, t have equal tradeoffs comparisons, i.e., for any two pairs of amounts $x^1 < x^2$ and $x^3 < x^4$ in I, if society is willing to tradeoff more to increase x_s from x^1 to x^2 than to increase x_s from x^3 to x^4 in period s, then society is willing to tradeoff more to increase x_t from x^1 to x^2 than to increase x_t from x^3 to x^4 in period t.

Lemma A1. A planning model satisfies the conditions (A)-(F) on tradeoffs if and only if the preference relation $\gtrsim$ restricted to the set C^* of consequences is represented by a value function (1) as described in Lemma 1 where for an intermediate amount x^* the range of the tradeoffs function v is $(-\infty, \infty)$, for a least preferred amount x^* the range of v is $[0, \infty)$, and for a most preferred amount x^* the range of v is $(-\infty, 0]$.

Proof. For a model having an intermediate standard amount, it is shown in Harvey (1986a) that conditions (A)-(E) are equivalent to the existence of a value function as described in Lemma 1. For a model having an extreme standard amount, a similar argument can be used to establish this equivalence. Thus, for each type of planning model, it remains to show that condition (F) is equivalent to the property that the range of v is the appropriate interval $(-\infty, \infty)$, $[0, \infty)$, or $(-\infty, 0]$.

For a model having an intermediate standard amount, there exists amounts $x > x^*$ and $x < x^*$ in the open interval I. Condition (F) implies that for any consequence $c = (x_1, x_2, x^*_3, x^*_4, \ldots)$ there exists an x'_1 in I such that $(x'_1) \sim c$, that is, $v(x'_1) = v(x_1) + a_2 v(x_2)$. Choosing a fixed $x_2 < x^*$ or $x_2 > x^*$, it follows that the range of v is unbounded above and below. Thus, since v is continuous the range of v is $(-\infty, \infty)$. Conversely, if the range of v is $(-\infty, \infty)$, then for any consequence $c = (x_1, x_2, \ldots)$ with a convergent value function $V(c)$ there exists a consequence

(x'_1) with $v(x'_1) = V(c)$, and thus condition (F) is satisfied.

For a model having an extreme standard amount, there exists amounts $x > x^*$ or $x < x^*$ respectively in the half-open interval I. By an argument similar to that above, condition (F) is satisfied if and only if the range of v is $[0,\infty)$ or $(-\infty,0]$.

Proof of Theorem 1. First, consider a model having an intermediate standard amount. Assume conditions (A)-(G). Then, there is defined on the set C^* a value function $V(x_1,x_2,\ldots)$ as in Lemma 1 with $v(I) = (-\infty,\infty)$ and a utility function $U(x_1,x_2,\ldots)$. It follows that $U(x_1,x_2,\ldots) = f(V(x_1,x_2,\ldots))$ for some strictly increasing function f defined on $(-\infty,\infty)$. The second part of condition (G), i.e., that every lottery has a certainty equivalent, implies that the range of f is an interval. Since f is increasing, it follows that f is continuous. Hence, U is of the form (2).

Suppose that $u(x_1)$ denotes the conditional first-period utility function defined by $u(x_1) = U(x_1,x^*_2,x^*_3,\ldots)$. For any consequence $c = (x_1,x_2,\ldots)$ in C^*, $v^{-1}(V(c))$ is a well-defined amount in I and $c \sim (v^{-1}(V(c)),\ x^*_2,x^*_3,\ldots)$. Therefore, $U(c) = U(v^{-1}(V(c)),x^*_2,x^*_3,\ldots) = u \circ v_{-1}(V(c))$, and thus U is also of the form (3). For any consequence $c = (x_1,x^*_2,x^*_3,\ldots)$, $u(x_1) = U(c) = f(v(x_1))$. Therefore, $v(x) = f^{-1} \circ u(x)$ for all x in I, and thus U is of the form (4).

Conversely, if there exists a value function V and a utility function U as described in Theorem 1, then conditions (A)-(E) follow from the value function and condition (G) follows from the utility function. Therefore, it remains to infer condition (F). The form (3) of U implies that the range of V is contained in the domain of v^{-1}. However, the domain of v^{-1} is by definition the range of v. Thus, for any consequence $(x_1,x_2,\ldots)$ in C^* there exist an amount x'_1 in I such that $V(x_1,x^*_2,x^*_3,\ldots) = v(x'_1)$, and condition (F) is established.

Second, consider a model having an extreme standard amount. Most of the arguments are similar to those for a model having an intermediate standard amount, and thus it suffices to be brief except for points of difference. Assume conditions (A)-(G). Then, there is defined on the set C^* a value function V with $v(I) = [0,\infty)$ or $v(I) = (-\infty,0]$, and there is defined on the set $C^* - \{(x^*_1,x^*_2,\ldots)\}$ a utility function U. It follows that $U = f \circ V$ for some continuous, strictly increasing function f defined on $(0,\infty)$ or $(-\infty,0)$. Thus, U is of the form (2). The forms (3) and (4) now follow with the first-period utility function $u(x_1) = U(x_1, x^*_2, x^*_3, \ldots)$ defined on the interior of I. The converse arguments are similar to those for a model having an intermediate standard amount.

Appendix B: Proofs of Results on Preference Conditions

This appendix contains the proofs of Lemma 2, 3 and Theorems 2-5 in Sections 4, 5, and Theorems 6-9 in Section 8. Some of these proofs use the following implications for a function g that is defined and continuous on an interval: $g(\frac{1}{2}x_1 + \frac{1}{2}x_2) = \frac{1}{2}g(x_1) + \frac{1}{2}g(x_2)$ for all x_1, x_2 if and only if g is linear, $g(\frac{1}{2}x_1 + \frac{1}{2}x_2) > \frac{1}{2}g(x_1) + \frac{1}{2}g(x_2)$ for all $x_1 \neq x_2$ if and only if g is strictly concave (see, e.g., Hardy et al. 1934), and $g(x_1) = \frac{1}{2}g(x_2) + \frac{1}{2}g(x_3)$ implies $g(x_1 + h) = \frac{1}{2}g(x_2 + h) + \frac{1}{2}g(x_3 + h)$ for all $x_1 + h, \ldots, x_3 + h$ if and only if g is linear or exponential (see, e.g., Pfanzagl 1959).

Proof of Lemma 2. By Lemma A1, the range of the tradeoffs function x is $(-\infty,\infty)$, $[0,\infty)$, or $(-\infty,0]$. Thus, for two amounts $x_t < x_t'$ in a period t, the amount $\bar{x}_t$ defined by

$$v(\bar{x}_t) = \frac{1}{2}v(x_t) + \frac{1}{2}v(x_t')$$

is the tradeoffs midvalue of x_t, x_t' since for another period s there exist amounts $z < z'$ such that, for example, $a_s v(z') - a_s v(z) = a_t v(\bar{x}_t) - a_t v(x_t)$. Moreover, the tradeoffs midvalue of x_t, x_t' is unique since v is strictly increasing. The remaining parts of Lemma 2 may be established by similar arguments.

Proof of Theorem 2. Conditions (A)-(G) imply by Theorem 1 that preferences can be represented by utility functions of the forms (2) and (4). First, it will be shown that each of the preference conditions (a), (b) in parts I, II, is equivalent to the corresponding property (c) of the multiperiod risk function f in (2).

Consider condition (a). For a pair of amounts $x_t < x_t'$ in period t, let $v^0 = a_t v(x_t), v^1 = a_t v(x_t')$ and $\bar{v} = a_t v(\bar{x}_t)$. Then, $\bar{v} = \frac{1}{2}v^0 + \frac{1}{2}v^1$. Thus, condition (I.a) is equivalent to $f(\frac{1}{2}v^0 + \frac{1}{2}v^1) = \frac{1}{2}f(v^0) + \frac{1}{2}f(v^1)$ for all $v^0 < v^1$ in the interval $v(I)$ which is equivalent to f being linear on $v(I)$. Similarly, (II.a) is equivalent to f being strictly concave on $v(I)$.

Consider condition (b). For pairs of amounts x_s, x_s' and x_t, x_t' as described, let $v^0 = a_s v(x_s) = a_t v(x_t)$ and $v^1 = a_s v(x_s') = a_t v(x_t')$. Then, condition (I.b) is equivalent to $f(v^0 + v^1) = \frac{1}{2}f(2v^0) + \frac{1}{2}f(2v^1)$ for all $v^0 < v^1$ in $v(I)$ which is equivalent to f being linear on $v(I)$. Similarly, (II.b) is equivalent to f being strictly concave on $v(I)$.

Next, it will be shown that property (I.c) is equivalent to property (I.d). If there exists an additive utility function $U(x_1, x_2, \ldots)$ as in (I.c), then $u(x_1) = U(x_1, x^*_2, x^*_3, \ldots)$ equals $v(x_1)$, and thus there exists an additive utility function as in (I.d). The converse argument is similar.

Proof of Theorem 3. First, it will be shown that each of the conditions (a), (b) is equivalent to property (c) of the multiperiod risk function f.

Consider condition (a). Let $v_w = a_t v(w_t), \ldots, v_y' = a_t v(y_t')$. Then, $v_w' = v_w + h$, $v_x' = v_x + h$, and $v_y' = v_y + h$ for some constant h. There is absolute multiperiod risk constancy if and only if $f(v_x) = \frac{1}{2}f(v_w) + \frac{1}{2}f(v_y)$ implies $f(v_x + h) = \frac{1}{2}f(v_w + h) + \frac{1}{2}f(v_y + h)$. This implication is satisfied if and only if f is as described in (c).

Consider condition (b). For tradeoffs amounts z, z' in a period s, let $h = a_s v(z)$ and $h' = a_s v(z')$. Period t is utility independent of the period s if and only if $f(v_x + h) \gtreqq \frac{1}{2}f(v_w + h) + \frac{1}{2}f(v_y + h)$ implies $f(v_x + h') \gtreqq \frac{1}{2}f(v_w + h') + \frac{1}{2}(v_y + h')$ for any h'. This implication is satisfied if and only if f is as described in (c).

Next it will be shown that (c) is equivalent to (d) with the cases $r > 0$, $r = 0$, and $r < 0$ in (c) corresponding to the three cases in (d). The case $r = 0$ is already treated in Theorem 2.

Proof of Theorem 4. First, it will be shown that each of the preference conditions (a), (b) in parts I, II is equivalent to the corresponding property (c) of the multiperiod risk function f in (2).

Consider condition (a). For a pair of amounts $x_t < x_t'$ not equal to x^* let $v^0 = a_t v(x_t)$, $v^1 = a_t v(x_t')$, and $\hat{v} = a_t v(\hat{x}_t)$. Then, $\hat{v} = (v^0 v^1)^{1/2}$. Thus, condition (I.a) is equivalent to $f((v^0 v^1)^{1/2}) = \frac{1}{2} f(v^0) + \frac{1}{2} f(v^1)$ for all $v^0 < v^1$ in $v(I) = (0,\infty)$ which is equivalent to f being logarithmic on $v(I)$. Similarly, condition (II.a) is equivalent to $f \circ \exp$ being strictly concave on the interval $(-\infty,\infty)$.

Consider condition (b). For a period t (actual or imaginary) with an arbitrary timing weight, $0 < a_t < 1$, and an amount $x > x^*$, let $v^1 = v(x_1)$, $v^t = a_t v(x_t)$, and $\hat{v} = a_m v(x_m)$ where $x = x_1 = x_m = x_t$. Any two numbers, $v^1 > v^t > 0$, can be represented in this manner. Moreover, $\hat{v} = (v^1 v^t)^{1/2}$. Thus, condition (I.b) is equivalent to $f((v^1 v^t)^{1/2}) = \frac{1}{2} f(v^1) + \frac{1}{2} f(v^t)$ for all $v^1 > v^t$ in $(0,\infty)$ which is equivalent to f being logarithmic. Similarly, condition (II.b) is equivalent to $f \circ \exp$ being strictly concave on the interval $(-\infty,\infty)$.

Next, it will be shown that condition (I.c) is equivalent to condition (I.d). If there exists a utility function $U(x_1,x_2,\ldots)$ as in (I.c), then the function $\hat{u}(x) = \log v(x)$ is a first-period utility function with $\hat{u}(x^*) = -\infty$ and $\hat{u}(x^+) = +\infty$. Suppose that $u(x)$ is an assessed first-period utility function. Then, $\hat{u}(x) = cu(x) + b$ some constants $c > 0$ and b, and thus by substitution,

$$U(x_1,x_2,\ldots) = \log(e^{cu(x_1)} + a_2 e^{cu(x_2)} + \cdots) + b \quad .$$

Conversely, if there exists a utility function $U(x_{1},_{2},\ldots)$ as in (I.d), then $\hat{v}(x) = \exp(cu(x))$ is a tradeoffs function with $\hat{v}(x^*) = 0$ and $\hat{v}(x^+) = +\infty$. Suppose that v is an assessed tradeoffs function with $v(x^*) = 0$. Then, $\hat{v}(x) = av(x)$ for some constant $a > 0$, and thus by substitution

$$U(x_1,x_2,\ldots) = \log(v(x_1) + a_2 v(x_2) + \cdots) + \log a \quad .$$

Proof of Theorem 5. First, it will be shown that each of the conditions (a), (b) is equivalent to property (c) of the multiperiod risk function f.

Consider condition (a). Let $v_w = a_t v(w_t),\ldots,v_y' = a_t v(y_t')$. Then $v_w' = pv_w, v_x' = pv_x$, and $v_y' = pv(v_y)$ for the same constant $p > 0$. There is relative multiperiod risk constancy if and only if $f(v_x) = \frac{1}{2}f(v_w) + \frac{1}{2}f(v_y)$ implies $f(pv_x) = \frac{1}{2}f(pv_w) + \frac{1}{2}f(pv_y)$. This implication is satisfied if and only if f is as described in (c).

Consider condition (b). For two actual or imaginary periods s and t, let $p = a_t$ and $p' = a_s$. Compare a consequence (x) and a lottery $<(w), (y)>$ in periods t and s. The periods are timing independent if and only if $f(pv_x) \gtreqqless \frac{1}{2}f(pv_w) + \frac{1}{2}f(pv_y)$ implies $f(p'v_x) \gtreqqless \frac{1}{2}f(p'v_w) + \frac{1}{2}f(p'v_y)$ for all positive p, p', v_w, v_x, v_y. Defining g by $f(v) = g(\log v)$, it follows that the multiplications $pv_x,\ldots,p'v_y$ can be replaced by the additions $\log p + \log v_x,\ldots,\log p' + \log v_y$. This modified implication is satisfied if and only if, with a positive linear transformation, g is of the form $g(z) = e^{rz}, r > 0$; $z, r = 0$; or $-e^{rz}, r < 0$. Thus, the implication involving f is satisfied if and only if f is as described in (c).

Next, it will be shown that (c) is equivalent to (d) with the cases $r > 0$, $r = 0$, and $r < 0$ in (c) corresponding to the three cases in (d). The case $r = 0$ is already treated in Theorem 4.

Assume that there exists a utility function $U(x_1,x_2,\ldots)$ as in (c) with $r > 0$. Then the function $\hat{u}(x) = (v(x))^r$ is a first-period utility function with non-negative values and $\hat{u}(x^*) = 0$. Moreover,

$$U(x_1,x_2,\ldots) = (u(x_1)^{1/r} + a_2 u(x_2)^{1/r} + \cdots)^r \quad .$$

Suppose that u is an assessed first-period utility function with $u(x^*) = 0$. Then, $\hat{u}(x) = au(x)$ for some constant $a > 0$, and thus by substitution into the above formula for $U(x_1,x_2,\ldots)$ the first case of (d) follows. The argument that (c) with $r < 0$ implies the third case of (d) is similar.

Conversely, assume that there is a utility function $U(x_1,x_2,\ldots)$ as in the first case of (d). Then, $\hat{v}(x) = (u(x))^r$ is defined for all x in $I = [x^*,x^+)$, and $v(x^*) = 0$. Moreover,

$$U(x_1,x_2,\ldots) = (\hat{v}(x_1) + a_2\hat{v}(x_2) + \cdots)^r \quad .$$

Therefore, $\hat{v}$ is a tradeoffs function. Suppose that v is an assessed tradeoffs function with $v(x^*) = 0$. Then, $\hat{v}(x) = \alpha v(x)$ for some constant $\alpha > 0$. By substitution into the above formula for $U(x_1,x_2,...)$, the case (c) with $r > 0$ follows. The argument that the third case of (d) implies the third case of (c) is similar.

Proof of Theorem 6. The functions f, v, and u in Theorem 1 are related by $U = f \circ v$. Thus, if any two of these functions are linear, then so is the third.

Proof of Theorem 7. If the multiperiod risk function f is linear, then except for a positive linear transformation $u = v$. If the tradeoffs function v is linear, then except for a positive linear transformation $u = f$. if the single-period risk function u is linear, then the two strictly increasing functions f and v cannot both be strictly concave.

Proof of Theorem 8. The preference conditions (1)-(3) are equivalent to $f(V) = -e^{rV}$ with $r < 0, v(x) = \log(x + c)$, and $u(x) = -(x + c)^q$ with $q < 0$, respectively (where f, v, and u are each determined up to a positive linear transformation). It may be verified that since $u = f \circ v$ any two of these forms implies the third and that all three of these forms are equivalent to the form (13) of $U(x_1,x_2,...)$.

Proof of Theorem 9. The preference conditions (1)-(3) are equivalent to $f(V) = \log V, v(x) = (x - x^*)^q$ with $q > 0$, and $u(x) = \log(x - x^*)$, respectively. Here, the forms $v(x) = \log(x - x^*)$ and $v(x) = -(x - x^*)^q$ with $q < 0$ are excluded since it must be possible to normalize v so that $v(x^*) = 0$. It may be verified that since $u = f \circ v$ any two of these forms implies the third and that all three of these forms are equivalent to the form (14) of $U(x_1,x_2,...)$.

References

Allais, M. and O. Hagen (Eds.). 1979. *Expected Utility Hypotheses and the Allais Paradox*. Reidel, Dordrecht, Holland.

Arrow, K.J. 1971. *Essays in the Theory of Risk-Bearing*. Markham Publishing Company, Chicago.

Atkinson, A.B. 1970. On the Measurement of Inequality. *Journal of Economic Theory* 2, 244-263.

Barrager, S.M. 1980. Assessment of Simple Joint Time/Risk Preference Functions. *Management Science* 26, 620-632.

Bell, D.E. 1974. Evaluating Time Streams of Income. *Omega* 2, 691-699.

Bell, D.E. 1977. A Utility Function for Time Streams Having Inter-Period Dependencies. *Operations Research* 25, 448-458.

Bell, D.E. 1983. Risk Premiums for Decision Regret. *Management Science* 29, 1156-1166.

Bell, D.E. 1985. Putting a Premium on Regret. *Management Science* 31, 117-120.

Bodily, S.E. 1981. Stability, Equality, Balance, and Multivariate Risk. In *Organizations: Multiple Agents with Multiple Criteria*, J.N. Morse (Ed.). Springer-Verlag, New York.

Bordley, R.F. 1985. Intertemporal Utility Functions and Tversky's Objections. Working Paper, General Motors Research Labs, Warren, Mich.

Broome, J. 1982. Equity in Risk Bearing. *Operations Research* 30, 412-414.

Chew, S.H. 1983. A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox. *Econometrica* 51, 1065-1092.

Chew, S.H. and K.R. MacCrimmon. 1979. Alpha-Nu Choice Theory: A Generalization of Expected Utility Theory. Working Paper, University of British Columbia, Vancouver, British Columbia, Canada.

Dyer, J.S. and R.K. Sarin. 1982. Relative Risk Aversion. *Management Science* 28, 875-886.

Farquhar, P.H. 1984. Utility Assessment Methods. *Management Science* 30, 1283-1300.

Fishburn, P.C. 1965a. Independence in Utility Theory with Whole Product Sets. *Operations Research* 13, 28-45.

Fishburn, P.C. 1965b. Markovian Dependence in Utility Theory with Whole Product Sets. *Operations research* 13, 238-257.

Fishburn, P.C. 1968. Utility Theory. *Management Science* 14, 335-378.

Fishburn, P.C. 1970. *Utility Theory for Decision Making.* Wiley, New York.

Fishburn, P.C. 1982. Nontransitive Measurable Utility. *J. Math. Psychology* 26, 31-67.

Fishburn, P.C. 1983. Transitive Measurable Utility. *J. Economic Theory* 31, 293-317.

Fishburn, P.C. and A. Rubinstein. 1982. Time Preference. *International Economic Review* 23, 677-694.

Fleming, M. 1952. A Cardinal Concept of Welfare. *Quarterly J. of Economics* 66, 366-384.

Gorman, W.M. 1968. The Structure of Utility Functions. *Review of Economic Studies* 35, 367-390.

Hardy, G.E., J.E. Littlewood, and G. Polya, 1934. *Inequalities.* Cambridge University Press, Cambridge, U.K.

Harvey, C.M. 1981. Conditions on Risk Attitude for a Single Attribute. *Management Science* 27, 190-203. (Erratum in Vol. 30, No. 12, p. 1548.)

Harvey, C.M. 1984. Assessment of Risk Attitudes by a Utility Dependence Condition. *Operations Research* 32, 457-464.

Harvey, C.M. 1985a. Assessment of Preferences by Conditions on Pricing-Out Amounts. *Operations Research* 33, 443-454.

Harvey, C.M. 1985b. Decision Analysis Models for Social Attitudes Toward Inequity. *Management Science* 31, 1199-1212.

Harvey, C.M. 1985c. Preference Functions for Catastrophe and Risk Inequity. *Large Scale Systems* 8, 131-146.

Harvey, C.M. 1986a. Value Functions for Infinite-Period Planning. *Management Science* 32, 1123-1139.

Harvey, C.M. 1986b. Special Conditions on Risk Attitudes. Working Paper, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Herstein, I.N. and J. Milnor. 1953. An Axiomatic Approach to Measurable Utility. *Econometrica* 21, 291-297.

Kahneman D. and A. Tversky. 1979. Prospect Theory: An Analysis of Decision Under Risk. *Econometrica* 47, 263-291.

Keeney, R.L. 1968. Quasi-Separable Utility Functions. *Naval Research Logistics Quarterly* 15, 551-565.

Keeney, R.L. 1974. Multiplicative Utility Functions. *Operations Research* 22, 22-34.

Keeney, R.L. 1980. Equity and Public Risk. *Operations Research* 28, 527-534.

Keeney, R.L. and H. Raiffa. 1976. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Wiley, New York.

Keeney, R.L. and R.L. Winkler. 1985. Evaluating Decision Strategies for Equity of Public Risks. *Operations Research* 33, 955-970.

Kirkwood, C.W. 1979. Pareto Optimality and Equity in Social Decision Analysis. *IEEE Transactions on Systems, Man, and Cybernetics* 9, 89-91.

Kirkwood, C.W. and R.K. Sarin. 1980. Preference Conditions for Multiattribute Value Functions. *Operations Research* 28, 225-232.

Koopmans, T.C. 1960. Stationary Ordinal Utility and Impatience. *Econometrica* 28, 287-309.

Koopmans, T.C. 1972. Representation of Preference Orderings over Time. *Decision and Organization*, C.B. McGuire and R. Radner (Eds.). North-Holland, Amsterdam.

Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky. 1971. *Foundations of Measurement*, Vol. 1. Academic Press, New York.

Machina, M.J. 1982. "Expected Utility" Analysis without the Independence Axiom. *Econometrica* 50, 277-323.

Markowitz, H.M. 1959. *Portfolio Selection: Efficient Diversification of Investments*. John Wiley, New York.

Meyer, R.F. 1970. On the Relationship among the Utility of Assets, the Utility of Consumption, and Investment Strategy in an Uncertain, but Time Invariant World. In *OR 69: Proceedings of the Fifth International Conference on Operational Research*, J. Lawrence (Ed.). Tavistock Publications, London.

Meyer, R.F. 1972. Some Notes on Discrete Multivariate Utility. Working Paper, Harvard Business School, Boston.

Meyer, R.F. 1977. State Dependent Time Preference. In *Conflicting Objectives in Decisions*, D.E. Bell, R.L. Keeney and H. Raiffa (Eds.) Wiley (Interscience), New York.

Mishan, E.J. 1976. *Cost-Benefit Analysis*. Praeger Publishers, New York.

Pfanzagl, J. 1959. A General Theory of Measurement Applications to Utility. *Naval Research Logistics Quarterly* 6, 283-294.

Pollak, R.A. 1967. Additive Von Neumann-Morgenstern Utility Functions. *Econometrica* 35, 485-494.

Pratt, J.W. 1964. Risk Aversion in the Small and in the Large. *Econometrica* 32, 122-136.

Richard, S.F. 1975. Multivariate Risk Aversion, Utility Independence and Separable Utility Functions. *Management Science* 22, 12-21.

Schoemaker, P.J.H. 1982. The Expected Utility Model: Its Variants, Purposes, Evidence, and Limitations. *J. Economic Literature* 20, 529-563.

Sarin, R.K. 1985. Measuring Equity in Public Risk. *Operations Research* 33, 210-217.

Sugden, R. and A. Williams. 1978. *The Principles of Practical Cost-Benefit Analysis*. Oxford University Press, Oxford.

Weibull, J.W. 1986. Discounted-Value Representations of Temporal Preferences. *Mathematics of Operations Research* 10, 244-250.

Williams, A.C. and J.I. Nassar. 1966. Financial Measurement of Capital Investments. *Management Science* 12, 851-864.