Working Paper

IMPLICIT FUNCTION THEOREMS FOR MULTI-VALUED MAPPINGS

B.N. Pshenichny

September 1986
WP-86-45

International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

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FOREWORD

This paper deals with the proof of implicit function theorems for certain classes of set-valued functions. The techniques applied here are mainly based on the duality theory of convex analysis. The paper was written during a visit of Professor Pshenichny to the System and Decision Sciences Program of IIASA.

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IMPLICIT FUNCTION THEOREMS FOR MULTI-VALUED MAPPINGS

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Let us consider three Banach spaces X, Y, Z and operator $F: X \times Y \to Z$. We are interested in the solution of the following equation:

$$F(x, y) = 0 \qquad (1)$$

Suppose that the points (x_0, y_0) satisfy this equation. Then implicit function theorems yield certain sufficient conditions for solvability of the equation (1) with respect to y for all x from the certain neighborhood of x_0.

Let us somewhat reformulate the problem now. Define

$$a(x) = \{y : F(x, y) = 0\}$$

Then implicit function theorems give conditions for $a(x) \neq \phi$ in the neighborhood of x_0. We shall be concerned mostly with this reformulation.

Let us introduce some notations. Take $Z = X \times Y$. Dual spaces of continuous linear functionals will be denoted by X^*, Y^*, Z^*. Pair of points from X and Y will be defined (x, y) while $\langle x, x^* \rangle$ is reserved for the value of the functional x^* at the point x. Taking $z = (x, y)$, $z^* = (x^*, y^*)$ we obtain

$$\langle z, z^* \rangle = \langle x, x^* \rangle + \langle y, y^* \rangle$$

Multivalued mapping transforms each point $x \in X$ into set $a(x) \subseteq Y$. The set $a(x)$ may be empty.

Some more notations:

$$\text{gf}\, a = \{(x, y) : y \in a(x)\}$$

$$\text{dom}\, a = \{x : a(x) \neq \phi\}$$

Mapping a is called convex if gf a is a convex set and closed if gf a is closed. Some more notations will be introduced in due course. Terminology is close to [1].

1. IMPLICIT FUNCTION THEOREMS FOR CONVEX MAPPINGS

Let us start with some definitions. For any convex set M exist

$$\text{con}\, M = \{\lambda x : \lambda > 0,\ x \in M\}$$

which is convex cone associated with the set M. For convex mapping a let us define

$$K_a(z) = \text{con}(\text{gf}\, a - z)\ .$$

Suppose that $z \in \text{gf}\, a$. Then

$$a_z(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K_a(z)\}$$

$$a_z^*(y^*) = \{x^* : (-x^*, y^*) \in K_a^*(z)\}\ ,$$

where $K_a^*(z)$ is cone dual to $K_a(z)$. Thus,

$$x^* \in a_z^*(y^*) \quad \text{if and only if}$$

$$-\langle \bar{x}, x^* \rangle + \langle \bar{y}, y^* \rangle \geq 0,\ (\bar{x}, \bar{y}) \in K_a(z)$$

Let us prove two auxiliary lemmas:

LEMMA 1 $a_z^*(0) = -(\text{dom}\, a_z)^*$.

PROOF The mapping a_z is a positively homogenious convex mapping and therefore dom a_z is convex cone and dual cone to this cone exists. According to definition of $a_z^*(0)$ it contains those and only those elements x^* which satisfy the following condition:

$$-\langle \bar{x}, x^* \rangle + \langle \bar{y}, 0 \rangle \geq 0,\ (\bar{x}, \bar{y}) \in K_a(z)\ ,$$

which is equivalent to

$$\langle \bar{x}, -x^* \rangle \geq 0,\ \bar{x} \in \text{dom}\, a_z$$

The last inequality means that

$$-x^* \in (\text{dom}\, a_z)^*$$

The proof is completed.

LEMMA 2 *Suppose that* K *is cone in* X, int $K \neq 0$ *and* $K^* = \{0\}$. *Then* $K = X$. *If* $X = R^n$ *then requirement* int $k \neq \phi$ *can be dropped.*

PROOF It follows from the well-known theorems of convex analysis that $K^{**} = \mathrm{cl}\, K$, where symbol cl defines closure of the set. The fact that $K^* = \{0\}$ implies $\mathrm{cl}\, K = K^{**} = X$. Therefore cone X is dense everywhere in X and it is left to prove that it coincides with X. Let us assume the opposite, namely suppose that exists x_0 such that $x_0 \bar{\in} K$. Take $x_1 \in \mathrm{int} K$ and select $\tilde{x} = 2x_0 - x_1$. The cone K is dense everywhere in X, thus exist sequence $\tilde{x}_k \in K$ such that $\tilde{x}_k \to \tilde{x}$. Let us take $x_{1k} = 2x_0 - \tilde{x}_k$. It is clear that $x_{1k} \to X$ and for large k we have $x_{1k} \in K$. According to definition $x_0 = \frac{1}{2}x_k + \frac{1}{2}\tilde{x}_k$, and due to convexity of K we obtain $x_0 \in K$. The proof is completed because in the finite-dimensional case assumption int$K \neq \phi$ is fulfilled automatically, which can be verified in the standard way. These two lemmas lead to the following result:

THEOREM 1 *Suppose that* a *is a convex mapping,* $z = (x_0, y_0) \in \mathrm{gf}\, a$ *and the following conditions are satisfied:*

1 int dom $a \neq \{0\}$

2 $a_z^*(0) = \{0\}$

Then for any element $\bar{x}$ *exist such number* $\delta > 0$, *that* $a(x_0 + \lambda\bar{x}) \neq \phi$ *for all* $\lambda \in [0, \delta]$. *If* X *is of finite dimension then the first requirement can be dropped and in addition, exist such a member* $\delta > 0$ *that* $a(x_0 + \bar{x}) \neq \phi$ *for all* $\bar{x}$, $\|\bar{x}\| \geq \delta$.

Before starting the proof let us make one comment. We have in the statement of the theorem certain point $y_0 \in a(x_0)$ and it is not defined how to select it. It might seem therefore that the result of the theorem depends on appropriate selection of this point. Let us show that this is not the case. Firstly we shall introduce the following notations [1]:

$$W_a(x, y^*) = \inf_y \{<y, y^*>: y \in G(x)\} ,$$

$$a(x, y^*) = \{y \in a(x): <y, y^*> = W_a(x, y^*)$$

Notice that $W_a(x, y^*) = +\infty$ if $a(x) = \phi$ and the function $W_a(x, y^*)$ is convex with respect to x. Let us take

$$a^*_{(x,y)}(y^*) = \begin{cases} \phi & \text{if } y \bar{\in} a(x, y^*) \\ \partial_* W_a(x, y^*) & \text{if } y \in a(x, y^*) \end{cases} .$$

If $y^* = 0$ then

$$W_a(x, 0) = \delta(x \mid \mathrm{dom}\, a) = \begin{cases} 0 & \text{if } x \in \mathrm{dom}\, a \\ +\infty & \text{if } x \bar{\in} \mathrm{dom}\, a \end{cases}$$

$$a(x, 0) = a(x)$$

Therefore $\partial_x W_a(x, 0) = \partial\, \delta(x \mid \mathrm{dom}\, a) = -[\mathrm{con}(\mathrm{dom}\, a - x)]^*$. Comparison of these statements leads to the conclusion that

$$a^*_{(x,y)}(0) = \partial_x W_a(x, 0) = -[\mathrm{con}(\mathrm{dom}\, a - x)]^*$$

for any point $y \in a(x)$. Therefore we can take an arbitrary point in the statement of the theorem 1. Let us start the proof now. If assumption 1 is satisfied then it is easy to get that $\mathrm{dom}\, a_z = X$. Let us select arbitrary $\bar{x}$. We have $\bar{x} \in \mathrm{dom}\, a_z$ and therefore exist vector $\bar{y}$ such that $(\bar{x}, \bar{y}) \in K_a(z)$ i.e. $\bar{x} = \gamma(x - x_0)$, $\bar{y} = \gamma(y - y_0)$, $\gamma > 0$, $(x, y) \in \mathrm{gf}\, a$ Taking now $\delta = \gamma^{-1}$ we obtain

$$(x_0 + \lambda \bar{x}, y_0 + \lambda \bar{y}) = ((1 - \lambda\gamma)x_0 + \lambda\gamma x, (1 - \lambda\gamma)y_0 + \lambda\gamma y) \in \mathrm{gf}\, a$$

for $\lambda \in [0, \delta]$ i.e. $a(x_0 + \lambda \bar{x}) \neq 0$. In the case of a finite dimension the first assumption is not necessary. Furthermore, if $X = R^n$ then it is possible to find such vectors $\bar{x}_i$, $i = 1, \ldots, n+1$ that simplex $S = \{\lambda_1 \bar{x}_1 + \cdots + \lambda_{n+1} \bar{x}_{n+1} : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1\}$ contains 0 as inner point. If we reduce the length of the vectors $\bar{x}_i$ appropriately we can obtain that all sets $a(x_0 + \bar{x}_i)$ are not empty. Therefore any point $\bar{x}$ from the certain neighborhood of zero can be represented as follows:

$$\bar{x} = \lambda_1 \bar{x}_1 + \cdots + \lambda_{n+1}, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 ,$$

This together with properties of the convex maps implies

$$a(x_0 + \bar{x}) = a\left\{\sum_{i=1}^{n+1} \lambda_i (x_0 + \bar{x}_i)\right\} \geq \sum_{i=1}^{n+1} \lambda_i\, a(x_0 + \bar{x}_i) \neq \phi .$$

The proof is completed.

This proof is fairly simple, but the result is quite interesting. Let us illustrate this with some examples:

EXAMPLE 1 Take $X = R^n$, $Y = R^m$, A and B are matrices $r \times n$ and $r \times m$. Define $\mathrm{gf}\, a = \{(x, y) : Ax - By = 0\}$ and select $x_0 = 0$, $y_0 = 0$. Then $K_a(0) = \{(\bar{x}, \bar{y}) : A\bar{x} - B\bar{y} = 0\}$,

$$K_a^*(0) = \{(x^*, y^*) : x^* = A^*u, y^* = -B^*u, u \in R^r\}$$

Therefore $a_0^*(y^*) = \{A^*u = B^*u, u \in R^r\}$. Condition $a_0^*(0) = \{0\}$ means now that $B^*u = 0$ which implies $A^*u = 0$. Thus, for solvability of the system of equation $Ax - By = 0$ with respect to y for all x, it is sufficient that Kern $B^* \subseteq$ Kern A^*. Here, as usual, Kern $C = \{v : Cv = 0\}$

EXAMPLE 2 Take the same assumption as in the previous example and consider gf $a = \{(x, y) : Ax - By \le 0, y \ge 0\}$. Now we have

$$K_a^*(0) = \{(-A^*u, B^*u + v) : u \in R_+^r, v \in R_+^m\} \ ;$$

where R_+^r and R_+^m are positive orthants. This gives

$$a_0^*(y^*) = \{A^*u : y^* = B^*u + v, u \in R_+^r, u \in R_+^m\} \ .$$

Hence condition $a_0^*(y^*) = \{0\}$ implies in this case that from inequalities $B^*u \le 0$, $u \ge 0$ the equality $A^*u = 0$ follows. Therefore for solvability of the system $Ax - By \le 0$, $y \ge u$ for all x it is sufficient that the following inclusion is satisfied: Kern A^* >= $\{u \ge 0 : B^*u \le 0\}$.

EXAMPLE 3 Suppose now that X, Y, W are Banach spaces, $Z = X \times Y$, F-convex multivalued map from Z into W, i.e. $F(x, y) \subseteq W$, M-convex subset of W. Define $a(x) = \{y : F(x, y) \cap M \ne \phi\}$. It is clear that points $(x, y) \in$ gf a if and only if exist such point w that $(x, y, w) \in$ gf F, $w \in M$. Let us select the point x_0 such that $a(x_0) \ne \phi$, $y_0 \in a(x_0)$ i.e. $(x_0, y_0) \in$ gf a. Suppose that w_0 is such a point from M that $(x_0, y_0, w_0) \in$ gf$F$. Let us denote $z_0 = (x_0, y_0)$ and define $a_{z_0}^*(y^*)$ as follows. According to definition $x^* \in a_{z_0}^*(y^*)$ if and only if $-\langle x - x_0, x^*\rangle + \langle y - y_0, y^*\rangle \ge 0$, $(x, y) \in$ gf a. Taking into account the description of gf a given above, we arrive at the conclusion that the last inequality is equivalent to the following:

$$-\langle x - x_0, x^*\rangle + \langle y - y_0, y^*\rangle + \langle w - w_0, 0_w^*\rangle \ge 0$$

for all $(x, y, w) \in$ gf$F$, $w \in M$, where 0_w^* is zero of the space W^*. But the last inequality is equivalent to the following:

$$(-x^*, y^*, 0_w^*) \in \{(\text{con}(\text{gf}F - (x_0, y_0, w_0)) \cap (\text{con}(X \times Y \times M$$

$-(x_0, y_0, u_0))$

Suppose now that exist the point $(x, y, y) \in \mathrm{gf}F$ such that (2) $u \in \mathrm{int}M$. This implies $\mathrm{gf}F \cap \mathrm{int}(X \times Y \times M) \neq 0$. It is consequence of the well-known results of convex analysis that under this assumption the cone in the right hand side of (2) is equal to the sum of convex cones dual to the intersected ones. Taking into account easily established relations

$$(\mathrm{con}(X \times Y \times M - (x_0, y_0, w_0)))^* = (0_x^*, 0_y^*, (\mathrm{con}(M - w_0))^*)$$

we obtain that (2) is equivalent to the following statement:

$$(-x^*, y^*, 0_w^*) \in K_F^*(x_0, y_0, w_0) + (0_x^*, 0_y^*, (\mathrm{con}(M - w_0))^*)$$

or, in other words, exist functional $w^* \in (\mathrm{con}(M - w_0))^*$ such that the following inclusion holds:

$$(-x^*, y^*, -w^*) \in K_F^*(x_0, y_0, w_0) \ .$$

And, finally using a definition of the conjugate mapping introduced before we obtain that $x^* \in G_{z_0}^*(y^*)$ only if for some $w^* \in (\mathrm{con}(M - w_0))^*$ the following inclusion holds:

$$(x^*, -y^*) \in F_{(z_0, w_0)}^*(-w^*) \ ,$$

$$\text{i.e. } (x^*, -y^*) \in F_{(z_0, w_0)}^*(-(\mathrm{con}(M - w_0))^*) \ .$$

Hence $a_{z_0}^*(0) = \{0\}$ only if the following inclusion

$$(x^*, 0_y^*) \in F_{(z_0, w_0)}^*(-\mathrm{con}(M - w_0))^*)$$

implies equality $x^* = 0_x^*$, i.e.

$$F_{(z_0, w_0)}^*(-(\mathrm{con}(M - x_0))^*) = \{(0_x^*, 0_y^*)\}$$

THEOREM 2 *Suppose that* X, Y, W *are Banach spaces,* M *is a convex set which belongs to* W, F *is convex multivalued mapping from* $X \times Y$ *to* W *and*

$$a(x) = \{y : F(x, y) \cap M\} \neq \phi \ .$$

suppose that $y_0 \in a(x_0)$ *and* $w_0 \in M$ *are points such that* $(x_0, y_0, w_0) \in \mathrm{gf}\,F$. *Then the following conditions are sufficient for existence of number* $\delta > 0$ *for any* $\bar{x}$ *such that* $a(x_0 + \lambda \bar{x}) \neq \phi$, $\lambda \in [0, \delta]$:

1 $\quad$ $\mathrm{int\,dom}\, a \neq \phi$;

2 $\quad$ $\mathrm{gf}\,F \cap (X \times Y \times \mathrm{int}\, M) \neq \phi$;

3 $\quad$ $F^*_{(z_0, w_0)}(-(\mathrm{con}\,(M - w_0))^*) = \{0^*_x, 0^*_y\}$.

If X is of finite dimension then it is not necessary to check the first condition. The proof follows directly from theorem 1 and the above argument.

EXAMPLE 4 Suppose that b and c are convex multivalued mappings from X to Y and $a(x) = b(x) \cap c(x)$.

THEOREM 3 *Suppose that* $y_0 \in a(x_0)$ *and the following conditions are satisfied:*

1 $\quad$ $\mathrm{int\,dom}\, a \neq \phi$;

2 $\quad$ $\mathrm{gf}\, b \cap \mathrm{int\,gf}\, c \neq \phi$;

3 $\quad$ *for any* y^* *either one of the sets* $b^*_{z_0}(y^*)$, $c^*_{z_0}(-y^*)$ *is empty, or exists such functional* x^* *that* $b^*_{z_0}(y^*) = \{x^*\}$, $c^*_{z_0}(-y^*) = \{-x^*\}$. *Then for any* $\bar{x} \in X$ *exist* $\delta > 0$ *such that* $a(x_0 + \lambda \bar{x}) \neq \phi$ *for* $\lambda \in [0, \delta]$. *If X is of finite dimension then it is not necessary to check condition 1.*

The proof of the theorem follows from theorem 1, the fact that $\mathrm{gf}\, a = \mathrm{gf}\, b \cap \mathrm{gf}\, c$ and direct calculation of $a^*_{z_0}(0)$ by using convex analysis techniques.

Let us show how to use this theorem by the following example. Take $b(x) = \{x : Ax - By \leq 0\}$, $c(x) = M$ where M is a fixed convex set in R^m, A and B are matrices of dimension $r \times n$ and $r \times m$, $X = R^n$, $Y = R^m$. Suppose that $0 \in M$ and exist points x_1, y_1 such that $Ax_1 - By_1 \leq 0$, $y_1 \in \mathrm{int}\, M$. A straightforward argument shows that

$$K_b(0) = \{(\bar{x}, \bar{y}) : A\bar{x} - B\bar{y} \leq 0\}$$

$$K^*_b(0) = \{(x^*, y^*) : x^* = -A^* u,\ y^* = B^* u,\ u \geq 0\}$$

$$b^*_0(y^*) = \{A^* u : y^* = B^* u,\ u \geq 0\}$$

$$K_c(0) = X \times \operatorname{con} M$$

$$K_c^*(0) = \{0\} \times (\operatorname{con} M)^* \quad ,$$

$$c_0^*(y^*) = \begin{cases} \{0\} & \text{if } y^* \in (\operatorname{con} M)^* \; , \\ \phi & \text{if } y^* \overline{\in} (\operatorname{con} M)^* \; . \end{cases}$$

We can obtain more from the following result by applying theorem 3. In order that the system of inequalities $Ax - By \leq 0$, $y \in M$ has a solution with respect to y for all x from some neighborhood of zero it is sufficient that conditions

$$u \geq 0, \; -B^* u \in (\operatorname{con} M)^*$$

imply equality $A^* u = 0$. In fact $b_0^*(y^*) = \phi$ if y^* can not be represented in the form $y^* = B^* u$, $u \geq 0$. In case if $y^* = B^* u$, $u \geq 0$, and $-B^* u \overline{\in} (\operatorname{con} M)^*$. If, however,

$$-B^* u \in (\operatorname{con} M)^*, \; y \geq 0 \; ,$$

then for $f^* = B^* u$ we have $c_0^*(-y^*) = \{0\}$. Therefore according to the theorem 3 the set $b_0^*(y^*)$ also should contain only one point i.e. zero. It follows from representation of the set $b_0^*(y^*)$ that $A^* u = 0$.

If $M = R_+^m$ i.e. M consists of nonnegative vectors then this result coincides with one obtained from example 2.

2. LOCALLY SMOOTH MAPS

Let K be a convex cone in a Banach space X. It is obvious that

$$\operatorname{Lin} K = K - K$$

is the minimal linear manifold containing K. If M is a convex set, then

$$\operatorname{Lin} M = \operatorname{con}(M - x) - \operatorname{con}(M - x) \; ,$$

where x is an arbitrary point of M. It is not difficult to show that $\operatorname{Lin} M$ does not depend upon $x \in M$.

We shall say that the point x belongs to the relative interior of M (denote it $\mathrm{ri}\,M$), if for some $\varepsilon > 0$

$$x + (\varepsilon B) \cap \mathrm{Lin}\,M \subseteq M \ ,$$

where B is the unit ball of the space X with the center in the origin.

If $X = R^n$, then it is well known [2], [3], that $\mathrm{ri}\,M \neq \phi$. In general this result is not true.

Let M be an arbitrary set of X.

DEFINITION *Call the set K marquee for M at the point $x \in M$, if K is the convex cone and for each $\bar{x} \in K$ exist values $\varepsilon > 0$, $\delta > 0$ and continuously differentiable in the neighborhood of the origin function $\Psi: X \rightarrow X$ exist such that*

1 $\Psi(0) = 0$, $\Psi'(0) = I$ *(unit operator);*

2 $x + \Psi(\bar{y}) \in M$ *for all*

$$\bar{y} \in [\mathrm{con}(\bar{x} + (\varepsilon B) \cap \mathrm{Lin}\,K)] \cap (\delta B) \ .$$

This definition is based on ideas of V.G. Boltyanski, he developed them for n-dimensional space. But these ideas can not be generalized on infinite dimensional space without changes. For this reason the introduced definition differs from V.G. Boltyanski's definition [3].

Further it is convenient to suppose without reducing generality that point x coincide with the origin.

THEOREM 4 *Let M_0 is a set, $0 \in M_0$, K_0 the marquee for M_0 at point $x = 0$ and functions $f_i(x)$, $i = 1, \ldots, K$ satisfy the conditions:*

1 $f_i(0) = 0$, $i = , \ldots, K$;

2 *f_i are continuously Freshet-differentiable in the neighborhood of zero and derivatives $f_i'(0)$ are linearly independent on $L_0 = \mathrm{Lin}\,K_0$.*

Then

$$K = \{\bar{x} \in K_0 : f_i'(0)\bar{x} = 0, \ i = 1, \ldots, k\} \tag{3}$$

is the local marquee at the point $x = 0$ for the set

$$M = \{x : x \in M_0, \ f_i(x) = 0, \ i = 1, \ldots, k\} \ . \tag{4}$$

PROOF Since $f_i'(0)$ are linearly independent on linear manifold L_0, then vectors $e_j \in L_0$ exist [1], such that

$$f_i'(0)e_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$i, j = 1, \ldots, k \quad .$$

Let $\bar{x} \in K$. Then

$$f_i'(0)\bar{x} = 0, \; i = , \ldots, k$$

and smooth in the neighborhood of zero. Therefore exists function Ψ_0 such that $\Psi_0(0) = 0$, $\Psi_0'(0) = I$,

$$\Psi_0(\bar{y}) \in M$$

for all

$$\bar{y} \in [\mathrm{con}(\bar{x} + (\varepsilon B) \cap L_0)] \cap (\delta B) \quad , \tag{5}$$

for $\varepsilon > 0$, $\delta > 0$.

Let

$$g_i(\bar{y}, z) = f_i\left[\Psi_0\left[\bar{y} + \sum_{j=1}^{k} z_j e_j\right]\right] - f_i'(0)\bar{y} \quad , \tag{6}$$

where z is the vector with components z_j, $j = 1, \ldots, k$.

Let us consider the system of equations

$$g_i(\bar{y}, z) = 0, \; i = 1, \ldots, k \tag{7}$$

with respect to z. It is evident

$$g_i(0, 0) = 0, \; i = 1, \ldots, k \quad .$$

Since g_i are compositions of smooth functions, then they are smooth functions. Using rules for differentiation of the complex function it is not difficult to get

$$g_{iy}'(0, 0) = f_i'(0)\Psi_0'(0) - f_i'(0) = 0 \quad , \tag{8}$$

Because $\Psi_0'(0) = I$. Further

$$g_{iz_j}'(0, 0) = f_i'(0)\Psi_0'(0)e_j = \delta_{ij} \quad . \tag{9}$$

Thus, the matrix with elements g'_{iz_j} (0, 0) is non-degenerate and according to the implicit functions theorem [4] the solution $z(\bar{y})$ of the system (7) exists, such that $z(0) = 0$. Further according to the same theorem $z(\bar{y})$ is continuously differentiable in the neighborhood of the origin of coordinates and taking into account (8)

$$z'(0) = 0 \tag{10}$$

Let now

$$\Psi(\bar{y}) = \Psi_0\left[\bar{y} + \sum_{j=1}^{k} z_j(\bar{y})e_j\right] . \tag{11}$$

Then $\Psi(0) = 0$, and

$$\Psi'(0) = \Psi_0'(0)\left[I + \sum_{j=1}^{k} e_j\, z_j'(0)\right] = I$$

according to (10). Denote

$$\tilde{y} = \bar{y} + \sum_{j=1}^{k} z_j(\bar{y})e_j .$$

Let us choose $\varepsilon_1 > 0$, $\delta_1 > 0$ sufficiently small so that the inclusion

$$\bar{y} \in [\mathrm{con}\,(\bar{x} + (\varepsilon_1 B) \cap L_0)] \cap (\delta_1 B) \tag{12}$$

ensure that $\tilde{y}$ satisfies the inclusion (5). It is possible, because (10), but if $\tilde{y}$ satisfies (5) then

$$\Psi(\bar{y}) = \Psi_0(\tilde{y}) \in M_0 .$$

Denote

$$L_1 = \{\bar{x} : f_i'(0)\bar{x} = 0,\ i = 1, \ldots, K\} .$$

It is easy to see that

$$\mathrm{Lin}\,K \subseteq L_0 \cap L_1 .$$

Let now

$$\bar{y} \in [\mathrm{con}\,(\bar{x} + (\varepsilon_1 B) \cap \mathrm{Lin}\,K)] \cap (\delta_1 B) . \tag{13}$$

The set from the right part of (13) is the subset of set from the right part of (12). For this reason (13) implies the inclusion $\Psi(\bar{y}) \in M_0$. Further, since $\bar{x} \in L_1$, then

$\bar{y} \in L_1$. Therefore due to (7) and (6)

$$f_i(\Psi(\bar{y})) = 0,\ i = 1, \ldots, k \ .$$

Thus (13) assumes $\Psi(\bar{y}) \in M$ and consequently K is the local marquee for M.

Since the proofs of the following theorems repeat the similar parts of the theorem 4 proof the details will be omited.

THEOREM 5 *Let M_1 and M_2 be two sets, $M = M_1 \cap M_2$, $0 \in M$ and K_1, K_2 be respectively marquees for M_1 and M_2. If the linear manifolds exist such that:*

1 $L_1 \subseteq \mathrm{Lin}\,K_1$, $L_2 \subseteq \mathrm{Lin}\,K_2$;

2 $L_1 + L_2 = X$;

3 *for all $x_1 \in L_1$, $x_2 \in L_2$*

$$\|x_1 + x_2\| \geq c(\|x_1\| + \|x_2\|),\ c \in (0, 1) \ ,$$

then $K = K_1 \cap K_2$ is the marquee for M at the point 0.

PROOF In accordance to the condition 2 arbitrary $x \in X$ can be represented as follows

$$x = x_1 + x_2,\ x_1 \in L_1,\ x_2 \in L_2 \ .$$

That representation is unique, because if different representation exists

$$x = x_1' + x_2',\ x_1' \in L_1,\ x_2' \in L_2 \ ,$$

then

$$(x_1 - x_1') + (x_2 - x_2') = 0 \ ,$$

$$0 \geq c(\|x_1 - x_1'\| + \|x_2 - x_2'\|) \ ,$$

i.e. $x_1 = x_1'$, $x_2 = x_2'$.

Consequently operators P_1 and P_2 are defined such that

$$P_1 x \in L_1,\ P_2 x \in L_2,\ P_1 x + P_2 x = x \ .$$

Due to the condition 3

$$\|x\| \geq c(\|P_1 x\| + \|P_2 x\|) \ ,$$

i.e.

$$\|P_i x\| \le \frac{1}{c}\|x\| ,$$

it is easy to see that operators P_i are linear and consequently they are linear and continuous operators.

Let now $\bar{x} \in K_1 \cap K_2$ and ε, δ_i, Ψ_i correspond to $\bar{x}$ in the marquee K_i. Consider the equation

$$g(\bar{y}, z) \equiv \Psi_1(\bar{y} + P_1 z) - \Psi_2(\bar{y} - P_2 z) = 0 , \tag{14}$$

with respect to z. It is easy to see

$$g'_{\bar{y}}(0, 0) = \Psi'_1(0) - \Psi'_2(0) = 0$$

$$g'_z(0, 0) = \Psi'_1(0)P_1 + \Psi_2(0)P_2 = P_1 + P_2 = I .$$

In accordance to the implicit functions theorem the smooth function $z(\bar{y})$ is defined in the neighborhood of zero and $z(0) = 0$, $z'(0) = 0$.

Suppose

$$\Psi(\bar{y}) = \Psi_1(\bar{y} + P_1 z(\bar{y})) = \Psi_2(\bar{y} - P_2 z(\bar{y})) . \tag{15}$$

Taking into account $z'(0) = 0$, i.e.

$$\frac{z(\bar{y})}{\|\bar{y}\|} \to 0, \quad \text{for } \|\bar{y}\| \to 0$$

it is not difficult to prove that Ψ is the desired function for $\bar{x}$.

This proof is based on the fact that for $\bar{y}$ close to the direction $\bar{x}$

$$\Psi_1(\bar{y} + P_1 z(\bar{y})) \in M_1 ,$$

$$\Psi_2(\bar{y} + P_2 z(\bar{y})) \in M_2 ,$$

and consequently due to (15) $\Psi(\bar{y}) \in M_1 \cap M_2$. In this connection if $\bar{y} \in \operatorname{Lin} K_i$, $i = 1, 2$, then $\bar{y} + P_1 z(\bar{y}) \in L_1$, $\bar{y} - P_2 z(\bar{y}) \in L_2$ and taking into account the smallness of $z(\bar{y})$ all conditions related with the choice of ε, δ can be satisfied. Besides

$$\Psi(0) = \Psi_1(0) = 0 ,$$

$$\Psi'(0) = \Psi'_1(0)[I + P_1 z'(0)] = I_1$$

because $\Psi'_1(0) = I$, $z'(0) = 0$. The proof is completed.

REMARK Conditions 1–3 for $X = R^n$ can be replaced by the condition

$$\mathrm{Lin}\, K_1 + \mathrm{Lin}\, K_2 = X \ .$$

In a general case that condition is not sufficient because condition 2 means that the sets M_1 and M_2 have sufficiently large dimensions. The condition 3 means that linear manifolds intersect at some angle. Indeed, if X is a Hilbert space with inner product $[x, y]$ then condition 3 is equivalent to condition

$$[x_1, x_2] \le 1 - 2c^2, \ \|x_1\| = 1, \ \|x_2\| = 1,$$

$$x_1 \in L_1, \ x_2 \in L_2 \ .$$

Let us consider now the theorem about implicit functions for nonconvex multivalued maps.

THEOREM 6 *Let a be a multivalued map, K be a marquee for $gf\ a \subseteq X \times Y$ at the point $z_0 = (x_0, y_0)$,*

$$a_{z_0}(\bar{x}) = \{\bar{y} : (\bar{x}, \bar{y}) \in K\} \ ,$$

$$a^*_{z_0}(y^*) = \{x^* : (-x^*, y^*) \in K\} \ .$$

If the following conditions are satisfied

1 int dom $a_{z_0} \ne \phi$;

2 exist a linear restricted operator $P : X \longrightarrow Y$ such that $(x, Px) \in \mathrm{Lin}\, K$;

3 $a^*_{z_0}(0) = \{0\}$, then for any $\bar{x}$ exists $\delta > 0$ such that

$$a(x_0 + \lambda \bar{x}) \ne \phi \quad \text{for} \quad \lambda \in [0, \delta] \ .$$

If X and Y have finite dimensions then the first two conditions are satisfied and besides value $\delta > 0$ exists such that $a(x_0 + \bar{x}) \ne \phi$ for all $\bar{x} \in (\delta B)$.

PROOF Without loss of the generality we consider $x_0 = 0$, $y_0 = 0$.

Lemma 2 implies

$$\mathrm{dom}\, a_{z_0} = X \ ,$$

i.e. for any $\bar{x} \in X$ the vector $\bar{y}$ exists such that $(\bar{x}, \bar{y}) \in K$. Consequently for any $\bar{x} \in X$ the vector $\bar{y} \in Y$ exists such that $(\bar{x}, \bar{y}) \in \mathrm{Lin}\, K$.

Thus, let $\bar{x}_0$ be a vector from X and $z_0 = (\bar{x}_0, \bar{y}_0) \in K$. Since K is the marquee then $\varepsilon > 0$, $\delta > 0$ and exists smooth function $\Psi(0) = 0$, $\Psi'(0) = I_z$, where I_z is the unit operator in Z and $\Psi(\bar{z}) \in gf\ \boldsymbol{a}$ for all

$$\bar{z} \in [\mathrm{con}(\bar{z}_0 + (\varepsilon B_z) \cap \mathrm{Lin}\,K)] \cap (\delta B_z) \ , \tag{16}$$

where B_z is the unit ball in Z. Since $Z = X \times Y$, then

$$\Psi(\bar{z}) = \begin{bmatrix} \Psi_1(\bar{x}, \bar{y}) \\ \Psi_2(\bar{x}, \bar{y}) \end{bmatrix} ,$$

and condition $\Psi'(0) = I_z$ can be rewritten as follows

$$\Psi'(0) = \begin{bmatrix} I_x & 0 \\ 0 & I_y \end{bmatrix} ,$$

i.e. $\Psi'_{1\bar{x}}(0, 0) = I_x$, $\Psi'_{1\bar{y}}(0, 0) = 0$, $\Psi'_{2\bar{x}}(0, 0) = 0$, $\Psi'_{2\bar{y}}(0, 0) = I_y$.

Consider the system of equations

$$g(\lambda, r) = \Psi_1(\lambda\bar{x}_0 + r, \lambda\bar{y}_0 + Pr) - \lambda\bar{x}_0 = 0 \ , \tag{17}$$

where $\lambda \in R^1$, $z \in X$. Taking into account previous relations we get

$$g'_\lambda(0, 0) = \Psi'_{1\bar{x}}(0, 0)\bar{x}_0 + \Psi'_{1\bar{y}}(0, 0)\bar{y}_0 - \bar{x}_0 = 0 \ ,$$

$$g'_r(0, 0) = \Psi'_{1\bar{x}}(0, 0) + \Psi'_{1\bar{y}}P = I_x \ .$$

Thus due to the theorem about implicit functions the system (17) has the solution $r(\lambda)$ and also

$$r(0) = 0, \ r'(0) = 0 \ .$$

Let us consider now the point

$$\bar{\bar{z}}(\lambda) = (\lambda\bar{\bar{x}}_0 + r(\lambda), \lambda\bar{\bar{y}}_0 + Pr(\lambda)) \ .$$

Taking into account $r'(0) = 0$, $r(\lambda) = o(\lambda)$ for sufficiently small $\lambda > 0$ inclusion (16) is satisfied, because by definition of the operator P we have $\bar{z}(\lambda) \in \mathrm{Lin}\,K$. Consequently

$$\Psi(\bar{z}(\lambda)) \in \mathrm{gf}\ \boldsymbol{a}$$

for small λ. But

$$\Psi(\bar{z}(x)) = \begin{bmatrix} \Psi_1(\bar{z}(\lambda)) \\ \Psi_z(\bar{z}(\lambda)) \end{bmatrix} = \begin{bmatrix} \lambda\bar{x}_0 \\ \Psi_2(\bar{z}(\lambda)) \end{bmatrix} ,$$

here the condition (17) was used. Thus

$$(\lambda\bar{x}_0,\ \Psi_2(\bar{z}(\lambda))) \in \mathrm{gf}(a)\ ,$$

i.e. $a(\lambda\bar{x}_0) \neq \phi$ for small λ. Thus the first part of the theorem is proved. Let us consider the case with the finite dimension. Condition 1 can be omitted in accordance to lemma 1 and 2. We will show that condition 2 can be omitted too.

Indeed because $\mathrm{Lin}K \subseteq R^n \times R^m$ then exist the matrices A and B with dimensions $r \times n$ and $r \times m$ respectively such that the points $(x, y) \in \mathrm{Lin}K$ and only they satisfy the equations

$$Ax - By = 0\ , \tag{18}$$

where rows of the matrix $(A, -B)$ with the dimension $r \times (n + m)$ are linearly independent. This follows from the fact that in the finite dimensional space linear manifold can be described as set of solution of some linear equations system.

Since the rows of the matrix $(A, -B)$ are linearly independent then exist non-degenerate submatrix of this matrix with dimension $r \times r$. Consequently system (18) has solution y for arbitrary x. For this reason (see example 1 § 1)

$$\mathrm{Kern}B^* \subseteq \mathrm{Kern}A^* \tag{19}$$

However $\mathrm{Kern}B^* = \{0\}$. Indeed let $y^* \in R^r$, $y^* \neq 0$ and $B^*y^* = 0$, then due to (19) $A^*y^* = 0$. This means that exist the non-zero vector y^* orthogonal to all columns of the matrix $(A, -B)$. The last statement contradicts the existence of non-degenerated $r \times r$ submatrix of the matrix $(A, -B)$. Thus $\mathrm{Kern}B^* = \{0\}$, i.e. columns of the matrix B^* are linearly independent. For this reason B^* (and consequently B) contains non-degenerated submatrix B_1 with rank $r \times r$. Let $B = (B_1, B_2)$. Consider the vectors y

$$y = \begin{bmatrix} y_1 \\ 0_{m-r} \end{bmatrix},$$

where $y_1 \in R^r$ and 0_{m-r} is the non-zero vector with dimension $m - r$. It is clear that $By = B_1y_1$. If $y_1 = B_1^{-1}Ax$, then vector

$$y = \begin{bmatrix} B_1^{-1}Ax \\ 0_{m-r} \end{bmatrix}$$

satisfies (18). It is obvious that the linear operator

$$P_x = \begin{bmatrix} B_1^{-1}Ax \\ 0_{m-r} \end{bmatrix}$$

satisfies condition 2 of the theorem. Q.E.D.

REMARK Let us consider the condition 2 in general. Let $L \subset X \times Y$ is a linear manifold. Denote

$$l(x) = \{y : (x, y) \in L\} .$$

It is not difficult to see that

1 $l(x)$ is an affine manifold;

2 $l(x_1 + x_2) = l(x_1) + l(x_2)$;

3 $l(\lambda x) = \lambda l(x)$ for $\lambda \neq 0$.

Let in accordance to the theorem 6 $\text{dom}\, l = X$. Thus the map l from the space X to the set of affine manifolds of y is linear. Condition 2 of theorem 6 implies that exist a linear continuous map P such that $Px \in l(x)$. Since l is a linear map then existence of P is a natural condition. As was shown earlier in a finite dimensional space, this operator exists. It would be interesting in general to formulate an additional condition for L guaranteeing existence of a continuous linear selector of P.

Let us consider now the questions connected with the construction of a marquee for a convex set.

THEOREM 7 *Let M be a convex set in Banach space X, $0 \in M$ and* $\text{ri}\, M \neq \phi$. *Then the cone*

$$K = \text{con}(\text{ri}\, M)$$

is the marquee for set M at the point $x = 0$.

PROOF It is evident that $\text{ri}\, M \subseteq M$, for this reason $\text{Lin}(\text{ri}\, M) \subseteq \text{Lin}\, M$ and $\text{Lin}\, L \subseteq \text{Lin}\, M$.

Let $\bar{x} \in K$ i.e. $\bar{x} = \gamma x$, $\gamma > 0$, $x \in \text{ri}\, M$. If $x \neq 0$ then for some $\varepsilon > 0$ $x + (\varepsilon B) \cap \text{Lin}\, M \subseteq M$ in accordance to the definition of the relative interior of a set M.

Let

$$\bar{y} = \lambda(\bar{x} + \varepsilon_1 y) = \lambda(\gamma x + \varepsilon_1 y) = \lambda\gamma(x + \frac{\varepsilon_1}{\gamma} y) \ , \tag{20}$$

where y is an arbitrary point of $\text{Lin}\, K$, $\|y\| \leq 1$. If

$$\frac{\varepsilon_1}{y} < \varepsilon, \ \lambda\gamma < 1 \ ,$$

then $\bar{y} \in M$ due to the definition ri$M$. Since $\bar{x} \neq 0$, then

$$\|\bar{y}\| = \lambda \|\bar{x} + \varepsilon_1 y\| \geq \lambda(\|\bar{x}\| - \varepsilon_1) \ .$$

For this reason if $\varepsilon_1 < \|\bar{x}\|$ then

$$\lambda \leq \frac{\|\bar{y}\|}{\|\bar{x}\| - \varepsilon_1} \ .$$

Thus if $\varepsilon_1 > 0$, $\delta > 0$ are chosen so that

$$\frac{\delta\gamma}{\|\bar{x}\| - \varepsilon_1} \leq 1, \ \varepsilon_1 \leq \|\bar{x}\|, \ \varepsilon_1 < \gamma\varepsilon \ ,$$

then for $\bar{y}$ satisfying condition

$$\bar{y} \in [\mathrm{con}(\bar{x} + (\varepsilon_1 B) \cap \mathrm{Lin}K)] \cap (\delta B)$$

the following inclusion is true: $\bar{y} \in M$. Thus, in this case $\Psi(\bar{y}) = \bar{y}$ can be taken.

If $\bar{x} = 0$ then in accordance with the definition of the relative interior exists such $\varepsilon > 0$ that

$$(\varepsilon B) \cap \mathrm{Lin}K \subseteq M$$

i.e. any point of $\mathrm{Lin}K \subseteq K$ with norm less than ε belongs to M. It is clear that in this case $\Psi(\bar{y}) = \bar{y}$ too. Q.E.D.

Let us now consider some applications of these results. In particular it is interesting for us to generalize the implicit functions theorem in cases when solutions belong to some set M. It is formulated below.

THEOREM 8 *Let the functions $f_i(z)$, $i = 1, \ldots, K$ be defined on the space $Z = X \times Y$, these functions be smooth in the neighborhood of the origin of coordinates, M be a convex set containing 0. Let in addition:*

1 *gradients $f_i'(z_0)$ are linearly independent on subspace* Lin$M$;

2 *exist point $\bar{z}$ such that*

$$f_i'(z_0)\bar{z} = 0, \ i = 1, \ldots, k, \ \bar{z} \in \mathrm{ri}M \ ;$$

3 *for any vector* $u \in R^k$ *the set*

$$\{x^* : (x^*, f_y'^*(z_0)u) \in [\mathrm{con}(M - z_0)]^*\}$$

is empty or consists from the unique vector $f_x'^* u$.

Then for any vector $\bar{x}$, $\|\bar{x}\| < \delta$ *exists vector* $\bar{y}$ *such that*

$$f_i(x_0 + \bar{x}, y_0 + \bar{y}) = 0, \; i = 1, \ldots, k \;,$$

$$(x_0 + \bar{x}, y_0 + \bar{y}) \in M \;.$$

PROOF Define

$$a(x) = \{y : f_i(x, y) = 0, \; i = 1, \ldots, k, \; (x, y) \in M\} \;.$$

In accordance with the theorems 4 and 7 the cone

$$K = \{\bar{z} : f_i'(z_0)\bar{z} = 0, \; 1 = 1, \ldots, k, \; \bar{z} \in \mathrm{con}(\mathrm{ri}M)\}$$

is the marquee for gf a at the point $z_0 = (x_0, y_0)$. Taking into account assumptions and well known theorems of convex analysis we get

$$K^* = (\mathrm{con}(M - z_0))^* + \{f_z'^*(z_0)u : u \in R^m\} \;,$$

where $f_z'(z_0)$ is the Freshet derivative of the map $f : R^{n+m} \to R^k$, i.e. matrix with rows $f_{iz}'(z_0) \in R^{n+m}$. Condition 3 of the theorem 6 means that relations

$$(x'^*, y^*) \in (\mathrm{con}(M - z_0))^*$$

$$y^* + f_y'^*(z_0)u = 0$$

assume the equality

$$x^* + f_y'^*(z_0)u = 0 \;.$$

The last condition is equivalent to condition 3 of the theorem.

THEOREM 9 *Let* $Z = R^n \times R^m$, $f_i(z)$, $i = 1, \ldots, k$ *be a smooth function and* U *be a convex set in* R^n. *If* (x_0, y_0) *is a point such that*

$$f_i(x_0, y_0) = 0, \; i = 1, \ldots, k, \; y_0 \in U$$

then for the existence of the value $\delta > 0$ *such that for any* $\bar{x} \in R^n$ *exist vector* $\bar{y} \in R^m$ *satisfying*

$$f_i(x_0 + \bar{x}, y_0 + \bar{y}) = 0, i = 1, \ldots, k ,$$

$$y_0 + \bar{y} \in U$$

it is sufficient

1 *the vectors* $f'_{ix}(z_0)$ *are linearly independent;*

2 *exist vector* $(\bar{x}_1, \bar{y}_1)$ *such that*

$$f'_{ix}(z_0)\bar{x}_1 + f'_{iy}(z_0)\bar{y}_1 = 0, \bar{y}_1 \in (\mathrm{ri}\,U - y_0)$$

3 *The set*

$$\{u : f'^*_y(z_0)u \in [\mathrm{con}(U - y_0)]^* \}$$

contains only zero.

The proof follows directly from the previous theorem, taking into account the equivalence of equalities $f'_x(z_0)u = 0$ and $u = 0$ which, in turn follows from linear independence of vectors $f'_{ix}(z_0)$, $i = 1, \ldots, k$

Let us consider now the solvability of the system of inequalities

$$f_i(x, y) \le 0, i = 1,\ldots, r, x \in R^n, y \in R^m$$

for any x from vicinity of some point x_0. Suppose that the point (x_0, y_0) is one of the solutions of this system.

This problem can be reduced to the previous one by introducing auxiliary variables w_i, $i = 1, \ldots, k$ and considering the following system:

$$f_i(x, y) + w_i = 0, i = 1, \ldots, k$$

$$w_i \ge 0, i = 1, \ldots, k$$

The theorem 8 can be applied now. To do this let us take $X = R^n$ and the space Y from this theorem will be the space of pairs $(y, w) \in R^m \times R^k$. The set M is now the set (R^n, R^m, R^k_+). Therefore $\mathrm{Lin}M = (R^n, R^m, R^k)$. Let us note that in the conditions (21) each new variable corresponds to separate equality, therefore condition 1 of theorem 8 is true. Furthermore, we can assume without loss of generality that

$$f_i(x_0, y_0) = 0, i = 1, \ldots, k$$

This assumption will considerably simplify the argument. What is needed now for fulfillment of the second condition of the theorem is existence of the vector $\bar{z}_1 = (\bar{x}_1, \bar{y}_1)$ such that

$$f_z'(z_0)\bar{z}_1 = 0 \ .$$

Due to the fact that $M = (R^n, R^m, R_+^k)$ we have

$$[\operatorname{con}(M - Z_0)]^* = (0_n, 0_m, R_+^k) \ .$$

The third condition of theorem 8 easily follows now from the assumption that conditions $u \geq 0$, $f_y'^*(z_0)u = 0$ *imply* $f_y'^*(z_0)u = 0$. Or in other words

$$\text{Kern}\, f_x'^*(z_0) \supseteq (\text{Kern}\, f_y'^*(z_0)) \cap R^k$$

Thus, we have obtained the following result:

THEOREM 10 *Suppose that* $x \in R^n$, $y \in k^m$, *functions* $f_i(z)$, $i = 1, \ldots, k$ *are smooth for* $z = (x, y)$ *and the point* $z_0 = (x_0, y_0)$ *is such that*

$$f_i(x_0, y_0) = 0, \ i = 1, \ldots, k \ .$$

Let us take in addition the following assumptions:

1 *Exists vector* $\bar{z}_1 = (\bar{x}_1, \bar{y}_1)$ *such that*

$$f_z'(z_0)\bar{z}_1 < 0$$

2 $\text{Kern}\, f_x'^*(z_0) \supseteq (\text{Kern}\, f_y'(z_0)) \cap R_+^k$

Then exists $\delta > 0$ *such that for any* $\bar{x}$, $\|\bar{x}\| < \delta$ *exists* $\bar{y}$ *such that*

$$f_i(x_0 + \bar{x}, y_0 + \bar{y}) \leq 0, \ i = 1, \ldots, r \ .$$

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