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ON APPROXIMATE VECTOR OPTIMIZATION

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PREFACE

The roots of current interest in the theory of approximate solutions of optimization problems lie in approximation theory and nondifferentiable optimization. In this paper an approximate saddle point theory is presented for vector valued convex optimization problems. The considerations cover different possible types of approximate optimality, including both the efficient, or Pareto-type, which is more frequently used in practical decision making applications, and the absolute, or strict type, which is more of theoretical interest. The saddle point theorems are used to study duality in the context of approximate solutions. The approach of the paper also provides for a unified view of a number of results achieved either in approximate scalar optimization or exact vector optimization.

Alexander B. Kurzhanski
Chairman
Systems and Decision Sciences Area

CONTENTS

ON APPROXIMATE VECTOR OPTIMIZATION

István Vályi

1. INTRODUCTION

The aim of the present paper is to give the proofs of the theory presented at the IIASA Workshop on Nondifferentiable Optimization held between 17 and 22 September, 1984, Sopron, Hungary. (See Vályi (1985a)),but some more recent related results are also included.

The central results are Hurwitz-type saddle point theorems corresponding to approximate solutions extending the theory developed by Zowe (1977) for one type of optima, or by Tanino and Sawaragi (1980) for another. By these theorems then we investigate the respective duality type problems. The study of this subject was started by Hiriart-Urruty (1982) and Strodiot et al. (1983) in the scalar case, and by Kutateladze (1978) and Loridan (1984) in the vector valued case.

The paper is divided into two parts according to the type of optimality considered.

Chapter 2. covers the case of strict, or non-Pareto optimality. This type of optimization in ordered spaces is regarded by many as having little practical use. This criticism, however is of less force in the approximate case, since for some value of the approximation error we may find solutions even if exact solutions do not exist (like the so called utopia point so often used in the Pareto case). Anyway, interest in it appears to be lasting as is shown e. g. by the recent paper of Azimov (1982).

Section 2.1. is devoted to some basic properties of approximate extremal elements in ordered vector spaces and in Section 2.2. the main results are proved. Applications expounded in Section 2.3. clarify the relationships between approximate saddle points and approximate solutions of the primal and dual problems associated to the original problem. In this we also show the connections to analogous results, namely the corresponding Kuhn-Tucker theorems based on the ε-subgradient calculus, obtained by Kutateladze (1978). In such a way the analogy will be complete with the theory developed for the scalar case in the paper by Strodiot et al. (1983). Finally we give a partial generalization to the vector valued case of

Golshtein's duality theorem dealing with generalized solutions of convex optimization problems and of Tuy's result characterizing well posed problems, i. e. those where the primal and dual values coincide. (See 14H in Holmes (1975)).

Chapter 3. deals with Pareto, or nondominated optimality. Here we define different types of approximate efficient solutions to vector optimization problems and develop the corresponding saddle point theorems along the logics of Tanino and Sawaragi (1980) or Luc (1984).

Section 3.1. is devoted to definitions and some basic properties of approximate extremal elements in ordered vector spaces and in Section 3.2. the saddle point theorems are proved. As for applications, in Section 3.3. we show the equivalence between approximate saddle points and the corresponding primal-dual pairs of solutions.

As a consequence of the fact that the notion of approximate solution coincides with that of (exact) solution in the case when the approximation error is zero, our results reduce to those obtained in the above mentioned papers. Throughout the paper we rely on a knowledge of convex analysis and the theory of ordered vector spaces, and therefore basic notions and facts are used without special explanation. If needed see e. g. Peressini (1967), Holmes (1975) or Akilov and Kutateladze (1978).

All the vector spaces appearing in the paper are real and ordering cones are supposed to be convex, pointed and algebraically closed. In the presence of a topological structure we suppose compatibility, i. e. that the ordering cone is closed. We denote by X and V vector spaces and by (Z,K) an ordered vector space with $core(K)\neq\phi$, where *core* refers to the algebraic interior. Similarly, *rcore* denotes the relative algebraic interior. (Y,C) is an order complete space, i. e. a vector lattice where every nonvoid set with a lower bound possesses an infimum. In order to ensure the existence of infima, resp. suprema for every (i.e. nonbounded) sets, we supplement the space (Y,C) with the elements ∞ and $-\infty$ using the notation $\bar{Y}=Y\cup\{-\infty,\infty\}$, and suppose that the usual algebraic and ordering properties hold. Hence for the set $H\subset Y$, which is not bounded from below, we have $inf\,(H)=-\infty$ and $inf\,(\phi)=\infty$. The dual space of Y is Y' while the topological dual is Y^*. The cone of positive functionals with respect to the cone $C\subset Y$, or the dual of C is C^+. The functional $y^*\in Y'$ denotes an element of C^+. $L^+(Z,Y)\subset L(Z,Y)$, or $\Lambda^+(Z,Y)\subset\Lambda(Z,Y)$ stands for the cone of positive linear, or continuous positive linear maps from Z to Y, respectively.

We recall now that for the various ordering relationships between two elements of an ordered vector space we shall use the following notations for example in (Y, C):

$$
\begin{aligned}
y_2 \geqq y_1 \quad &\text{iff} \quad y_2 - y_1 \in C \\
y_2 \geq y_1 \quad &\text{iff} \quad y_2 - y_1 \in C \setminus 0 \\
y_2 > y_1 \quad &\text{iff} \quad y_2 - y_1 \in \mathit{core}(C)
\end{aligned}
$$

To denote opposite relations we use symbols like $\not\geqq$ and $\not>$. Accordingly

$$y_2 \geqq y_1 \text{ or } y_2 \not\geqq y_1$$

refer to the fact that $y_1 \in Y$ dominates or does not dominate $y_2 \in Y$ from below, respectively.

The vectors e, $e_n, e_\gamma \in Y$ and the scalars $\varepsilon, \varepsilon_n \in \mathbb{R}$ represent the approximation error, of them we suppose that $e \geqq 0$, $e_n \geqq 0$ and $e_\gamma \geqq 0$ holds and similarly that $\varepsilon, \varepsilon_n$ are nonnegative.

Now the usual definition of the main subject of study in this paper follows, i. e. that of the convex minimization problen and of the corresponding vector valued Lagrangian (see e. g. Zowe (1976)).

Definition 1.1.

Let

$$f : X \to Y \cup \{\infty\}$$

$$h : X \to Z \cup \{\infty\}$$

proper convex functions with $\Delta = \mathbf{dom}\ f \cap \mathbf{dom}\ h$, and $l \in L(X, V)$. We define the minimization problem (MP) by way of the set of solutions:

$$MIN(MP) = \{x_0 \in F : f(x_0) \in MIN\ \{f(F)\}\} \tag{MP}$$

where

$$F = \{x \in X : x \in \Delta,\ h(x) \leqq 0,\ l(x) = 0\}$$

is called the feasibility set of the problem (MP).

The algebraic Lagrangian of the convex minimization problem (MP)

$$\Phi_L : X \times L(Z, Y) \times L(V, Y) \to \bar{Y}$$

is defined by the equality

$$\Phi_L(x,R,S) = \begin{cases} \infty & \text{if } x \notin \Delta \\ f(x) + R \cdot h(x) + S \cdot l(x) & \text{if } x \in \Delta \text{ and } R \in L^+(Z,Y) \\ -\infty & \text{if } x \in \Delta \text{ and } R \notin L^+(Z,Y) \end{cases}$$

with the set

$$dom\, \Phi_L = \{(x,R,S) \in X \times L(Z,Y) \times L(V,Y) : x \in \Delta,\ R \in L^+(Z,Y)\}$$

called the domain of Φ_L .

The element $(x_0,R_0,S_0) \in dom\, \Phi_L$ is a saddle point of the Lagrangian Φ_L if the following is met

(i) $\Phi_L(x_0,R_0,S_0) \in MIN\{\Phi_L(x,R_0,S_0) : x \in X\}$

(ii) $\Phi_L(x_0,R_0,S_0) \in MAX\{\Phi_L(x_0,R,S) : (R,S) \in L(Z,Y) \times L(V,Y)\}$.

Instead of the symbol *MIN* or *MAX*, one has to substitute one of the approximate (or exact) notions of minimality or maximality from the later following respective definitions. Depending on this choice, we call the elements of *MIN*(*MP*) solutions of the problem (*MP*) of the corresponding approximate (or exact) type.

The continuous Lagrangian Φ_Λ is defined as the restriction of Φ_L to $X \times \Lambda(Z,Y) \times \Lambda(V,Y)$ and the notion of saddle point of the continuous Lagrangian Φ_Λ is defined in a corresponding manner.

The above notations and conditions are supposed to be valid throughout the paper and will not be mentioned again.

2. STRICT OPTIMA

2.1. Approximate Extremal Points and Approximate Solutions

Now we start with the definition of strict extremal and strict approximate extremal points, and then we formulate some simple relationships between approximate extremal points corresponding to different values of the approximation parameter.

Definition 2.1.1.

Suppose that $H \subset \bar{Y}$. Then an element $y \in H$ is called a strict minimal element of H, or

$$y \in S-MIN(H) \qquad \text{if} \qquad H \subset y + C \quad .$$

The set

$$S(e)-MIN(H) = \{y \in H : H \subset y - e - C\}$$

is called the set of strict e-approximate minimai or $S(e)$-minimal points of H.

By convention we say that

$$S-MIN(\phi) = S(e)-MIN(\phi) = \{\infty\}$$

and if $H \subset Y$ is not bounded from below

$$S-MIN(H) = S(e)-MIN(H) = \{-\infty\}$$

Remark 2.1.1.

By the pointedness of the cone $C \subset Y$ the set $S-MIN(H)$ cannot have more than one element. If it has one, this obviously means that $inf(H) = S-MIN(H)$.

The notions of $S(e)$-maximal and S-maximal elements are to be defined in a corresponding manner.

The statements in the following proposition are straightforward consequences of the definitions.

Proposition 2.1.1.

(a) $S-MIN(H) = S(0)-MIN(H)$

(b) If $0 \leqq e_1 \leqq e_2$, then $S(e_1)-MIN(H) \subset S(e_2)-MIN(H)$.

(c) If $H \subset Y$ is bounded from below then $S(e)-MIN(H)=(inf\,(H)+e-C) \cap H$.

Corollary 2.1.1.

Let (Y,C) be equipped with a topological structure and $H \subset Y$ closed. Suppose that a net $\{e_\gamma \in C : \gamma \in \Gamma\}$ decreasing to $e \in Y$ exists with

(a) $S(e_\gamma)-MIN(H) \cap Y \neq \phi \quad \forall\, \gamma \in \Gamma$ and

(b) $S(e_{\gamma_0})-MIN(H) \subset Y$ is compact for some $\gamma_0 \in \Gamma$.

Then $$S(e)-MIN(H) \neq \phi.$$

Proof.

As a consequence of the closedness of $C \subset Y$, we have that $e \in C$, and so $S(e)-MIN(H)$ is well defined. $S(e_\gamma)-MIN(H) \neq \phi$ obviously implies $inf\,(H) \neq \infty$, and so we can apply (b) and (c) in Proposition 2.1.1. to conclude that $S(e)-MIN(H)$ is the intersection of nonvoid compact sets.

∎

Proposition 2.1.2.

Let (Y,C) be equipped with a topological structure and $\{e_\gamma \in C : \gamma \in \Gamma\}$ a decreasing net that converges to $e \in Y$.

Then

$$S(e)-MIN(H) = \cap \{S(e_\gamma)-MIN(H) : \gamma \in \Gamma\} \quad . \tag{2.1}$$

Proof.

By Corollary 3.2., Chap. 2. in Peressini (1967) we have $e = inf\,\{e_\gamma \in C : \gamma \in \Gamma\}$. Hence by Proposition 2.1.1. the left hand side in (2.1) is a subset of the right hand side.

For the reverse inclusion let $y \in Y$ be an element of $S(e_\gamma)-MIN(H)$ for each $\gamma \in \Gamma$. This means that $y \in H$ and for each fixed $h \in H$, the net $\{h-y+e_\gamma \in Y : \gamma \in \Gamma\}$ is contained in the closed cone $C \subset Y$, hence $h-y+e \in C$ also holds.

∎

Corollary 2.1.2.

Let (Y,C) be equipped with a weakly sequential complete topology, the ordering cone $C \subset Y$ normal and suppose that $\{e_n \in C : n \in N\}$ is a decreasing sequence.

Then

$$e = inf\,\{e_n \in C : n \in N\} = lim\,\{e_n \in C : n \in N\} \in Y$$

exists and

$$S(e)-MIN(H) = \cap\,\{S(e_n)-MIN(H) : n \in N\}$$

Proof.

The statement is a consequence of our Proposition 2.1.2. and the Corollary 3.5. Chap. 2. in Peressini (1967).

∎

Remark 2.1.2.

As a consequence of Proposition 2.1.1. the case with $e=0$ provides for conditions ensuring the existence of exact extremal points based on information about approximate ones in the previous Proposition and Corollaries.

Now, using Propositions 2.1.1. and 2.1.2. we formulate a few simple properties of the approximate solutions.

Corollary 2.1.3.

Suppose, that (Y,C) is equipped with a topological structure.

Then the following hold:

(a) $S-MIN(MP) = S(0)-MIN(MP)$

(b) $0 \leqq e_1 \leqq e_2$ implies $S(e_1)-MIN(MP) \subset S(e_2)-MIN(MP)$.

(c) If $\{e_\gamma \in C : \gamma \in \Gamma\}$ is a decreasing net that converges to $e \in Y$, then $\cap\,\{S(e_\gamma)-MIN(MP) : \gamma \in \Gamma\} = S(e) - MIN(MP)$.

(d) If the topology of (Y,C) is weakly sequentially complete, the cone $C \subset Y$ is normal, $\{e_n \in C : n \in N\}$ is a decreasing sequence with the infimum $e \in Y$, then $\cap\{S(e_n)-MIN(MP) : n \in N\} = S(e)-MIN(MP)$.

(e) Suppose that the set $f(F) \in Y$ is closed, $\{e_n \in C : n \in N\}$ is a decreasing sequence that converges to $e \in Y$, $x_\gamma \in X$ is an $S(e_\gamma)$-solution of (MP) for each $\gamma \in \Gamma$ and there is a $\gamma_0 \in \Gamma$ such that the set $S(e_{\gamma_0})-MIN\{f(x) \in Y : x \in F\} \subset Y$

is compact. Then (MP) has an $S(e)$-minimal solution.

2.2. The Saddle Point Theorems

The present section is closely related to Zowe's results both as far as proofs and notions are concerned (Zowe (1976) and Zowe (1977)). As there, in the case of exact solutions, one implication between the existence of solutions and saddle points is valid under fairly general conditions while we need additional assumptions in the case of the other.

Proposition 2.2.1.

If $(x_0,R_0,S_0) \in dom\ \Phi_L$ is a $S(e)$-saddle point of the Lagrangian Φ_L, then

$$-e \leqq R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \leqq 0$$

Proof.

Follows from $x_0 \in F$ and (ii) of Definition 1.1. if one considers the case $(R,S)=(0,0)$ in $S(e)-MAX$.

∎

Theorem 2.2.1.

If $(x_0,R_0,S_0) \in dom\ \Phi_L$ is an $S(e)$-saddle point of the Lagrangian Φ_L, then $x_0 \in X$ is an $S(2 \cdot e)$-solution of (MP).

Proof.

First we prove that this implies that $x_0 \in F$.

$x_0 \in D$ follows from the relation $(x_0,R_0,S_0) \in dom\ \Phi_L$. Using the choice $(x,R,S)=(x_0,R,S_0)$ in (ii) of Definition 1.1. we obtain that

$$R \cdot h(x_0) \leqq R_0 \cdot h(x_0) + e \qquad \forall\ R \in L^+(Z,Y) \tag{2.2}$$

and using $(x,R,S)=(x_0,R_0,S)$ that

$$S \cdot l(x_0) \leqq S_0 \cdot l(x_0) + e \qquad \forall S \in L(V,Y) \tag{2.3}$$

If $h(x_0) \not\in -K$, then the strict separation theorem applied to the singleton set $\{h(x_0)\} \subset Z$ and the algebraically closed convex cone $K \subset Z$ with a nonempty core (Köthe Sect. 17, 5, (2)) yields a $z^* \in K^+$ with

$$sup \ \{ \ <z^*, -k> : k \in K \ \} = 0 \ < \ <z^*, h(x_0)>.$$

Let $c \in C \setminus \{0\}$ be a fixed vector, to be specified later and let us define $\bar{R} \in L^+(Z,Y)$ with the equation

$$\bar{R} \cdot z \ = \ \frac{<z^*, z>}{<z^*, h(x_0)>} \cdot c.$$

By inequality (2.2) now we have

$$0 \neq c = \bar{R} \cdot h(x_0) \leqq R_0 \cdot h(x_0) + e \tag{2.4}$$

Selecting first any $c \in C \setminus \{0\}$, we see that $R_0 \cdot h(x_0) + e \neq 0$ holds, therefore we are allowed to set at a second step

$$c = 2 \cdot (R_0 \cdot h(x_0) + e)$$

leading to a contradiction with (2.4).

A similar argument shows the impossibility of $l(x_0) \neq 0$, and so we can conclude that $x_0 \in F$.

Again, by the definition of Φ_L and the $S(e)$-saddle point, we have for each $(x, R, S) \in dom \ \Phi_L$:

$$f(x_0) + R \cdot h(x_0) + S \cdot l(x_0) - e \leqq f(x) + R_0 \cdot h(x) + S_0 \cdot l(x) + e$$

As a consequence of $(x_0, R_0, S_0) \in dom \ \Phi_L$ the relation $R_0 \in L^+(Z,Y)$ holds and therefore a substitution $(x, R, S) = (x, 0, 0)$ completes the proof.

■

Using the topological version of the strict separation theorem in the above proof, we readily obtain the following for the continuous Lagrangian Φ_Λ.

Theorem 2.2.2.

Suppose that (Y,C), (Z,K) and V are equipped with a topological structure.

If $(x_0, R_0, S_0) \in dom\ \Phi_\Lambda$ is an $S(e)$-saddle point of the Lagrangian Φ_Λ then $x_0 \in X$ is a $S(2 \cdot e)$-solution of (MP).

Definition 2.2.1.

We say that the problem (MP) meets the algebraic Slater-Uzawa constraint qualification if either

(i) there exists an $x_1 \in rcore(\Delta)$ with $h(x_1) \in -rcore(K)$ and $l(x_1)=0$,

or

(ii) no linear constraint is present and there exists an $x_1 \in \Delta$ with

$$h(x_1) \in -rcore\ \{h(x) + k \in Z : x \in \Delta,\ k \in K\}.$$

Definition 2.2.2.

The problem (MP), where (Y,C), (Z,K) and V are topological spaces meets the topological Slater-Uzawa constraint qualification if there exists an $x_1 \in int(\Delta)$ with $h(x_1) \in -int(K)$ and $l(x_1)=0$.

Now for the convenience of the reader we quote from Zowe (1976) the algebraic and topological vector valued versions of the Farkas-Minkowski lemma.

Theorem 2.2.3.

Suppose that the minimization problem (MP) meets the algebraic Slater-Uzawa constraint qualification.

Then the following statements are equivalent:

(a) $f(x) \geqq 0 \quad \forall x \in F$

(b) there exist operators $R \in L^+(Z,Y)$, $S \in L(V,Y)$ such that

$$f(x) + R \cdot h(x) + S \cdot l(x) \geqq 0 \qquad \forall x \in \Delta.$$

Theorem 2.2.4.

Let (Y,C) and (Z,K) be equipped with a topological structure, X a completely metrizable topological vector space and the cone $C \subset Y$ normal. Let further V be a Hilbert space with $l(X) \subset V$ a closed subspace and suppose that the minimization problem (MP) meets the topological Slater-Uzawa constraint qualification.

Then the following statements are equivalent:

(a) $f(x) \geqq 0 \quad \forall x \in F$

(b) there exist operators $R \in \Lambda^+(Z,Y)$, $S \in \Lambda(V,Y)$ such that

$$f(x) + R \cdot h(x) + S \cdot l(x) \geqq 0 \qquad \forall x \in \Delta.$$

Now we are able to formulate and prove the converse statements to Theorems 2.2.1. and 2.2.2.

Theorem 2.2.5.

Under the assumptions of Theorem 2.2.3. the following holds:

If $x_0 \in X$ is an $S(e)$-solution of the problem (MP), then there exist operators $R_0 \in L^+(Z,Y)$ and $S_0 \in L(V,Y)$, such that $(x_0,R_0,S_0) \in \boldsymbol{dom}\ \Phi_L$ is an $S(e)$-saddle point of the Lagrangian Φ_L.

Proof.

As x_0 is an $S(e)$-solution, we can apply Theorem 2.2.3. for the function f_1, where

$$f_1 : X \to Y$$

$$f_1(x) = f(x) - f(x_0) + e$$

instead of the original f. Therefore there exist operators such that

$$f_1(x) + R_0 \cdot h(x) + S_0 \cdot l(x) \geqq 0 \qquad \forall\, x \in \Delta \tag{2.5}$$

From Proposition 2.2.1. and (2.5) now we have

$$f(x) + R_0 \cdot h(x) + S_0 \cdot l(x) + e \geqq f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \qquad \forall\, x \in \Delta,$$

on one hand, and by

$$0 \geqq R \cdot h(x_0) + S \cdot l(x_0) \qquad \forall\, R \in L^+(Z,Y),\ S \in L(V,Y) \quad ,$$

$$f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \geqq f(x_0) + R \cdot h(x_0) + S \cdot l(x_0) - e \qquad \forall\, x \in \Delta,$$

on the other, completing the proof.

∎

Repeating the above procedure with the topological Theorem 2.2.4. instead of Theorem 2.2.3. we obtain:

Theorem 2.2.6.

Under the assumptions of Theorem 2.2.4. the following holds:

If $x_0 \in X$ is an $S(e)$-solution of the minimization problem (MP), then there exist operators $R_0 \in \Lambda^+(Z,Y)$ and $S_0 \in \Lambda(V,Y)$, such that $(x_0,R_0,S_0) \in \boldsymbol{dom}\ \Phi_\Lambda$ is an $S(e)$-saddle point of the continuous Lagrangian Φ_Λ.

Remark 2.2.1.

As a consequence of Proposition 2.1.1. (a) our results reduce to those of Zowe (1977) and Zowe (1976) in the case when $e = 0$.

2.3. Primal and Dual Problems

In this section we place the results of Section 3. in the context of some related results and apply them to analyze the primal and dual problems associated with the problem (MP).

Definition 2.3.1.

Consider the following functions:

$$P : X \to \bar{Y}$$

$$P(x) = sup \ \{ \ \Phi_L(x, R, S) \in \bar{Y} : R \in L(Z, Y), \ S \in L(V, Y) \ \}$$

and

$$D : L(Z, Y) \times L(V, Y) \to \bar{Y}$$

$$D(R, S) = inf \ \{ \ \Phi_L(x, R, S) \in \bar{Y} : x \in X \ \}$$

which we call the (algebraic) strict primal and dual functions of the minimization problem (MP), respectively. The vectors defined as

$$v = inf \ \{ \ P(x) \in \bar{Y} : x \in X \ \}$$

and

$$v^* = sup \ \{ \ D(R, S) \in \bar{Y} : R \in L(Z, Y), \ S \in L(V, Y) \ \}$$

are the (algebraic) strict value and dual value, respectively.

The algebraic strict primal and dual problems are formulated by way of the sets of solutions:

$$S(e)\text{-}MIN(P) = \{ x \in X : P(x) \in S(e)\text{-}MIN(P(X)) \}. \tag{P}$$

and

$$S(e)\text{-}MAX(D) = \{ (R,S) \in L(Z,Y) \times L(V,Y) :$$
$$D(R,S) \in S(e)\text{-}MAX(D(L(Z,Y) \times L(V,Y))) \}. \tag{D}$$

The relationship between the original minimization problem (MP) and its primal problem (P) is shown by the following proposition, namely that the latter is just the reformulation of a constrained problem into a nonconstrained one.

Proposition 2.3.1.

If the space (Y,C) is Archimedean then the problem (P) is equivalent to (MP), i.e.

$$P(x) = \begin{cases} f(x) & \text{if } x \in F \\ \infty & \text{if } x \notin F \end{cases}$$

Proof.

If $x \in F$, then we have

$$R \cdot h(x) + S \cdot l(x) \leqq 0 \qquad \forall R \in L^+(Z,Y),\ S \in L(V,Y)$$

and therefore

$$\Phi_L(x,R,S) \leqq f(x),$$

but the equality is valid in the case $(R,S) = (0,0)$.

If $x \notin F$ because of $x \notin \Delta$, then $\Phi_L(x,R,S)=\infty$, and hence $P(x)=\infty$. If $x \notin F$ because of $h(x) \nleq 0$, then by the separation argument in the proof of Theorem 2.2.1. ensures the existence of a $z^* \in K^+$ with $\langle z^*, h(x)\rangle > 0$. This enables us to construct a sequence of operators $\{R_n \in L^+(Z,Y) : n \in N\}$ with

$$sup\ \{\Phi_L(x,R_n,0) : n \in N\} = \infty \ . \tag{2.6}$$

Let, namely be $c \in C \setminus \{0\}$ a fixed vector and define $R_n \in L^+(Z,Y)$ by the equation

$$R_n \cdot z = n \cdot \langle z^*, z\rangle \cdot c.$$

From the Archimedean property of (Y,C) now (2.6) follows, and a similar argument shows $P(x)=\infty$ in the case of $l(x) \neq 0$.

Proposition 2.3.2.

The primal function P is convex and the dual function D is concave.

Proof.

The first statement directly follows from Proposition 2.3.1. The domain of the dual function is clearly convex, and concavity follows from the superadditivity of the *inf* operation.

Proposition 2.3.3.

(a) The primal value is greater or equal than the dual value.

(b) If $x_0 \in X$ is an $S(e)$-solution of the primal problem (P) and $(R_0,S_0) \in L(Z,Y) \times L(V,Y)$ is that of the dual problem (D) then $P(x_0) \geqq D(R_0,S_0)$.

(c) Let us have for some $x_0 \in X$, $(R_0,S_0) \in L(Z,Y) \times L(V,Y)$

$$P(x_0) \leqq D(R_0,S_0) + e.$$

Then x_0 is an $S(e)$-solution of the primal problem (P) and (R_0,S_0) is an $S(e)$-solution of the dual problem (D).

Proof:

The statement is an obvious consequence of the definitions.

Now we turn to the consideration of the connection between our Hurwitz-type results and those obtained by Kutateladze (1978). In this we shall rely on the notion of perturbation function and approximate subgradients.

Definition 2.3.2.

The function

$$p : Z \times V \to \bar{Y}$$

$$p(z,v) = inf \ \{ f(x) \in \bar{Y} : x \in F(z,v) \}$$

where

$$F(z,v) = \{ x \in X : h(x) \leqq z, \ l(x) = v \}$$

is called the perturbation function associated with the problem (MP).

Proposition 2.3.4.

Suppose that $R \in L^+(Z,Y)$, then

$$inf \ \{ p(z,v) + R \cdot z + S \cdot v : (z,v) \in Z \times V \} = D(R,S)$$

Proof.

The following equation is a direct consequence of the definitions;

$$inf \ \{ p(z,v) + R \cdot z + S \cdot v : (z,v) \in Z \times V \} =$$

$$= inf \ \{ f(x) + R \cdot z + S \cdot v : x \in X, \ z \geqq h(x), \ v = l(x) \}$$

Therefore we only have to prove that the right hand side equals with $D(R,S)$. To do this consider the inclusion

$$\{ f(x) + R \cdot z + S \cdot v : x \in X, \ z \geqq h(x), \ v = l(x) \} \supset$$

$$\supset \{ f(x) + R \cdot h(x) + S \cdot l(x) : x \in X \}.$$

By this and the definition of the dual function D the relation $\leq$ always holds. On the other hand $R \in L^+(Z,Y)$ implies

$$R \cdot z + S \cdot v \geqq R \cdot h(x) + S \cdot l(x)$$

and hence we also have the opposite relation.

■

The definition of approximate, or e-subgradient and the following theorem is taken from Kutateladze (1978).

Definition 2.3.3.

The set

$$\partial_e f(x_0) = \{ T \in L(X,Y) : T \cdot (x - x_0) \leqq f(x) - f(x_0) + e, \ x \in X \}$$

is called the approximate, or e-subdifferential of f at $x_0 \in X$.

Remark 2.3.1.

The statement that $0 \in \partial_e f(x_0)$ is obviously equivalent to the relation $x_0 \in S(e)-MIN(MP)$ if there are no feasibility constraints.

Theorem 2.3.1.

Suppose that the problem (MP) meets the algebraic Slater-Uzawa constraint qualification, then

$$x_0 \in S(e)-MIN(MP)$$

if and only if

there exist $R_0 \in L^+(Z,Y)$, $S_0 \in L(V,Y)$ and $e_1 \geqq 0$, $e_2 \geqq 0$ with

$$e_1+e_2+R_0 \cdot h(x_0) \leqq e,$$

such that

$$0 \in \partial_{e_1} f(x_0) + \partial_{e_2}(R_0 \cdot h)(x_0) + S \cdot l$$

Theorem 2.3.2.

Suppose that $S(e)-MIN(MP) \neq \phi$ and for $(R_0,S_0) \in L^+(Z,Y) \times L(V,Y)$

$$-(R_0,S_0) \in \partial_e p(0,0) \tag{2.7}$$

holds.

Then

$$(R_0,S_0) \in S(2 \cdot e)-MIN(D) \tag{2.8}$$

Proof.

As the conditions ensure there is a $x_0 \in S(e)-MIN(MP)$.

Now using this and the definition of the e-subgradient we obtain

$$f(x_0) - 2 \cdot e \leqq p(z,v) + (R_0,S_0) \cdot (z,v) \qquad \forall\, (z,v) \in Z \times V.$$

Proposition 2.3.4. yields

$$f(x_0) - 2 \cdot e \leqq D(R_0,S_0)$$

and by feasibility the proof is complete.

■

Theorem 2.3.3.

Suppose that the problem (MP) meets the algebraic Slater-Uzawa constraint qualification, and suppose that $S(e)-MIN(MP) \neq \phi$.

If for $(R_0,S_0) \in L^+(Z,Y) \times L(V,Y)$

$$(R_0,S_0) \in S(e)-MIN(D) \tag{2.9}$$

holds then

$$-(R_0,S_0) \in \partial_{3\cdot e} p(0,0) \tag{2.10}$$

Proof.

By the conditions we have an $x_0 \in S(e)-MIN(MP)$ and hence

$$f(x_0) - e \leqq p(0,0).$$

Theorem 2.2.5. ensures the existence of a pair $(R_1,S_1) \in L^+(Z,Y) \times L(V,Y)$ such that $(x_0,R_1,S_1) \in \mathit{dom}\ \Phi_L$ is a $S(e_2)$-saddle point for the Lagrangian Φ_L, that is by the definition of the problem (D) and (i) of Definition 1.1. this means that

$$f(x_0) + R_1 \cdot h(x_0) + S_1 \cdot l(x_0) \leqq D(R_1,S_1) + e.$$

Now Proposition 2.3.1. implies

$$f(x_0) - 2 \cdot e \leqq D(R_1,S_1)$$

and as $(R_0,S_0) \in S(e)-MIN(D)$, we also have

$$p(0,0) - 3 \cdot e \leqq f(x_0) - 3 \cdot e \leqq D(R_0,S_0).$$

From here by Proposition 2.3.4. the statement follows.

∎

Theorem 2.3.4.

Suppose that the problem (MP) meets the Slater-Uzawa constraint qualification and consider the following statements.

(a) $(x_0,R_0,S_0) \in \mathit{dom}\ \Phi_L$ is an $S(e_1)$-saddle point for Φ_L.

(b) For $(x_0,R_0,S_0) \in \mathit{dom}\ \Phi_L$ we have

$$0 \in \partial_{e'} f(x_0) + \partial_{e''}(R_0 \cdot h)(x_0) + S_0 \cdot l \tag{2.11}$$

with

$$e' \geqq 0,\ e'' \geqq 0 \text{ and } 0 \leqq e' + e'' \leqq R_0 \cdot h(x_0) + e_2 \tag{2.12}$$

Then (a) implies (b) with $e_2 = 2 \cdot e_1$, and (b) implies (a) with $e_1 = 2 \cdot e_2$.

Proof.

If (a) holds then according to Theorem 2.2.1., $x_0 \in S(2 \cdot e)-MIN(MP)$ and so Theorem 2.3.1. ensures (2.11) and (2.12). On the other hand, by Proposition 2.2.1. and the saddle point property we have

$$f(x_0) - 2 \cdot e_1 \leqq \Phi_L(x_0, R_0, S_0) \leqq D(R_0, S_0)$$

and hence

$$p(0,0) - 2 \cdot e_1 \leqq D(R_0, S_0).$$

From here by Proposition 2.3.4. (b) follows.

Suppose now that (b) holds, that is again by Proposition 2.3.4. on one hand we have

$$p(0,0) - e_2 \leqq D(R_0, S_0).$$

As by Theorem 2.3.1. $x_0 \in S(e_2)-MIN(MP)$, implying

$$\Phi_L(x_0, R_0, S_0) - 2 \cdot e_2 \leqq inf \ \{ \ \Phi_L(x, R_0, S_0) : x \in X \ \} .$$

On the other, by feasibility and (2.12):

$$\Phi_L(x_0, R, S) \leqq f(x_0) \leqq \Phi_L(x_0, R_0, S_0) + e_2 \qquad \forall \ R \in L(Z,Y),\ S \in L(V,Y).$$

∎

In view of Proposition 2.3.1. the above can be reformulated as:

Corollary 2.3.1.

Suppose that the problem (MP) meets the Slater-Uzawa constraint qualification and consider the following statements:

(a) $(x_0, R_0, S_0) \in dom\ \Phi_L$ is an $S(e_1)$-saddle point for the Lagrangian Φ_L associated with the problem (MP).

(b) For $(x_0, R_0, S_0) \in dom\ \Phi_L$ we have

  (i) $x_0 \in X$ is a $S(e_2)$-solution of the primal problem (P) and

  (ii) $(R_0, S_0) \in L^+(Z,Y) \times L(V,Y)$ is a $S(e_2)$-solution of the dual problem (D).

Then (a) implies (b) with $e_2 = 4 \cdot e_1$, and (b) implies (a) with $e_1 = 6 \cdot e_2$.

Now we turn to the consideration of generalized solutions.

Definition 2.3.4.

Suppose that (Y,C) is eqiupped with a topological structure, and $\{e_\gamma \in C : \gamma \in \Gamma\}$ is a decreasing net that converges to $0 \in Y$. Let further $x_\gamma \in X$ be an $S(e_\gamma)$-solution of (MP) for each $\gamma \in \Gamma$. Then we call the net $\{x_\gamma \in X : \gamma \in \Gamma\}$ a generalized strict solution of the minimization problem (MP).

Suppose, in addition that there exists a net $\{(R_\gamma,S_\gamma) \in L(Z,Y) \times L(V,Y) : \gamma \in \Gamma\}$ with the property that $(x_\gamma,R_\gamma,S_\gamma)$ is an $S(e_\gamma)$-saddle point for the Lagrangian Φ_L. Then we call the net $\{(x_\gamma,R_\gamma,S_\gamma) \in X \times L(Z,Y) \times L(V,Y) : \gamma \in \Gamma\}$ a generalized strict saddle point of the Lagrangian Φ_L.

Proposition 2.3.5.

For the strict value of (MP), $v \in \bar{Y}$ we have

$$v = inf \{ f(x_\gamma) \in Y : \{x_\gamma \in X : \gamma \in \Gamma\} \text{ a generalized solution, } \gamma \in \Gamma \}.$$

Proof.

The equality is a direct consequence of the definitions.

Definition 2.3.5.

Suppose that (Y,C) is equipped with a topological structure. We call the problem (MP) well posed if there exists a net $\{(x_\gamma,R_\gamma,S_\gamma) : \gamma \in \Gamma\}$ such that

$$lim \ \{ \Phi_L(x_\gamma,R_\gamma,S_\gamma) : \gamma \in \Gamma \} = v = v^*.$$

Remark 2.3.2.

By the definition of infimum and supremum, our definition coincides with the single requirement of $v = v^*$ in the scalar valued case.

Theorem 2.3.5.

Suppose that (Y,C) is equipped with a topological structure and that the cone $C \subset Y$ is normal. If the Lagrangian Φ_L has a generalized saddle point, then the problem (MP) is well posed.

Proof.

As a consequence of Proposition 2.3.3. (a) we only have to prove $v \leq v^*$. By the definition of the generalized saddle point, there exist a decreasing net $\{e_\gamma \in C : \gamma \in \Gamma\}$, that converges to $0 \in Y$ such that

$$P(x_\gamma) - e_\gamma \leqq \Phi_L(x_\gamma, R_\gamma, S_\gamma) \leqq D(R_\gamma, S_\gamma) + e_\gamma \qquad \forall\, \gamma \in \Gamma.$$

Hence for every fixed $\delta \in \Gamma$ we have

$$inf\ \{ P(x) : \gamma \in \Gamma \} - e_\delta \leqq sup\ \{ D(R_\gamma, S_\gamma) : \gamma \in \Gamma \} + e_\delta,$$

and, by the normality of the cone $C \subset Y$, from here the statement follows.

■

Corollary 2.3.1.

Under the conditions of Theorems 2.2.5. and 2.3.5. the existence of a generalized strict solution to the problem (MP) implies that the problem is well posed.

Proof.

Easily follows from the combination of the quoted theorems.

■

Remark 2.3.3.

It is worth noting that the reverse implication seems not to hold in the vectorial case while it is trivial for scalars.

Similarly to the preceding, notions and statements of Section 2.3. can also be formulated in a purely topological way. Proofs are analogous, but of course relying on Theorem 2.2.6. instead of Theorem 2.2.5.

3. NON-DOMINATED OPTIMA

3.1. Approximate Non-dominated Elements

Definition 3.1.1.

The vector $y \in H$ is a $P(e)$-minimal element of $H \subset \bar{Y}$ or approximately Pareto minimal, in notation

$$y \in P(e)-MIN(H),$$

if

$$(y - e - C) \cap H \subset \{y - e\},$$

$WP(e)$-minimal, in notation

$$y \in WP(e)-MIN(H),$$

if

$$(y - e - core(C)) \cap H = \phi.$$

Here, of course, we need the condition that $core(C) \neq \phi$ and speaking about WP-minimality, we always suppose it.

and $P(y^*, \varepsilon)$-minimal, in notation

$$y \in P(y^*, \varepsilon)-MIN(H),$$

if

$$<y^*, y> - \varepsilon \leqq <y^*, h> \quad \forall h \in H.$$

By convention, we say that all kinds of minima of the void set consist of the single element $\infty \in \bar{Y}$. The approximately maximal elements are to be defined in a corresponding manner.

Remark 3.1.1.

Our definitions, in the case of $e=0$, or $\varepsilon=0$, reproduce the usual exact notions of minimality. Weak approximate minimality means the corresponding approximate minimality with respect to the (algebraically non-closed) cone $C'=\{0\}\cup core(C)$.

The notion of $y\in Y$ being $P(y^*,\varepsilon)$-minimal means that $<y^*,y>\in\mathbb{R}$ is a $P(\varepsilon)$-minimal element of the set $y^*(H)=\{<y^*,h>\in\mathbb{R} : h\in H\}$.

Remark 3.1.2.

In the scalar case the different notions of approximate solutions for the minimization problem (MP) coincide and there we simply speak of ε-solutions or ε-saddle points.

Let us formulate some simple facts that are easy consequences of the definitions but are still interesting because they clarify the relationships between the different notions of minimal solution. Omitted proofs are trivial.

Proposition 3.1.1.

Suppose that $e_1\leqq e_2$ and $\varepsilon_1\leqq\varepsilon_2$. Then we have

$$P(e_1) - MIN(MP) \subset P(e_2) - MIN(MP)$$

$$WP(e_1) - MIN(MP) \subset WP(e_2) - MIN(MP)$$

$$P(y^*,\varepsilon_1) - MIN(MP) \subset P(y^*,\varepsilon_2) - MIN(MP)$$

Proposition 3.1.2.

(a) Suppose that we have $<y^*,e> > 0$. Then

$$P(y^*,\varepsilon)-MIN(MP) \subset P(e')-MIN(MP)$$

with

$$e' = \frac{\varepsilon}{<y^*,e>}\cdot e$$

(b) $WP(e)-MIN(MP) = \cup\{P(y^*,<y^*,e>)-MIN(MP) : y^*\in C^+\setminus\{0\}\}$.

Proposition 3.1.3.

Suppose that (Y,C) is equipped with such a weakly sequentially complete topology that the ordering cone $C\subset Y$ is normal. Consider a sequence $\{e_n\in C : n\in\mathbb{R}\}$ decreasing to $e\in C$.

Then

$$P(e)-MIN(MP) \subset \cap\{P(e_n)-MIN(MP) : n \in N\} \subset WP(e)-MIN(MP)$$

and

$$\cap\{WP\ (e_n)-MIN(MP) : n \in N\} = WP(e)-MIN(MP)$$

Proof.

The first inclusion is obvious.

For the second let us reason by contradiction and suppose that the element $x_0 \in F$ is not $WP(e)$-minimal. This means that we can find another $x_1 \in F$ with

$$f(x_1) < f(x_0) - e. \tag{3.1}$$

By normality $int(C) \neq \phi$ and therefore $int(C)=core(C)$. Hence the formula under (3.1) is equivalent to

$$f(x_0) - e - f(x_1) \in int(C).$$

As a consequence of Corollary 3.5. Chap. 2. in Peressini (1967) for the sequence we have

$$lim\ \{f(x_0) - e_n - f(x_1) : n \in N\} = f(x_0) - e - f(x_1) \in int(C),$$

and so, there exists an $m \in N$ with

$$f(x_0) - e_m - f(x_1) \in int(C).$$

This means that $f(x_1)$ dominates the element $f(x_0)-e_m \in Y$ from below.

The proof of the second statement is analogous.

∎

Proposition 3.1.4.

Suppose that the sequence $\{\varepsilon_n \in \mathbb{R}^+ : n \in N\}$ decreases to $\varepsilon \in \mathbb{R}^+$.

Then

$$\cap\{P(y^*, \varepsilon_n)-MIN(MP) : n \in N\} = P(y^*, \varepsilon)-MIN(MP).$$

3.2. Saddle Point Theorems

Proposition 3.2.1.

The element $(x_0,R_0,S_0) \in dom\, \Phi_L$ is a $P(e)$-saddle point of the Lagrangian Φ_L, iff

(a) $\Phi_L(x_0,R_0,S_0) \in P(e)-MIN\ \{\Phi_L(x,R_0,S_0) \in \bar{Y} : x \in X\}$

(b) $x_0 \in F$

(c) $-e \not\geq R_0 \cdot h(x_0) \leqq 0.$

Proof.

Condition (a) is identical with the first part of the definition. Suppose now that $(x_0,R_0,S_0) \in dom\, \Phi_L$ is a $P(e)$-saddle point. The definition of $dom\, \Phi_L$ immediately yields (b), and we have

$$\Phi_L(x_0,R_0,S_0) = f(x_0) + R_0 \cdot h(x_0).$$

From the definition of the $P(e)$-saddle point we also know that

$$\Phi_L(x_0,R,S) \not\geq \Phi_L(x_0,R_0,S_0) + e \tag{3.2}$$

for each $(R,S) \in L(Z,Y) \times L(V,Y)$. Selecting $S = S_0$ and $R = R_0$ we obtain

$$(R - R_0) \cdot h(x_0) \not\geq e \qquad \forall R \in L^+(Z,Y) \tag{3.3}$$

and

$$(S - S_0) \cdot l(x_0) \not\geq e \qquad \forall S \in L(V,Y) \tag{3.4}$$

respectively. Suppose now that $h(x_0) \leqq 0$ does not hold. Then by the strict algebraic separation theorem (see Kothe (1966) Section 17.5. (2)) applied for the sets $\{h(x_0)\} \subset Z$ and $-K \subset Z$, the existence of a functional $z^* \in K^+$ is guaranteed with

$$<z^*, h(x_0)> \ > 0.$$

Let $c \geq 0$, $c \in Y$ be an arbitrary, fixed element, and define the map $R \in L(Z,Y)$ as

$$R : z \to \frac{<z^*, z>}{<z^*, h(x_0)>} \cdot (e + c) + R_0 z.$$

For this operator R we obviously have $R \in L^+(Z,Y)$ and

$$(R - R_0) \cdot h(x_0) = e + c.$$

in contradiction with (3.3). A similar argument leads to contradiction with (3.4), if we suppose $l(x_0) \neq 0$. Here we define an operator $S \in L(V,Y)$ as

$$S : v \rightarrow \frac{\langle v^*, v\rangle}{\langle v^*, l(x_0)\rangle} \cdot (e + c) + S_0 v.$$

The last inequality in (c) is a consequence of $x_0 \in F$ and $R_0 \in L^+(Z,Y)$, while the first follows from (3.2) if we choose $(R,S)=(0,0)$.

To prove the reverse implication, suppose that (a), (b) and (c) are valid. From the last two we have the following relations:

$$f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) + e \nleq f(x_0) \geqq f(x_0) + R \cdot h(x_0) + S \cdot l(x_0)$$

for each $(R,S) \in L^+(Z,Y) \times L(V,Y)$ implying the missing relationship for $(x_0,R_0,S_0) \in dom\ \Phi_L$ to be a $P(e)$-saddle point.

∎

Remark 3.2.1.

The property stated in Proposition 3.2.1. is as much negative as positive, and therefore is a first sign of the problems to be seen in the sequel. Point (c), namely, turns into the well-known complementarity condition

$$R_0 \cdot h(x_0) = 0$$

in the case of exact saddle points. In general, however it only means that

$$R_0 \cdot h(x_0) \in (-C \setminus \{-e - C\}) \cup \{-e\},$$

and the right hand side here is an unbounded set.

The proof of the following two statements is analogous.

Proposition 3.2.2.

The element $(x_0,R_0,S_0) \in dom\ \Phi_L$ is a $WP(e)$-saddle point of the Lagrangian Φ_L iff

(a) $\Phi_L(x_0,R_0,S_0) \in WP(e)-MIN\ \{\Phi_L(x,R_0,S_0) \in \bar{Y} : x \in X\}$

(b) $x_0 \in F$

(c) $-e \not> R_0 \cdot h(x_0) \leqq 0$.

Proposition 3.2.3.

The element $(x_0,R_0,S_0) \in dom\ \Phi_L$ is a $P(y^*,\varepsilon)$-saddle point of the Lagrangian Φ_L iff

(a) $\Phi_L(x_0,R_0,S_0) \in P(y^*,\varepsilon)-MIN\ \{\Phi_L(x,R_0,S_0) \in \bar{Y} : x \in X\}$

(b) $x_0 \in F$

(c) $-\varepsilon \leqq \langle y^*, R_0 \cdot h(x_0) \rangle \leqq 0$,

Theorem 3.2.1.

Suppose that the point $(x_0, R_0, S_0) \in dom\ \Phi_L$ is a $P(e)$-saddle point / $WP(e)$-saddle point / $P(y^*, \varepsilon)$-saddle point of the Lagrangian Φ_L.

Then $x_0 \in X$ is an approximate solution of the minimization problem (MP) in the respective sense where the approximation error is

$$e' = e - R_0 \cdot h(x_0)$$

in the first and second, and

$$\varepsilon' = 2 \cdot \varepsilon \tag{3.5}$$

in the last case.

Proof.

By Proposition 3.2.1. $x_0 \in X$ is a feasible point. If $x \in F$ is another, then for the same reason we have

$$f(x) \geqq f(x) + R_0 \cdot h(x) + S_0 \cdot l(x) \not\leq f(x_0) + R_0 \cdot h(x_0) + S_0 \cdot l(x_0) - e$$

and this means

$$f(x) \not\leq f(x_0) - (e - R_0 \cdot h(x_0) - S_0 \cdot l(x_0))$$

By feasibility $l(x_0)=0$, and so the first case is proved.

The proof of the rest is analogous, with the additional use in the last case of the transitivity of the relation $\leqq$ on $\mathbb{R}$.

∎

Remark 3.2.2.

Instead of the relation (3.5) for the approximation error $e' \in Y$ we have

$$0 \leqq e' \not\geq 2 \cdot e \quad \text{and} \quad 0 \leqq e' \not> 2 \cdot e,$$

as a consequence of the points (c) in Proposition 3.2.1. and 3.2.2., respectively. However, unlike the scalarized case, transitivity for the relation of non-domination or weak non-domination does not hold, and so we cannot claim in Theorem 3.2.1. that $x_0 \in X$ is a $P(2 \cdot e)$-solution or $WP(2 \cdot e)$-solution.

Theorem 3.2.2.

Suppose that the problem (MP) meets the algebraic Slater-Uzawa constraint qualification. If $x_0 \in X$ is a $P(y^*, \varepsilon)$-approximate solution of the problem, then there exist operators $R_0 \in L^+(Z,Y)$ and $S_0 \in L(V,Y)$ such that $(x_0, R_0, S_0) \in dom\ \Phi_L$ is a $P(y^*, \varepsilon)$-saddle point of the Lagrangian Φ_L.

Proof.

It is supposed that $x_0 \in X$ is an ε-solution of the scalar valued optimization problem

$$min\ \{ <y^*, f(x)> \in \mathbb{R} : x \in \Delta,\ h(x) \leqq 0,\ l(x) = 0 \}$$

By Theorem 2.2.5. in the scalar valued case, there exist functionals $r^*_0 \in K^+$ and $s^*_0 \in V'$ ensuring that (x_0, r^*_0, s^*_0) is an ε-saddle point for the Lagrangian corresponding to the above scalar problem, i.e.

$$<y^*, f(x_0)> + <r^*, h(x_0)> + <s^*, l(x_0)> - \varepsilon \leqq$$

$$\leqq <y^*, f(x_0)> + <r^*_0, h(x_0)> + <s^*_0, l(x_0) \leqq$$

$$\leqq <y^*, f(x)> + <r^*_0, h(x)> + <s^*_0, l(x)> + \varepsilon$$

If $c \in C$ is an element with $<y^*, c> = 1$, then defining $R_0 \in L^+(Z,Y)$ and $S_0 \in L(V,Y)$ with the following correspondences,

$$R_0 : z \rightarrow c \cdot <r^*_0, z>$$

$$S_0 : v \rightarrow c \cdot <s^*_0, v>$$

the theorem is proved.

■

Theorem 3.2.3.

Suppose that the problem (MP) meets the Slater-Uzawa constraint qualification, and $core(C) \neq \phi$. If $x_0 \in X$ is a $WP(e)$-solution of the problem (MP) then there exist operators $R_0 \in L^+(Z,Y)$ and $S_0 \in L(V,Y)$ such that $(x_0, R_0, S_0) \in dom\ \Phi_L$ is a $WP(e)$-saddle point of the Lagrangian Φ_L.

Proof.

By point (c) in Proposition 3.1.2. there exists an $y^* \in C^+$ such that $x_0 \in X$ is a $P(y^*, <y^*, e>)$-solution of (MP) and so Theorem 3.2.2. implies that there exist a $P(y^*, <y^*, e>)$-saddle point for Φ_L. Now, obviously $y^* \in C^+$ is strictly positive for the cone $C_1 = core(C) \cup \{0\}$. From an argument similar to the one used in the proof of

(b) in Proposition 3.1.2. we can conclude that this $P(y^*, \langle y^*, e \rangle)$-saddle point is a $WP(e)$-saddle point as well.

■

Remark 3.2.3.

A respective theorem concerning $P(e)$-solutions cannot be stated as a $y^* \in C^+$, which is strictly positive for the whole cone $C \subset Y$, does not always exist.

3.3. Primal and Dual Functions

In this final section we only deal with the scalarized case, i. e. $P(y^*, \varepsilon)$-type minimality, as otherwise being the solution of the respective approximate primal problem carries little information, as is indicated in Remark 3.3.1.

Definition 3.3.1.

We call the following set valued maps the approximate primal and dual functions of the minimization problem (MP):

$$P(y^*, \varepsilon) : X \to 2^{\bar{Y}}$$

$$P(y^*, \varepsilon) : x \to P(y^*, \varepsilon)-MAX\{\Phi_L(x, R, S) : (R, S) \in L(Z, Y) \times L(V, Y)\}$$

and

$$D(y^*, \varepsilon) : L(Z, Y) \times L(V, Y) \to 2^{\bar{Y}}$$

$$D(y^*, \varepsilon) : (R, S) \to P(y^*, \varepsilon)-MIN\{\Phi_L(x, R, S) : x \in X\}.$$

The approximate primal and dual problems $(P(y^*, \varepsilon))$ and $(D(y^*, \varepsilon))$ are defined in terms of the functions $P(y^*, \varepsilon)$ and $D(y^*, \varepsilon)$. Accordingly $x_0 \in X$ or $(R_0, S_0) \in L^+(Z, Y) \times L(V, Y)$ is a solution of the approximate primal or dual problems, if

$$P(y^*, \varepsilon)(x_0) \cap P(y^*, 3\varepsilon)-MIN\{ \cup P(y^*, \varepsilon)(x) : x \in X\} \neq \phi$$

or

$$D(y^*,2\varepsilon)(R_0,S_0) \cap P(y^*,\varepsilon)\text{-}MAX\{(y^*,\varepsilon)(R,S): R \in L(Z,Y), S \in L(V,Y)\} \neq \phi$$

respectively.

Proposition 3.3.1.

$$P(y^*,\varepsilon)(x) \subset P(y^*,\varepsilon)\text{-}MAX\{f(x) - C\} \qquad \forall x \in F$$

$$P(y^*,\varepsilon)(x) = \{\infty\} \qquad \forall x \in X \setminus F$$

Proof.

For $x \in F$ we have

$$\{\Phi_L(x,R,S) \in Y \cup \{-\infty,\infty\} : R \in L^+(Z,Y), S \in L(V,Y)\} =$$

$$= \{f(x) + R \cdot h(x) : R \in L^+(Z,Y)\} \subset f(x) - C.$$

■

Remark 3.3.1.

If we define e.g. the approximate primal problem $(P(e))$ in a corresponding manner to Definition 3.3.1. then the analogue of Proposition 3.3.1. is valid, and in such a way that the set $P(e)(x)$ is not bounded from below if $x \in F$ and $h(x) \neq 0$. As a consequence, it would have only $-\infty$ as a solution. As we know from e.g. Luc (1984) this irregularity disappears if $e=0$.

Proposition 3.3.2.

(a) If $x_0 \in X$ is a $P(y^*,\varepsilon)$-solution of the problem (MP)
then it is a solution of the problem $(P(y^*,\varepsilon))$.

(b) If $x_0 \in X$ is a solution of the problem $(P(y^*,\varepsilon))$
then it is a $P(y^*,4\varepsilon)$ solution of the problem (MP).

Proof.

(a) By Proposition 3.3.1. we have for all $x \in F$ that

$$F(x) \in P(y^*,\varepsilon)(x).$$

Therefore it is sufficient to prove that

$$f(x_0) \in P(y^*,3\varepsilon)\text{-}MIN\{\cup\{P(y^*,\varepsilon)(x) : x \in X\}\} \tag{3.6}$$

Again by the last proposition:

$$\cup \{ P(y^*,\varepsilon)(x) : x \in F \} \subset \cup \{P(y^*,\varepsilon)-MAX \{f(x) - C\} \} =$$

$$= \{y \in \bar{Y} : \exists\, x \in F, y \leqq f(x),\ <y^*,y> \geqq <y^*,f(x)> - \varepsilon\}.$$

Hence by the definition of $P(y^*,3\varepsilon)-MIN$, the validity of (3.6) follows from the inequality:

$$<y^*,f(x_0)> - 3\varepsilon \leqq <y^*,f(x)> - \varepsilon \qquad \forall\, x \in F.$$

And this is a consequence of the relation we supposed.

(b) Let us suppose now that $x_0 \in X$ solves $(P(y^*,\varepsilon))$, i.e. there exists an

$$y_0 \in P(y^*,\varepsilon)(x_0) \cap P(y^*,3\varepsilon)-MIN\{ \cup \{P(y^*,\varepsilon)(x) : x \in X\} \}$$

Belonging to the first set means that

$$y_0 = f(x_0) - c_0 \tag{3.7}$$

where $c_0 \in C$ and $0 \leqq <y^*,c_0> \leqq \varepsilon$. As we have for all $x \in X \setminus F$ that $P(y^*,\varepsilon)(x) = \{\infty\}$, it is enough to consider $x \in F$, implying

$$f(x) \in P(y^*,\varepsilon)(x).$$

Hence belonging to the second set implies:

$$<y^*,y_0> - 3\varepsilon \leqq <y^*,f(x)> \qquad \forall\, x \in X,$$

and by (3.7)

$$<y^*,f(x_0)> - 4\varepsilon \leqq <y^*,f(x)> \qquad \forall\, x \in X.$$

■

Definition 3.3.2.

The element $(x_0,R_0,S_0) \in X \times L(Z,Y) \times L(V,Y)$ is called a $P(y^*,\varepsilon)$-dual pair of solutions if

(i)

$x_0 \in X$ is a solution of the problem $(P(y^*,\varepsilon))$

and

(ii)

$$f(x_0) \in D(y^*,2\varepsilon)(R_0,S_0) \cap P(y^*,\varepsilon)-MAX\{ \cup \{D(y^*,\varepsilon)(R,S) : R \in L(Z,Y),\ S \in L(V,Y)\}$$

Remark 3.3.2.

The definition could equivalently be formulated as: $x_0 \in X$ and $(R_0,S_0) \in L(Z,Y)\times L(V,Y)$ is a solution of the primal and the dual problem respectively, where the latter is valid by way of $f(x_0) \in Y$.

Theorem 3.3.1.

(a) If $(x_0,R_0,S_0) \in \boldsymbol{dom}\ \Phi_L$ is a $P(y^*,\varepsilon)$-saddle point of the Lagrangian Φ_L, then it is a $P(y^*,\varepsilon)$-dual pair of solutions.

(b) If $(x_0,R_0,S_0) \in X\times L(Z,Y)\times L(V,Y)$ is a $P(y^*,\varepsilon)$-dual pair of solutions then it is a $P(y^*,2\varepsilon)$-saddle point of the Lagrangian Φ_L.

Proof.

(a) On one hand by Proposition 3.3.1. we have

$$f(x_0) \in P(y^*,\varepsilon)(x_0).$$

On the other, by Theorem 3.2.1. we know that $x_0 \in X$ is a $P(y^*,2\varepsilon)$-solution of the problem (MP). Together with Proposition 3.3.1. this yields the relation

$$<y^*, f(x_0)> - 3\varepsilon \leqq <y^*, y> \qquad \forall\ y \in P(y^*,\varepsilon)(x)$$

i.e.

$$f(x_0) \in P(y^*,3\varepsilon)-MIN\{P(y^*,\varepsilon) : x \in X\}.$$

This proves the first requirement of $(x_0,R_0,S_0) \in \Phi_L$ being a $P(y^*,\varepsilon)$-dual pair of solutions. If $(x_0,R_0,S_0) \in \boldsymbol{dom}\ \Phi_L$ is a $P(y^*,\varepsilon)$-saddle point then by (c) in Proposition 3.2.3. we have

$$<y^*, f(x_0)> - \varepsilon \leqq <y^*, \Phi_L(x_0,R_0,S_0)>,$$

and also by the definition of the saddle point

$$\Phi_L(x_0,R_0,S_0) \in P(y^*,\varepsilon)-MIN\{\Phi_L(x,R_0,S_0) : x \in X\}.$$

If we combine these two relations then we obtain

$$<y^*, f(x_0)> - 2\varepsilon \leqq <y^*, \Phi_L(x,R_0,S_0)> \qquad \forall\ x \in X,$$

and as a consequence

$$f(x_0) \in D(y^*,2\varepsilon)(R_0,S_0).$$

We also have to prove that

$$F(x_0) \in P(y^*,\varepsilon)\text{-}MAX\{D(y^*,\varepsilon)\,(R,S) : R \in L(Z,Y),\ S \in L(V,Y)\}.$$

If this is not so then there exist $R \in L(Z,Y)$, $S \in L(V,Y)$ and $y_1 \in D(y^*,\varepsilon)(R,S)$ such that

$$<y^*,y_1> \ > \ <y^*,f(x_0)> + \varepsilon \tag{3.8}$$

Here it is necessary that $R \in L^+(Z,Y)$ be valid because otherwise $D(y^*,\varepsilon)(R,S)=\{-\infty\}$ and consequently $<y^*,y_1>=-\infty$. Therefore

$$D(y^*,\varepsilon)(R,S) = P(y^*,\varepsilon)\text{-}MIN\{f(x) + R\cdot h(x) + S\cdot l(x) : x \in X\} \tag{3.9}$$

i.e.

$$y_1 = f(x_1) + R \cdot h(x_1) + S \cdot l(x_1)$$

for some $x_1 \in F$. Using (c) in Proposition 3.2.3 and the formula under (3.8), we obtain

$$<y^*,y_1> \ > \ <y^*,f(x_0)+R\cdot h(x_0)+S\cdot l(x_0)>+\varepsilon$$

This, and $y_1 \in D(y^*,\varepsilon)(R,S)$, however, contradict to (3.9). So the second requirement is proved.

(b) By the first part of the definition of the $P(y^*,\varepsilon)$-dual pair of solutions, the conditions imply that $\infty \notin P(y^*,\varepsilon)(x_0)$, and therefore $x_0 \in F$. By the second we know that $-\infty \notin D(y^*,2\varepsilon)(R_0,S_0)$ and therefore $R_0 \in L^+(Z,Y)$. Hence, $(x_0,R_0,S_0) \in dom\ \Phi_L$ holds. As a consequence of $x_0 \in F$ we have

$$<y^*,\Phi_L(x_0,R_0,S_0)> \leqq <y^*,f(x_0)>,$$

and so

$$f(x_0) \in D(y^*,2\varepsilon)(R_0,S_0) \tag{3.10}$$

implies

$$\Phi_L(x_0,R_0,S_0) \in P(y^*,2\varepsilon)\text{-}MIN\{\Phi_L(x,R_0,S_0) : x \in X\} \tag{3.11}$$

From (3.10) it also follows that

$$<y^*,f(x_0)> - 2\varepsilon \leqq <y^*,f(x_0) + R_0\cdot h(x_0) + S_0\cdot l(x_0)>$$

i.e.

$$-2\varepsilon \leqq \langle y^*, R_0 \cdot h(x_0) + S_0 \cdot l(x_0) \rangle \tag{3.12}$$

By (3.11) (3.12) and the relation $x_0 \in F$, Proposition 3.2.3. holds and therefore $x_0 \in F$ is a $P(y^*, 2\varepsilon)$-saddle point of Φ_L.

∎

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