Working Paper

TIME SERIES IN LINEAR PROGRAMS WITH RANDOM RIGHT-HAND SIDES

Tomáš Cipra

May 1986
WP-86-21

International Institute for Applied Systems Analysis
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FOREWORD

One of the theoretical problems in stochastic optimization which can have important consequences for practical implementation consists in investigating programs whose coefficients are observable in time as time series. Some conclusions derived in this paper for linear programs under relatively simple statistical assumptions on the random right-hand sides stimulate further research in this direction.

The work was carried out within the Adaptation and Optimization Project of the System and Decision Sciences Program during the stay of the author as a guest scholar at IIASA.

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ABSTRACT

Linear programs such that the right-hand sides of their restrictions have the form of multivariate time series may be useful in practical applications. Behavior of the processes formed by the optimal values of the corresponding objective functions is investigated in the following cases: the right-hand side process is (*i*) a normal white noise; (*ii*) a normal white noise with a linear trend; (*iii*) a normal random walk. Some basic probability characteristics of such processes are calculated explicitly.

CONTENTS

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Tomáš Cipra

1. INTRODUCTION

Let us consider linear programs of the form

$$\{\min c'x : Ax = b_t, \ x \geq 0\}, \ t = \cdots, -1, 0, 1, \ldots \ , \tag{1.1}$$

where the matrix $A(m, n)$ and the vector $c(n, 1)$ are deterministic and $\{b_t\}$ is a m-dimensional process. Such general model may be applicable in various practical situations. The optimal values $\varphi(b_t)$ of (1.1) (if they exist) form obviously a scalar process the behavior of which we shall investigate.

Let us denote

$$S = \{b \in R^m : \varphi(b) \text{ is finite } \} \ . \tag{1.2}$$

Then according to [6] or [7] the function $\varphi(b)$ is convex, continuous and piecewise linear on S. Moreover, S can be decomposed to a finite number of convex polyhedral cones S_i $(i = 1, \ldots, k)$ with the vertices in the origin such that the interiors of S_i are mutually disjunct and $\varphi(b)$ is linear on each S_i. One can write

$$S_i = \{b \in R^m : H^i b \geq 0\}, \ i = 1, \ldots, k \tag{1.3}$$

and

$$\varphi(b) = g^{i\prime} b, \ b \in S_i, \ i = 1, \ldots, k \ , \tag{1.4}$$

where H^i are regular (m, m) matrices and g^i are $(m, 1)$ vectors (the vectors g^i need not be mutually different). One can also write

$$\varphi(b) = \max_{i=1, \ldots, k} \{g^{i\prime} b\}, \ b \in S \ . \tag{1.5}$$

The explicit form of H^i and g^i can be found by means of various algorithmic procedures (see e.g. [4], [5, p.276], [8], [9]).

EXAMPLE 1 (see [5]). In the program

$$\min\{x_2 + x_3 + 3x_4 \colon \begin{array}{l} x_1 - 2x_2 + x_3 - x_4 + x_5 = b_1 \\ 2x_1 + 3x_2 - x_3 + 2x_4 + x_6 = b_2 \\ -x_1 + 2x_2 + 3x_3 - 3x_4 + x_7 = b_3 \\ x_1, \ldots, x_7 \geq 0\} \end{array} \tag{1.6}$$

one can take

$$H^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad H^3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

$$H^4 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad H^5 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 0 \end{pmatrix}, \quad H^6 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix},$$

$$H^7 = \begin{pmatrix} -1 & 0 & -1 \\ 3 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad H^8 = \begin{pmatrix} -1 & 0 & -1 \\ -3 & 0 & 1 \\ 13 & 8 & 1 \end{pmatrix}, \quad H^9 = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 3 & 2 \\ 1 & -1 & -1 \end{pmatrix},$$

$$g^1 = g^2 = g^3 = g^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad g^5 = \begin{pmatrix} -1/2 \\ 0 \\ 0 \end{pmatrix}, \quad g^6 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

$$g^7 = \begin{pmatrix} -3/4 \\ 0 \\ -3/4 \end{pmatrix}, \quad g^8 = \begin{pmatrix} -9/8 \\ 0 \\ -5/8 \end{pmatrix}, \quad g^9 = \begin{pmatrix} 0 \\ -3/4 \\ -3/4 \end{pmatrix}.$$

The process $\{\varphi(b_t)\}$ originates as a piecewise linear (i.e. nonlinear in general) transformation of the process $\{b_t\}$. If one investigates stationarity of $\{\varphi(b_t)\}$ in dependence on stationarity of $\{b_t\}$ then it is obvious that $\{\varphi(b_t)\}$ need not be weakly stationary when $\{b_t\}$ has this property (i.e. when Eb_t and $\mathrm{cov}(b_t, b_{t-s})$ do not depend on t).

EXAMPLE 2 Let $m = 1$, $\varphi(b) = b^+ - 2b^-$ for $b \in R^1$ (where $b^+ = \max\{0, b\}$, $b^- = \min\{0, b\}$) and b_t be independent random variables such that

$$P(b_{2r+1} = 1) = P(b_{2r+1} = -1) = \frac{1}{2},$$

$$P(b_{2r} = 2) = P(b_{2r} = -2) = \frac{1}{8}, \; P(b_{2r} = 0) = \frac{3}{4}$$

for arbitrary integer r. Then

$$Eb_t = 0, \; \mathrm{var}\, b_t = 1, \; \mathrm{cov}(b_t, b_{t-s}) = 0 \text{ for } s \neq 0$$

for all t (i.e. $\{b_t\}$ is weakly stationary) but

$$E\varphi(b_{2r+1}) = \frac{3}{2}, \; E\varphi(b_{2r}) = \frac{3}{4}.$$

If $\{b_t\}$ is strongly stationary (i.e. the joint probability distribution of $(b_{t_1}, \ldots, b_{t_i})$ is equal to that of $(b_{t_1+s}, \ldots, b_{t_i+s})$ for all $i, t_1, \ldots, t_i, s$) then $\{\varphi(b_t)\}$ should have the same property but one must bear in mind that $\varphi(b_t)$ is not finite for $b_t \notin S$. Moreover, the explicit calculation of basic probability characteristics of $\{\varphi(b_t)\}$ (e.g. the mean value and autocovariances) may be very difficult even in simple situations. In order to demonstrate it the case with a two-dimensional normal white noise $\{b_t\}$ is studied in section 2. The derived formulas for $E\,\varphi(b_t)$ and var $\varphi(b_t)$ are so complicated that it turns up reasonable to recommend the simulation approach of Deák [3] for a more general case. The case of a stationary process $\{b_t\}$ with a constant mean value seems to be not very useful in practical situations. Therefore a m-dimensional normal process $\{b_t\}$ with a linear trend is considered in section 3. Finally, in order to provide potential generalization to the nonstationary integrated processes of Box and Jenkins which are capable to model trends in a stochastic way (see [1]) we deal with a one-dimensional normal random walk $\{b_t\}$ in section 4.

The following denotation will be used in the paper: a' and A' for the transpose of a vector a and a matrix A; $\|a\| = \sqrt{a'\,a}$ for $a \in R^m$; det A for the determinant of a square matrix A; $sgn\ (x) = 1$ for $x > 0$, $= 0$ for $x = 0$ and $= -1$ for $x < 0$; $x^+ = \max\ \{0, x\}$, $x^- = \min\ \{0, x\}$.

2. NORMAL WHITE NOISE

Let $\{b_t\}$ be a two-dimensional normal white noise, i.e.

$$b_t \sim iid\ N_2(0, \Sigma)\ , \tag{2.1}$$

where Σ is a positive definite variance matrix. Let T be a lower triangular matrix with positive elements on the main diagonal such that

$$\Sigma = TT' \tag{2.2}$$

(Cholesky decomposition) and let us denote

$$Q^i = H^i T\ , \tag{2.3}$$

where the matrix Q^i has the elements denoted as q^i_{uv} and the row vectors of the type (2,1) denoted as q^i_u $(u, v = 1,2)$.

LEMMA 1 *It holds*

$$P(b_t \in S) = (2\pi)^{-1} \sum_{i=1}^{k} \arccos \left[- \frac{q_1^{i\prime} q_2^i}{\|q_1^i\| \|q_2^i\|} \right] . \tag{2.4}$$

PROOF If using the method of substitution we have

$$P(b_t \in S) = \sum_{i=1}^{k} \iint_{\{x,\, y : Q^i (x,\, y)' \geq 0\}} (2\pi)^{-1} \exp\{-(x^2 + y^2)/2\}\, dx\, dy$$

$$= \sum_{i=1}^{k} \iint_{\{r,\, \vartheta : r \geq 0,\ Q^i (r \cos \vartheta,\, r \sin \vartheta)' \geq 0\}} (2\pi)^{-1} r \exp(-r^2/2) dr\, d\vartheta$$

$$= (2\pi)^{-1} \sum_{i=1}^{k} \int_{\{\vartheta : Q^i (\cos \vartheta,\, \sin \vartheta)' \geq 0\}} d\vartheta$$

The last integrals are equal to the values of the convex angles between q_1^i and $-q_2^i$ so that (2.4) is obvious now.

We can proceed to the calculation of $E\varphi(b_t)$ and var $\varphi(b_t)$. Since the probability (2.4) can be less than one in general the conditional values $E(\varphi(b_t)|b_t \in S)$ and var $(\varphi(b_t)|b_t \in S)$ have sense only.

THEOREM 1 *Under the previous assumptions it holds*

$$E(\varphi(b_t)|b_t \in S) = \frac{1}{P(b_t \in S)} (2\sqrt{2\pi})^{-1} \sum_{i=1}^{k} g^{i\prime}\, T(q_1^i / \|q_1^i\| + q_2^i / \|q_2^i\|) , \tag{2.5}$$

where $P(b_t \in S)$ *is given in (2.4).*

PROOF We can write

$$P(b_t \in S) E(\varphi(b_t) | b_t \in S) = \sum_{i=1}^{k} \int_{\{b : H^i b \geq 0\}} g^{i\prime} b (2\pi)^{-1} (\det \Sigma)^{-1/2}$$

$$\exp(-b' \Sigma^{-1} b / 2) db$$

$$= \sum_{i=1}^{k} g^{i\prime} T \int_{\{x,\, y : Q^i (x,\, y)' \geq 0\}} (x,\, y)' (2\pi)^{-1} \exp\{-(x^2 + y^2)/2\} dx\, dy$$

$$= \sum_{i=1}^{k} g^{i\prime} T \int_{\{r,\, \vartheta : r \geq 0,\ Q^i (r \cos \vartheta,\, r \sin \vartheta)' \geq 0\}} (\cos \vartheta,\, \sin \vartheta)' (2\pi)^{-1} r^2$$

$$\exp(-r^2/2) dr\, d\vartheta$$

$$= (2\sqrt{2\pi})^{-1} \sum_{i=1}^{k} g^{i\prime} T \int_{\{\vartheta : Q^i (\cos \vartheta,\, \sin \vartheta)' \geq 0\}} (\cos \vartheta,\, \sin \vartheta)' d\vartheta .$$

The variable ϑ is bounded by the angles corresponding to the couples of vectors $(q_{22}^i, -q_{21}^i)'$ and $(-q_{12}^i, q_{11}^i)'$ (if det $Q^i = q_{11}^i q_{22}^i - q_{12}^i q_{21}^i > 0$) or $(q_{12}^i, -q_{11}^i)'$ and $(-q_{22}^i, q_{21}^i)'$ (if det $Q^i < 0$). Since $\int \cos\vartheta\, d\vartheta = \sin\vartheta$ and $\int \sin\vartheta\, d\vartheta = -\cos\vartheta$ we have

$$P(b_t \in S)E(\varphi(b_t)|b_t \in S)$$

$$= (2\sqrt{2\pi})^{-1} \sum_{i=1}^{k} g^{i\prime} T(q_{11}^i/\|q_1^i\| + q_{21}^i/\|q_2^i\|,\ q_{12}^i/\|q_1^i\| + q_{22}^i/\|q_2^i\|)'$$

which is equivalent to (2.5).

REMARK 1 The formulas (2.4) and (2.5) can be rewritten to the form

$$P(b_t \in S) = (2\pi)^{-1}\arccos\{-h_1^{i\prime}\Sigma h_2^i / \left[h_1^{i\prime}\Sigma h_1^i h_2^{i\prime}\Sigma h_2^i\right]^{1/2}\} \ , \qquad (2.6)$$

$$E(\varphi(b_t)|b_t \in S) = \frac{1}{P(b_t \in S)}(2\sqrt{2\pi})^{-1} \qquad (2.7)$$

$$\sum_{i=1}^{k} g^{i\prime}\Sigma\{h_1^i\left[h_1^{i\prime}\Sigma h_1^i\right]^{-1/2} + h_2^i\left[h_2^{i\prime}\Sigma h_2^i\right]^{-1/2}\} \ ,$$

where h_u^i $(u = 1, 2)$ are the row vectors of the type (2,1) of H^i.

It is obvious that random variables $\varphi(b_t)$ are mutually independent; the following theorem evaluates their (conditional) variance.

THEOREM 2 *Under the previous assumptions it holds*

$$\text{var}\,(\varphi(b_t)|b_t \in S) = E(\varphi(b_t)^2|b_t \in S) - \{E(\varphi(b_t)|b_t \in S)\}^2 \ , \qquad (2.8)$$

where

$$E(\varphi(b_t)^2|b_t \in S) = \frac{1}{P(b_t \in S)}(2\pi)^{-1}\sum_{i=1}^{k} g^{i\prime} T\left[sgn\ (\det H^i)\left\{\frac{q_1^i p_1^{i\prime}}{q_1^{i\prime} q_1^i} \qquad (2.9)\right.\right.$$

$$\left.\left. + \frac{q_2^i p_2^{i\prime}}{q_2^{i\prime} q_2^i}\right\} + \arccos\left(-\frac{q_1^{i\prime} q_2^i}{\|q_1^i\|\,\|q_2^i\|}\right) I\right] T' g^i \ ,$$

$E(\varphi(b_t)|b_t \in S)$ is given in (2,5), *I is the* (2,2) *unit matrix and* $p_1^i = (-q_{12}^i, q_{11})'$, $p_2^i = (q_{22}^i, -q_{21}^i)'$.

PROOF One can write analogously as in the proof of Theorem 1

$$P(b_t \in S)E(\varphi(b_t)^2 | b_t \in S)$$

$$= \sum_{i=1}^{k} g^{i\prime} T \iint_{\{x,\, y\, :\, Q^i(x,\, y)' \geq 0\}} (x,\, y)'(x,\, y)(2\pi)^{-1}$$

$$\exp\{-(x^2 + y^2)/2\} \mathrm{d}x\, \mathrm{d}y\; T' g^i$$

$$= (\pi)^{-1} \sum_{i=1}^{k} g^{i\prime} T \int_{\{\vartheta\, :\, Q^i(\cos\vartheta,\, \sin\vartheta)' \geq 0\}} (\cos\vartheta,\, \sin\vartheta)'(\cos\vartheta,\, \sin\vartheta) d\vartheta\, T' g^i \ .$$

Since

$$\int \cos^2 \vartheta \,\mathrm{d}\vartheta = \frac{1}{2}(\sin\vartheta \cos\vartheta + \vartheta), \ \int \sin^2 \vartheta \,\mathrm{d}\vartheta = \frac{1}{2}(-\sin\vartheta\cos\vartheta + \vartheta) \ ,$$

$$\int \sin\vartheta \cos\vartheta \,\mathrm{d}\vartheta = \frac{1}{2} \sin^2\vartheta = \frac{1}{2} - \frac{1}{2}\cos^2\vartheta \tag{2.10}$$

and $sgn\,(\det Q^i) = sgn\,(\det H^i \det T) = sgn\,(\det H^i)$ we shall get (2.9) similarly as in the proof of Theorem 1.

3. PROCESS WITH LINEAR TREND

Let $\{b_t\}$ be a m-dimensional process of the form

$$b_t = b + at + \varepsilon_t,\ t = 0, 1, \dots \ , \tag{3.1}$$

where a and b are $(m, 1)$ fixed vectors ($a \neq 0$) and $\{\varepsilon_t\}$ is a m-dimensional normal white noise, i.e.

$$\varepsilon_t \sim iid\, N_m(0, \Sigma),\ \Sigma > 0 \ . \tag{3.2}$$

The linear model (3.1) is the usual model of multivariate time series used frequently in practice.

It is obvious that in this situation the behavior of the process $\{\varphi(b_t)\}$ depends substantially on the position of the vector a with respect to the sets S_i. If it is $a \notin S$ then obviously after certain time the process $\{\varphi(b_t)\}$ will not be finite with a large probability. We shall exclude this case from further considerations.

Now let us investigate the situation when a is an interior point of a set S_i. Then due to the properties of the convex polyhedral cone S_i when time t proceeds the process $\{\varphi(b_t)\}$ will have the form $\{g^{i\prime} b_t\}$ with a probability which grows in time and it enables to draw some conclusions on the behavior of this process. The following theorem evaluates the time period after which it is guaranteed with a given probability that $\{\varphi(b_t)\}$ lies in S_t. Let the denotation (2.2) and (2.3) be preserved.

THEOREM 3 *Let* $0 < \alpha < 1$ *be a given number, let* a *be an interior point of* S_t *and let* $\chi^2_m(\alpha)$ *be the critical value of the chi-squared distribution with* m *degrees of freedom on the level* α *(i.e.* $P(\chi^2_m \geq \chi^2_m(\alpha)) = \alpha$*). Then for* t *fulfilling*

$$t \geq \max_{u=1,\ldots,m} \{(\sqrt{\chi^2_m(\alpha)}\, \|q^t_u\| - h^{t\prime}_u b)/(h^{t\prime}_u a)\} \tag{3.3}$$

the values b_t *lie in* S_t *(i.e.* $\varphi(b_t) = g_t' \ b_t$*) with the probability at least* $1 - \alpha$ *(one can also use* $\|q^t_u\| = \sqrt{(h^{t\prime}_u \Sigma h^t_u)}$*).*

PROOF It holds for all t

$$P\{(b_t - b - at)' \Sigma^{-1} (b_t - b - at) \leq \chi^2_m(\alpha)\} = 1 - \alpha \ . \tag{3.4}$$

According to [2, Theorem 1] the $(1 - \alpha)100$ per cent confidence region

$$P(\alpha) = \{b_t : (b_t - b - at)' \Sigma^{-1} (b_t - b - at) \leq \chi^2_m(\alpha)\} \tag{3.5}$$

lies in S_t if and only if

$$h^{t\prime}_u (b + at) - \sqrt{\chi^2_m(\alpha)}\, \|q^t_u\| \geq 0, \ u = 1, \ldots, m \ . \tag{3.6}$$

Since a is the interior point of S_t it is $h^{t\prime}_u a > 0$ for $u = 1, \ldots, m$ and (3.6) is equivalent to

$$t \geq (\sqrt{\chi^2_m(\alpha)}\, \|q^t_u\| - h^{t\prime}_u b)/(h^{t\prime}_u a), \ u = 1, \ldots, m$$

so that the theorem is proved.

REMARK 2 Theorem 3 can be formulated for more general types of processes for which one is capable to calculate the confidence region in the form of an elipsoid as in (3.5) and the trend of which stays in a convex cone with the vertex in the origin contained (with except of the vertex) in the interior of S_t. Specially such natural generalization may be derived for the processes the trend of which has been estimated by means of the regression technique (see [2]).

If α is very small then for t fulfilling (3.3) one can approximate the probability characteristics of the process $\{\varphi(b_t)\}$ by the ones of the process $\{g^{t\prime} b_t\}$, e.g.

$$E(\varphi(b_t)|b_t \in S) \sim E g^{t\prime} b_t = g^{t\prime}(b + at) \ , \tag{3.7}$$

$$\operatorname{var}(\varphi(b_t)|b_t \in S) \sim \operatorname{var} g^{i\prime} b_t = g^{i\prime} \sum g^i \quad . \tag{3.8}$$

In some situations one can also desire the evaluation of the accuracy of such approximations. In the following theorem such evaluation is derived for the approximation (3.7) of the mean value.

THEOREM 4 *Let under the assumptions of Theorem 3 t fulfill (3.3). Let us denote* $c = \chi^2_m(\alpha)$, Φ *the distribution function of the standard normal distribution* $N(0, 1)$ *and*

$$v = \max_{j=1,\ldots,k} \sqrt{g^{j\prime}\Sigma g^j} \quad . \tag{3.9}$$

Then it holds

$$(1-\alpha)g^{i\prime}(b+at) \le E(\varphi(b_t)|b_t \in S) \le g^{i\prime}(b+at) \quad ,$$

$$+ \frac{1}{1-\alpha}\left[\alpha \max_{j=1,\ldots,k} \{g^{j\prime}(b+at)\} + vV_m(c)\right] \tag{3.10}$$

where

$$V_2(c) = [1 - 2\{\Phi(\sqrt{c}) - 1/2\}] + \sqrt{c}\ \exp(-c/2) \quad ,$$

$$V_m(c) = (\pi/2)^{1/2}\frac{(m-1)(m-3)\cdots 1}{(m-2)(m-4)\cdots 2}[1 - \{\Phi(\sqrt{c}) - 1/2\}]$$

$$+ \frac{1}{(m-2)(m-4)\cdots 2}\sqrt{c}\,\{c^{(m-2)/2} + (m-1)c^{(m-4)/2}$$

$$+ (m-1)(m-3)c^{(m-6)/2} + \cdots$$

$$\cdots + (m-1)(m-3)\cdots 3\}\exp(-c/2)\ \textit{for even}\ m \ge 4 \quad ,$$

$$= (\pi/2)^{1/2}\frac{1}{(m-2)(m-4)\cdots 1}\{c^{(m-1)/2} + (m-1)c^{(m-3)/2}$$

$$+ (m-1)(m-3)c^{(m-5)/2} + \cdots$$

$$\cdots + (m-1)(m-3)\cdots 2\}\exp(-c/2)\ \textit{for odd}\ m \ge 3 \quad .$$

PROOF Let us denote f_t the probability density of the distribution $N_m(b + at, \Sigma)$. As the lower bound in (3.10) is concerned it is obviously

$$E(\varphi(b_t)|b_t \in S) = \frac{1}{P(b_t \in S)}\int_S \varphi(b_t)f_t(b_t)\mathrm{d}b_t \ge \int_S g^{i\prime} b_t f_t(b_t)\mathrm{d}b_t$$

$$\ge \int_{P(\alpha)} g^{i\prime} b_t f_t(b_t)\mathrm{d}b_t = (1-\alpha)g^{i\prime}(b+at) \quad .$$

The upper bound in (3.10) can be derived in the following way:

$$E(\varphi(b_t)|b_t \in S) \le \frac{1}{1-\alpha}\int_{R^m} \varphi(b_t) f_t(b_t)\mathrm{d}b_t$$

$$= \frac{1}{1-\alpha}\left[\int_{P(\alpha)} g^{i\prime} b_t f_t(b_t)\mathrm{d}b_t + \int_{R^m/P(\alpha)} \max_{j=1,\dots,k}\{g^{j\prime} b_t\} f_t(b_t)\mathrm{d}b_t\right]$$

$$\le \frac{1}{1-\alpha}\left[(1-\alpha)g^{i\prime}(b+\alpha t) + \int_{R^m/P(\alpha)} \max_{j=1,\dots,k}\{g^{j\prime}(b+\alpha t)\} f_t(b_t)\mathrm{d}b_t\right.$$

$$\left. + \int_{R^m/P(\alpha)} \max_{j=1,\dots,k}\{g^{j\prime}(b_t - b - \alpha t)\} f_t(b_t)\mathrm{d}b_t\right]$$

$$= g^{i\prime}(b+\alpha t) + \frac{1}{1-\alpha}\left[\alpha \max_{j=1,\dots,k}\{g^{j\prime}(b+\alpha t)\}\right.$$

$$\left. + \int_{\{y\in R^m: y'y > c\}} \max_{j=1,\dots,k}\{g^{j\prime} Ty\}(2\pi)^{-m/2}\exp\{-y'y/2\}\mathrm{d}y\right]$$

$$\le g^{i\prime}(b+\alpha t) + \frac{1}{1-\alpha}\left[\alpha \max_{j=1,\dots,k}\{g^{j\prime}(b+\alpha t)\}\right.$$

$$+ \iint_{\{r,\vartheta_1,\dots,\vartheta_{m-1}: r\ge\sqrt{c},\ 0\le\vartheta_1<2\pi,\ -\pi/2\le\vartheta_2\le\pi/2,\dots,-\pi/2\le\vartheta_{m-1}\le\pi/2\}} vr(2\pi)^{-m/2}$$

$$\left.\exp(-r^2/2)r^{m-1}\cos\vartheta_2\cos^2\vartheta_3\cdots\cos^{m-2}\vartheta_{m-1}\mathrm{d}r\,\mathrm{d}\vartheta_1\cdots\mathrm{d}\vartheta_{m-1}\right],$$

where the last inequality holds due to the fact that outside the elipsoid $\{y \in R^m : y'y \le c\}$ the graph of the function max $\{g^{j\prime} Ty\}$ can be dominated by the surface of the cone C in R^{m+1} with the vertex in the origin of the form

$$C = \{y \in R^m,\ z \in R^1 : z = \max_{j=1,\dots,k}\{\|T'g^j\|\}\|y\|\},$$

where max $\{\|T'g^j\|\}$ = max $\sqrt{g^{j\prime}\Sigma g^j} = v$ (the description of the mentioned surface in the polar coordinates is used with the Jacobian r^{m-1} $\cos\vartheta_2\cdots\cos^{m-2}\vartheta_{m-1}$). The final form of the upper bound can be derived using the formulas

$$\int_{\{0\le\vartheta_1<2\pi,\ -\pi/2\le\vartheta_2\le\pi/2,\dots,-\pi/2\le\vartheta_{m-1}\le\pi/2\}} \cos\vartheta_2\cos^2\vartheta_3 \cdots\cos^{m-2}\vartheta_{m-1}\mathrm{d}\vartheta_1\cdots\mathrm{d}\vartheta_{m-1} = 2^{m-1}\pi a_1 a_2\cdots a_{m-2}, \qquad (3.11)$$

where

$$a_i = \int_0^{\pi/2} \cos^i x \, dx \quad ,$$

i.e. $a_1 = 1$, $a_2 = \pi/4$ and

$$a_i = (i-1)(i-3)\cdots 2/\{i(i-2)\cdots 1\} \text{ for odd } i \geq 3 \quad ,$$

$$= (\pi/2)(i-1)(i-3)\cdots 1/\{i(i-2)\cdots 2\} \text{ for even } i \geq 4 \quad ;$$

$$\int_{\sqrt{c}}^{\infty} r^m \exp(-r^2/2) dr \tag{3.12}$$

$$= \{c^{(m-1)/2} + (m-1)c^{(m-3)/2} + (m-1)(m-3)c^{(m-5)/2} + \cdots$$

$$\cdots + (m-1)(m-3)\cdots 2\} \exp(-c/2) \text{ for odd } m \geq 1 \quad ,$$

$$= (m-1)(m-3)\cdots 1(\pi/2)^{1/2}[1 - 2\{\Phi(\sqrt{c}) - 1/2\}]$$

$$+ \sqrt{c}\{c^{(m-2)/2} + (m-1)c^{(m-4)/2} + (m-1)(m-3)c^{(m-6)/2} + \cdots$$

$$\cdots + (m-1)(m-3)\cdots 3\} \exp(-c/2) \text{ for even } m \geq 2 \quad .$$

REMARK 3 For $m = 1$ one can calculate $E(\varphi(b_t)|b_t \in S)$ exactly. If e.g. $S = R^1$ (i.e. $S_1 = (-\infty, 0]$ and $S_2 = [0, \infty)$), $\varphi(b) = g_1 b^- + g_2 b^+$ for $b \in R^1$ (where $g_1, g_2 \in R^1$), $b_t \sim N(\mu, \sigma^2)$ (where $\mu = b + at$) and $P(\alpha) = [c_1, c_2]$ (where $-\infty < c_1 < c_2 < \infty$) then

$$E(\varphi(b_t)|b_t \in S) = E\varphi(b_t)$$

$$= g_1[\mu\{\Phi(C_2) - \Phi(C_1)\} + \sigma(2\pi)^{-1/2}\{\exp(-C_1^2/2) - \exp(-C_2^2/2)\}] \text{ for } c_2 \leq 0 \quad ,$$

$$= g_1[\mu\{1/2 - \Phi(C_1)\} + \sigma(2\pi)^{-1/2}\{\exp(-C_1^2/2) - \exp(-(\mu/\sigma)^2/2\}]$$

$$+ g_2[\mu\{\Phi(C_2) - 1/2\} +$$

$$\sigma(2\pi)^{-1/2}\{\exp(-(\mu/\sigma)^2/2 - \exp(-C_2^2/2)\}] \text{ for } c_1 < 0 < c_2 \quad ,$$

$$= g_2[\mu\{\Phi(C_2) - \Phi(C_1)\} + \sigma(2\pi)^{-1/2}\{\exp(-C_1^2/2) - \exp(-C_2^2/2)\}] \text{ for } c_1 \geq 0 \quad ,$$

where $C_i = (c_i - \mu)/\sigma$, $i = 1, 2$.

In Table 1 there are given $V_m(c)$ for some values m if $\alpha = 0.05$ and $\alpha = 0.01$ ($c = \chi_m^2(\alpha)$). For larger even m the first term in the corresponding formula for the calculation of $V_m(c)$ can be omitted since then $\Phi(\sqrt{c}) \sim 1$ (e.g. for $m \geq 4$ if $\alpha = 0.05$).

Table 1 Values $V_m(c)$ if a) $\alpha = 0.05$ and b) $\alpha = 0.01$.

a) $\alpha = 0.05$:

m	c	$V_m(c)$
2	5.99	0.141
3	7.81	0.157
4	9.49	0.167
5	11.07	0.183
10	18.31	0.230
25	37.65	0.323
50	67.50	0.427
90	113.15	0.547

b) $\alpha = 0.01$:

m	c	$V_m(c)$
2	9.21	0.033
3	11.34	0.037
4	13.28	0.039
5	15.09	0.042
10	23.21	0.051
25	44.31	0.069
50	76.15	0.090
90	124.12	0.114

Now let us consider the case when a is a relative interior point of a $(m-1)$-dimensional face in which two cones $S_i = \{x \in R^m : H^i x \geq 0\}$ and $S_j = \{x \in R^m : H^j x \geq 0\}$ adjoin. One can assume (renumbering the rows of H^i and H^j if it is necessary) that this face has the form

$$\{x \in R^m : h_1^{i\prime} x = 0,\ h_u^{i\prime} x \geq 0,\ h_u^{j\prime} x \geq 0,\ u = 2, \ldots, m\}\ , \tag{3.13}$$

where $h_1^i = \lambda h_1^j$ for some negative scalar λ.

EXAMPLE 3 In the situation described in Example 1 e.g. the vector $a = (0, 2, 5)'$ is the relative interior point of the two-dimensional face $\{x \in R^3 : x_1 = 0, x_2 \geq 0, x_3 \geq 0\}$ in which the cones S_1 and S_5 adjoin. In this case it is $h_1^1 = -h_1^5 = (1, 0, 0)'$ so that it is not necessary to renumber the rows of the matrices H^1 and H^5.

The following theorem can be proved quite analogously as Theorem 3.

THEOREM 5 *Let* $0 < \alpha < 1$ *be a given number and let* a *be a relative interior point of the* $(m-1)$*-dimensional face (3.13) in which two cones* S_i *and* S_j *adjoin. Then for* t *fulfilling*

$$t \geq \max_{\nu = i, j}\ \max_{u = 1, \ldots, m} \left\{ \left[\sqrt{\chi_m^2(\alpha)}\, \|q_u^\nu\| - h_u^{\nu\prime} b \right] / h_u^{\nu\prime} a \right\} \tag{3.14}$$

the values $\varphi(b_t)$ *lie in* S_i *or* S_j *with the probability at least* $1 - \alpha$.

This theorem enables again to approximate the probability characteristics of the process $\{\varphi(b_t)\}$ for t fulfilling (3.14) if α is small. E.g. we can write for the mean value

$$E(\varphi(b_t) | b_t \in S) \sim \int_{\{b_t \in R^m : h_1^i b_t \geq 0\}} g^{i\prime} b_t\, f_t(b_t) \mathrm{d} b_t$$

$$+ \int_{\{b_t \in R^m : h_1^{t\prime} b_t \leq 0\}} g^{j\prime} b_t f_t(b_t) \mathrm{d}b_t \quad ,$$

where $f_t(b_t)$ is the density function of $N_m(\mu, \Sigma)$ with $\mu = b + at$. Let R be a (m, m) matrix such that $\Sigma = RR'$ and the first row of R^{-1} has the same direction as the vector h_1^t (such matrix R can be always constructed). Then $h_1^{t\prime} R$ has the same direction as the vector $(1, 0, \dots, 0)$. Let us denote

$$\nu = R^{-1}\mu, \; d^t = R' g^t \quad . \tag{3.15}$$

Then it holds e.g.

$$\int_{\{b_t \in R^m : h_1^{t\prime} b_t \geq 0\}} g^{t\prime} b_t f_t(b_t) \mathrm{d}b_t$$

$$= \int_{\{x \in R^m : x_1 \geq -\nu_1\}} d^{t\prime}(x + \nu)(2\pi)^{-m/2} \exp(-x'x/2) \mathrm{d}x$$

$$= d_1^t \int_{-\nu_1}^{\infty} x_1 (2\pi)^{-1/2} \exp(-x_1^2/2) \mathrm{d}x_1 + g^{t\prime}\mu \int_{-\nu_1}^{\infty} (2\pi)^{-1/2} \exp(-x_1^2/2) \mathrm{d}x_1$$

$$= (2\pi)^{-1/2} d_1^t \exp(-\nu_1^2/2) + g^{t\prime}\mu\{1 - \Phi(-\nu_1)\} \quad .$$

Altogether we shall obtain

$$E(\varphi(b_t) | b_t \in S) \sim (2\pi)^{-1/2}(d_1^t - d_1^j) \exp(-\nu_1^2/2) \tag{3.16}$$

$$+ [\{1 - \Phi(-\nu_1)\}] g^t + \Phi(-\nu_1) g^j]' \mu$$

and similarly

$$\mathrm{var}(\varphi(b_t) | b_t \in S) \sim (2/\pi)^{1/2}(d_1^t g^t - d_1^j g^j)' \mu \exp(-\nu_1^2/2) \tag{3.17}$$

$$+ \{1 - \Phi(-\nu_1)\}\{(g^{t\prime}\mu)^2 + g^{t\prime} \Sigma g^t\} + \Phi(-\nu_1)\{(g^{j\prime}\mu)^2 + g^{j\prime} \Sigma g^j\}$$

$$+ (2\pi)^{-1/2}\{(d_1^t)^2 - (d_1^j)^2\} |\nu_1| \exp(-\nu_1^2/2) - \{E(\varphi(b_t) | b_t \in S)\}^2 \quad .$$

4. RANDOM WALK

Random walk is the simplest case of the integrated processes ARIMA of Box and Jenkins. These processes are nonstationary but this nonstationarity can be removed easily by differencing the original process. Since these processes are very useful for practical purposes it is important to investigate whether this type of nonstationarity is preserved also for the processes $\{\varphi(b_t)\}$. We shall confine ourselves to the one-dimensional case with a normal random walk $\{b_t\}$ of the form

$$b_t = \sum_{i=1}^{t} \varepsilon_i, \ t = 1, 2, \dots \ , \tag{4.1}$$

where $\{\varepsilon_t\}$ is a normal white noise, i.e.

$$\varepsilon_t \sim iid\, N(0, \sigma^2) \ . \tag{4.2}$$

Let the function $\varphi(b)$ be finite for all $b \in R^1$ so that it has the form

$$\varphi(b) = g_1 b^- + g_2 b^+, \ b \in R^1 \ , \tag{4.3}$$

where g_1 and g_2 are given real numbers. We shall investigate the behavior of the process $\{\varphi(b_{t+1}) - \varphi(b_t)\}$ (the process $\{b_{t+1} - b_t\} = \{\varepsilon_{t+1}\}$ is stationary).

THEOREM 6 *Under the previous assumptions it holds*

$$E\{\varphi(b_{t+1}) - \varphi(b_t)\} = \sigma(2\pi)^{-1/2}(g_1 + g_2)\left[\sqrt{t+1} - \sqrt{t}\right] \to 0 \tag{4.4}$$

when $t \to \infty$.

PROOF It is

$$\sum_{i=1}^{t} \varepsilon_i \sim N(0, t\sigma^2) \tag{4.5}$$

so that

$$E\left[\sum_{i=1}^{t} \varepsilon_i\right]^+ = E\left[\sum_{i=1}^{t} \varepsilon_i\right]^- = \sigma(2\pi)^{-1/2} t^{1/2} \ .$$

Hence the assertion of the theorem follows.

THEOREM 7 *Under the previous assumptions it holds for arbitrary* $k \geq 0$

$$\operatorname{cov}\{\varphi(b_{t+k+1}) - \varphi(b_{t+k}), \ \varphi(b_{t+1}) - \varphi(b_t)\} \tag{4.6}$$

$$= (g_1^2 + g_2^2)\{C(t + k + 1, t + 1) + C(t + k, t)$$

$$- C(t+k+1, t) - C(t+k, t+1)\}$$

$$+ 2g_1g_2\{D(t+k+1, t+1) + D(t+k, t)$$

$$- D(t+k+1, t) - D(t+k, t+1)\}$$

$$- \sigma^2/(2\pi)(g_1+g_2)^2(\sqrt{t+k+1} - \sqrt{t+k})(\sqrt{t+1} - \sqrt{t}) ,$$

where

$$C(t+k, t) = \frac{\sigma^2 t}{2} + \frac{\sigma^2 k\sqrt{kt}}{2\pi(t+k)} + \frac{\sigma^2 t}{2\pi}\left[\frac{\sqrt{kt}}{t+k} - \arcsin\sqrt{\frac{k}{t+k}}\right] , \quad (4.7)$$

$$D(t+k, t) = -\frac{\sigma^2 k\sqrt{kt}}{2\pi(t+k)} + \frac{\sigma^2 t}{2\pi}\left[\frac{\sqrt{kt}}{t+k} - \arcsin\sqrt{\frac{k}{t+k}}\right] \quad (4.8)$$

and $C(t, t+k) = C(t+k, t)$, $D(t, t+k) = D(t+k, t)$. *Moreover, it is when* $t \to \infty$

$$\mathrm{cov}\,\{\varphi(b_{t+k+1}) - \varphi(b_{t+k}), \varphi(b_{t+1}) - \varphi(b_t)\} \to (\sigma^2/2)(g_1^2 + g_2^2) \text{ for } k = 0 , \quad (4.9)$$

$$\to 0 \qquad \text{for } k > 0 .$$

REMARK 4 Specially it holds

$$\mathrm{var}\,\{\varphi(b_{t+1}) - \varphi(b_t)\} \quad (4.10)$$

$$= (g_1^2 + g_2^2)\left\{\frac{\sigma^2}{2} - \frac{\sigma^2\sqrt{t}}{\pi(t+1)} + \frac{\sigma^2 t}{\pi}\left[\arcsin\sqrt{\frac{1}{t+1}} - \frac{\sqrt{t}}{t+1}\right]\right\}$$

$$+ 2g_1g_2\left\{\frac{\sigma^2\sqrt{t}}{\pi(t+1)} + \frac{\sigma^2 t}{\pi}\left[\arcsin\sqrt{\frac{1}{t+1}} - \frac{\sqrt{t}}{t+1}\right]\right\}$$

$$- \sigma^2/(2\pi)(g_1+g_2)^2\left[\sqrt{t+1} - \sqrt{t}\right]^2 .$$

PROOF Let us denote

$$C(t+k, t) = EY^+Z^+ ,$$

where

$$Y = \sum_{i=1}^{t+k} \varepsilon_i , \; Z = \sum_{i=1}^{t} \varepsilon_i .$$

The joint density $f(y, z)$ of Y and Z has the form

$$f(y, z) = g(y \mid z) h(z) \quad ,$$

where

$$g(y \mid z) = (2\pi k \sigma^2)^{-1/2} \exp \{-(y - z)^2 / (2k\sigma^2)\}$$

is the conditional density of Y for fixed Z and

$$h(z) = (2\pi t \sigma^2)^{-1/2} \exp \{-z^2 / (2t\sigma^2)\}$$

is the marginal density of Z. Hence it is

$$C(t + k, t) = (2\pi\sigma^2)^{-1}(kt)^{-1/2} \int_0^\infty \int_0^\infty yz$$

$$\exp \{-(y - z)^2 / (2k\sigma^2) - z^2 / (2t\sigma^2)\} \mathrm{d}y \, \mathrm{d}z$$

$$= (2\pi\sigma^2)^{-1}(kt)^{-1/2} \int_0^\infty \int_0^\infty yz \exp \{-(1/2)(y, z)\Sigma^{-1}(y, z)'\} \mathrm{d}y \, \mathrm{d}z \quad ,$$

where

$$\Sigma^{-1} = (kt\sigma^2)^{-1} \begin{bmatrix} t & -t \\ -t & t + k \end{bmatrix} .$$

We can write

$$\Sigma = \sigma^2 \begin{bmatrix} t + k & t \\ t & t \end{bmatrix} = TT' \quad ,$$

where

$$T = \sigma \begin{bmatrix} \sqrt{t + k} & 0 \\ t / \sqrt{t + k} & \sqrt{kt / (t + k)} \end{bmatrix}$$

If using the method of substitution and the formulas (2.10) and (3.12) we can further write

$$C(t + k, t)$$

$$= \sigma^2 / (2\pi) \iint_{\{u, v : (\sqrt{t + k}) u \geq 0, \ \left[t / \sqrt{t + k}\right] u + \sqrt{kt / (t + k)} v \geq 0\}}$$

$$(\sqrt{t + k}) u \left\{ \left[t / \sqrt{t + k} \right] u + (\sqrt{kt / (t + k)}) v \right\} \exp \{-(u^2 + v^2)/2\} \mathrm{d}u \, \mathrm{d}v$$

$$= \sigma^2 t/(2\pi) \iint_{\{u,\, v : u \ge 0,\, v \ge -(\sqrt{t/k})u\}} u^2 \exp\{-(u^2+v^2)/2\} du\, dv$$

$$+ \sigma^2 \sqrt{kt}/(2\pi) \iint_{\{u,\, v : u \ge 0,\, v \ge -(\sqrt{t/k})u\}} uv \exp\{-(u^2+v^2)/2\} du\, dv$$

$$= \sigma^2 t/(2\pi) \int_0^\infty r^3 \exp(-r^2/2) dr \int_{-\operatorname{arctg}\sqrt{t/k}}^{\pi/2} \cos^2 \vartheta \, d\vartheta$$

$$+ \sigma^2 \sqrt{kt}/(2\pi) \int_0^\infty u \exp(-u^2/2)\{\int_{-(\sqrt{t/k})u}^{\infty} v \exp(-v^2/2) dv\} du$$

$$= \sigma^2 t/(2\pi)\{\pi - \arcsin\sqrt{k/(t+k)} + \sqrt{kt}/(t+k)\} + \sigma^2 k\sqrt{kt}/\{2\pi(t+k)\} ,$$

which coincides with (4.7). It can be shown similarly that $D(t+k, t)$ defined as

$$D(t+k,t) = E\left[\sum_{i=1}^{t+k} \varepsilon_i\right]^+ \left[\sum_{i=1}^{t} \varepsilon_i\right]^-$$

coincides with (4.8). If we notice that

$$E\left[\sum_{i=1}^{t+k} \varepsilon_i\right]^- \left[\sum_{i=1}^{t} \varepsilon_i\right]^- = E\left[\sum_{i=1}^{t+k} \varepsilon_i\right]^+ \left[\sum_{i=1}^{t} \varepsilon_i\right]^+ ,$$

$$E\left[\sum_{i=1}^{t+k} \varepsilon_i\right]^- \left[\sum_{i=1}^{t} \varepsilon_i\right]^+ = E\left[\sum_{i=1}^{t+k} \varepsilon_i\right]^+ \left[\sum_{i=1}^{t} \varepsilon_i\right]^-$$

then after some algebraic manipulation the formula (4.6) follows. Finally, it is possible to show (e.g. by means of l'Hospital rule) that

$$\lim_{t\to\infty} \{C(t+k,t) - \sigma^2 t/2\} = \lim_{t\to\infty} D(t+k,t) = 0$$

so that (4.9) follows.

One can summarize that the process $\{\varphi(b_{t+1}) - \varphi(b_t)\}$ where $\{b_t\}$ is the random walk (4.1) is not (weakly) stationary but approximately for large t one can take it as the (stationary) white noise with the variance $(\sigma^2/2)\,(g_1^2 + g_2^2)$.

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