DUALITY AND GEOMETRIC PROGRAMMING

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1. - Introduction and Motivation

The theory of production refers basically to the problem of optimal allocation of resources or factors of production such that the total cost of producing a certain output is minimized. If y is an aggregate measure of output that can be produced from a given set of inputs $(x_1, \ldots, x_n)$ in certain amounts specified by the technical characteristics of the production function $y = F(x_1, \ldots, x_n)$, and $p_1, \ldots, p_n$ are the prices of the inputs, the problem is mathematically formulated by [1, p. 60]

$$\min_{x} \quad C = A + \sum_{i=1}^{n} p_i x_i$$

$$\text{subject to } F(x_1, \ldots, x_n) = \bar{y} \text{ (constant)} \tag{1.1}$$

If the production possibility set allows an output to be produced by an infinity of combinations of productive factors, it would be impossible, without any other considerations, to determine the total cost uniquely for each output. However, the minimization problem in (1.1) eliminates indeterminacy, so that by solving (1.1), an optimum value for each factor can be obtained as a function of input prices and output.

$$x_i^* = g_i^*(y, p_1, \ldots, p_n), \; i = 1, \ldots, n \tag{1.2}$$

This gives

$$C^*(y, p) = A + \sum_{i=1}^{n} p_i x_i^* = A + \sum_{i=1}^{n} p_i g_i^*(y, p_1, \ldots, p_n) \tag{1.3}$$

[1, p. 59]

The cost minimization problem gives a total cost function. At the same time, it has pointed out the existence of a dual problem which would allow for determination of production structures from cost curves [2, p. 159]. Extensions of the dual relationship are given by Uzawa [3] and Diewert [4].

The solution of the cost minimization problem is conditioned by the form of the production function representing the underlying technology. The first form of the production function was that of Cobb-Douglas [5]. In 1961, Arrow et. al. [6] introduced the Constant Elasticity of Substitution (CES) production function. More recently, Christensen et. al. [7], [8] proposed a form for the production function based on a second order Taylor's expansion, evaluated at $x_i = 1$, of any arbitrary explicit production function. For example, the two inputs one output formulation would be

$$\log y = \log \beta_0 + \beta_1 \log x_1 + \beta_2 \log x_2 + \beta_3(\log x_1 - \log x_2)^2 \qquad (1.4)$$

The rationale for the Transcendental Logarithmic Frontier, as it is called, is based on the argument of generality and absence of assumptions that were included in previous representations of production functions. This absence allows the assumptions made in previous forms to be subjected to statistical tests. The hypotheses which have been tested are those derived from the theory of production (homogeneity, symmetry, and normalization), and others included implicitly in the Cobb-Douglas and CES forms (additivity and separability of inputs and outputs) [8].

Two main problems arise from the use of the Transcendental Logarithmic form:

1. - For practical and estimation purposes, the authors take the approximating function as the true function and include any possible source of error in the error term of the regression equation. This implies that there is no way of telling whether the results are affected by stochastic or approximation error.

2. - The Cobb-Douglas and the CES production function have the property of "self duality", i.e., both the production and the cost forms are members

of the same family of functional forms. This makes irrelevant the choice of representation of the technology by the production or cost functions. The Transcendental Logarithmic Form when taken as the true form for the primal (dual) problem and then taken again as the true form of the dual (primal), makes one of the selections arbitrary since the form is not self-dual. This point is treated by Burgess [9] who shows with empirical results the consequences of choosing the cost or the production Transcendental Logarithmic form as a representation of the underlying technology.

This paper is addressed to the possible solution of these two problems while still being able to work with more general production functions. We propose for the consideration of the economists interested in the Theory of Production, the Geometric Programming (GP) method of solving cost minimization problems which is extensively used in engineering. The similarities observed in both fields also indicate the possible benefits of closer communication among them. In the coming sections, we give an introduction to GP and illustrate with examples using the Cobb-Douglas, CES, and a more general explicit production function.

2. - Introduction to Geometric Programming

The field of Geometric Programming can be considered initiated with the work of Duffin, Peterson, and Zener, which is summarized in their book, "Geometric Programming" [13]. Another valuable reference is Wilde and Beightler [14, especially Chapter 4], and more recent discussions can be found in [15].

As is pointed out in [13, Chapter 1]: GP "has developed with problems of engineering design ... (as) an attempt to develop a rapid systematic method of designing for minimum costs... The basis of the method is a relentless exploitation of the properties of inequalities."

The method of GP is particularly suitable for cost functions having polynomial form, with each term of the polynomial being the joint product of a set of variables raised to arbitrary powers. For example, in engineering design, the total cost c is a sum of component cost u_i; i.e.,

$$c = \sum_{i=1}^{n} u_i \tag{2.1}$$

where each u_i is a positive function of the design variables $t_1, \ldots, t_n$, of the form

$$u_i = c_i \prod_{j=1}^{n} t_j^{a_{ij}} \tag{2.2}$$

The c_i and a_{ij} are specified parameters. Generalized polynomial inequality constraints of either sense can also be handled [14, Chapter 4].

The problem of minimizing a polynomial c subject to polynomial constraints is termed a <u>primal program</u>. If a solution to the primal problem exists, there exists a related maximization problem which is called a <u>dual program</u>.

The relation between the primal and dual programs is precisely the result of the geometric inequality [13, pp. 4 and 5]

$$\sum_{i=1}^{m} \delta_i U_i \geq \prod_{i=1}^{m} U_i^{\delta_i} \tag{2.3}$$

where U_i are arbitrary non negative numbers and δ_i are positive weights satisfying

$$\sum_{i=1}^{m} \delta_i = 1 \tag{2.4}$$

If we let $u_i = \delta_i U_i$, then (2.3) converts to

$$\sum_{i=1}^{m} u_i \geq \prod_{i=1}^{m} \left(\frac{u_i}{\delta_i}\right)^{\delta_i} \tag{2.5}$$

and if u_i is of the form given in (2.2), the right side of (2.5) can be written as

$$\prod_{i=1}^{m} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{j=1}^{p} t_j^{\sum_{i=1}^{m} \delta_i a_{ij}} \qquad (2.6)$$

If the δ_i are chosen so that $\sum_{i=1}^{m} \delta_i a_{ij} = 0$, for all j, then the function obtained is independent of t_j and is called the <u>dual function</u>, $v(\delta)$:

$$v(\delta) = \prod_{i=1}^{m} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \qquad (2.7)$$

For any set of δ_i satisfying the normality ($\sum_{i=1}^{m} \delta_i = 1$) and the orthogonality ($\sum_{i=1}^{m} \delta_i a_{ij} = 0$) conditions, the value of $v(\delta)$ is a lower bound of the total cost c, and for the δ_i values resulting from maximizing (2.7) subject to normality and orthogonality conditions, the values of the primal and dual objectives are the same (see [13], [14] for the proofs).

It is of interest for the subsequent development to summarize the dual GP problem of a primal minimization problem with constraints:

Suppose a cost function $g_0(t)$ is to be minimized subject to a set of constraints $g_k(t) \leq 1$, $k = 1, \ldots, p$, $t_j > 0$ where the $g_k(t)$ are of the form[1]

$g_k(t) = \sum_{i \in [k]} c_i \prod_{j=1}^{p} t_j^{a_{ij}}$. If m_k $(k = 0, \ldots, p)$ is the number of terms in function k and if all c_i are positive

as [13, p. 78]

$$\max_{\delta} v(\delta) = \prod_{i=1}^{m} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{k=1}^{p} \lambda_k(\delta)^{\lambda_k(\delta)} \qquad (2.8)$$

where $\lambda_k(\delta) = \sum_{i \in [k]} \delta_i$, $k = 1, \ldots, p$

and $m = \sum_{k=0}^{p} m_k$

subject to $\sum_{i \in [0]} \delta_i = 1$

$$\sum_{i=1}^{m} a_{ij}\delta_i = 0, \; j = 1, \ldots, n \quad ,$$

The relationship between primal and dual variables at their respective optimum values is given by [13, p. 81]

$$c_i \prod_{j=1}^{n} t_j^{a_{ij}} = \begin{cases} \delta_i^* v(\delta^*) & i \in [0] \\ \dfrac{\delta_i^*}{\lambda_k(\delta^*)} & i \in [k] \end{cases} \qquad (2.9)$$

where δ^* means evaluated at optimum.

Note that the logarithm of $v(\delta)$ is a concave function. Hence, the GP duality theory allows the use of a linearly constraint concave dual maximization problem to solve the nonlinear nonconvex primal. Therein resides the real power of the method. The effort required to solve the dual is related to a parameter called degree of difficulty of such a program, which is given by the number m - n - 1, where m is the total number of terms and n the rank of the exponent matrix. This degree of difficulty is in fact the difference between the number of variables and constraints in the dual program. When the degree of difficulty is zero, the solution is directly obtained by solving the system of constraints of the dual program. For higher degrees of difficulty, the optimal solution is not as straight-forward, but formalized procedures have been developed to either approximate upper and lower bounds to the cost function [13, pp. 81, 101] or else to iteratively search for the maximum.

As pointed out in [13, p. 13], the dual problem is not just a mathematical artifice but has engineering interpretations. The weights δ_i have a one to one correspondence with the polynomial terms of the primal problem, and the optimal δ_i^* provides the relative size of these terms. The dual problem also has intrinsic features which supply qualitative information about the primal.

We hope to confirm this in the next sections when we use GP to derive some results of the Economic Theory of Production.

3. - Application of GP to the problems of the Economic Theory of Production

3.1 Illustration with the Cobb-Douglas Production Function

A Cobb-Douglas Production Function is of the form

$$y = F(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \tag{3.1.1}$$

where y is considered an aggregate measure of output, x_i is the value of the input i, and α_i are parameters satisfying the condition $\sum_{i=1}^{n} \alpha_i = 1$, in order for the function to be homogeneous of degree one.

If it is assumed that the behavior of a firm is directed to minimize the input cost to produce a certain level of output, y, the firm's cost minimization problem can be formulated as (primal problem)

$$\begin{aligned} &\min \sum_{i=1}^{n} p_i x_i \\ &\text{subject to: } \prod_{i=1}^{n} x_i^{\alpha_i} \geq y, \; x_i \geq 0 \end{aligned} \tag{3.1.2}$$

If we transform the constraint to its equivalent form:

$$y \prod_{i=1}^{n} x_i^{-\alpha_i} \leq 1$$

then we can construct the GP dual:

$$\max v(\delta) = \prod_{i=1}^{n} \left(\frac{p_i}{\delta_i}\right)^{\delta_i} \left(\frac{y}{\delta_{n+1}}\right)^{\delta_{n+1}} \delta_{n+1}^{\;\delta_{n+1}} \tag{3.1.3}$$

subject to:

$$\sum_{i=1}^{n} \delta_i = 1 \tag{3.1.4}$$

$$\delta_i - \alpha_i \delta_{n+1} = 0 \qquad i = 1, \ldots, n \tag{3.1.5}$$

$$\delta_i \geq 0 \tag{3.1.6}$$

Summing over constraints (3.1.5), we have:

$$\sum_{i=1}^{n} \delta_i - \delta_{n+1}\left(\sum_{i=1}^{n} \alpha_i \right) = 0 \tag{3.1.7}$$

which implies that $\delta_{n+1} = 1$ and $\delta_i{}^* = \alpha_i$ (3.1.8)

and by the property of equality between primal and dual objectives at optimality,

$$\sum_{i=1}^{n} p_i x_i = y \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i} \tag{3.1.9}$$

where $c(p) = \prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$ would be the unit cost.

From the correspondence between primal and dual variables, we see that

$$\delta_i{}^* = \frac{p_i x_i}{\sum_{i=1}^{n} p_i x_i} = \alpha_i, \text{ and also } \delta_i{}^* = \alpha_i = \frac{\partial \log y}{\partial \log x_i}$$

or the optimal cost share is equal to the output elasticity with respect to the input i.

3.2 Illustration with the CES Production Function

The primal cost minimization problem for this case will be

$$\min_x \sum_{i=1}^{n} p_i x_i$$
$$\text{subject to } F(x) = \left[\sum_{i=1}^{n} a_i x_i{}^{-b} \right]^{-1/b} \geq y \tag{3.2.1}$$

where $\sum_{i=1}^{n} a_i = 1$.

Formulated as a GP primal, the problem becomes

$$\min_{x} \sum_{i=1}^{n} p_i x_i$$

subject to (3.2.2)

$$y^b \left(\sum_{i=1}^{n} a_i x_i^{-b} \right) \leq 1$$

And the corresponding GP dual is

$$\max_{\delta} \prod_{i=1}^{n} \left(\frac{p_i}{\delta_i}\right)^{\delta_i} \prod^{n} \left(\frac{a_i y^b}{\delta_{n+i}}\right)^{\delta_{n+i}} \left(\sum_{i=1}^{n} \delta_{n+i}\right)^{\sum_{i=1}^{n} \delta_{n+i}} \qquad (3.2.3)$$

subject to $\sum_{i=1}^{n} \delta_i = 1$

$$\delta_i - b\delta_{n+i} = 0 \quad i = 1, \ldots, n$$

Summing over i in the second constraint and making use of the first one, we have $\sum_{i=1}^{n} \delta_{n+i} = \frac{1}{b}$.

The problem simplifies to

$$\max_{\delta} y \prod_{i=1}^{n} \left(\frac{p_i}{\delta_i}\right)^{\delta_i} \prod_{i=1}^{n} \left(\frac{a_i}{\delta_i}\right)^{\delta_i/b} \equiv \max_{\delta} y \prod_{i=1}^{n} \left(\frac{p_i a_i^{1/b}}{\delta_i^{(1+b)/b}}\right)^{\delta_i}$$

subject to $\sum_{i=1}^{n} \delta_i = 1$. (3.2.4)

The solution to problem (3.2.4), obtained via the generalized arithmetic/geometric mean inequality as shown in Appendix 1, is,

$$\delta_i^* = \frac{p_i^{b/1+b} a_i^{1/1+b}}{\sum_{i=1}^{n} p_i^{b/1+b} a_i^{1/1+b}} \qquad (3.2.5)$$

For this optimal value, as shown in Appendix 1, the minimum of the primal problem is given by

$$C(y,p) = y \left(\sum_{i=1}^{n} (p_i a_i^{1/b})^{b/1+b}\right)^{1+b/b} \qquad (3.2.6)$$

and hence

$$\sum_{i=1}^{n} p_i x_i^1 = \left(\sum_{i=1}^{n} (p_i a_i^{1/b})^{b/1+b}\right)^{1+b/b} \qquad (3.2.7)$$

where $\sum_{i=1}^{n} p_i x_i^1$ is now the normalized unit cost $(x_i^1 = x_i/y)$.

Note that the results above can be generalized to any homogeneous production function of the form

$$F(x) = \left[\sum_{i=1}^{n} a_i x_i^{-b/v}\right]^{-w/b} \qquad (3.2.8)$$

where v and w are positive parameters.

3.3. More general results on primal dual relationships

In the previous sections, we have illustrated the use of GP in solving cost minimization problems under the different production technologies long used in the study of the theory of production. The effectiveness of the method is particularly clear in the Cobb-Douglas form. In that case, the dual problem has zero degrees of difficulty which allows the dual cost function to be obtained merely from the solution of the constraints of the intermediate GP dual. The GP formulation also illustrates that the optimal cost shares are independent of the input prices and proportional to the elasticity of output to input, α_i. The price independence is generalizable to all the cases of zero degrees of difficulty as is also shown in [16].

For the CES form of the production function, the dual problem does not have zero degrees of difficulty, but we can still solve for the optimal δ_i^* by making use of one of the commonly used GP relationships. The optimal δ_i^* may also be considered a form of writing the demand equation for factor i which in this case is dependent on the inputs prices.

The previous results also suggest more general relationships between the primal and dual problems and further extensions of the role of the intermediate GP dual in solving for the input demand equations.

If we write the Kuhn-Tucker necessary conditions for optimality for the dual GP problem as stated in (2.8) but with $v(\delta)$ replaced by $\log v(\delta)$, we obtain

$$\begin{aligned} \frac{\partial \log v(\delta)}{\partial \delta} - \lambda \frac{\partial q(\delta)}{\partial \delta} &\leq 0 \\ \left(\frac{\partial \log v(\delta)}{\partial \delta} - \lambda \frac{\partial q(\delta)}{\partial \delta}\right)\delta &= 0 \\ q(\delta) &= 0 \\ \delta &\geq 0 \end{aligned} \tag{3.3.1}$$

where $q(\delta)$ represents the set of normality and orthogonality constraints; and λ is the associated vector of multipliers. Now, since $\log v(\delta)$ is a concave function, the problem of maximizing $\log v(\delta)$ subject to the linear dual constraints is a concave program. Consequently, the Kuhn-Tucker conditions are also sufficient for optimality, and the solution of equations (3.3.1) will be a global maximizing point. In fact, providing that the dual constraints are linearly independent, it will be the unique global maximizing point... Next, since it is easily shown that $v(\delta)$ and $\log v(\delta)$ have the same set of maximizing points, [13, Theorem 3.2], it follows that the solution to equations (3.3.1) will be the global maximizing point of the dual GP problem (2.8). Finally, from the duality theory of GP, such a solution will exist providing that the primal constraints possess an interior point and that a feasible minimizing solution to the primal exists. Moreover, at their respective optimia, the primal cost dual objective function values will be the same and the respective variables will be related via equations (2.9).

In the case of our cost minimization problem, the objective function is always linear and positive, and the problem always involves only a single posynomial constraint. Hence, an interior point can always be found, and a minimizing solution will exist providing the problem is bounded. Hence, under reasonable conditions, a solution to equations (3.3.1) can always be found. In general, that solution, $\delta*$, will be a function of p, although only in special situations will it be possible to solve (3.3.1) to obtain an explicit functional form $\delta* = f(p)$. If such a functional form can be determined, then when $\delta*$ is substituted into the GP dual objective function, the dual cost function in the Shephard sense, C(y,p) will be obtained. From the GP duality theory, we have assurance that this dual objective function value will be exactly equal to the primal objective value evaluated at its minimizing point, i.e.,

$$\sum_{i=1}^{n} p_i x_i = v(\delta*) = C(y,p) \ . \tag{3.3.2}$$

Taking the derivatives with respect to p in 3.3.2, we have

$$\frac{\partial v(\delta*)}{\partial \delta*} \cdot \frac{\partial \delta}{\partial p} = \frac{\partial C(y,p)}{\partial p} = x \ . \tag{3.3.3}$$

Dividing both sides by $v(\delta*)$ and multiplying by p, we obtain,[2]

$$p\left(\frac{\partial \log v(\delta*)}{\partial \delta*} \cdot \frac{\partial \delta}{\partial p}\right) = \delta_0^* \tag{3.3.4}$$

where δ_0^* is the subvector consisting of the first n components of $\delta*$. From the equivalence between the primal and dual solutions, δ_0^* will be the same as the first n components of the $\delta*$ obtained by solving (3.3.1).[3]

As an illustration, we can take the CES case. Equation (3.3.2) for that case is written as

$$\sum_{i=1}^{n} p_i x_i = y\left(\sum_{i=1}^{n} (p_i a_i{}^{1/b})^{b/1+b}\right)^{\frac{1+b}{b}} . \tag{3.3.5}$$

Taking derivatives with respect to p_i, we have

$$x_i = \left(\sum_{i=1}^{n} (p_i a_i^{1/b})^{b/1+b}\right)^{\frac{1+b}{b} - 1} \left(p_i a_i^{1/b}\right)^{\frac{b}{1+b} - 1} a_i^{\frac{1}{b}} \qquad (3.3.6)$$

or

$$\delta_i = \frac{p_i x_i}{\sum_{i=1}^{n} p_i x_i} = \frac{(p_i a_i^{1/b})^{\frac{b}{1+b}}}{\sum_{i=1}^{n} (p_i a_i^{1/b})^{\frac{b}{1+b}}} \qquad (3.3.7)$$

which is the same as (3.2.5).

The results above show how the intermediate GP dual can provide the equivalent demand equations for factor i without having to actually write the explicit cost function and then take the derivative with respect to p. They can easily be extended to the case when (3.3.1) does not have a solution for δ in terms of p only, because of non linearities in (3.3.1) which do not permit the elimination of λ. In this case, λ will appear in $C(y,p,\lambda)$, and the composite function may even be difficult to write explicitly. But since the results (3.3.2) to (3.3.4) still apply, and if we are interested in the form of the demand equation, as most empirical studies are [8], [9], then $C(y,p,\lambda)$ does not have to be computed since we show that the same result is obtainable by simply using the intermediate GP dual. Note that the results are independent of the condition of self-duality of Production and Cost functional forms which in fact, restricts attention to only a particular class of functions.

3.3.1 Illustration with a General Production Function

Consider the concave production function

$$y = F(x) = \left[\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}\right]^{-1/\alpha} \qquad (3.3.1.1)$$

where y represents again aggregate output and x_i are the input factors $(i = 1, \ldots, n)$. The c_{ij} and α are parameters, and the function is homogeneous of degree one.

The reasons for selecting the above form are:

1 - It has input interaction terms that will allow for testing some assumptions implicit in other production functions (like separability on inputs of the Cobb-Douglas and CES).

2 - It has the property of approaching in the limit a Cobb-Douglas form when $\alpha \to 0$.

From (3.3.1), the elasticity of output to input x_i would be

$$\frac{\partial \log y}{\partial \log x_i} = \frac{x_i}{y} \frac{\partial y}{\partial x_i} = \frac{\frac{1}{2}\left(\sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}\right)}{\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}} \tag{3.3.1.2}$$

The cost minimization problem under (3.3.1.1) would actually be

$$\min_{x} \sum_{i=1}^{n} p_i x_i$$
$$\text{subject to} \left[\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}\right]^{-1/\alpha} \geq y \tag{3.3.1.3}$$
$$x_i \geq 0 \quad i = 1, \ldots, n$$

where p_i are the input prices ($p_i > 0$ for all i). The competitive equilibrium conditions would be, using (3.3.1.2)

$$\frac{p_i x_i}{p_k x_k} = \frac{\sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}}{\sum_{j=k}^{n} c_{kj} x_k^{-\alpha/2} x_j^{-\alpha/2}} \tag{3.3.1.4}$$

Equation (3.3.1.4) could be used in estimating the parameters c_{ij} and α and in testing certain assumptions on them. However, this would require the use of non-linear estimation procedures.

The GP dual of (3.3.1.3) would be written as

$$\max_{\delta} \prod_{i=1}^{n} \left(\frac{p_i}{\delta_i}\right)^{\delta_i} \prod_{i=1}^{n} \left(\prod_{j=i}^{n} \left(y^{\alpha} \frac{c_{ij}}{\delta_{ij}}\right)^{\delta_{ij}}\right) \left(\sum_{i=1}^{n} \sum_{j=i}^{n} \delta_{ij}\right)^{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij}\right)}$$

$$\text{subject to } \sum_{i=1}^{n} \delta_i = 1 \qquad (3.3.1.5)$$

$$\delta_i - \alpha\delta_{ii} - \frac{\alpha}{2} \sum_{(h,\ell)\in J(i)} \delta_{h\ell} = 0 \quad i = 1, \ldots, n$$

$$\delta_i \geq 0, \ \delta_{ij} \geq 0 \qquad i,j = 1, \ldots, n \text{ and } j \geq i$$

where $J(i)$ is a subset of the set of subscripts pairs (h,ℓ) with $\ell \geq h$, such that either $h = i$ or $\ell = i$ but not both.

More explicitly, for the three input case of the form

$$y = F(x) = \left[c_{11}x_1^{-\alpha} + c_{22}x_2^{-\alpha} + c_{33}x_3^{-\alpha} + c_{12}(x_1 \cdot x_2)^{-\alpha/2} + c_{13}(x_1 \cdot x_3)^{-\alpha/2} + c_{23}(x_2 \cdot x_3)^{-\alpha/2}\right]^{-1/\alpha}, \qquad (3.3.1.6)$$

(3.3.1.5) would read

$$\max_{\delta} \prod_{i=1}^{3} \left(\frac{p_i}{\delta_i}\right)^{\delta_i} \prod_{i=1}^{3} \left(\prod_{j=i}^{3} \left(\frac{y^{\alpha}c_{ij}}{\delta_{ij}}\right)^{\delta_{ij}}\right) \left(\sum_{i=1}^{3} \sum_{j=i}^{3} \delta_{ij}\right)^{\left(\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij}\right)} \qquad (3.3.1.7)$$

$$\text{subject to } \sum_{i=1}^{3} \delta_i = 1 \qquad (3.3.1.8)$$

$$\delta_1 - \frac{\alpha}{2}\left(2\delta_{11} + \delta_{12} + \delta_{13}\right) = 0 \qquad (3.3.1.9)$$

$$\delta_2 - \frac{\alpha}{2}\left(2\delta_{22} + \delta_{12} + \delta_{23}\right) = 0 \qquad (3.3.1.10)$$

$$\delta_3 - \frac{\alpha}{2}\left(2\delta_{33} + \delta_{13} + \delta_{23}\right) = 0 \qquad (3.3.1.11)$$

$$\delta_i \geq 0, \ \delta_{ij} \geq 0 \qquad (3.3.1.12)$$

Summing over constraints (3.3.1.9) to (3.3.1.11) and using (3.3.1.8), we have the result

$$\sum_{i=1}^{3} \delta_i - \alpha\left(\sum_{i=1}^{3} \sum_{j=i}^{3} \delta_{ij}\right) = 0 \qquad (3.3.1.13)$$

and $$\sum_{i=1}^{3} \sum_{j=i}^{3} \delta^*_{ij} = \frac{1}{\alpha} \,. \qquad (3.3.1.14)$$

Considering the equivalence between primal and dual variables:

$$p_i x_i = \delta^*_i\left(\sum_{i=1}^{3} p_i x_i\right) \qquad i = 1, 2, 3 \qquad (3.3.1.15)$$

$$c_{ii} x_i^{-\alpha} = \frac{\delta^*_{ii}}{\sum_{i=1}^{3} \sum_{j=i}^{3} \delta^*_{ij}} \qquad i = 1, 2, 3 \qquad (3.3.1.16)$$

$$c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2} = \frac{\delta^*_{ij}}{\sum_{i=1}^{3} \sum_{j=i}^{3} \delta^*_{ij}} \qquad i, j = 1, 2, 3 \;, \; j \geq i \qquad (3.3.1.17)$$

we can show that constraints (3.3.1.9) to (3.3.1.11) are in fact the competitive equilibrium conditions as expressed by (3.3.1.4).

Note from the equivalence relations, (3.3.1.15 - .17), that $x_i > 0$, $i = 1, 2, 3$ if and only if $\delta_i, \delta_{ij} > 0$, $i,j = 1, 2, 3$, $j \geq i$. Consequently, for $x_i > 0$, the Kuhn-Tucker conditions (3.3.1) collapse to the conventional Lagrangian necessary conditions. For the case of problem (3.3.1.7), these are,

$$\log p_i - \log \delta_i - \lambda_1 - \lambda_2^i - 1 = 0 \qquad i = 1, 2, 3 \qquad (3.3.1.18)$$

$$\log c_{ii} - \log \delta_{ii} - \alpha\lambda_2^i - 1 = 0 \qquad i = 1, 2, 3 \qquad (3.3.1.19)$$

$$\log c_{ij} - \log \delta_{ij} - \frac{\alpha}{2}\left(\sum_{h \in H(i,j)} \lambda_2^h\right) - 1 = 0 \qquad i,j = 1, 2, 3 \;, \; j \geq i \qquad (3.3.1.20)$$

together with equations,

(3.3.1.8), (3.3.1.9), (3.3.1.10), (3.3.1.11), (3.3.1.12) (3.3.1.21) where λ_1 and λ_2^i are the Lagrange multipliers **and where H(i,j) is the set of all h such that (i,j) is in J(h).**

If we solve (3.3.1.18) to (3.3.1.21) in terms of δ_i, i = 1, 2, 3 and use the results in equations (3.3.1.9) to (3.3.1.11), we obtain the following system of equations in δ_i, i = 1, 2, 3 and p_i, i = 1, 2, 3. (See Appendix 3 for the derivation.)

$$\beta_{11} + \beta_{12}p_1 + \beta_{13}\left(\frac{\delta_2}{p_2}\right) + \beta_{14}\left(\frac{\delta_3}{p_3}\right) = 0$$

$$\beta_{21} + \beta_{22}p_1 + \beta_{23}\left(\frac{\delta_1}{p_1}\right) + \beta_{24}\left(\frac{\delta_3}{p_3}\right) = 0 \qquad (3.3.1.22)$$

$$\beta_{31} + \beta_{32}p_3 + \beta_{33}\left(\frac{\delta_1}{p_1}\right) + \beta_{34}\left(\frac{\delta_2}{p_2}\right) = 0$$

with the restrictions

$$\beta_{12} = \beta_{22} = \beta_{32}$$

$$\beta_{13} - \beta_{23} = 0$$

$$\beta_{14} - \beta_{33} = 0 \qquad (3.3.1.23)$$

$$\beta_{24} - \beta_{34} = 0$$

where

$$\beta_{ii} = c_{ii}e^{-(\alpha-1)\lambda_2^i} \qquad i = 1, 2, 3$$

$$\beta_{i4} = -\alpha e^{\lambda_1} \qquad i = 1, 2, 3 \qquad (3.3.1.24)$$

$$\beta_{ij} = c_{ij}e^{-(\alpha/2 - 1)(\sum_{h \in H(i,j)} \lambda_2^h) + \lambda_1 + 1} \qquad i,j = 1, 2, 3 \text{ and } j > i$$

The system of equations (3.3.1.22) allows us to solve for δ_i, i = 1, 2, 3 in terms of p_i, i = 1, 2, 3, λ_1 and λ_2^i, i = 1, 2, 3. The final system is, however, non-linear in λ_1 and λ_2^i, and these variables can not be eliminated in such a way that δ_i becomes a function of p_i alone. If we assume the economy is operating at optimum, by taking data on δ_i, cost share, and p_i, factor prices, and treating λ_1 and λ_2^i as parameters, we can use system (3.3.1.22) with the restrictions (3.3.1.23) to estimate β_{ij} by statistical procedures using other functional forms [8], [9]. Likewise, statistical tests on assumptions about production technology could be performed. For example, to test the assumption on input separability, we would test for $c_{12} = c_{13} = c_{23} = 0$ or $\beta_{13} = \beta_{14} = \beta_{24} = 0$.

4. - Conclusion

In this paper, we have studied the problems of finding the dual cost function associated with a particular production technology and the derivation of the demand function of a factor i with the methodology of GP. We used some results from GP to illustrate the primal dual relationships and to show how the intermediate GP dual can replace the so-called Shephard's dual cost function C(y,p) for empirical studies concerned with the demand equation for factor i. An explicit production function with interaction terms among the factors has been used to illustrate some of the results and to show how an explicit general form can be used to test some of the assumptions of the theory that before had been tested with approximated and, in some way, arbitrary forms.

It is also important to point out that if we start with the cost function explicitly and write the dual problem (2)

$$\max \left\{ y \mid \inf_p \left[\sum_{i=1}^{n} p_i x_i \mid C(y,p) \geq 1 \right] \right\} \tag{4.1}$$

the GP method that in the paper has been used to solve the primal problem to (4.1), would still be applicable to the solution of the above problem.

As a corollary, the paper supports an idea introduced already in [17] of the utility of using some of the concepts developed in the engineering field to model the Economic system, since a closer look reveals that both fields are looking at similar problems.

5. - Footnotes

1. $i \in [k]$ indicates the range of values for i in the kth constraint; that is, i going from $\sum_{k=0}^{k-1} m_k + 1$ till $\sum_{k=0}^{k} m_k$, where m_k is the number of terms of the kth constraint.

2. The multiplication by p is in the form of Kroeneker product $\otimes$; that is, we multiply by p_i, i = 1, ..., n, the respective ith element of the vectors at both sides of equation (3.3.3).

3. The remaining components of δ are the dual variables associated with the constraint terms. In the C-D and CES cases, these could be eliminated by means of the dual constraints. In general, they are of course always available as part of the optimal dual solution.

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7. - Appendix 1

Solution of the problem (3.2.4) in section 3.2 using the Generalized Arithmetic/geometric mean inequality

Given $x_i > 0$ and $\alpha_i \geq 0$, $i = 1, \ldots, n$, $\sum_i^n \alpha_i = 1$, for any $r > 0$

$$\left[\sum_{i=1}^{n} \alpha_i x_i^{\,r}\right]^{1/r} \geq \prod_{i=1}^{n} x_i^{\,r}, \tag{A1.1}$$

with equality if and only if all x_i are equal. For any $s > 0$, defining $y_i = x_i \alpha_i^{\,s}$, $i = 1, \ldots, n$ (A1.1) converts to

$$\left[\sum_i \alpha_i^{\,1-rs}\, y_i^{\,r}\right]^{1/r} \geq \prod_i^{n} \left(\frac{y_i}{\alpha_i^{\,s}}\right)^{\alpha_i} . \tag{A1.2}$$

Since $\sum_i \alpha_i = 1$, (A1.2) is satisfied as equality if and only if for $i = 1, \ldots, n$,

$$\alpha_i = \frac{y_i^{\,1/s}}{\sum_i y_i^{\,1/s}} \quad \text{for any } s. \tag{A1.3}$$

Substituting this value of α_i into the left side of (A1.2),

$$\left[\frac{\left(\sum_i y_i^{\frac{1-rs}{s}}\, y_i^{\,r}\right)}{\left(\sum_i y_i^{\,1/s}\right)^{1-rs}}\right]^{1/r} = \frac{\left(\sum_i y_i^{\,1/s}\right)^{1/r}}{\left(\sum_i y_i^{\,1/s}\right)^{1/r-s}} = \left(\sum_i y_i^{\,1/s}\right)^{s} \tag{A1.4}$$

In problem (3.2.4), $s = \frac{1+b}{b}$ and $y_i = p_i a_i^{\,1/b}$, which when substituted into (A1.3) and (A1.4), justifies the results shown in the main text, respectively, (3.2.5) and (3.2.6).

Appendix 2

Limit results for function (3.3.1)

We have the function:

$$y = F(x) = \left[\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}\right]^{-1/\alpha} \tag{A2.1}$$

Taking logarithms,

$$\log y = -\frac{\log\left[\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2}\right]}{\alpha} \tag{A2.2}$$

If $\alpha \to 0$, $\lim_{\alpha\to 0} (\log y) = \frac{0}{0}$, if

$$\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} = 1.$$

For resolving the indeterminacy, we use l'Hôpital's rule

$$\lim_{\alpha\to 0} (\log y) \equiv \lim_{\alpha\to 0} \left(\frac{-\partial/\partial\alpha \left(\log\left[\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}\right]\right)}{\partial\alpha/\partial\alpha}\right) \tag{A2.3}$$

Taking the derivatives:

$$\log y = \lim_{\alpha\to 0} \left(\frac{\frac{1}{2}\left(\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2} \log (x_i \cdot x_j)\right)}{\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} x_i^{-\alpha/2} x_j^{-\alpha/2}}\right) \tag{A2.4}$$

which is equal to

$$\log y = \sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} \log (x_i \cdot x_j) \tag{A2.5}$$

which can always be written as a Cobb-Douglas by choosing c_{ij} such that

$$\sum_{i=1}^{n} \sum_{j=i}^{n} c_{ij} = 1.$$

Appendix 3

Solving δ_i in terms of λ_1, λ_2^i, and p_i in (3.3.1.9) to (3.3.1.11)

From (3.3.1.18) and (3.3.1.19), we have

$$\delta_i = p_i e^{-\lambda_i} - \lambda_2^i - 1 \qquad i = 1, 2, 3 \qquad \text{(A3.1)}$$

$$\delta_{ii} = c_{ii} e^{-\alpha\lambda_2^i} - 1 \qquad i = 1, 2, 3$$

which implies that

$$\delta_{ii} = c_{ii} e^{-(\alpha-1)\lambda_2^i + \lambda_1 \frac{\delta_i}{p_i}} \qquad i = 1, 2, 3 \qquad \text{(A3.2)}$$

Also, from (3.3.1.19) and (3.3.1.20),

$$\delta_{ii}\delta_{jj} = c_{ii}c_{jj} e^{-\alpha(\lambda_2^i + \lambda_2^j) - 2} \qquad i,j = 1, 2, 3 \qquad \text{(A3.3)}$$

$$j \geq i$$

and

$$\delta_{ij} = \frac{c_{ij}}{c_{ii}c_{jj}} (\delta_{ii}\delta_{jj}) e^{\alpha/2(\lambda_2^i + \lambda_2^j) + 1} \qquad \text{(A3.4)}$$

or, using (A3.2) above,

$$\delta_{ij} = c_{ij} e^{-(\alpha/2-1)(\lambda_2^i + \lambda_2^j) + 2\lambda_1 + 1 \frac{\delta_i\delta_j}{p_i p_j}} \qquad i,j = 1, 2, 3 . \qquad \text{(A3.5)}$$

$$j \geq i$$

With these results, the system (3.3.1.9) to (3.3.1.11) becomes,

$$\delta_1 - \alpha c_{11} e^{-(\alpha-1)\lambda_2^1 + \lambda_1 \frac{\delta_1}{p_1}} - \alpha c_{12} e^{-(\alpha/2-1)(\lambda_2^1 + \lambda_2^2) + 2\lambda_1 + 1 \frac{\delta_1\delta_1}{p_1p_2}}$$

$$-\alpha c_{13} e^{-(\alpha/2-1)(\lambda_2^1 + \lambda_2^3) + 2\lambda_1 + 1 \frac{\delta_1\delta_3}{p_1p_2}} = 0. \qquad \text{(A3.6)}$$

$$\delta_2 - \alpha c_{22} e^{-(\alpha-1)\lambda_2^2 \frac{\delta_2}{p_2}} - \alpha c_{12} e^{-(\alpha/2-1)(\lambda_2^1 + \lambda_1^2) + 2\lambda_1 + 1 \frac{\delta_1\delta_2}{p_1p_2}}$$

$$-\alpha c_{23} e^{-(\alpha/2-1)(\lambda_2^2 + \lambda_2^3) + 2\lambda_1 + 1 \frac{\delta_2\delta_3}{p_2p_3}} = 0.$$

$$\delta_3 - \alpha c_{33} e^{-(\alpha-1)\lambda_2^3 + \lambda_1} \frac{\delta_3}{p_3} - \alpha c_{13} e^{-(\alpha/2-1)(\lambda_2^1 + \lambda_2^3) + 2\lambda_1 + 1} \frac{\delta_1 \delta_3}{p_1 p_3}$$

$$-\alpha c_{23} e^{-(\alpha/2-1)(\lambda_2^2 + \lambda_3^3) + 2\lambda_1 + 1} \frac{\delta_2 \delta_3}{p_2 p_3} = 0$$

which corresponds to (3.3.1.22) with (3.3.1.23) and (3.3.1.24) in the main text after dividing each equation in A3.6 by $\delta_{ii} \alpha e^{\lambda_1}$.