CONNECTIVITY AND STABILITY IN ECOLOGICAL AND ENERGY SYSTEMS

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1. Introduction

One of the more enduring topics of methodological interest at IIASA has been the problem of ascertaining and describing stability characteristics for large-scale systems. These points have been of particular applied interest in the ecology and energy areas where the terms "resilience" and "hypotheticality" have been used to intuitively characterize the type of stability of greatest practical interest [1,2].

Our primary purpose in this note is to present some new results in stability theory which have great relevance to the aforementioned studies. These results deal with the problem of "connective" stability, in which the basic question is how large a perturbation in structure the system can withstand and still remain asymptotically stable. In many ways, these results are similar in spirit to structural stability questions in which the invariance of the topological features of the system trajectory is the central issue. However, the two theories are not the same as connective stability deals with the stability of a *point* under structural perturbation, while structural stability is concerned with *trajectories*. In addition, connective stability is a quantitative theory as precise numerical estimates can be given for the magnitude of the allowed perturbation, while structural stability is primarily qualitative. Therefore, we feel justified in presenting these results in order to provide systems analysts with another tool to probe the stability characteristics of applied systems.

A secondary objective of this note is to point out the connections between the notion of connective stability as defined in [3] and the idea of a system's connectivity pattern as discussed in [4,5]. All of these results will be illustrated with examples from energy and ecology.

2. Connective Stability

In this section, we briefly review the major conclusions of the important paper [3]. We consider a dynamical process described by the equation

$$\dot{x} = A(t,x)x \quad , \qquad x(0) = x_0 \quad , \tag{1}$$

where x is an n-vector, while A is continuous on $(0,\infty) \times R^n$.

To consider the connective aspects of (1), write the elements a_{ij} of the matrix A in (1) as

$$a_{ij}(t,x) = -\delta_{ij}\psi_i(t,x) + e_{ij}\psi_{ij}(t,x) \quad , \tag{2}$$

where δ_{ij} is the Kronecker symbol and the ψ_i, ψ_{ij}, are continous on $(0,\infty) \times R^n$. In (2), the elements e_{ij} are the components of the $n \times n$ *connection* matrix E and are such that

$$e_{ij} = \begin{cases} 1, \text{ if the variable } x_j \text{ influences } \dot{x}_i \\ 0, \text{ otherwise} \end{cases} .$$

The notion of connective stability is then given by

Definition 1. The equilibrium state $x = 0$ of (1) is *connectively asymptotically stable* in the large if and only if it is asymptotically stable in the large for all interconnection matrices E.

To establish conditions for connective asymptotic stability, we clearly need to impose some constraints on the function ψ_i and ψ_{ij}. Assume that there exist numbers $\alpha_i > 0$, $\alpha_{ij} \geq 0$ such that $\alpha_i > \alpha_{ii}$ and

$$\psi_i(t,x)|x_i| \geq \alpha_i\phi_i(|x_i|) \quad , \qquad \psi_{ij}(t,x)x_j \leq \alpha_{ij}\phi_j(|x_j|) \tag{3}$$

for $i, j = 1,2,\ldots,n$, and for all $t \geq 0$, $x \in R^n$. The functions $\phi_i(\cdot)$ are continuous functions such that $\phi_i(0) = 0$, and $\phi_i(r_i) < \phi_i(r_2)$ for all r_1, r_2 such that $0 \leq r_1 < r_2 < \infty$.

Define the constant matrix $\bar{A} = [\bar{a}_{ij}]$ where

$$a_{ij} = -\delta_{ij}\alpha_i + e_{ij}\alpha_{ij} \quad , \qquad i,j = 1,2,\ldots,n \quad . \tag{4}$$

The following result is now available

Theorem 1 [3]. The equilibrium state x = 0 of (1) is connectively asymptotically stable in the large if the matrix $\bar{A}$ satisfies the (Metzler) conditions

$$(-1)^k \det \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1k} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{a}_{k1} & \bar{a}_{k2} & \cdots & \bar{a}_{kk} \end{bmatrix} > 0 \quad , \qquad k = 1,2,\ldots,n \quad . \tag{5}$$

Remarks: (1) A consequence of $\bar{A}$ satisfying the conditions (5) is that there exists a vector $d = (d_1,\ldots,d_n)'$, all of whose elements are positive, and a positive number π such that

$$|\bar{a}_{jj}| - d_j^{-1} \sum_{i \neq j} d_i |\bar{a}_{ij}| \geq \pi \quad , \qquad j = 1,2,\ldots,n \quad . \tag{6}$$

This result is of some importance in estimating the size of the domain of connective stability if a global result cannot be obtained.

(2) The proof of Theorem 1 utilizes the Lyapunov function

$$\nu(x) = \sum_{i=1}^{n} d_i |x_i| \quad , \tag{7}$$

where the d_i are as in (6).

If we choose the comparison functions $\phi_i(|x_i|) = |x_i|$, then a result on uniform, exponential stability can be obtained since, in this case, the conditions on ψ_i, ψ_{ij} take the form

$$\psi_i(t,x) \leq \alpha_i \quad , \qquad |\psi_{ij}(t,x)x_j| \leq \alpha_{ij}|x_j| \quad , \qquad i,j = 1,2,\ldots,n \quad . \tag{8}$$

The basic result is

Theorem 2. The equilibrium state $x = 0$ is connectively, absolutely, and exponentially stable if and only if the matrix $\bar{A}$ corresponding to the conditions (8), satisfies (5).

Remarks. (1) In contrast to Theorem 1, we now have both necessary and sufficient conditions in Theorem 2.

(2) Exponential stability means that

$$||x(t)|| \le \Pi||x_0||\exp(-\pi t)$$

for π as in (6) and $\Pi = \sqrt{n}\, d_M d_m^{-1}$, where $d_M = \max_i d_i$, $d_m = \min_i d_i$. Thus, Theorem 2 gives us an estimate of the rate at which the system trajectory approaches equilibrium for any type of perturbation (measured by the magnitude of the d_i's).

In the common case when the bounds (3) (or (8)) do not hold for all $x \in R^n$ but only for a region $\mathscr{M} \subset R^n$, we are faced with a problem of estimating the domain of attraction of the origin. More precisely, we have

Definition 2. A set $\mathscr{M} \subset R^n$ is a *region of connective asymptotic stability* for the origin if and only if for all interconnection matrices E,x,=,0 is stable in the sense of Lyapunov, and $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \mathscr{M}$.

To study this situation in more detail, assume that the (possibly) nonlinear functions ψ_i, ψ_{ij} satisfy the conditions (3) for all $t > 0$, $x \in \mathscr{N} \subset R^n$, where

$$\mathscr{N} = \{x \in R^n : |x_i| < \mu_i,\ i = 1,2,\ldots,n\} ,$$

for some numbers $\mu_i > 0$. Then consideration of the Lyapunov function (7) leads to the result

Theorem 3. The region $\mathscr{M}$ defined by

$$\mathscr{M} = \{x \in R^n : \nu(x) < \min_i d_i \mu_i\}$$

is a region of connective asymptotic stability corresponding to $x = 0$ for the systems (1).

Theorem 3 shows that, roughly speaking, maximizing the "size" of $\mathscr{M}$ is equivalent to finding the elements d_i in (6) such that the smallest d_i is as large as possible.

It should be noted that the region $\mathscr{M}$ of Theorem 3 is the largest region of connective stability available with the Lyapunov function (7) and the constraints imposed on ψ_i and ψ_{ij}. However, larger stability regions might be obtained using other Lyapunov functions as discussed in [8].

3. Applications

We employ the methodology sketched above to analyze some recent IIASA work in ecology and energy. Specifically, we shall focus attention on the dispersal linked ecological models discussed in [6] and on the societal equations postulated in [7]. In both cases, it will be seen that connective stability can play a significant role in understanding the amount of structural uncertainty which the models can tolerate and still maintain their stability properties. It might be argued, although we will not do so, that the degree to which the postulated models satisfy the conditions of the foregoing theorems could be used as a quantitative measure of the so-called "resilience" of the system.

Ecology: We begin with the four species predator/prey model analyzed in [6]. The dynamical equations are

$$\dot{x}_1 = a_1x_1 + b_1x_1x_2 - D_1(x_1) + D_3(x_3) \quad ,$$

$$\dot{x}_2 = a_2x_2 + b_2x_2x_1 - D_2(x_2) + D_4(x_4) \quad ,$$

$$\dot{x}_3 = a_3x_3 + b_3x_3x_4 - D_3(x_3) + D_1(x_1) \quad ;$$

$$\dot{x}_4 = a_4x_4 + b_4x_4x_3 - D_4(x_4) + D_2(x_2) \quad , \qquad (9)$$

where the functions $D_i(x_i)$ represent dispersal rates for species i, while the a's and b's are constants. In [6], of the seven functional forms given for the $D_i(x_i)$, six have the structure

$$D_i(x_i) = x_i f_i(x_i) \quad , \tag{10}$$

which we shall assume for the remainder of our analysis. Our objective will be to state conditions on the constants a_i, b_i, and functions f_i which insure the connective stability of the origin* for the system (9).

Under the condition (10), we see that (9) is equivalent to the nonlinear matrix system (1) with

$$A(t,x) = \begin{bmatrix} a_1+b_1x_2-f_1(x_1) & 0 & f_3(x_3) & 0 \\ 0 & a_2+b_2x_1-f_2(x_2) & 0 & f_4(x_4) \\ f_1(x_1) & 0 & a_3+b_3x_4-f_3(x_3) & 0 \\ 0 & f_2(x_2) & 0 & a_4+b_4x_3-f_4(x_4) \end{bmatrix} . \tag{11}$$

Thus, the interconnection matrix E for this problem is

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} .$$

The appropriate functions ψ_i and ψ_{ij} are

$$\psi_1 = -(a_1 + b_1x_2) - f_1(x_1)) \quad , \qquad \psi_2 = -(a_2 + b_2x_1 - f_2(x_2)) \quad ,$$

$$\psi_3 = -(a_3 + b_3x_4) - f_3(x_3)) \quad , \qquad \psi_3 = -(a_4 + b_4x_3 - f_4(x_4)) \quad ,$$

$$\psi_{13} = f_3(x_3) \quad , \qquad \psi_{24} = f_4(x_4) \quad ,$$

$$\psi_{31} = f_1(x_1) \quad , \qquad \psi_{42} = f_2(x_2) \quad ,$$

*If the origin is not the physically interesting equilibrium point, the usual transformation of coordinates will make it so without affecting our arguments in any essential way.

all other $\psi_{ij} = 0$, $i,j = 1,\ldots,4$.

The conditions (8) and the fact that

$$|\psi_{ij}(t,x)|\,|x_j| \geq |\psi_{ij}(t,x)x_j|$$

implies the second condition in (8) if $|\psi_{ij}(t,x)| \leq \alpha_{ij}$, for some $\alpha_{ij} \geq 0$, gives by Theorem 2 that $x = 0$ is connectively, absolutely, and exponentially stable for the ecological process (9) if and only if we can find α_i, α_{ij}, $i,j = 1,\ldots,4$, such that

$$\begin{aligned} &a_1 + b_1x_2 - f_1(x_1) \leq -\alpha_1 < 0 \\ &a_2 + b_2x_1 - f_2(x_2) \leq -\alpha_2 < 0 \\ &a_3 + b_3x_4 - f_3(x_3) \leq -\alpha_3 < 0 \\ &a_4 + b_4x_3 - f_4(x_4) \leq -\alpha_4 < 0 \\ &|f_1(x_1)| \leq \alpha_{31} \quad , \qquad |f_2(x_2)| \leq \alpha_{42} \quad , \\ &|f_3(x_3)| \leq \alpha_{13} \quad , \qquad |f_4(x_4)| \leq \alpha_{24} \quad . \end{aligned} \qquad (12)$$

The conditions of Theorem 2 will be satisfied if

i) $\alpha_1 > 0$,

ii) $\alpha_1\alpha_2 > 0$,

iii) $\alpha_1\alpha_3 > \alpha_{13}\alpha_{31}$,

iv) $\alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_{13}\alpha_{31}\alpha_{24}\alpha_{42} > \alpha_1\alpha_3\alpha_{24}\alpha_{42} + \alpha_2\alpha_4\alpha_{13}\alpha_{31}$.

Thus, conditions (i) - (iv) define regions in (x_1,x_2,x_3,x_4) space for which the origin is asymptotically stable for all perturbing $f_i(x_i)$ and all (a_i,b_i) satisfying the conditions (12).

The preceding results are fairly general, not distinguishing between the behavior of the functions f_i and the values of the parameters a_i, b_i. If the problem under investigation is to study the stability behavior as the parameters change, for *fixed* functions f_i, then more precise information can be obtained.

For example, choosing the functions $D_i(x_i)$ in the form

$$D_i(x_i) = k_i x_i \quad , \qquad i = 1,\ldots,4 \quad ,$$

we see that the functions $f_i(x_i) = k_i$. Thus,

$$\alpha_{31} = |k_1| \quad , \qquad \alpha_{42} = |k_2| \quad , \qquad \alpha_{13} = |k_3| \quad , \qquad \alpha_{24} = |k_4| \quad ,$$

and the admissible set of a_i, b_i values (for a fixed region in R^n) is given by the first set of inequalities in (12), e.g.

$$a_1 + b_1 x_2 - k_1 \leq -\alpha_1 < 0$$

or

$$a_1 + b_1 x_2 \leq k_1 - \alpha_1 \quad ,$$

with x_2 constrained to some region given in advance.

On the other hand, the analysis may be for fixed values of the parameters a_i, b_i with the objective to determine the regions of phase space for which $x = 0$ is the attractor. Again for the case $D_i(x_i) = k_i x_i$, we see that the foregoing inequalities yield this information and, in fact, show that any perturbing functions f_i satisfying $|f_i| \leq k_i$ will give the same domain of attraction. Most of the other dispersal rate functions $D_i(x_i)$ from [6] can be analyzed in a similar fashion.

Energy: Turning now to the mathematical model for society developed in [7], we have the equations

$$\dot{g} = \mu g(1 - g/g_A) \quad ,$$

$$\dot{P} = P[a_p(1 - P/P_A) - a_c a_v g] \quad ,$$

$$\dot{E} = \frac{N(g,P,E)}{D(g,P,E)} \quad , \tag{13}$$

where

$$N(g,P,E) = gP(1 - a_v) - (K - K_0)E$$

$$-\frac{M}{\beta}[\mu(1 - g/g_A) + (1 - \alpha)(a_p(1 - P/P_A) - a_c a_v g]$$

$$D(g,P,E) = \frac{\gamma E}{\beta M} - i_0[(g/g_0)^2 - 1] \quad ,$$

with

$$M = \left[\frac{gP^{1-\alpha}}{AE\gamma}\right]^{\frac{1}{\beta}} \quad .$$

The numbers μ, g_A, a_p, P_A, a_c, a_v, K, K_0, α, β, γ, i_o, A, and g_0 are parameters, while the dependent variables g, p, e represent the per capita gross national product, the population, and the total energy demand, respectively. The system (13) cannot be put directly into the form (1) since an additional forcing term enters the picture, i.e. (13) has the form

$$\dot{x} = A(t,x)x + b(t,x) \quad , \tag{14}$$

where b is a continuous vector function of its arguments. The vector function b(t,x) has components of the form

$$b_i(t,x) = \ell_i \theta_i(t,x) \quad , \qquad i = 1,\ldots,n$$

where the ℓ_i are components of an interconnection vector

$$\ell = (\ell_1, \ell_2, \ldots, \ell_n)' \quad ,$$

and each $\ell_i = 0$ or 1, depending upon whether or not the forcing function $b(t,x)$ influences $\dot{x}_i$, i.e.

$$b_i(t,x) \not\equiv 0 \Rightarrow \ell_i = 1 \quad .$$

For systems of the form (14), it is more natural to speak of boundedness with respect to a region $\mathscr{L}$ than about the stability of a point. Specifically, we have

Definition 3. The solution $x(t)$ of the system (14) is *connectively, exponentially, and ultimately bounded in the large* with respect to the region

$$\mathscr{L} = \{x \in R^n : ||x|| \leq \xi\}$$

if and only if there exist three positive numbers $\gamma < \xi$, Π, and π, independent of the initial state x_0, such that

$$||x(t)|| \leq \gamma + \Pi ||x_0|| \exp[-\pi t]$$

for all $t > 0$ and for all interconnection matrices E and interconnection vectors ℓ.

Thus, a system satisfying Definition 3 would ultimately have its trajectory belonging to set $\mathscr{L}$ and the approach to $\mathscr{L}$ would be exponential for all interconnection matrices and vectors. Surprisinly, the forcing vector $b(t,x)$ plays very little role in establishing ultimate boundedness for (14) as the following result shows.

Theorem 4[3]. The solutions $x(t)$ of (14) are connectively, exponentially, and ultimately bounded in the large if the matrix $\bar{A} = (\bar{a}_{ij})$, corresponding to the conditions (8), satisfies conditions (5).

Remarks. (1) Note that Theorem 4 gives only a sufficient condition for ultimate boundedness.

(2) As before, if conditions (8) are not satisfied for all $x_0 \in R^n$, then we can estimate the domain of attraction of $\mathscr{L}$ from conditions (5).

(3) If the numbers d_m, d_M, Π and π, are as defined in Remark 2 following Theorem 2, we can estimate the region $\mathscr{L}$ by choosing

$$\xi = d_m^{-1}\pi^{-1}\eta + \varepsilon \quad ,$$

where $\varepsilon > 0$ is an arbitrarily small number, and

$$\xi = \sum_{i=1}^{n} d_i \bar{b}_i \quad ,$$

where if

$$|\theta_i(t,x)| \leq \beta_i \quad ,$$

then

$$\bar{b}_i = \ell_i \beta_i \quad .$$

The time t_1 necessary to reach the region $\mathscr{L}$ can be estimated as

$$t_1 = \pi^{-1}\log(\Pi||x_0||\ \varepsilon^{-1}) \quad .$$

Returning to the societal model (13), we see that the equations may be expressed in the form (14) upon identifying

$$A(t,x) = \begin{bmatrix} \mu(1 - g/g_A) & 0 & 0 \\ 0 & [a_p(1 - P/P_A) - a_c a_v g] & 0 \\ \dfrac{P(1 - a_v)}{D(g,P,E)} & 0 & \dfrac{-(k - k_0)}{D(g,P,E)} \end{bmatrix} ,$$

$$b(t,x) = \begin{pmatrix} 0 & 0 & -\frac{M}{\beta}\,[\mu(1 - g/g_A) + (1 - \alpha)(a_p(1 - P/P_A) \\ & & - a_c a_v g]/D(g,P,E) \end{pmatrix}' .$$

The interconnection matrix E is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} ,$$

while the interconnection vector ℓ assumes the form

$$\ell = (0 \quad 0 \quad 1)' \quad .$$

Applying Theorem 4, in much the same way as Theorem 2 was applied for the predator/prey model, yields information on the range of parameter values and/or the range of initial conditions for which ultimate boundedness of (13) can be assured.

4. Structural Connectivity and Polyhedral Dynamics

The connectivity matrix E and vector ℓ introduced above allow us to make contact with the quite different ideas of algebraic connectivity presented in [5] under the name "polyhedral dynamics." As is clear from their definitions, the matrix E represents the influence of the j^{th} system variable on the i^{th} time derivative, while the vector ℓ gives information about the influence of external perturbations on the rate of change of the i^{th} variable. Roughly speaking, we might say that E is indicative of internal coupling within the system, while ℓ shows the coupling of the system to external disturbances.

Without wishing to belabor the point, we only note that the matrix E may be identified with an incidence matrix of a binary relation as in [4] and a simplicial complex geometrically characterizing the system connectivity pattern may be constructed. Thereafter, all the points illustrated in [4,5] may be applied to analyze the *algebraic*-topological character of the process in contrast to the earlier discussions of this note, which are primarily analytic in character.

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