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THE EFFECTS OF AGGREGATION ON THE PERRON ROOT AND ITS CORRESPONDING EIGENVECTOR

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FOREWORD

The issue of aggregation in economics in general and in the case of the widely used input-output models is an important problem not yet satisfactory solved. This paper addresses precisely this question and constitutes a contribution to existing literature.

The classical economic problem which is connected with the results discussed in the paper relates to the aggregation of the data of an input-output table into a single sector to determine the rate of profit. This can be done under certain conditions, which are discussed in the paper.

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The Effects of Aggregation on the Perron Root and its Corresponding Eigenvector

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ABSTRACT

In this paper the behaviour of the Perron root and its corresponding eigenvector is examined, when the underlying matrix is being aggregated. Bounds are presented for the Perron root and the elements of the Perron vector of the resultant matrix. The bounds are mainly expressed in terms of the Perron root and vector of the original matrix. As an application aggregation in input-output analysis is considered.

1. Introduction.

Eigenvalues and eigenvectors have become useful tools in economic analysis. Especially the Perron root and its corresponding eigenvector play an essential role. In empirical work however are the underlying matrices often exposed to errors or obtained after aggregation. It therefore seems important to investigate how this affects the Perron root and its vector. The impacts of small changes in matrixelements are thoroughly discussed in Varga [1962] and Wilkinson [1965]. When the matrix is changed substantially or is aggregated, relatively less is known although, of course, bounds can be given for the Perron root and its vector (see e.g. Berman and Plemmons [1979] and Seneta [1981], also for bibliographies). Unfortunately, these bounds are in general expressed in terms of matrixelements, so that nothing can be concluded about the actual change of the Perron vector. Our primary interest goes to obtaining bounds, for the new Perron vector, that are given in terms of the old one. Recently Elsner, Johnson and Neumann [1982] examined the case where the i^{th} row of a nonnegative, irreducible matrix is increased while its j^{th} row is decreased. In the present paper we shall consider changes in the Perron root and vector when the matrix is being aggregated.

Aggregation has been a topic in economics for long. For instance any study based on input-output tables, uses data that are somehow aggregated over products and industries or over regions. Our results may therefore be applied directly to some of the dynamic Leontief-type models (see e.g. Takayama [1985]). For further applications we may think of Seton's eigenprices (see e.g. Seton [1985]) and Saaty's priority concept (see e.g. Saaty [1980], see Steenge [1986] for links with the Leontief framework).

In the next section we shall discuss non-weighted aggregation, also referred to as consolidation, where rows and columns are simply added. In section 3 we shall consider the consolidation of an input-output table which leads to a weighted aggregation of the matrix of input-output coefficients. The summary and conclusions shall be presented in section 4.

2. Non-weighted aggregation.

In this section we consider an nxn nonnegative, irreducible matrix A. With ρ, y and p we denote its Perron root and the right- and left-hand eigenvector corresponding to ρ. To avoid unnecessary notational inconveniences we present and proof our results for the simplest case where only the first and second sector are taken together. The aggregated (n-1)x(n-1) nonnegative, irreducible matrix is denoted by $\tilde{A}$, its Perron root and vectors by $\tilde{\rho}$, z and q. Generalizations of the theorems are obtained straightforwardly and are presented without proofs in the appendix.

Definitions. [1)]

$$Ay = \rho y \qquad \rho > 0,\ y \gg 0,\ y' = (y_1, y_2, \ldots, y_n)$$

$$p'A = \rho p' \qquad p \gg 0,\ p' = (p_1, p_2, \ldots, p_n)$$

$$\tilde{A}z = \tilde{\rho}z \qquad \tilde{\rho} > 0,\ z \gg 0,\ z' = (z_2, \ldots, z_n)$$

$$q'\tilde{A} = \tilde{\rho}q' \qquad q \gg 0,\ q' = (q_2, \ldots, q_n)$$

$$\tilde{A} = GAG' \qquad \text{with } G = \begin{bmatrix} 1 & 1 & 0' \\ 0 & 0 & I_{n-2} \end{bmatrix} \tag{1}$$

where I_n denotes the nxn identity matrix. Throughout this section we assume $y_1 \geq y_2$.

Theorem 2.1.

$$\rho < \tilde{\rho} < \rho\frac{y_1+y_2}{y_2} \tag{2}$$

Proof. According to the well known Subinvariance Theorem, the left-hand side is proved when we find a vector $x > 0$ for which $\tilde{A}x > \rho x$. Let $(\tilde{A}x)_i$ denote the i^{th} element of the vector $\tilde{A}x$, note that in our notation $i = 2,\ldots,n$. Take $x' = (y_1+y_2, y_3, \ldots, y_n)$ then

$$(\tilde{A}x)_2 = (a_{11} + a_{12} + a_{21} + a_{22})(y_1 + y_2) + \sum_{j=3}^{n}(a_{1j} + a_{2j})y_j$$

$$\geq (a_{11} + a_{21})y_1 + (a_{12} + a_{22})y_2 + \sum_{j=3}^{n}(a_{1j} + a_{2j})y_j$$

$$= \rho(y_1 + y_2) = \rho x_2.$$

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1) For vectors and matrices we adopt the following notation in order to describe their nonnegativity. Let x be a n-element vector, then $x \geq 0$ means $x_i \geq 0$ for each i, $x > 0$ means $x \geq 0$ and $x \neq 0$, $x \gg 0$ means $x_i > 0$ for each i. With 0 we denote the n-element null vector.

$$(\tilde{A}x)_i = (a_{i1} + a_{i2})(y_1 + y_2) + \sum_{j=3}^{n} a_{ij}y_j$$

$$\geq a_{i1}y_1 + a_{i2}y_2 + \sum_{j=3}^{n} a_{ij}y_j = \rho y_i = \rho x_i \quad \text{for } i = 3,..,n.$$

Thus $\tilde{A}x \geq \rho x$, but equality can only occur when $a_{i1} = a_{i2} = 0$ for $i = 1,..,n$ which would contradict the irreducibility of A. Therefore $\tilde{A}x > \rho x$.

To prove the right-hand side we take $x' = (y_2, y_3, .., y_n)$.

$$(\tilde{A}x)_2 = (a_{11} + a_{12} + a_{21} + a_{22})y_2 + \sum_{j=3}^{n} (a_{1j} + a_{2j})y_j$$

$$\leq (a_{11} + a_{21})y_1 + (a_{12} + a_{22})y_2 + \sum_{j=3}^{n} (a_{1j} + a_{2j})y_j$$

$$= \rho(y_1 + y_2) = \frac{y_1 + y_2}{y_2}\rho x_2$$

$$(\tilde{A}x)_i = (a_{i1} + a_{i2})y_2 + \sum_{j=3}^{n} a_{ij}y_j$$

$$\leq \rho y_i = \rho x_i < \frac{y_1 + y_2}{y_2}\rho x_i \quad \text{for } i = 3,..,n.$$

From $\tilde{A}x < \frac{y_1 + y_2}{y_2}\rho x$ it follows that $\tilde{\rho} < \rho\frac{y_1 + y_2}{y_2}$.

We now present our basic theorem which states that the relative increase in the elements of the Perron vector is the largest for the sector which is aggregated. The proof essentially is a refinement of the one used in Elsner, Johnson and Neumann [1982] for perturbations of a single row[2].

Theorem 2.2.

$$\frac{z_i}{y_i} \leq \frac{\rho}{\tilde{\rho}}\frac{z_2}{y_2} \qquad \text{for } i = 3,..,n. \tag{3}$$

Proof. Suppose to the contrary that there exists an index $m > 2$ for which

$$\frac{z_m}{y_m} = \max_{i>2} \frac{z_i}{y_i} > \frac{\rho}{\tilde{\rho}}\frac{z_2}{y_2} \quad \text{,hence}$$

$$0 < \rho\frac{z_m}{y_m} = \tilde{\rho}\frac{\rho}{\tilde{\rho}}\frac{z_m}{y_m}$$

$$= \frac{\rho}{\tilde{\rho}}\frac{1}{y_m}\left[(a_{m1} + a_{m2})z_2 + \sum_{j=3}^{n} a_{mj}z_j\right]$$

$$= \frac{1}{y_m}\left[\frac{\rho}{\tilde{\rho}}\frac{z_2}{y_2}\frac{y_2}{y_1}a_{m1}y_1 + \frac{\rho}{\tilde{\rho}}\frac{z_2}{y_2}a_{m2}y_2 + \sum_{j=3}^{n}\frac{\rho}{\tilde{\rho}}\frac{z_j}{y_j}a_{mj}y_j\right]$$

2) Using our refinement, the bounds in their Theorem 2.1 can also be sharpened according to (3).

$$< \frac{1}{y_m}\left[a_{m1}y_1 + a_{m2}y_2 + \sum_{j=3}^{n} a_{mj}y_j\right]\frac{z_m}{y_m} = \rho\frac{z_m}{y_m}$$

which is not possible. Note that equality would imply $a_{mi} = 0$ for $i = 1,..,n$ which would contradict irreducibility.

The inequality in (3) is strict when additionally the irreducibility is assumed of the submatrix defined by a_{ij} for $i,j = 3,..,n$. Alternative proofs of (3) can be obtained by using Fiedler and Pták [1962; Th. 4.2] or by applying the framework of Courtois and Semal [1984]. Both approaches yield $z_i/y_i \leq z_2/y_2$ for $i = 3,..,n$ from which $\tilde{\rho} z_i/y_i \leq \rho z_2/y_2$ easily follows. Because the non-weighted aggregation in this section simply means that the first two rows and columns are added, both theorems also hold for the left-hand Perron vectors. Thus in (2) and (3) we may replace y_i and z_i by p_i and q_i respectively (for $i = 1,..,n$).

3. Weighted aggregation.

In this section we first consider aggregation in an input-output framework and present results comparable with theorems 2.1 and 2.2. Secondly, we show that aggregation of a homogeneous Markov Chain simply implies aggregation of the stationary distribution.

The problems which may arise when in an input-output model sectors are aggregated, were first recognized by Leontief [1951]. Hatanaka [1952] and McManus [1956] focus on necessary and sufficient conditions for the aggregation scheme to be acceptable. Starting point is an input-output table X, with its typical element x_{ij} denoting the deliveries from sector i to sector j. If, for the sake of simplicity again, we aggregate sectors 1 and 2, the new table $\tilde{X}$ is obtained as GXG', with G as defined in (1). The matrix of input-output coefficients is defined as $A \equiv X\hat{x}^{-1}$, where $\hat{x}$ denotes the diagonal matrix with the output vector x on its main diagonal. Hence $\tilde{A} = GXG'(G\hat{x}G')^{-1} = GAH'$ with $H' \equiv \hat{x}G'(G\hat{x}G')^{-1}$, or equivalently

$$H = \begin{bmatrix} w_1 & w_2 & 0' \\ 0 & 0 & I_{n-2} \end{bmatrix} \quad \text{with } w_1 = \frac{x_1}{x_1 + x_2} \quad \text{and } w_2 = \frac{x_2}{x_1 + x_2} \tag{4}$$

The weights w_1 and w_2 denote which fraction of total output of sectors 1 and 2 comes from the sectors seperately[3]. Final demand vectors are denoted by f and $\tilde{f} = Gf$. The input-output equations are given by

$$x = Ax + f \quad \Longrightarrow \quad x = (I - A)^{-1}f \tag{5}$$

$$\tilde{x} = \tilde{A}\tilde{x} + \tilde{f} \quad \Longrightarrow \quad \tilde{x} = (I - \tilde{A})^{-1}\tilde{f} \tag{6}$$

The aggregation is called acceptable if $\tilde{x} = Gx$ for each final demand vector f. From (5) and (6) it follows that under acceptability $\tilde{x} = (I - \tilde{A})^{-1}Gf = G(I - A)^{-1}f$ must hold for all f, which implies $\tilde{A}G = GA$. Ara

[3] The theorems to be presented below also hold for the more general case in which it is only assumed that $0 < w_1, w_2 < 1$ and $w_1 + w_2 = 1$.

[1959] first took eigenvectors into consideration and showed that under acceptability the Perron vector of the aggregated matrix is the aggregated Perron vector of the original matrix. Moreover, the Perron root does not change: $\rho y = Ay$ implies $\rho Gy = GAy = \tilde{A}Gy$. The condition of acceptability is quite severe and it is unlikely that it will be fullfilled in practical work[4]. It also is not necessary for Gy to be the Perron vector of $\tilde{A}$, with ρ the Perron root.

Lemma 3.1. $\rho Gy = \tilde{A}Gy$ if there exists a vector $t \gg 0$ such that $y = \hat{x}G't$

Proof. $\rho Gy = GAy = GA\hat{x}G't = \tilde{A}G\hat{x}G't = \tilde{A}Gy$

This lemma implies that when we start from the matrix A instead of from the input-output table X, we can always find weights w_1 and w_2 that provide an aggregation which results in Gy being the new Perron vector. The condition $y = \hat{x}G't$ states that $y_1/x_1 = y_2/x_2$, and thus the weights become $w_1 = y_1/(y_1 + y_2)$ and $w_2 = y_2/(y_1 + y_2)$. In practical work, starting from X, it is unlikely that this condition is met although it is weaker than acceptability. In general the Perron root will change, bounds for which are given by the following theorem.

Theorem 3.2.

$$\rho(y_1 + y_2) \min_{j=1,2}\left[\frac{w_j}{y_j}\right] \le \tilde{\rho} \le \rho(y_1 + y_2) \max_{j=1,2}\left[\frac{w_j}{y_j}\right] \tag{7}$$

Proof. We only show the left-hand side and again use the Subinvariance Theorem. We construct a positive vector $z' = (z_2, z_3, .., z_n)$ for which

$$\tilde{A}z \ge \rho(y_1 + y_2) \min_j\left[\frac{w_j}{y_j}\right] z$$

First take $z_i = y_i$ for $i = 3,..,n$ then

$$(\tilde{A}z)_2 = w_1(a_{11} + a_{21})z_2 + w_2(a_{12} + a_{22})z_2 + \sum_{j=3}^{n}(a_{1j} + a_{2j})z_j$$

$$= \rho(y_1 + y_2) - (a_{11} + a_{21})(y_1 - w_1 z_2) - (a_{12} + a_{22})(y_2 - w_2 z_2)$$

Now taking $z_2 = \max\left[\frac{y_1}{w_1}, \frac{y_2}{w_2}\right]$ gives

$$(\tilde{A}z)_2 \ge \rho(y_1 + y_2) = \frac{\rho(y_1 + y_2)}{\max_j (y_j/w_j)} z_2 = \rho(y_1 + y_2)\min_j\left[\frac{w_j}{y_j}\right] z_2$$

$$(\tilde{A}z)_i = w_1 a_{i1} z_2 + w_2 a_{i2} z_2 + \sum_{j=3}^{n} a_{ij} z_j$$

$$= \rho y_i - a_{i1}(y_1 - w_1 z_2) - a_{i2}(y_2 - w_2 z_2)$$

$$\ge \rho y_i \ge \rho(y_1 + y_2)\min_j\left[\frac{w_j}{y_j}\right] z_i$$

4) See Theil [1957] and Ara [1959] for conditions under which a matrix can acceptably be aggregated.

which completes the proof.

For an input-output matrix the weights are defined as output fractions and (7) can be restated as

$$\rho\frac{y_1+y_2}{x_1+x_2}\min_j\left[\frac{x_j}{y_j}\right] \le \tilde{\rho} \le \rho\frac{y_1+y_2}{x_1+x_2}\max_j\left[\frac{x_j}{y_j}\right] \quad \text{with } j=1,2$$

Note that when $x_1/y_1 = x_2/y_2$ we find $\tilde{\rho} = \rho$ and $z_2 = y_1/w_1 = y_2/w_2 = y_1 + y_2$ which conforms with lemma 3.1.

The non-weighted aggregation simply meant addition of the first and second row and then summing the first two columns. Therefore we could in our theorems replace right- by left-hand Perron vectors. Weighted aggregation implies addition of the first two rows and then taking the weighted sum of the first and second column. We thus need different expressions for the left-hand Perron vector. We first present the equivalent of (7), we assume throughout this section that $p_1 \ge p_2$.

Theorem 3.3.

$$\rho(w_1 + \frac{p_2}{p_1}w_2) \le \tilde{\rho} \le \rho(w_1\frac{p_1}{p_2} + w_2) \tag{8}$$

Proof. To prove the left-hand side take $q' = (p_1, p_3, \ldots, p_n)$ then

$$(q'\tilde{A})_2 \ge \rho(w_1p_1 + w_2p_2) = \rho(w_1 + \frac{p_2}{p_1}w_2)q_2$$

$$(q'\tilde{A})_i \ge \rho q_i \ge \rho(w_1 + \frac{p_2}{p_1}w_2)q_i$$

Thus $q'\tilde{A} \ge \rho(w_1 + \frac{p_2}{p_1}w_2)q'$. For the right-hand side take $q' = (p_2, p_3, \ldots, p_n)$.

We now present the bounds for the elements of the Perron vector, theorem 3.4 for the right-hand and 3.5 for the left-hand vector. In both theorems we have to distinguish between an increasing, decreasing or constant Perron root.

Theorem 3.4. For $i = 3, \ldots, n$ and $j = 1,2$

$$\text{if } \tilde{\rho} > \rho: \quad \frac{z_i}{y_i} \le \max_j\left[\frac{w_j}{y_j}\right]\frac{\rho}{\tilde{\rho}}z_2 \tag{9}$$

$$\text{if } \tilde{\rho} < \rho: \quad \frac{z_i}{y_i} \ge \min_j\left[\frac{w_j}{y_j}\right]\frac{\rho}{\tilde{\rho}}z_2 \tag{10}$$

$$\text{if } \rho = \tilde{\rho}: \quad \min_j\left[\frac{w_j}{y_j}\right]z_2 \le \frac{z_i}{y_i} \le \max_j\left[\frac{w_j}{y_j}\right]z_2 \tag{11}$$

Proof. We first prove (9). Suppose to the contrary that there exists an index $m > 2$ such that

$$\frac{z_m}{y_m} = \max_i\frac{z_i}{y_i} > \max_j\left[\frac{w_j}{y_j}\right]\frac{\rho}{\tilde{\rho}}z_2 \quad \text{,then}$$

$$0 < \rho \frac{z_m}{y_m} = \tilde{\rho} \frac{\rho}{\tilde{\rho}} \frac{z_m}{y_m} = \frac{1}{y_m} \frac{\rho}{\tilde{\rho}} \left[w_1 a_{m1} z_2 + w_2 a_{m2} z_2 + \sum_{j=3}^{n} a_{mj} z_j \right]$$
$$= \frac{1}{y_m} \left[z_2 \frac{w_1}{y_1} \frac{\rho}{\tilde{\rho}} a_{m1} y_1 + z_2 \frac{w_2}{y_2} \frac{\rho}{\tilde{\rho}} a_{m2} y_2 + \sum_{j=3}^{n} \frac{\rho}{\tilde{\rho}} \frac{z_j}{y_j} a_{mj} y_j \right]$$
$$< \frac{1}{y_m} \left[a_{m1} y_1 + a_{m2} y_2 + \sum_{j=3}^{n} a_{mj} y_j \right] \frac{z_m}{y_m} = \rho \frac{z_m}{y_m}$$

which is not possible. (10) is proved analogously. Finally we prove only the right-hand side of (11). Suppose to the contrary that there exists an index $m > 2$ such that

$$\frac{z_m}{y_m} = \max_i \frac{z_i}{y_i} > \max_j \left[\frac{w_j}{y_j} \right] z_2$$

and suppose furthermore that the sectors $3,..,n$ are re-ordered such that

$$m = n \, , \; \frac{z_i}{y_i} = \frac{z_n}{y_n} \text{ for } i = k,..,n \, , \; \frac{z_j}{y_j} < \frac{z_n}{y_n} \text{ for } j = 3,..,k-1 \text{ with } k = 3,..,n.$$

Then for $i = k,..,n$

$$0 < \rho \frac{z_i}{y_i} = \tilde{\rho} \frac{z_i}{y_i} = \frac{1}{y_i} \left[z_2 \frac{w_1}{y_1} a_{i1} y_1 + z_2 \frac{w_2}{y_2} a_{i2} y_2 + \sum_{j=3}^{n} \frac{z_j}{y_j} a_{ij} y_j \right]$$
$$\le \rho \frac{z_n}{y_n} = \rho \frac{z_i}{y_i}$$

Strict inequality must hold for at least one i because equality for $i = k,..,n$ would imply $a_{i1} = a_{i2} = a_{ij} = 0$ with $j = 3,..,k-1$, which contradicts with the irreducibility of A.

Note that when $w_1/y_1 = w_2/y_2$ the left- as well as the right-hand side of (11) equal $z_2/(y_1 + y_2)$. From lemma 3.1 it follows that $z_2 = y_1 + y_2$ and $z_i = y_i$.

Theorem 3.5. For $i = 3,..,n$

$$\text{if } \tilde{\rho} > \rho: \quad \frac{q_i}{p_i} \le \frac{\rho}{\tilde{\rho}} \frac{q_2}{p_2} \tag{12}$$

$$\text{if } \tilde{\rho} < \rho: \quad \frac{q_i}{p_i} \ge \frac{\rho}{\tilde{\rho}} \frac{q_2}{p_1} \tag{13}$$

$$\text{if } \tilde{\rho} = \rho: \quad \frac{q_2}{p_1} \le \frac{q_i}{p_i} \le \frac{q_2}{p_2} \tag{14}$$

Proof. Analogous to the proof of Theorem 3.4.

In the Appendix theorems 3.2 - 3.5 are presented for three generalized types of aggregation.

As a further application we next consider the aggregation of a homogeneous Markov Chain (MC). We show that this results in a non-homogeneous MC and therefore eigenvectors no longer play a role. The stationary distribution of the aggregated MC however equals the aggregated stationary distribution of the original MC.

Let the finite homogeneous MC be described by $\pi(t)' = \pi(0)'P^t$, where $\pi(t)$ denotes the probability distribution at time t, $\pi(0)$ the initial distribution and P the transition matrix. Let the stationary distribution be denoted by v, with $v'1 = 1$ where $1' = (1,..,1)$. When P is primitive $P^t \rightarrow 1v'$ elementwise for $t \rightarrow \infty$. Consequently $\pi(t)' \rightarrow \pi(0)'1v' = v'$ elementwise for $t \rightarrow \infty$. By using conditional probabilities it is easily seen that, when the first two states are aggregated, the new MC becomes $\tilde{\pi}(t+1)' = \tilde{\pi}(t)'\tilde{P}(t+1)$. Here $\tilde{P}(t+1)$ is defined as $\tilde{P}(t+1) = H(t)PG'$ with G as defined in (1) and $H(t)$ as defined in (4) with the following weights.

$$w_1 = \frac{\pi(t)_1}{\pi(t)_1 + \pi(t)_2} \text{ and } w_2 = \frac{\pi(t)_2}{\pi(t)_1 + \pi(t)_2}$$

where $\pi(t)_i$ denotes the ith element of the probability vector $\pi(t)$. The weight w_1 now is the probability to be in state 1 at time t, given that one is in state 1 or 2. Thus, the transition matrix $\tilde{P}$ is no longer independent from t. If $\tilde{\pi}(t) = G\pi(t)$ then $\tilde{\pi}(t+1)' = \pi(t)'G'H(t)PG' = \pi(t)'PG' = \pi(t+1)'G'$. Therefore, when the distribution of the aggregated MC equals the aggregated distribution of the original MC at time t, it also does at time t+1. This obviously is the case for the initial probability vector, $\tilde{\pi}(0) = G\pi(0)$. We then obtain for $t \rightarrow \infty$: $\tilde{\pi}(t)' = \pi(t)'G' \rightarrow v'G' \equiv \tilde{v}'$. This also is the stationary distribution as follows from $\tilde{v}' = \tilde{v}'\bar{P}$, where $\bar{P}$ is the aggregation of P using weights $w_1 = v_1/(v_1 + v_2)$ and $w_2 = v_2/(v_1 + v_2)$.

4. Summary and conclusions.

In this paper we have derived bounds for the Perron root and for the elements of its corresponding eigenvector when the underlying matrix is being aggregated. We have distinguished two types of aggregation. First the case where aggregation simply meant adding rows and columns and secondly the case where aggregation was applied within an input-output framework which led to the use of weighted sums. The bounds are mainly expressed in terms of the original Perron root and vector. As such these bounds provide important information on the behaviour of the Perron root and vector under aggregation.

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6. Appendix.

Throughout this paper we have only examined the simplest case of aggregation where the first two sectors were taken together. Consequently most notational inconveniences could be avoided. Our results however can be adapted to other forms of aggregation without undue efforts. Below we present the equivalences of theorems 3.2 - 3.5 for three types of aggregation. The proofs are omitted as they are basically the same as the proofs presented for the original theorems.

Type *i*: aggregation of k sectors into a single new sector, n sectors remain unchanged. We shall use the following indexation of the sectors.

$$\text{old}: 1,2,\ldots,n,\underbrace{n+1,\ldots,n+k}$$

$$\text{new}: 1,2,\ldots,n,\quad n+1$$

Type *ii*: aggregation of k_s sectors into s new sectors, n sectors remain the same.

old : $1,2,\ldots,n,\underbrace{n+1,\ldots,n+k_1}_{},\ldots,\underbrace{n+k_{r-1}+1,\ldots,n+k_r}_{},\ldots,\underbrace{n+k_{s-1}+1,\ldots,n+k_s}_{}$

new: $1,2,\ldots,n,\quad n+1 \quad ,\ldots,\quad n+r \quad ,\ldots,\quad n+s$

Type *iii* : all k_s sctors are aggregated into s new sectors.

old : $\underbrace{1,\ldots,k_1}_{},\ldots,\underbrace{k_{r-1}+1,\ldots,k_r}_{},\ldots,\underbrace{k_{s-1}+1,\ldots,k_s}_{}$

new: $1 \quad ,\ldots, \quad r \quad ,\ldots, \quad s$

Theorem 3.2.

i. $$\rho\left[\sum_j y_j\right]\min_j\left[\frac{w_j}{y_j}\right] \le \tilde{\rho} \le \rho\left[\sum_j y_j\right]\max_j\left[\frac{w_j}{y_j}\right]$$

with $j = n+1,\ldots,n+k$

ii. $$\rho\min_r\left[\left[\sum_j y_j\right]\min_j\left[\frac{w_j}{y_j}\right]\right] \le \tilde{\rho} \le \rho\max_r\left[\left[\sum_j y_j\right]\max_j\left[\frac{w_j}{y_j}\right]\right]$$

with $r = 1,\ldots,s$ and $j = n+k_{r-1}+1,\ldots,n+k_r$, where $k_0 = 0$

iii. as *ii* with $j = k_{r-1}+1,\ldots,k_r$

Theorem 3.3.

i. $$\rho\frac{\sum_j w_j p_j}{\max_j (p_j)} \le \tilde{\rho} \le \rho\frac{\sum_j w_j p_j}{\min_j (p_j)}$$

with $j = n+1,\ldots,n+k$

ii. $$\rho\min_r\left[\frac{\sum_j w_j p_j}{\max_j (p_j)}\right] \le \tilde{\rho} \le \rho\max_r\left[\frac{\sum_j w_j p_j}{\min_j (p_j)}\right]$$

with $r = 1,\ldots,s$ and $j = n+k_{r-1}+1,\ldots,n+k_r$, where $k_0 = 0$

iii. as *ii* with $j = k_{r-1}+1,\ldots,k_r$

Theorem 3.4. We only present the expressions for the case where $\tilde{\rho} > \rho$.

i. $$\frac{z_i}{y_i} \le \frac{\rho}{\tilde{\rho}}\max_j\left[\frac{w_j}{y_j}\right]z_{n+1}$$

with $i = 1,..,n$ and $j = n+1,..,n+k$

ii. $$\frac{z_i}{y_i} \le \frac{\rho}{\tilde{\rho}} \max_r \left[z_{n+r} \max_j \left[\frac{w_j}{y_j} \right] \right]$$

with $i = 1,..,n$, $r = 1,..,s$ and $j = n+k_{r-1}+1,..,n+k_r$, where $k_0 = 0$

iii. $$\frac{\rho}{\tilde{\rho}} \min_i \left[z_i \min_j \left[\frac{w_j}{y_j} \right] \right] \le \frac{z_i}{\sum_j y_j} \le \frac{\rho}{\tilde{\rho}} \max_i \left[z_i \max_j \left[\frac{w_j}{y_j} \right] \right]$$

with $i = 1,..,s$ and $j = k_{i-1}+1,..,k_i$

Theorem 3.5. If $\tilde{\rho} > \rho$ then

i. $$\frac{q_i}{p_i} \le \frac{\rho}{\tilde{\rho}} \frac{q_{n+1}}{\min_j (p_j)}$$

with $i = 1,..,n$ and $j = n+1,..,n+k$

ii. $$\frac{q_i}{p_i} \le \frac{\rho}{\tilde{\rho}} \max_r \left[\frac{q_{n+r}}{\min_j (p_j)} \right]$$

with $i = 1,..,n$, $r = 1,..,s$ and $j = n+k_{r-1}+1,..,n+k_r$, where $k_0 = 0$

iii. $$\frac{\rho}{\tilde{\rho}} \min_i \left[\frac{q_i}{\max_j (p_j)} \right] \le \frac{q_i}{\sum_j w_j p_j} \le \frac{\rho}{\tilde{\rho}} \max_i \left[\frac{q_i}{\min_j (p_j)} \right]$$

with $i = 1,..,s$ and $j = k_{i-1}+1,..,k_i$.

If in this last theorem we replace q_i and p_i by z_i and y_i respectively, furthermore set $w_j = 1$ for all j, then we obtain the generalizations of theorem 2.2 for non-weighted aggregation.

Note that the bounds become weaker when aggregation results in more than one new sector. Consider for instance the generalization of theorem 3.5. For aggregation of type *i*. we may choose q_{n+1} as the numéraire after which bounds for all other elements of the Perron vector are given. In case of type *ii*. aggregation we may set q_{n+r} at unity for each $r = 1,..,s$ and thus obtain s sets of bounds. When s is small this may still provide useful information, although the sets of bounds can not be compared with each other as long as q_{n+r} is unknown. The expressions for aggregation of type *iii*. are given for the sake of completeness, their practical use however is little.