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STABILITY AND SENSITIVITY ANALYSIS IN CONVEX VECTOR OPTIMIZATION

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Preface

In this paper stability and sensitivity of the efficient set in convex vector optimization are considered. The perturbation map is defined as a set-valued map which associates, with each parameter vector, the set of all minimal points of the parametrized feasible set with respect to an ordering cone in the objective space. Sufficient conditions for the upper and lower semicontinuity of the perturbation map are obtained. Because of the convexity assumptions, the conditions obtained are fairly simple if compared with those in the general case. Moreover, a complete characterization of the contingent derivative of the perturbation map is obtained under some assumptions. It provides a quantitative information on the behavior of the perturbation map and allows to investigate the sensitivity of the efficient set with respect to the perturbations of the problem parameters.

Contents

1. Introduction	1
2. Convexity assumption and preliminary results	2
3. Upper semicontinuity of the perturbation map	6
4. Lower semicontinuity of the perturbation map	8
5. Contingent derivative of the perturbation map	15
6. Conclusion	23
References	24

STABILITY AND SENSITIVITY ANALYSIS IN CONVEX VECTOR OPTIMIZATION

Tetsuzo Tanino

1. Introduction

In this paper we consider a family of parametrized vector optimization problems:

$$\begin{cases} P\text{-minimize} & f(x,u) = (f_1(x,u),\dots,f_p(x,u)) \\ \text{subject to} & x \in X(u) \subset R^n \end{cases} \tag{1.1}$$

Here x is an n-dimensional decision variable, u is an m-dimensional parameter vector, f_i $(i = 1,\dots,p)$ is a real valued objective function on $R^n \times R^m$, X is a set-valued map from R^m to R^n, which specifies a feasible decision set and P is a nonempty pointed closed convex ordering cone in R^p. We can define another set-valued map Y from R^m to R^p by

$$Y(u) := \{y \in R^p \mid y = f(x,u),\ x \in X(u)\} \ . \tag{1.2}$$

$Y(u)$ is the parametrized feasible set in the objective space. The cone P induces a partial order $\leqq_P$ on R^p, that is, we define the relation $\leqq_P$ by

$$y \leqq_P y' \leftrightarrow y' - y \in P \quad \text{for} \quad y, y' \in R^p \ . \tag{1.3}$$

This relation $\leqq_P$ is reflexive, antisymmetric and transitive. In the problem (1.1), we aim to obtain all the minimal points of the feasible set $Y(u)$ with respect to the order $\leqq_P$. In other words, the solution set in the objective space to the problem (1.1) is given by

$$\begin{aligned} Min_P Y(u) &= \{\hat{y} \in Y(u) \mid \text{there exists no} \quad y \neq \hat{y} \quad \text{such that} \quad y \leqq_P \hat{y}\} \ , \\ &= \{\hat{y} \in Y(u) \mid (Y(u) - \hat{y}) \cap (-P) = \{0\}\} \end{aligned} \tag{1.4}$$

Therefore, we can define another set-valued map W from the parameter space R^m to the objective space R^p by

$$W(u) := Min_P Y(u) \quad . \tag{1.5}$$

W is often called the perturbation map for (1.1).

In usual scalar optimization where $p = 1$ and $P = R_+$ (= the set of nonnegative real numbers) W is at most single-valued and so it can be identified with the function

$$w(u) := \min \{f(x,u) \mid x \in X(u)\} \quad . \tag{1.6}$$

And the stability and sensitivity analysis in scalar optimization is mainly a study of continuity properties and derivatives of the function w. In case of vector optimization, we investigate the behavior of the set-valued map W.

Some results for general vector optimization problems from this point of view can be seen, for example, in [2], [7] for stability and in [6] for sensitivity. In this paper we consider the case in which convexity is assumed. It is shown that the convexity assumption considerably simplifies the sufficient conditions for the semicontinuity of the perturbation map W and also makes it possible to characterize the contingent derivative of W completely.

2. Convexity assumption and preliminary results

Throughout this paper we assume the following convexity on the feasible decision set map X and the objective function f.

Convexity Assumption (CA)

(1) The set-valued map X is convex, i.e., the graph of X which is defined by

$$graph X = \{(u,x) \mid x \in X(u)\} \tag{2.1}$$

is a convex set in $R^m \times R^n$. In other words, for any $u^1, u^2 \in R^m$ and any α, $0 \leqq \alpha \leqq 1$,

$$\alpha X(u^1) + (1-\alpha)X(u^2) \subset X(\alpha u^1 + (1-\alpha)u^2) \quad . \tag{2.2}$$

(2) The function f is P-convex, i.e. for any (x^1,u^1), $(x^2,u^2) \in R^n \times R^m$ and any α, $0 \leqq \alpha \leqq 1$,

$$\alpha f(x^1,u^1) + (1-\alpha)f(x^2,u^2) \in f(\alpha x^1 + (1-\alpha)x^2, \alpha u^1 + (1-\alpha)u^2) + P \quad .$$

Lemma 2.1. If P is a pointed closed convex cone and f is P-convex, then f is continuous.

(Proof). Since P is a pointed closed convex cone, the interior of the negative polar cone P^o of P is not empty.† It is easy to prove that $-< \mu, f(x,u) >$ is convex as a function of (x,u) for $\mu \in P^o$. Hence $< \mu, f(\cdot,\cdot) >$ is continuous ([3], Corollary 10.1.1). Take $\bar{\mu} \in int\ P^o$ and $\bar{\mu} + \delta e^i \in P^o$ for sufficiently small $\delta > 0$, where e^i is the ith unit vector in R^p. Then both $< \bar{\mu}, f(\cdot,\cdot) >$ and $< \bar{\mu} + \delta e^i, f(\cdot,\cdot) >$ are continuous and hence $f_i(\cdot,\cdot)$ is continuous ($i = 1,...,p$). Namely f is continuous. ■

Proposition 2.1. Under the convexity assumption (CA), the set-valued map Y defined by (1.2) is P-convex, i.e., for any $u^1, u^2 \in R^m$ and α, $0 \leqq \alpha \leqq 1$,

$$\alpha Y(u^1) + (1-\alpha)Y(u^2) \subset Y(\alpha u^1 + (1-\alpha)u^2) + P \quad . \tag{2.3}$$

In other words, the graph of the set-valued map $Y + P$ is convex. Here $Y + P$ is defined by

$$(Y + P)(u) := Y(u) + P \quad \text{for each} \quad u \in R^m \quad . \tag{2.4}$$

(Proof). This proposition can be easily proved. ■

Now we introduce concepts of semicontinuity of set-valued maps. Let F be a set-valued map from R^m to R^p hereafter in this section.

Definition 2.1. 1) F is said to be upper semicontinuous at $\hat{u} \in R^m$ if $u^k \to \hat{u}, y^k \in F(u^k)$ and $y^k \to \hat{y}$ all imply that $\hat{y} \in F(\hat{u})$.

2) F is said to be lower semicontinuous at $\hat{u} \in R^m$ if $u^k \to \hat{u}$ and $\hat{y} \in F(\hat{u})$ imply the existence of an integer K and a sequence $\{y^k\} \subset R^p$ such that $y^k \in F(u^k)$ for $k \geqq K$ and $y^k \to \hat{y}$.

3) F is said to be continuous at $\hat{u} \in R^m$ if it is both upper and lower semicontinuous at $\hat{u}$.

Remark 2.1. F is upper semicontinuous on R^m if and only if *graph* F is a closed set in $R^m \times R^p$.

† $P^o = \{\mu \in R^p \mid < \mu, d > \leqq 0$ for all $d \in P\}$, where $< \cdot, \cdot >$ denotes the inner product.

We shall provide lemmas concerning the semicontinuity of convex set-valued maps. Given F and $\hat{y} \in R^p$, we define the function ρ from R^m to $R \cup \{+\infty\}$ by

$$\rho(u) = dist(\hat{y}, F(u)) := inf \ \{\|y - \hat{y}\| \mid y \in F(u)\} \quad . \tag{2.5}$$

If $F(u) = \phi$, let $\rho(u) = +\infty$. The domain of the set- valued map F is defined and denoted by

$$dom \ F := \{u \in R^m \mid F(u) \neq \phi\} \quad . \tag{2.6}$$

Clearly $dom \ \rho = \{u \in R^m \mid \rho(u) < +\infty\} = dom \ F$.

Lemma 2.2. If F is convex, then the function ρ defined by (2.5) is a convex function.

(Proof). Let $u^1, u^2 \in dom \ \rho$, which is a convex set, and $0 \leqq \alpha \leqq 1$. Since F is convex,

$$\alpha F(u^1) + (1-\alpha) F(u^2) \subset F(\alpha u^1 + (1-\alpha) u^2)$$

and hence

$$\begin{aligned}
\rho(\alpha u^1 + (1-\alpha)u^2) &= inf \ \{\| y - \hat{y} \| \mid y \in F(\alpha u^1 + (1-\alpha)u^2)\} \\
&\leqq inf \ \{\| y - \hat{y} \| \mid y \in \alpha F(u^1) + (1-\alpha)F(u^2)\} \\
&= inf \ \{\| \alpha y^1 + (1-\alpha)y^2 - \hat{y} \| \mid y^1 \in F(u^1),\ y^2 \in F(u^2)\} \\
&\leqq inf \ \{\alpha \| y^1 - \hat{y} \| + (1-\alpha) \| y^2 - \hat{y} \| \mid y^1 \in F(u^1),\ y^2 \in F(u^2)\} \\
&= \alpha \ inf \ \{\| y^1 - \hat{y} \| \mid y^1 \in F(u^1)\} + (1-\alpha) \ inf \ \{\|y^2 - \hat{y}\| \mid y^2 \in F(u^2)\} \\
&= \alpha\rho(u^1) + (1-\alpha) \ \rho(u^2) \quad .
\end{aligned}$$

Lemma 2.3. If F is convex and $\hat{u} \in int(dom \ F)$, then F is lower semicontinuous at $\hat{u}$.

(Proof). Let $u^k \to \hat{u}$ and $\hat{y} \in F(\hat{u})$. Define the function ρ by (2.5). Then, from Lemma 2.2, ρ is a convex function and $dom \, \rho = dom F$. Since $\hat{u} \in int(dom \, \rho)$ and $u^k \to \hat{u}$, there exists a number K such that $u^k \in dom \, \rho$ for any $k \geqq K$. For each $u^k (k \geqq K)$, from the definition of $\rho(u^k)$, there exists $y^k \in F(u^k)$ such that

$$\| y^k - \hat{y} \| < \rho(u^k) + \frac{1}{k} \quad .$$

Since the convex function ρ is continuous at $\hat{u} \in int \ (dom \, \rho)$ and $\rho(\hat{u}) = 0$, by taking the limit of the above inequality, $\| y^k - \hat{y} \| \to 0$ as $k \to \infty$. Namely $y^k \to \hat{y}$.

Therefore F is lower semicontinuous at $\hat{u}$.

■

Remark 2.2. Since the spaces considered here are all finite dimensional, the assumption in Lemma 2.3 is weaker than in the result of Aubin and Ekeland ([1], p. 131), where F is assumed to be not only convex but also upper semicontinuous.

Remark 2.3. The following example illustrates that the condition $\hat{u} \in int(domF)$ is essential in Lemma 2.3. Let $F : R^2 \rightrightarrows R$ be defined by

$$F(u) = \begin{cases} \{y \in R \mid y \geqq \alpha\} & \text{if } (u_1 - \alpha)^2 + (u_2)^2 = \alpha^2 \text{ for } \alpha > 0, u \neq (0,0) \\ \{y \in R \mid y \geqq 0\} & \text{if } u = (0,0) \\ \phi & \text{otherwise} \end{cases} .$$

Then, for $u^k = (1 - \cos\frac{\pi}{k}, \sin\frac{\pi}{k})$, $F(u^k) = \{y \mid y \geqq 1\}$ for all $k = 1,2,\ldots,$. Clearly $u^k \to (0,0)$. However, by taking $0 \in F(0,0)$, we can easily see that F is not lower semicontinuous at $\hat{u} = (0,0)$.

Lemma 2.4. If F is convex, $\hat{u} \in int(domF)$ and $F(\hat{u})$ is a closed set, then F is upper semicontinuous (and therefore continuous in view of Lemma 2.3) at $\hat{u}$.

(Proof). Let $u^k \to \hat{u}$, $y^k \in F(u^k)$ and $y^k \to \hat{y}$. Define ρ as in (2.5). Then ρ is a convex function from Lemma 2.2. Hence ρ is continuous at $\hat{u} \in int(domF) = int(dom\,\rho)$. On the other hand, taking the limit of the inequality

$$0 \leqq \rho(u^k) \leqq \| y^k - \hat{y} \| ,$$

as $k \to \infty$, we can prove that $\rho(\hat{u}) = 0$. Since $F(\hat{u})$ is a closed set, this implies $\hat{y} \in F(\hat{u})$. Hence F is upper semicontinuous at $\hat{u}$. ■

Remark 2.4. It is easily understood that the closedness of $F(\hat{u})$ is very important in the above lemma. The following example illustrates the inevitability of the condition $\hat{u} \in int(dom\ F)$. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$F(u) = \begin{cases} \{y \mid y \geqq 0\} & \text{if } u > 0 \\ \{y \mid y \geqq 1\} & \text{if } u = 0 \\ \phi & \text{if } u < 0 \end{cases} .$$

Then, for $u^k = \frac{1}{k}$, $y^k = 0 \in F(u^k)$ $(k = 1,2,\ldots,)$. However, the limit 0 of $\{y^k\}$ is not contained in $F(0)$.

3. Upper semicontinuity of the perturbation map

In this section we shall consider sufficient conditions for the upper semicontinuity of the perturbation map W. First we provide sufficient conditions in terms of the feasible set map Y.

Theorem 3.1. If the following three conditions are satisfied, then the perturbation map W is upper semicontinuous at $\hat{u} \in R^m$:

(1) $\hat{u} \in int\ (dom\ Y)$;

(2) Y is upper semicontinuous at $\hat{u}$;

(3) $W(\hat{u}) = w - Min_P Y(\hat{u})$, where $w - Min_P Y(\hat{u})$ is the set of all weakly P - minimal points of $Y(\hat{u})$, i.e.

$$w - Min_P Y(\hat{u}) := \{y \in Y(\hat{u}) \mid (Y(\hat{u}) - y) \cap (-int P) = \phi\} \quad . \tag{3.1}$$

(Proof). Let $u^k \to \hat{u}$, $y^k \in W(u^k)$ and $y^k \to \hat{y}$. Since Y is upper semicontinuous at $\hat{u}$, $\hat{y} \in Y(\hat{u})$. Hence, if we suppose that $\hat{y} \notin W(\hat{u}) = w - Min_P Y(\hat{u})$, then there exists $\bar{y} \in Y(\hat{u})$ such that $\hat{y} - \bar{y} \in int\ P$. Since $\hat{u} \in int\,(dom Y) = int\,(dom\,(Y + P))$ and $Y + P$ is convex, $Y + P$ is lower semicontinuous at $\hat{u}$ from Lemma 2.3. Namely there exist a sequence $\{\bar{y}^k\} \subset R^p$ and a number K such that

$$\bar{y}^k \to \bar{y} \quad \text{and} \quad \bar{y}^k \in Y(u^k) + P \quad \text{for} \quad k \geqq K \quad .$$

since $y^k - \bar{y}^k \to \hat{y} - \bar{y} \in int P$, $y^k - \bar{y}^k \in int P$ for all k sufficiently large. However, this contradicts that $y^k \in W(u^k) = Min_P Y(u^k) = Min_P(Y(u^k) + P)$. (See Proposition 3.1.2 in [5]). Therefore $\hat{y} \in W(\hat{u})$, as was to be proved. ∎

Remark 3.1. We can guarantee the upper semicontinuity of W under the following conditions without the convexity assumption (CA) ([7]):

(i) Y is continuous at $\hat{u}$;

(ii) $W(\hat{u}) = w - Min_P Y(\hat{u})$.

If we compare these conditions with Theorem 3.1, the following can be observed: we can replace the lower semicontinuity condition of Y by the weaker condition $\hat{u} \in int\,(dom Y)$ under the convexity assumption.

Now we shall derive sufficient conditions for the upper semicontinuity of W, which are described in terms of the feasible decision set map X and the objective function f. For the purpose we shall introduce a set-valued map $\tilde{X}$ from $R^m \times R^p$ to R^n as follows:

$$\tilde{X}(u,y) := \{x \in X(u) \mid f(x,u) = y\} \quad . \tag{3.2}$$

The following proposition provides sufficient conditions for the upper semicontinuity of Y at $\hat{u}$.

Proposition 3.1. If $\hat{u} \in int(domX)$, if $X(\hat{u})$ is a closed set and if the map $\tilde{X}$ is uniformly compact near $(\hat{u},\hat{y})$† for any $\hat{y} \in \{y \mid (\hat{u},y) \in cl(graph\ Y)\}$, then Y is upper semicontinuous at $\hat{u}$.

(Proof). Let $u^k \to \hat{u}, y^k \in Y(u^k)$ and $y^k \to \hat{y}$. Then there exists a sequence $\{x^k\} \subset R^n$ such that $x^k \in \tilde{X}(u^k,y^k)$ for all $k = 1,2,....$ Since $\tilde{X}$ is uniformly compact near $(\hat{u},\hat{y})$, $\{x^k\}$ has a convergent subsequence. By taking the subsequence if necessary, we may assume that $\{x^k\}$ converges to some $\hat{x}$. From Lemma 2.4, X is upper semicontinuous at $\hat{u}$ and so $\hat{x} \in X(\hat{u})$. On the other hand, since f is continuous from Lemma 2.1, $f(\hat{x},\hat{u}) = \hat{y}$. Therefore $\hat{y} \in Y(\hat{u})$ and Y is upper semicontinuous at $\hat{u}$. ∎

Remark 3.2. If X is uniformly compact near $\hat{u}$, then $\tilde{X}$ is clearly uniformly compact near $(\hat{u},y)$ for any $y \in R^p$.

Now we can prove the following theorem.

Theorem 3.2. If the following four conditions are satisfied, then the set-valued map W is upper semicontinuous at $\hat{u}$:

(1) $\hat{u} \in int(domX)$;

(2) $X(\hat{u})$ is a closed set;

(3) $\tilde{X}$ is uniformly compact near $(\hat{u},\hat{y})$ for any $\hat{y} \in \{y \mid (\hat{u},y) \in cl(graphY)\}$;

(4) $W(\hat{u}) = w - Min_P Y(\hat{u})$.

(Proof). From (1), $\hat{u} \in int(domY)$. From (1) - (3), in view of Proposition 3.1, Y is upper semicontinuous at $\hat{u}$. Hence W is upper semicontinuous at $\hat{u}$ by Theorem 3.1. ∎

Remark 3.3. The following examples illustrate that each condition in the above theorem is essential.

1) Take F in Remark 2.3 as X and let $f(x,u) = x$ and $P = R_+$. Then $\hat{u} = (0,0) \notin int(domX)$ and

$$W(u) = \begin{cases} \{\alpha\} & \text{if } (u_1 - \alpha)^2 + (u_2)^2 = \alpha^2 \text{ for } \alpha > 0, \; u \neq (0,0) \\ \{0\} & \text{if } u = (0,0) \\ \phi & \text{otherwise} \end{cases}$$

† A set-valued map F is said to be uniformly compact near $\hat{u}$ if there exists a neighborhood N of $\hat{u}$ such that $cl \bigcup_{u \in N} F(u)$ is a compact set.

which is not upper semicontinuous at (0,0).

2) Let $m = p = n = 1, P = R_+, f(x,u) = x$ and

$$X(u) = \begin{cases} \{x \mid x \geqq 0\} & \text{if } u \neq 0 \\ \{x \mid x > 0\} & \text{if } u = 0 \end{cases} .$$

Then $W(u) = \{0\}$ if $u \neq 0$ and $W(0) = \phi$. Hence W is not upper semicontinuous at 0.

3) Let $m = n = p = 1, P = R_+$ and $X(u) = R$ for any $u \in R$. Let C be a convex set in $R \times R$ defined by

$$C = \{(u,x) \mid ux \geqq 1,\ u > 0\}$$

and f be defined by

$$f(x,u) = d((u,x),C) = inf\ \{\|(u,x) - (u',x')\| (u',x') \in C\} \quad .$$

Then f is P-convex and

$$Y(u) = \begin{cases} \{y \in R \mid y \geqq 0\} & \text{if } u > 0 \\ \{y \in R \mid y > -u\} & \text{if } u \leqq 0 \end{cases} .$$

Hence

$$W(u) = \begin{cases} \{0\} & \text{if } u > 0 \\ \phi & \text{if } u \leqq 0 \end{cases}$$

which is not upper semicontinuous at 0.

4. Lower semicontinuity of the perturbation map

In this section we consider sufficient conditions for the lower semicontinuity of the map W. First we should introduce several concepts.

Definition 4.1. A set S in R^p is said to be P-minicomplete if

$$S \subset Min_P S + P \quad . \tag{4.1}$$

Remark 4.1. Since $Min_P S \subset S$, if S is P-minicomplete,

$$S + P = Min_P S + P \quad . \tag{4.2}$$

Definition 4.2. For a nonempty set S in R^p, its recession cone S^+ is defined by

$$S^+ = \{y \in R^p \mid \text{there exist sequences } \{\lambda_k\} \subset R \text{ and } \{y^k\} \subset R^p \text{ such that}$$

$$\lambda_k > 0,\ \lambda_k \to 0,\ \lambda_k y^k \to y \text{ and } y^k \in S \text{ for all } k\} \quad . \tag{4.3}$$

Remark 4.2. S^+ is a closed cone which contains the origin. Moreover, if S is a nonempty closed convex set, S^+ coincides with the set 0^+S which is defined by

$$0^+S = \{y \in R^p \mid \bar{y} + \lambda y \in S \text{ for } \forall \lambda \geqq 0 \ ,\ \forall \bar{y} \in S\} \tag{4.4}$$

$$= \{y \in R^p \mid S + y \subset S\}$$

and therefore it is a closed convex cone ([3] Theorem 8.2).

Lemma 4.1. (Sawaragi et al. [5], Lemma 3.2.1.) A nonempty set S is bounded if and only if $S^+ = \{0\}$.

Lemma 4.2. (Sawaragi et al. [5], Lemma 3.2.3.) Let S_1 and S_2 be nonempty closed sets. If $S_1^+ \cap (-S_2^+) = \{0\}$, then $S_1 + S_2$ is also a nonempty closed set.

In view of the above two lemmas, the following concept plays an important role in this section.

Definition 4.3. A nonempty set S in R^p is said to be P-bounded if

$$S^+ \cap (-P) = \{0\} \tag{4.5}$$

Lemma 4.3. (Sawaragi et al. [5], Theorem 3.2.12.) If $S \subset R^p$ is a nonempty closed convex set, the following statements are equivalent:

(1) S is P-bounded.

(2) $Min_P S \neq \phi$.

(3) S is P-minicomplete.

Lemma 4.4. Suppose that F is P-convex, $\hat{u} \in int\ (dom F)$, and $F(\hat{u})$ is P-bounded. Then there exists a neighborhood N of $\hat{u}$ such that $F(u)$ is P-bounded for all $u \in N$.

(Proof). If the conclusion of the lemma were not true, there would exist sequences $\{u^k\} \subset R^m$ and $\{d^k\} \subset R^p$ such that $u^k \to \hat{u}$, $d^k \neq 0$ and

$$-d^k \in [F(u^k)]^+ \cap (-P) \quad .$$

Since $[F(u^k)]^+ \cap (-P)$ is a cone, we may assume that $\|d^k\| = 1$ for all k. By taking a subsequence if necessary, we may assume that $\{d^k\}$ converges to some d. Since P is closed, $d \in P$. Moreover, $\|d\| = 1$ and so $d \neq 0$. Since $-d^k \in [F(u^k)]^+$, there exist sequences $\{\lambda_{kl}\} \subset R$, $\{d^{kl}\} \subset -F(u^k)$ such that $\lambda_{kl} > 0$,

$$\lambda_{kl} \to 0 \text{ and } \lambda_{kl} d^{kl} \to d^k \text{ as } l \to \infty \quad .$$

If we take l sufficiently large,

$$\lambda_{kl} < \frac{1}{k} \text{ and } |\lambda_{kl} d^{kl} - d^k| < \frac{1}{k} \quad .$$

By choosing those λ_{kl} and d^{kl} as $\bar{\lambda}_k$ and $\bar{d}^k$ respectively, we can construct sequences $\{\bar{\lambda}_k\}$ and $\{\bar{d}^k\}$ satisfying

$$-\bar{d}^k \in F(u^k),\ 0 < \bar{\lambda}_k < \frac{1}{k},\ |\bar{\lambda}_k \bar{d}^k - d^k| < \frac{1}{k} \quad .$$

When $k \to \infty, \bar{\lambda}_k \to 0$ and $\bar{\lambda}_k \bar{d}^k \to d$. Now take an arbitrary $\tilde{y} \in F(\hat{u})$. Since $2\hat{u} - u^k \to \hat{u}$ and $F + P$ is lower semicontinuous at $\hat{u}$ by Lemma 2.3, there exist a sequence $\{\tilde{y}^k\}$ and a number K such that

$$\tilde{y}^k \to \tilde{y} \text{ and } \tilde{y}^k \in F(2\hat{u} - u^k) + P \text{ for } k \geqq K \quad .$$

Since F is P-convex,

$$\frac{1}{2}(\tilde{y}^k - \bar{d}^k) \in F(\hat{u}) + P \text{ for } k \geqq K \quad .$$

Moreover, $2\bar{\lambda}_k \cdot \frac{1}{2}(\tilde{y}^k - \bar{d}^k) \to -d$. This implies that $-d \in [F(\hat{u}) + P]^+$ and hence $[F(\hat{u}) + P]^+ \cap (-P) \neq \{0\}$. In view of Lemma 3.2.4 of [5], this means that $F(\hat{u})$ is not P-bounded, which is a contradiction. Hence $F(u)$ is P-bounded for all u in a certain neighborhood of $\hat{u}$. ∎

Now we can obtain sufficient conditions for the lower semicontinuity of W.

Theorem 4.1. If the following the conditions are satisfied, then the perturbation map W is lower semicontinuous at $\hat{u}$:

(1) $\hat{u} \in int(dom\ Y)$.

(2) $Y + P$ is upper semicontinuous in a neighborhood of $\hat{u}$.

(Proof). If $W(\hat{u}) = \phi$, the theorem is trivial. Hence we suppose that $W(\hat{u}) \neq \phi$. Let $u^k \to \hat{u}$ and $\hat{y} \in W(\hat{u})$. From Lemma 2.3, $Y + P$ is lower semicontinuous at $\hat{u}$ and hence there exist a sequence $\{y^k\}$ and a number K_1 such that

$$y^k \to \hat{y} \text{ and } y^k \in Y(u^k) + P \text{ for all } k \geqq K_1 \quad .$$

Since $Y(\hat{u}) + P$ is a nonempty closed convex set and $Min_P(Y(\hat{u}) + P) = W(\hat{u}) \neq \phi$, $Y(\hat{u}) + P$ is P-bounded from Lemma 4.3. Therefore, in view of Lemma 4.4, $Y(u) + P$ is P-bounded for all u in a certain neighborhood N

of $\hat{u}$. (Note that $\hat{u} \in int(dom\ Y)$). From Lemma 4.3 and Remark 4.1, this implies that

$$W(u) + P = (Y(u) + P) + P = Y(u) + P$$

in a neighborhood of $\hat{u}$. Hence there exist a sequence $\{\hat{y}^k\}$ and a number $K_2 \geqq K_1$ such that

$$y^k - \hat{y}^k \in P \text{ and } \hat{y}^k \in W(u^k) \text{ for } k \geqq K_2 \quad .$$

First we will show that $\{\hat{y}^k\}$ is bounded. If this were not the case, from Lemma 4.1, we can take a subsequence of $\{\hat{y}^k\}$, for which there exist a sequence $\{\lambda_k\}$ of positive numbers and a nonzero vector $\tilde{y}$ such that $\lambda_k \to 0$ and $\lambda_k \hat{y}^k \to \tilde{y}$. Since $\lambda_k(y^k - \hat{y}^k) \in P$ and $y^k \to \hat{y}$, the limit $-\tilde{y}$ of $\{\lambda_k(y^k - \hat{y}^k)\}$ is contained in P. Take an arbitrary $\bar{y} \in Y(\hat{u}) + P$. Then there exist a sequence $\{\bar{y}^k\}$ and a number $K_3 \geqq K_2$ such that

$$\bar{y}^k \to \bar{y} \text{ and } \bar{y}^k \in Y(2\hat{u} - u^k) + P \text{ for } k \geqq K_3 \quad ,$$

since $Y + P$ is lower semicontinuous at $\hat{u}$. Then, from the convexity of $Y + P$,

$$\frac{1}{2}(\hat{y}^k + \bar{y}^k) \in Y(\hat{u}) + P \text{ for } k \geqq K_3 \quad .$$

Moreover, $\lambda_k(\hat{y}^k + \bar{y}^k) \to \tilde{y}$. This implies that $\tilde{y} \in [Y(\hat{u}) + P]^+$ and hence leads to a contradiction to the P-boundedness of $Y(\hat{u}) + P$. Therefore $\{\hat{y}^k\}$ must be bounded. Hence $\{\hat{y}^k\}$ has a cluster point, which is denoted by y'. Since $y^k - \hat{y}^k \in P$ and $y^k \to \hat{y}$, $\hat{y} - y' \in P$. Since $Y + P$ is upper semicontinuous at $\hat{u}$, $y' \in Y(\hat{u}) + P$. Recalling that $\hat{y} \in W(\hat{u})$, we can conclude that $y' = \hat{y}$. In other words, $\hat{y}$ is the unique cluster point for the bounded sequence $\{\hat{y}^k\} \to \hat{y}$. Therefore $\hat{y}^k \to \hat{y}$, which indicates that W is lower semicontinuous at $\hat{u}$. ■

Remark 4.3. We can generate the lower semicontinuity of W under the following conditions without the convexity assumption (CA) ([7]):

(i) Y is continuous at $\hat{u}$,

(ii) Y is uniformly compact near $\hat{u}$,

(iii) $Y(u)$ is P-minicomplete for every u near $\hat{u}$.

Theorem 4.1 considerably simplifies the above result.

The following proposition shows that $Y + P$ is often upper semicontinuous when $W(\hat{u})$ is not empty.

Proposition 4.1. If $X(u)$ is a nonempty closed set for every u near $\hat{u}, W(\hat{u}) \neq \phi$ and $\tilde{X}(\hat{u},\hat{y})$ is bounded for some $\hat{y} \in W(\hat{u})$, then $Y(u)$ is a P-bounded closed set in a neighborhood of $\hat{u}$. In this case $Y(u) + P$ is also a closed set by Lemma 4.2 and therefore the set-valued map Y is upper semicontinuous in a neighborhood of $\hat{u}$.

(Proof). a) First we shall prove that $Y(u)$ is a closed set in some neighborhood of $\hat{u}$. If this were not true, we can consider sequences $\{u^k\}$ and $\{y^k\}$ such that

$$u^k \to \hat{u} \text{ and } y^k \in clY(u^k)\setminus Y(u^k) \quad .$$

Corresponding to each y^k, there exists a sequence $\{x^{kl}\} \subset X(u^k)$ such that $f(x^{kl},u^k) \to y^k$ as $l \to \infty$. Take k sufficiently large so that $X(u^k)$ is closed. If $\{x^{kl}\}_{l=1,2,\dots}$ has a convergent subsequence, the limit x^k of it is contained in $X(u^k)$. Since f is continuous, $f(x^k,u^k) = y^k$, which contradicts that $y^k \notin Y(u^k)$. Hence, if k is sufficiently large, $\{x^{kl}\}_{l=1,2,\dots}$ has not a convergent subsequence and so $\|x^{kl}\| \to +\infty$ as $l \to \infty$. We may assume that the sequence $\left\{\frac{x^{kl}}{\|x^{kl}\|}\right\}_{l=1,2,\dots}$ converges to some $\bar{x}^k$ as $l \to \infty$. Furthermore, since $\|\bar{x}^k\| = 1$ for all k, we may also assume without loss of generality that $\{\bar{x}^k\}$ converges to a vector $\bar{x}$. In this case $\|\bar{x}\| = 1$, i.e. $\bar{x} \neq 0$. From the assumptions, we can take $\hat{y} \in W(\hat{u})$ for which $\tilde{X}(\hat{u},\hat{y})$ is bounded. Let $\hat{x} \in \tilde{X}(\hat{u},\hat{y})$. Since X is lower semicontinuous at $\hat{u}$ from Lemma 2.3, there exist a sequence $\{\hat{x}^k\}$ and a number K such that

$$\hat{x}^k \to \hat{x} \text{ and } \hat{x}^k \in X(u^k) \text{ for } k \geqq K \quad .$$

Let $k \geqq K$. For an arbitrary $\alpha \geqq 0$, $0 \leqq \frac{\alpha}{\|x^{kl}\|} \leqq 1$ for all k sufficiently large, for $\|x^{kl}\| \to +\infty$ as $l \to \infty$. Since X is convex,

$$\left[1 - \frac{\alpha}{\|x^{kl}\|}\right]\hat{x}^k + \frac{\alpha}{\|x^{kl}\|}x^{kl} \in X(u^k) \quad .$$

Taking the limit when $l \to \infty$, we obtain from the closedness of $X(u^k)$,

$$\hat{x}^k + \alpha\bar{x}^k \in X(u^k), \text{ for all } k \text{ sufficiently large} \quad . \tag{4.6}$$

Since f is P-convex,

$$f((1 - \frac{\alpha}{\|x^{kl}\|})\hat{x}^k + \frac{\alpha}{\|x^{kl}\|}x^{kl}, u^k)$$

$$\leqq_P \left[1-\frac{\alpha}{\|x^{kl}\|}\right] f(\hat{x}^k, u^k) + \frac{\alpha}{\|x^{kl}\|} f(x^{kl}, u^k) \quad .$$

Let $l \to \infty$. Then, since $f(x^{kl}, u^k) \to y^k$,

$$f(\hat{x}^k + \alpha \bar{x}^k, u^k) \leqq_P f(\hat{x}^k, u^k) \text{ for } k \text{ sufficiently large} \quad . \tag{4.7}$$

Take the limit of (4.6) and (4.7) as $k \to \infty$. Then, since X is upper semicontinuous at $\hat{u}$ from Lemma 2.4 and f is continuous, $\hat{x} + \alpha \bar{x} \in X(\hat{u})$ and

$$f(\hat{x} + \alpha \bar{x}, \hat{u}) \leqq_P f(\hat{x}, \hat{u}) = \hat{y} \quad .$$

Since $\hat{y} \in W(\hat{u})$, these imply that $f(\hat{x} + \alpha \bar{x}, \hat{u}) = \hat{y}$, i.e., $\hat{x} + \alpha \bar{x} \in \tilde{X}(\hat{u}, \hat{y})$ for all $\alpha \geqq 0$. However, this contradicts the boundedness of $\tilde{X}(\hat{u}, \hat{y})$. Hence $Y(u)$ must be a closed set for every u in a certain neighborhood of $\hat{u}$.

b) Next, we shall prove that $Y(\hat{u})$ is P-bounded. Let $y \in [Y(\hat{u})]^+ \cap (-P)$. There exist sequences $\{\lambda_k\} \subset R$ and $\{x^k\} \subset X(\hat{u})$ such that $\lambda_k > 0$, $\lambda_k \to 0$ and $\lambda_k f(x^k, \hat{u}) \to y$. Then, for all k sufficiently large, $\lambda_k x^k + (1-\lambda_k)\hat{x} \in X(\hat{u})$ and

$$f(\lambda_k x^k + (1-\lambda_k)\hat{x}, \hat{u}) \leqq_P \lambda_k f(x^k, \hat{u}) + (1-\lambda_k) f(\hat{x}, \hat{u}) \tag{4.8}$$

due to the P-convexity of f. The right-hand side of the inequality (4.8) converges to $y + \hat{y}$. First we assume that $\{\lambda_k x^k\}$ has no convergent subsequence. Then $\lambda_k \|x^k\| \to +\infty$. We may assume without loss of generality that $\{\frac{x^k}{\|x^k\|}\}$ converges to a vector $\tilde{x}$ with $\|\tilde{x}\| = 1$. For any $\alpha \geqq 0$, $0 \leqq \frac{\alpha}{\|x^k\|} \leqq 1$ for all k sufficiently large and so

$$\frac{\alpha}{\|x^k\|} x^k + (1-\frac{\alpha}{\|x^k\|})\hat{x} \in X(\hat{u})$$

from the convexity of $X(\hat{u})$. Since $X(\hat{u})$ is a closed set, the limit of the above relation implies that $\hat{x} + \alpha \tilde{x} \in X(\hat{u})$. Moreover, since f is P-convex,

$$f(\frac{\alpha}{\|x^k\|} x^k + (1-\frac{\alpha}{\|x^k\|})\hat{x}, \hat{u}) \leqq_P \frac{\alpha}{\lambda_k \|x^k\|} \lambda_k f(x^k, \hat{u}) + (1-\frac{\alpha}{\|x^k\|}) f(\hat{x}, \hat{u})$$

for all k sufficiently large. Thus, as the limit of the above inequality, we have

$$f(\hat{x} + \alpha \tilde{x}, \hat{u}) \leqq_P \hat{y} \quad .$$

Since $\hat{y} \in W(\hat{u})$, $f(\hat{x} + \alpha \tilde{x}, \hat{u}) = \hat{y}$. This implies that $\hat{x} + \alpha \tilde{x} \in \tilde{X}(\hat{u}, \hat{y})$ for all $\alpha \geqq 0$, which contradicts the boundedness of $\tilde{X}(\hat{u}, \hat{y})$. Hence $\{\lambda_k x^k\}$ necessarily has a convergent subsequence whose limit is denoted by x. We may assume that

$\lambda_k x^k \to x$ from the first. Since $X(\hat{u})$ is closed, from the limit of $\lambda_k x^k + (1-\lambda_k)\hat{x} \in X(\hat{u})$, $x + \hat{x} \in X(\hat{u})$. Therefore the limit of the left-hand side of (4.8), which is $f(x + \hat{x}, \hat{u})$, belongs to $Y(\hat{u})$. Since (4.8) leads to

$$f(x + \hat{x}, \hat{u}) \lneqq_P y + \hat{y}$$

when $k \to \infty$, $y \in -P$ and $\hat{y} \in W(\hat{u})$, y must be equal to the zero vector. Thus $Y(\hat{u})$ is P-bounded.

c) Finally, the result proved just above and Lemma 4.4 imply that $Y(u)$ is P-bounded in a neighborhood of $\hat{u}$. This completes the proof of the proposition. ∎

Now we can immediately obtain the following result by combining Theorem 4.1 and Proposition 4.1.

Theorem 4.2. If the following conditions are satisfied, then the perturbation map W is lower semicontinuous at $\hat{u}$:

(1) $\hat{u} \in int(domX)$,

(2) $X(u)$ is a closed set for every u near $\hat{u}$,

(3) When $W(\hat{u}) \neq \phi$, $\tilde{X}(\hat{u}, \hat{y})$ is bounded for some $\hat{y} \in W(\hat{u})$.

Remark 4.4. The following examples show that each condition in the above theorem is essential.

1) Consider the case in Remark 3.3, 1). Then we can easily understand that the condition $\hat{u} \in int(domX)$ is essential.

2) Let $m = n = p = 1, P = R_+$,

$$X(u) = \begin{cases} \{x \in R \mid x > u^2\} & \text{if } u \neq 0 \\ \{x \in R \mid x \geqq 0\} & \text{if } u = 0 \end{cases}$$

and $f(x, u) = x$. Then

$$W(u) = \begin{cases} \phi & \text{if } u \neq 0 \\ \{0\} & \text{if } u = 0 \end{cases} .$$

which is clearly not lower semicontinuous at $\hat{u} = 0$.

3) Let $m = n = p = 1, P = R_+, X(u) = R_+$ and

$$f(x, u) = \begin{cases} 0 & \text{if } u = 0 \\ |u| e^{-\frac{x}{|u|}} & \text{if } u \neq 0 \end{cases} .$$

Then

$$Y(u) = \begin{cases} \{0\} & \text{if } u = 0 \\ \{y \mid 0 < y \leq |u|\} & \text{if } u \neq 0 \end{cases}$$

and so

$$W(u) = \begin{cases} \{0\} & \text{if } u = 0 \\ \phi & \text{if } u \neq 0 \end{cases} .$$

$\tilde{X}(0,0) = R_+$, which is not bounded, and W is not lower semicontinuous at $\hat{u} = 0$.

5. Contingent derivative of the perturbation map

In this section we will show some quantitative results concerning the behavior of the perturbation map by using the concept of contingent derivatives of set-valued maps. The author has already provided an "inner" approximation of the contingent derivative of the perturbation map for general multiobjective optimization problems ([6]). In this paper, a complete characterization of the contingent derivative will be obtained under the convexity assumption (CA) and some additional conditions.

First we briefly review the concept of contingent derivatives for set- valued maps.

Definition 5.1. Let S be a nonempty subset of R^q and $\hat{v} \in R^q$. The set $T_S(\hat{v})$ defined by

$$T_S(\hat{v}) := \{v \in R^q \mid \text{there exist sequences } \{h_k\} \subset \mathring{R}_+ \text{ and } \{v^k\} \subset R^q$$
$$\text{such that } h_k \to 0, v^k \to v \text{ and } \hat{v} + h_k v^k \in S \text{ for all } k\} \tag{5.1}$$

is called the contingent cone to S at $\hat{v}$.

Definition 5.2. Let F be a set-valued map from R^m to R^p and $\bar{y} \in F(\bar{u})$. The set-valued map $DF(\bar{u},\bar{y})$ from R^m to R^p defined by the following is called the contingent derivative of F at $(\bar{u}, \bar{y})$:

$$y \in DF(\bar{u},\bar{y})(u) \quad iff \quad (u,y) \in T_{graphF}(\bar{u},\bar{y}) . \tag{5.2}$$

In other words, $y \in DF(\bar{u},\bar{y})(u)$ if and only if there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$ such that $h_k \to 0$, $u^k \to u$, $y^k \to y$ and

$$\bar{y} + h_k y^k \in F(\bar{u} + h_k u^k) \text{ for } \forall k ,$$

where $\mathring{R}_+$ is the set of all positive real numbers.

The purpose of this section is to provide a complete characterization of the contingent derivative of the perturbation map. Throughout this section let $\hat{y}$ be a P-minimal point of $Y(\hat{u})$, i.e. $\hat{y} \in W(\hat{u})$. First we can simplify Theorem 3.2 in [6] under the convexity assumption (CA) as in the following theorem.

Theorem 5.1. If $Y(u)$ is P-minicomplete for every u near $\hat{u}$, then

$$Min_P DY(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u) \quad \text{for } \forall u \in R^m \quad . \tag{5.3}$$

(Proof) Let $y \in Min_P DY(\hat{u},\hat{y})(u)$. Since $y \in DY(\hat{u},\hat{y})(u)$, there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$ such that $h_k \to 0$, $u^k \to u$, $y^k \to y$ and

$$\hat{y} + h_k y^k \in Y(\hat{u} + h_k u^k) \quad \text{for} \quad \forall k \quad .$$

Since $Y(u)$ is P-minicomplete for every u near $\hat{u}$, there exists a sequence $\{\bar{y}k\} \subset R^p$ such that

$$\hat{y} + h_k \bar{y}^k \in W(\hat{u} + h_k u^k) \text{ and } y^k - \bar{y}^k \in P \tag{5.4}$$

for all k sufficiently large. We may assume (5.4) for all k. Suppose that $\{\bar{y}^k\}$ has no convergent subsequence. Then $\|\bar{y}^k\| \to +\infty$. There exist sequences $\{x^k\}$ and $\{\bar{x}^k\}$ in R^n such that

$$\begin{cases} \hat{x} + h_k x^k \in X(\hat{u} + h_k u^k) \ , \\ f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) = \hat{y} + h_k y^k \end{cases}$$

$$\begin{cases} \hat{x} + h_k \bar{x}^k \in X(\hat{u} + h_k u^k) \\ f(\hat{x} + h_k \bar{x}^k, \hat{u} + h_k u^k) = \hat{y} + h_k \bar{y}^k \quad . \end{cases}$$

For any α satisfying $0 \leqq \alpha \leqq 1$, we have

$$\hat{x} + h_k(\alpha x^k + (1-\alpha)\bar{x}^k) \in X(\hat{u} + h_k u^k)$$

from the convexity of X. Moreover, from the P-convexity of f,

$$\begin{aligned} \hat{y} + h_k y^k(\alpha) &:= f(\hat{x} + h_k(\alpha x^k + (1-\alpha)\bar{x}^k), \hat{u} + h_k u^k) \\ &\leqq_P \hat{y} + h_k(\alpha y_k + (1-\alpha)\bar{y}^k) \\ &\leqq_P \hat{y} + h_k y^k \quad . \end{aligned}$$

And, since f is continuous,

$$\hat{y} + h_k y^k(\alpha) \to \hat{y} + h_k \bar{y}^k \quad \text{as } \alpha \to 0$$

$$\hat{y} + h_k y^k(\alpha) \to \hat{y} + h_k y^k \text{ as } \alpha \to 1 \ .$$

Since $\|\bar{y}^k\| \to +\infty$ and $y^k \to y$, by taking α_k appropriately close to 1, we have

$$\varepsilon h_k \leqq \|\hat{y} + h_k y^k - (\hat{y} + h_k y^k(\alpha_k))\| \leqq h_k, \qquad \text{for } \forall k \text{ sufficiently large}$$

where ε is a fixed number such that $0 < \varepsilon < 1$. Taking this $y^k(\alpha_k)$ as $\tilde{y}^k$, we see that

$$\varepsilon \leqq \|y^k - \tilde{y}^k\| \leqq 1 \qquad \text{for } \forall k \text{ sufficiently large} \ .$$

Since $y^k \to y$, the sequence $\{\tilde{y}^k\}$ is bounded and so we may assume without loss of generality that $\{\tilde{y}^k\}$ converges to a vector $\tilde{y}$. It is clear that $\tilde{y} \in DY(\hat{u}, \hat{y})(u)$. Since $\|y^k - \tilde{y}^k\| \geqq \varepsilon$ for all k sufficiently large, $\|y - \tilde{y}\| \geqq \varepsilon$, that is, $y \neq \tilde{y}$. Since $y^k - \tilde{y}^k \in P$, $y - \tilde{y} \in P$. However these contradict the assumption that $y \in Min_P DY(\hat{u}, \hat{y})(u)$. Therefore $\{\bar{y}^k\}$ always has a convergent subsequence. Hence we may assume from the first that $\bar{y}^k \to \bar{y}$. Then $\bar{y} \in DW(\hat{u}, \hat{y})(u) \subset DY(\hat{u}, \hat{y})(u)$ and $y^k - \bar{y}^k \to y - \bar{y} \in P$. Since $y \in Min_P DY(\hat{u}, \hat{y})(u)$, $y = \bar{y}$. This implies that $y \in DW(\hat{u}, \hat{y})(u)$, and completes the proof of the theorem. ■

Remark 5.1. We can see from the example in Remark 4.4, 3) that the P-minicompleteness condition is essential for Theorem 5.1. There, $DW(\hat{u}, \hat{y})(u) = \phi$ for $\hat{u} = 0$, $\hat{y} = 0$ and $u \neq 0$. However $DY(\hat{u}, \hat{y})(u) = [0, u]$ and $Min_P DY(\hat{u}, \hat{y})(u) = \{0\}$ for $u \neq 0$.

Next we consider sufficient conditions for the converse inclusion of (5.3).

Definition 5.3. Let S be a nonempty set in R^p and $\hat{v} \in R^p$. The normal cone $N_S(\hat{v})$ to S at $\hat{v}$ is the negative polar cone of the tangent cone $T_S(\hat{v})$, i.e.

$$N_S(\hat{v}) = [T_S(\hat{v})]^o = \{\mu \in R^p \mid \langle \mu, v \rangle \leqq 0 \text{ for } \forall v \in T_S(\hat{v})\} \ . \tag{5.5}$$

When S is a convex set and $\hat{v} \in S$,

$$N_S(\hat{v}) = \{\mu \in R^p \mid \langle \mu, \hat{v} \rangle \geqq \langle \mu, v \rangle \text{ for } \forall v \in S\} \ . \tag{5.6}$$

Definition 5.4. Let S be a nonempty P-convex set in R^p. If a point $\hat{y} \in Min_P S$ satisfies the condition

$$N_{S+P}(\hat{y}) \subset int\ P^o \cup \{0\} \ . \tag{5.7}$$

then $\hat{y}$ is called the normally P-minimal point of S.

Remark 5.2. A point $\hat{y} \in S$ is said to be the properly P-minimal point of S if

$$T_{S+P}(\hat{y}) \cap (-P) = \{0\}^{\dagger} \tag{5.8}$$

If $\hat{y}$ is a properly P-minimal point of a convex set, there exists a vector $\mu \in N_{S+P}(\hat{y}) \cap int\ P^0$. The relation (5.7) is a stronger requirement than the existence of such μ as long as $\hat{y} \in Min_P S$. In other words, the normal P-minimality is a stronger concept than the proper P-minimality. From the geometric viewpoint, the latter implies the existence of the supporting hyperplane to S at $\hat{y}$ with the normal vector μ in $int\ P^0$ and, on the other hand, the former implies that all the normal vectors of the supporting hyperplanes to S at $\hat{y}$ belong to $int\ P^0$. (The existence of such a hyperplane is guaranteed by the fact that $\hat{y} \in Min_P S$).

Remark 5.3. It is not difficult to show that the normal P-minimality of $\hat{y}$ to a convex set S is equivalent to the following condition:

$$int\ T_{S+P}(\hat{y}) \cup \{0\} \supset P \quad . \tag{5.9}$$

Theorem 5.2. If $\hat{u} \in int\,(dom\ Y)$ and $\hat{y}$ is a normally P-minimal point of $Y(\hat{u})$, then

$$DW(\hat{u},\hat{y})(u) \subset Min_P DY(\hat{u},\hat{y})(u) \qquad \text{for } \forall u \in R^m \tag{5.10}$$

(Proof) Let $y \in DW(\hat{u},\hat{y})(u)$. Of course $y \in DY(\hat{u},\hat{y})(u)$. Hence if we assume that $y \notin Min_P DY(\hat{u},\hat{y})(u)$, there exists $\bar{y} \in DY(\hat{u},\hat{y})(u)$ such that $y - \bar{y} \in P \setminus \{0\}$. Since $\bar{y} \in DY(\hat{u},\hat{y})(u)$, there exist sequences $\{\bar{h}_k\} \subset \mathring{R}_+$, $\{\bar{u}^k\} \subset R^m$ and $\{\bar{y}^k\} \subset R^p$ such that $\bar{h}_k \to 0$, $\bar{u}^k \to u$, $\bar{y}^k \to \bar{y}$ and

$$\hat{y} + \bar{h}_k \bar{y}^k \in Y(\hat{u} + \bar{h}_k \bar{u}^k) \qquad \text{for } \forall k \quad .$$

On the other hand, since $y \in DW(\hat{u},\hat{y})(u)$, there exist sequences $\{h_k\} \in \mathring{R}_+$, $\{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$ such that $h_k \to 0$, $u^k \to u$, $y^k \to y$ and

$$\hat{y} + h_k y^k \in W(\hat{u} + h_k u^k) \qquad \text{for } \forall k \quad .$$

Since $h_k \to 0$, we may assume that $h_k \leqq \bar{h}_k$ by taking a subsequence if necessary. Since $\hat{y} + h_k y^k \in W(\hat{u} + h_k u^k)$, $(\hat{u} + h_k u^k, \hat{y} + h_k y^k)$ is a boundary point of the convex set *graph* $(Y + P)$. Hence there exist a vector $(\lambda^k, \mu^k) \in R^m \times R^p$ such that

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† There are several definitions of the proper P-minimality (see, e.g. [5]). However they coincide under the convexity assumption.

$$< \lambda^k, \hat{u} + h_k u^k > + < \mu^k, \hat{y} + h_k y^k > \geqq < \lambda^k, u' > + < \mu^k, y' >$$

$$\text{for } \forall (u', y') \in \textit{graph } (Y + P) \tag{5.11}$$

for each k. Since we may normalize these vectors so that $\|(\lambda^k, \mu^k)\| = 1$, we may assume that $\{(\lambda^k, \mu^k)\}$ converges to a nonzero vector $(\lambda, \mu) \in R^m \times R^p$. By taking the limit of (5.11) as $k \to \infty$, we see that

$$< \lambda, \hat{u} > + < \mu, \hat{y} > \geqq < \lambda, u' > + < \mu, y' >$$

$$\text{for } \forall (u', y') \in \textit{graph } (Y + P) \ . \tag{5.12}$$

Since $\hat{u} \in \textit{int } (\textit{dom } Y)$, $\mu \neq 0$. Take an arbitrary $\tilde{y} \in Y(\hat{u}) + P$. From Lemma 2.3, the set-valued map $Y + P$ is lower semicontinuous at $\hat{u}$ and so there exist a sequence $\{\tilde{y}^k\} \subset R^p$ and a number $K > 0$ such that $\tilde{y}^k \to \tilde{y}$ and

$$\tilde{y}^k \in Y(\hat{u} + h_k u^k) + P \quad \text{ for } k \geqq K \ . \tag{5.13}$$

From (5.11), for $k \geqq K$

$$< \lambda^k, \hat{u} + h_k u^k > + < \mu^k, \hat{y} + h_k y^k > \geqq < \lambda^k, \hat{u} + h_k u^k > + < \mu^k, \tilde{y}^k > \ .$$

Letting $k \to \infty$, we have that

$$< \mu, \hat{y} > \geqq < \mu, \tilde{y} > \ .$$

This implies that $\mu \in N_{Y(\hat{u})+P}(\hat{y})$. Since $\hat{y}$ is a normally P-minimal point of $Y(\hat{u})$, $\mu \in \textit{int } P^0$. Since $y - \bar{y} \in P \setminus \{0\}$.

$$< \mu, y > < < \mu, \bar{y} > \tag{5.14}$$

Recalling that $\hat{y} + \bar{h}_k \bar{y}^k \in Y(\hat{u} + \bar{h}_k \bar{u}^k)$, $\bar{y} \in Y(\hat{u})$ and $h_k \geqq \bar{h}_k$, we obtain that

$$\hat{y} + h_k \bar{y}^k \in Y(\hat{u} + h_k \bar{u}^k) + P$$

from the P-convexity of Y. Hence, from (5.11),

$$< \lambda^k, \hat{u} + h_k u^k > + < \mu_k, \hat{y} + h_k y^k > \geqq < \lambda^k, \hat{u} + h_k \bar{u}^k > + < \mu^k, \hat{y} + h_k \bar{y}^k > \ ,$$

i.e.

$$< \lambda^k, u^k > + < \mu^k, y^k > \geqq < \lambda^k, \bar{u}^k > + < \mu^k, \bar{y}^k > \ .$$

By taking the limit as $k \to \infty$, we have that

$$< \lambda, u > + < \mu, y > \geqq < \lambda, u > + < \mu, \bar{y} >$$

i.e.

$$<\mu, y> \geqq <\mu, \bar{y}> \quad ,$$

which contradicts (5.14). Therefore $y \in Min_P DY(\hat{u}, \hat{y})(u)$, as was to be proved. ∎

Remark 5.4. The following examples show that the conditions in Theorem 5.2 are essential.

1) ($\hat{u} \notin int\,(dom\ Y)$). Let $m = 2$, $n = p = 1$, $P = R_+$,

$$X(u) = \begin{cases} \{x \mid x \geqq 0\} & \text{if } u_1 \geqq 0,\ u_2 > 0 \\ \{x \mid x \geqq u_1\} & \text{if } u_1 \geqq 0, u_2 = 0 \\ \phi & \text{otherwise} \end{cases}$$

and $f(x, u) = x$. Then $Y(u) = X(u)$ and

$$W(u) = \begin{cases} \{0\} & \text{if } u_1 \geqq 0,\ u_2 > 0 \\ \{u_1\} & \text{if } u_1 \geqq 0,\ u_2 = 0 \\ \phi & \text{otherwise} \end{cases}$$

Let $\hat{u} = (0,0) \notin int(dom\ Y)$, $\hat{y} = 0$ and $u = (1,0)$. Then $DW(\hat{u}, \hat{y})(\boldsymbol{u}) = \{0,1\}$ and $DY(\hat{u}, \hat{y})(\boldsymbol{u}) = \{y \mid y \geqq 0\}$. Hence $DW(\hat{u}, \hat{y})(u) \not\subset Min_P DY(\hat{u}, \hat{y})(u)$.

2) ($\hat{y}$ is not normally P-minimal). Let $m = 1$, $n = p = 2$, $P = R^2_+$,

$$X(u) = \begin{cases} \{x \in R^2 \mid x_1 \geqq 0,\ x_2 \geqq |u|e^{-\frac{x_1}{|u|}}\} & \text{if } u \neq 0 \\ \{x \in R^2 \mid x_1 \geqq 0,\ x_2 \geqq 0\} & \text{if } u = 0 \end{cases}$$

and $f(u, u) = (x_1, x_2)$. Then $Y(u) = X(u)$ and

$$W(u) = \begin{cases} \{y \in R^2 \mid y_1 \geqq 0,\ y_2 = |u|e^{-\frac{y_1}{|u|}}\} & \text{if } u \neq 0 \\ \{(0,0)\} & \text{if } u = 0 \end{cases}$$

Let $\hat{u} = 0$ and $\hat{y} = (0,0)$. Then $\hat{y}$ is not a normally P-minimal point of $Y(\hat{u})$, though it is properly P-minimal. In this case $(0,0) \in DW(\hat{u}, \hat{y})(0) \subset DY(\hat{u}, \hat{y})(0)$ and $(1,0) \in DW(\hat{u}, \hat{y})(0)$. Hence $DW(\hat{u}, \hat{y})(0) \not\subset Min_P DY(\hat{u}, \hat{y})(0)$.

Now we can consider the case in which every objective function f_i is differentiable.

Definition 5.5. Let F be a set-valued map from R^m to R^p and $\bar{y} \in F(\bar{u})$. F is said to be upper pseudo-Lipschitzian at $(\bar{u}, \bar{y})$ if there exist neighborhood N_1 and N_2 of $\bar{u}$ and $\bar{y}$ respectively, and a positive number M such that

$$F(u) \cap N_2 \subset F(\bar{u}) + M\|u - \bar{u}\|B \quad \text{for } \forall u \in N_1 \ . \tag{5.15}$$

Proposition 5.1. If $X(\hat{u})$ is a closed set and $\tilde{X}(\hat{u},\hat{y})$ is bounded, then $\tilde{X}$ is uniformly compact near $(\hat{u},\hat{y})$.

(Proof) Suppose that the conclusion of the proposition is not true. Then there exist sequences $\{u^k\} \subset R^m$, $\{y^k\} \subset R^p$ and $\{x^k\} \subset R^n$ such that $u^k \to \hat{u}$, $y^k \to y^k \to \hat{y}$, $\|x^k\| \to +\infty$ and

$$x^k \in X(u^k) \text{ and } f(x^k,u^k) = y^k \quad \text{for } \forall k \ .$$

We may assume without loss of generality that $\{\frac{x^k}{\|x^k\|}\}$ converges to a nonzero vector x. Let $\alpha > 0$. Since $\|x^k\| \to +\infty$, $0 < \frac{\alpha}{\|x^k\|} \leqq 1$ for all k sufficiently large. Hence, from the convexity of X,

$$(1-\frac{\alpha}{\|x^k\|})\hat{x} + \frac{\alpha}{\|x^k\|}x^k \in X((1-\frac{\alpha}{\|x^k\|})\,\hat{u} + \frac{\alpha}{\|x^k\|}u^k) \ . \tag{5.16}$$

Since X is upper semicontinuous at $\hat{u}$ from Lemma 2.4, by taking the limit of (5.16) as $k \to \infty$, we see that

$$\hat{x} + \alpha x \in X(\hat{u}) \ .$$

Since f is P-convex,

$$f((1-\frac{\alpha}{\|x^k\|})(\hat{x},\hat{u}) + \frac{\alpha}{\|x^k\|}(x^k,u^k)) \leqq_P (1-\frac{\alpha}{\|x^k\|})f(\hat{x},\hat{u}) + \frac{\alpha}{\|x^k\|}f(x^k,u^k) \ .$$

Letting $k \to \infty$, we have

$$f(\hat{x} + \alpha x,\hat{u}) \leqq_P \hat{y} \ .$$

Since $\hat{y} \in W(\hat{u})$, $f(\hat{x} + \alpha x,\hat{u}) = \hat{y}$. Hence $\hat{x} + \alpha x \in \tilde{X}(\hat{u},\hat{y})$ for any $\alpha > 0$. However this contradicts the boundedness of $\tilde{X}(\hat{u},\hat{y})$. Therefore $\tilde{X}$ is uniformly compact near $(\hat{u},\hat{y})$.

Proposition 5.2. If $X(\hat{u})$ is a closed set, if $\tilde{X}(\hat{u},\hat{y})$ is a singleton, i.e. $\tilde{X}(\hat{u},\hat{y}) = \{\hat{x}\}$ and if $\tilde{X}$ is upper pseudo-Lipschitzian at $(\hat{u},\hat{y},\hat{x})$, then

$$DY(\hat{u},\hat{y})\,(u) = \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})\,(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u \quad \text{for } \forall u \in R^m \tag{5.17}$$

(Proof) It has been already proved that

$$DY(\hat{u},\hat{y})\,(u) \supset \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x}) + \nabla_u f(\hat{x},\hat{u}) \cdot u$$

([6], Proposition 4.1). So we shall prove the converse inclusion here. Let $y \in DY(\hat{u},\hat{y})(u)$. Then there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$ such that $h_k \to 0$, $u^k \to u$, $y^k \to y$ and $\hat{y} + h_k y^k \in Y(\hat{u} + h_k u^k)$ for all k. Hence there exists another sequence $\{x^k\} \subset R^n$ such that

$$\hat{x} + h_k x^k \in \tilde{X}(\hat{u} + h_k u^k, \hat{y} + h_k y^k) \quad \text{for } \forall k \quad .$$

From Proposition 5.1, the sequence $\{h_k x^k\}$ is bounded and so has a convergent subsequence. We may assume from the first that $h_k x^k \to x \in R^n$. Since X is upper semicontinuous at $\hat{u}$ and f is continuous,

$$\hat{x} + x \in \tilde{X}(\hat{u},\hat{y}) \quad .$$

Since $\tilde{X}(\hat{u},\hat{y}) = \{\hat{x}\}$, $x = 0$. Namely $h_k x^k \to 0$. Since $\tilde{X}$ is upper pseudo-Lipschitzian at $(\hat{u},\hat{y},\hat{x})$, there exists $M > 0$ such that, for any k sufficiently large,

$$\|\hat{x} + h_k x^k - \hat{x}\| \leqq M\|(\hat{u} + h_k u^k,\ \hat{y} + h_k u^k) - (\hat{u},\hat{y})\|$$

i.e.

$$\|x^k\| \leqq M\|(u^k,y^k)\| \quad .$$

Since $u^k \to u$ and $y^k \to y$, $\{x^k\}$ is bounded. Hence we may assume that $x^k \to \bar{x}$. Then clearly $\bar{x} \in DX(\hat{u},\hat{y})(u)$ and

$$y = \lim_{k\to\infty} y^k = \lim_{k\to\infty} \frac{f(\hat{x}+h_k x^k, \hat{u}+h_k u^k) - f(\hat{x},\hat{u})}{h_k}$$

$$= \nabla_x f(\hat{x},\hat{u}) \cdot \bar{x} + \nabla_u f(\hat{x},\hat{u}) \cdot u \quad .$$

Therefore $y \in \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u$. This completes the proof. ∎

Thus, from Theorem 5.1, Theorem 5.2 and Proposition 5.2, we have the following theorem which provides a complete characterization of the contingent derivative of the perturbation map W.

Theorem 5.3. If the following conditions (1)-(5) are satisfied, then

$$DW(\hat{u},\hat{y})(u) = Min_P[\nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_k f(\hat{x},\hat{u}) \cdot u] \tag{5.18}$$

$$\text{for } \forall u \in R^m \quad .$$

(1) $\hat{u} \in int(dom\ Y)$,

(2) $\hat{y}$ is a normally P-minimal point of $Y(\hat{u})$,

(3) $X(u)$ is a closed set for every u in a neighborhood of $\hat{u}$,

(4) $\tilde{X}(\hat{u},\hat{y})$ is a singleton, i.e. $\tilde{X}(\hat{u},\hat{y}) = \{\hat{x}\}$,

(5) $\tilde{X}$ is upper pseudo-Lipschitzian at $(\hat{u},\hat{y},\hat{x})$.

Finally we briefly mention sufficient conditions for the pseudo-Lipschitzian property of $\tilde{X}$. The following proposition can be obtained by applying Theorem 4.12 in Rockafellar [4].

Proposition 5.3. If the following two conditions are satisfied, then $\tilde{X}$ is (upper) pseudo-Lipschitzian at $(\hat{u},\hat{y},\hat{x})$:

(1) $X(u)$ is a closed set for every u in a neighborhood of $\hat{u}$,

(2) If $\sum_{i=1}^{p} \alpha_i \nabla_x f_i(\hat{x},\hat{u}) + \nu = 0$ for some $(\lambda,\nu) \in N_{graph\ X}(\hat{u},\hat{x})$, then

$$\alpha_i = 0 \text{ for } i = 1,\dots,p \text{ and } \lambda = 0. \tag{5.19}$$

Remark 5.5. When $X(u)$ is specified by inequality constraints as

$$X(u) = \{x \in R^n | g(x) \leqq u\}$$

the above condition (5.19) is nothing but the Mangasarian-Fromovitz constraint qualification at $\hat{x}$ for the set

$$\tilde{X}(\hat{u},\hat{y}) = \{x \in R^n | f(x,\hat{u}) - \hat{y} = 0,\ g(x) - \hat{u} \leqq 0\} \quad .$$

In view of Proposition 5.3, we can replace the condition (5) in Theorem 5.3 by (5.19).

6. Conclusion

We have obtained sufficient conditions for the upper and lower semicontinuity of the perturbation map, which provides the set of all cone minimal points depending upon the parameter vector, in convex vector optimization. It has been shown that the convexity assumption considerably simplifies the results in the general case. We have also provided a complete characterization of the contingent derivative of the perturbation map when the nominal point is normally minimal.

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