NOT FOR QUOTATION
WITHOUT THE PERMISSION
OF THE AUTHOR

SENSITIVITY ANALYSIS IN MULTIOBJECTIVE OPTIMIZATION

Tetsuzo Tanino

February 1986
WP-86-5

Working Papers are interim reports on work of the International Institute for Applied Systems Analysis and have received only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute or of its National Member Organizations.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

PREFACE

Sensitivity analysis is both theoretically and practically useful in optimization. However, only a few results in this direction have been obtained for multiobjective optimization. In this paper, the issue of sensitivity analysis in multiobjective optimization is dealt with. Given a family of parametrized multiobjective optimization problems, the perturbation map is defined as the set-valued map which associates to each parameter value the set of minimal points of the perturbed feasible set with respect to a fixed ordering convex cone. The behavior of the perturbation map is analyzed quantitatively by using the concept of contingent derivative for set-valued maps. Particularly it is shown that the contingent derivative of the perturbation map for multiobjective programming problems with parametrized inequality constraints is closely related to the corresponding Lagrange multipliers.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

SENSITIVITY ANALYSIS IN MULTIOBJECTIVE OPTIMIZATION

Tetsuzo Tanino

1. INTRODUCTION

Stability and sensitivity analysis is not only theoretically interesting but also practically important in optimization theory. A number of useful results have been obtained in usual scalar optimization (see, for example, Fiacco [3] and Rockafellar [4]). Here, by stability we mean the quantitative analysis, that is, the study of various continuity properties of the perturbation (or marginal) function (or map) of a family of parametrized optimization problems. On the other hand, by sensitivity we mean the quantitative analysis, that is, the study of derivatives of the perturbation function.

For multiobjective optimization, the "optimal" value of a problem is not unique and hence we must consider not a function but a set-valued perturbation map. The author and Sawaragi [7] investigated some sufficient conditions for the semicontinuity of the perturbation map. However, their results are qualitative and therefore provide no quantitative information. In this paper, the behavior of the perturbation map will be studied quantitatively via the concept of contingent derivative introduced by Aubin [1]. Though several other concepts of derivatives of set-valued maps were proposed (see Aubin and Ekeland [2]. p. 493), the concept of the contingent derivative is the most adequate for our purpose. Because it depends on the point in the graph of a set-valued map and when we discuss the sensitivity of the perturbation map, we fix some point in its graph.

The contents of this paper are as follows. In Section 2, we introduce the concept of the contingent derivative of set-valued maps along with some basic properties which are necessary in the later sections. Section 3 is devoted to the analysis of the contingent derivative of the perturbation map, which is defined from a feasible set map by taking the set of minimal points with respect to a given closed convex cone. In Section 4, we analyze the sensitivity in general multiobjective optimization problems specified by feasible decision sets and objective functions which depend on a parameter vector. In Section 5, we concentrate on multiobjective programming problems in which only the right-hand side of inequality constraints is perturbed. It is shown that the sensitivity of the perturbation map is closely related with the Lagrange multipliers of the nominal problem.

2. CONTINGENT DERIVATIVES OF SET-VALUED MAPS

In this section we introduce the concept of the contingent derivative of set-valued maps. Throughout this section V and Z are two Banach spaces and F is a set-valued map from V to Z.

Definition 2.1. (Aubin and Ekeland [2]). Let C be a nonempty subset of a Banach space V and $\hat{v} \in V$. The set $T_C(\hat{v}) \subset V$ defined by

$$T_C(\hat{v}) = \bigcap_{\varepsilon > 0} \bigcap_{\alpha > 0} \bigcup_{0 < h \leqq \alpha} \left(\frac{1}{h}(C - \hat{v}) + \varepsilon B\right) \tag{2.1}$$

is called the contingent cone to C at $\hat{v}$, where B is the unit ball in V. In other words, $v \in T_C(\hat{v})$ if and only if there exist sequences $\{h_k\} \subset \mathring{R}_+$ and $\{v^k\} \subset V$ such that $h_k \to 0+$, $v^k \to v$ and

$$\hat{v} + h_k v^k \in C \quad \text{for} \quad \forall k \quad ,$$

where $\mathring{R}_+$ is the set of positive real numbers.

It is well known that $T_C(\hat{v})$ is a closed (but not always convex) cone.

The graph of a set-valued map F from V to Z is defined and denoted by

$$graphF = \{(v, z) \mid z \in F(v)\} \subset V \times Z \quad . \tag{2.2}$$

The contingent derivative of F is defined by considering the contingent cone to $graphF$.

Definition 2.2. (Aubin and Ekeland [2]) Let $(\hat{v}, \hat{z})$ be a point in $graphF$. We denote by $DF(\hat{v}, \hat{z})$ the set-valued map from V to Z whose graph is the contingent cone $T_{graphF}(\hat{v}, \hat{z})$ to the graph of F at $(\hat{v}, \hat{z})$, and call it the contingent derivative of F at $(\hat{v}, \hat{z})$. In other words, $z \in DF(\hat{v}, \hat{z})(v)$ if and only if $(v, z) \in T_{graph}F(\hat{v}, \hat{z})$.

$DF(\hat{v},\hat{z})$ is a positively homogeneous set-valued map with closed graph. Due to Definition 2.1, $z \in DF(\hat{v},\hat{z})(v)$ if and only if there exist sequences $\{h_k\} \subset \mathring{R}_+, \{v^k\} \subset V$ and $\{z^k\} \subset Z$ such that $h_k \to 0+$, $v^k \to v, z^k \to z$ and

$$\hat{z} + h_k z^k \in F(\hat{v} + h_k v^k) \quad \text{for} \quad \forall k \quad .$$

Now we consider a nonempty pointed† closed convex cone P in Z. This cone P induces a partial order on Z. We use the following notations: For $z, z' \in P$

$$y \leqq_P y' \qquad iff \qquad y' - y \in P \tag{2.3}$$

$$y \leq_P y' \qquad iff \qquad y' - y \in P \setminus \{0\} \quad . \tag{2.4}$$

We consider the set-valued map $F + P$ from V to Z defined by

$$(F + P)(v) = F(v) + P \quad \text{for} \quad \forall v \in V \quad .$$

The graph of $F + P$ is often called the P-epigraph of F (Sawaragi et al. [6], p. 23). The following result, which shows a relationship between the contingent derivatives of $F + P$ and F, is useful.

Proposition 2.1. Let $(\hat{v},\hat{z})$ belong to *graphF*. Then

$$DF(\hat{v},\hat{z})(v) + P \subset D(F + P)(\hat{v},\hat{z})(v) \quad \text{for} \quad \forall v \in V \quad . \tag{2.5}$$

(Proof). Let $z \in DF(\hat{v},\hat{z})(v)$ and $d \in P$. Then there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$ and $\{z^k\} \subset Z$ such that $h_k \to 0+$, $v^k \to v$, $z^k \to z$ and

$$\hat{z} + h_k z^k \in F(\hat{v} + h_k v^k) \quad \text{for} \quad \forall k \quad .$$

Let $\bar{z}^k = z^k + d$ for all k. Then $\bar{z}^k \to z + d$ and

$$\hat{z} + h_k \bar{z}^k = \hat{z} + h_k z^k + h_k d \in F(\hat{v} + h_k v^k) + P \quad \text{for} \quad \forall k \quad .$$

Hence $z + d \in D(F + P)(\hat{v},\hat{z})(v)$ and the proof is complete. ■

The converse inclusion relation of this proposition

$$D(F + P)(\hat{v},\hat{z})(v) \subset DF(\hat{v},\hat{z})(v) + P$$

does not generally hold. (See Proposition 3.1 and Examples 3.3 and 3.4).

† A cone P is said to be pointed if $P \cap (-P) = \{0\}$.

Since we deal with multiobjective optimization, we must introduce the concept of minimal points with respect to the cone P.

Definition 2.3. Given a subset S of Z, a point $\hat{z} \in S$ is said to be a P-minimal point of S if there exists no $z \in S$ such that $z \leq_P \hat{z}$. We denote the set of all P-minimal points of S by $Min_P S$, i.e.

$$Min_P S \in \{\hat{z} \in S \mid \text{there exists no } z \in S \text{ such that } z \leq_P \hat{z}\} \qquad (2.6)$$
$$= \{\hat{z} \in S \mid (S - \hat{z}) \cap (-P) = \{0\}\} \ .$$

The following theorem is fundamental.

Theorem 2.1. Let $(\hat{v}, \hat{z})$ belong to $graph F$ and suppose that Z is finite dimensional. Then, for any $v \in V$.

$$Min_P D(F + P)(\hat{v}, \hat{z})(v) \subset DF(\hat{v}, \hat{z})(v)$$

(Proof) Let $z \in Min_P D(F + P)(\hat{v}, \hat{z})(v)$. Since $z \in D(F + P)(\hat{v}, \hat{z})(v)$ there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{v^k\} \subset V$ and $\{z^k\} \subset Z$ such that $h_k \to 0+$, $v^k \to V$, $z^k \to z$ and

$$\hat{z} + h_k z^k \in F(\hat{v} + h_k v^k) + P \quad \text{for} \quad \forall k \ .$$

There also exists a sequence $\{d^k\} \subset P$ such that

$$\hat{z} + h_k z^k - d^k \in F(\hat{v} + h_k v^k) \quad \text{for} \quad \forall k \ .$$

We shall prove that $\dfrac{d^k}{h_k} \to 0$. If this were not the case, then for some $\varepsilon > 0$, we can choose a subsequence of the natural numbers $\{l_k\}$ satisfying

$$\frac{|d^{l_k}|}{h_{l_k}} \geqq \varepsilon \quad \text{for} \quad \forall k \ .$$

Taking and renumbering this subsequence, we may assume from the first that $\dfrac{|d^k|}{h_k} \geqq \varepsilon$ for all k. Set $\bar{d}^k = \dfrac{\varepsilon h_k}{|d^k|} d^k \in P$. Then $\bar{d}^k \leqq_P d^k$ and

$$\hat{z} + h_k z^k - \bar{d}^k \in F(\hat{v} + h_k v^k) + P \ .$$

Since $\dfrac{|\bar{d}^k|}{h_k} = \varepsilon$ for all k, we may assume without loss of generality that the sequence $\left\{\dfrac{\bar{d}^k}{h_k}\right\}$ converges to some vector $d \in Z$. Since P is closed, $d \in P$ and

$|d| = \varepsilon > 0$. Thus, $z^k \frac{\bar{d}^k}{h_k} \to z - d$ and hence $z - d \in D(F + P)(\hat{v},\hat{z})(v)$. However, this contradicts the assumption $z \in Min_P\, D(F + P)(\hat{v},\hat{z})(v)$, since $z - d \leq_P z$. Therefore we can conclude that $\frac{d^k}{h_k} \to 0$. This implies that

$$\hat{z} + h_k(z^k - \frac{d^k}{h_k}) \in F(\hat{v} + {}_k v^k)$$

and

$$z^k - \frac{d^k}{h_k} \to z.$$

Therefore $z \in DF(\hat{v},\hat{z})(v)$ and this completes the proof of the theorem. ■

The converse inclusion of this theorem is not valid generally. (See Example 3.2.)

3. CONTINGENT DERIVATIVE OF THE PERTURBATION MAP

In this section we consider a family of parametrized multiobjective optimization problems. Let Y be a set-valued map from U to R^p, where U is the Banach space of a perturbation parameter vector, R^p is the objective space and Y is considered as the feasible set map. Let P be a nonempty pointed closed convex cone in R^p. In the optimization problem corresponding to each parameter value u, we aim to find the set of P-minimal points of the feasible set $Y(u)$. Hence we define another set-valued map W from U to R^p by

$$W(u) = Min_P Y(u) \quad \text{for} \quad \forall u \in U \tag{3.1}$$

and call it the perturbation map. The purpose of this section is to investigate relationships between the contingent derivative of W and that of Y. Hereafter in this paper, we fix a nominal value of u as $\hat{u}$ and consider a point $\hat{y} \in W(\hat{u})$.

In view of Theorem 2.1, we have the following relationship:

$$Min_P D(W + P)(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u) \quad \text{for} \quad \forall u \in U \; . \tag{3.2}$$

Definition 3.1. We say that Y is P-minicomplete near $\hat{u}$ if

$$Y(u) \subset W(u) + P \quad \text{for} \quad \forall u \in N \tag{3.3}$$

where N is some neighborhood of $\hat{u}$.

Since $W(u) \subset Y(u)$, the P-minicompleteness of Y near $\hat{u}$ implies that

$$W(u) + P = Y(u) + P \quad \text{for} \quad \forall u \in N \quad . \tag{3.4}$$

Hence, if Y is P-minicomplete near $\hat{u}$, then $D(Y + P)(\hat{u},\hat{y}) = D(W + P)(\hat{u},\hat{y})$ for all $\hat{y} \in W(\hat{u})$. Thus we obtain the following theorem from (3.2).

Theorem 3.1. If Y is P-minicomplete near $\hat{u}$, then

$$Min_P D(Y + P)(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u) \quad \text{for} \quad \forall u \in U \quad . \tag{3.5}$$

The following example illustrates that the P-minicompleteness of Y is essential for the above theorem.

Example 3.1. (Y is not P-minicomplete near $\hat{u}$). Let $U = R$, $p = 1$, $P = R_+$ and Y be defined by

$$Y(u) = \begin{cases} \{y \mid y \geqq 0\} & \text{if } u = 0 \\ \{y \mid y > |u|\} & \text{if } u \neq 0 \end{cases} .$$

Then

$$W(u) = \begin{cases} \{0\} & \text{if } u = 0 \\ \phi & \text{if } u \neq 0 \end{cases} .$$

Let $\hat{u} = 0$. Then

$$D(Y + P)(\hat{u},\hat{y})(u) = DY(\hat{u},\hat{y})(u) = \{y \mid y \geqq |u|\} \quad \text{for} \quad \forall u \in R$$

$$Min_P D(Y + P)(\hat{u},\hat{y})(u) = Min_P DY(\hat{u},\hat{y})(u) = \{|u|\} \quad .$$

On the other hand

$$DW(\hat{u},\hat{y})(u) = \begin{cases} \{0\} & \text{if } u = 0 \\ \phi & \text{if } u \neq 0 \end{cases} .$$

Hence

$$Min_P D(Y + P)(\hat{u},\hat{y})(u) \not\subset DW(\hat{u},\hat{y})(u) \quad \text{for} \quad u \neq 0 \quad .$$

The converse inclusion of the theorem does not generally hold as is shown in the following example.

Example 3.2. Let $U = R$, $p = 2$ and Y be defined by

$$Y(u) = \begin{cases} \{(0,0)\} & \text{if } u \leqq 0 \\ \{y \in R^2 \mid y_2 = -y_1^2, 0 \leqq y_1 \leqq u\} & \text{if } u > 0 \end{cases}$$

Let $P = R^2_+$, $\hat{u} = 0$ and $\hat{y} = (0,0)$. Then $W(u) = Y(u)$ for every u and

$$T_{graph Y}(\hat{u},\hat{y}) = T_{graph W}(\hat{u},\hat{y})$$

$$= \{(u,y) \mid u \leqq 0,\ y = 0\} \cup \{(u,y) \mid u > 0,\ y_2 = 0,\ 0 \leqq y_1 \leqq u\} \ ,$$

$$T_{graph\,(Y+P)}(\hat{u},\hat{y}) = T_{graph\,(W+P)}(\hat{u},\hat{y}) = \{(u,y) \mid y \geqq 0\} \ ,$$

$$DW(\hat{u},\hat{y})(u) = DY(\hat{u},\hat{y})(u) = \begin{cases} \{(0,0)\} & \text{if } u \leqq 0 \\ \{y \mid y_2 = 0,\ 0 \leqq y_1 \leqq u\} & \text{if } u > 0 \end{cases}$$

$$D(Y+P)(\hat{u},\hat{y})(u) = D(W+P)(\hat{u},\hat{y})(u) = \{y \mid y \geqq 0\}$$

and

$$Min_P D(Y+P)(\hat{u},\hat{y})(u) = Min_P D(W+P)(\hat{u},\hat{y})(u) = \{(0,0)\} \ .$$

In order to obtain a relationship between DW and DY, we shall introduce the following property of Y.

Definition 3.2. (Aubin and Ekeland [2]) Y is said to be upper locally Lipschitz at $\hat{u}$ if there exist a neighborhood N of $\hat{u}$ and a positive constant M such that

$$Y(u) \subset Y(\hat{u}) + M \, |u - \hat{u}| \, B \quad \text{for} \quad \forall u \in N \tag{3.6}$$

Remark 3.1. If Y is upper locally Lipschitz at $\hat{u}$, then it is upper semicontinuous at $\hat{u}$, i.e., for any $\varepsilon > 0$, there exists a positive number δ such that

$$Y(u) \subset Y(\hat{u}) + \varepsilon B \quad \text{for} \quad \forall u, |u - \hat{u}| \leqq \delta \ .$$

Definition 3.3. Let S be a set in R^p and P be a nonempty closed convex cone in R^p. A point $\hat{y} \in S$ is said to be a properly P-minimal point of S if

$$[cl \bigcup_{\alpha > 0} \alpha(S - \hat{y})] \cap (-P) = \{0\} \tag{3.7}$$

Of course, every properly P-minimal point of S is P-minimal, since $S - \hat{y} \subset cl \bigcup_{\alpha > 0} \alpha(S - \hat{y})$.

Proposition 3.1. If $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$ and if Y is upper locally Lipschitz at $\hat{u}$, then

$$D(Y+P)(\hat{u},\hat{y})(u) = DY(\hat{u},\hat{y})(u) + P \quad \text{for} \quad \forall u \in U \ . \tag{3.8}$$

(Proof) In view of Proposition 2.1,

$$DY(\hat{u},\hat{y})(u) + P \subset D(Y+P)(\hat{u},\hat{y})(u) \quad \text{for} \quad \forall u \in U \ .$$

Hence we prove the converse inclusion. Let $y \in D(Y+P)(\hat{u},\hat{y})(u)$. From the definition there exist sequences $\{h_k\} \subset \mathring{R}_+$, $\{u^k\} \subset U$ and $\{y^k\} \subset R^p$ such that $h_k \to 0+$, $u^k \to u$, $y^k \to y$ and

$$\hat{y} + h_k y^k \in Y(\hat{u} + h_k u^k) + P \quad \text{for} \quad \forall k \ .$$

Therefore there exists a sequence $\{d^k\} \in P$ such that

$$\hat{y} + h_k y^k - d^k \in Y(\hat{u} + h_k u^k) \qquad \text{for} \quad \forall k \ ,$$

i.e.,

$$\hat{y} + h_k(y^k - \frac{d^k}{h_k}) \in Y(\hat{u} + h_k u^k) \qquad \text{for} \quad \forall k \ .$$

Suppose that the sequence $\{\frac{d^k}{h_k}\}$ has a convergent subsequence. In this case, we may assume without loss of generality that $\frac{d^k}{h_k} \to d$ for some d. Since P is a closed set, $d \in P$. Moreover, the convergence $y^k - \frac{d^k}{h_k} \to y - d$ implies that $y - d \in DY(\hat{u},\hat{y})(u)$, namely that $y \in DY(\hat{u},\hat{y})(u) + P$. Hence we have the conclusion of the proposition. Therefore it completes the proof of the proposition to show that $\{\frac{d^k}{h_k}\}$ necessarily has a convergent subsequence. If this were not the case, then $\frac{|d^k|}{h_k} \to +\infty$. Since Y is upper locally Lipschitz at $\hat{u}$, there exist a neighborhood N of $\hat{u}$ and a positive number M satisfying (3.6). Since $\hat{u} + h_k u^k \to \hat{u}$, $\hat{u} + h_k u^k \in N$ for all k sufficiently large. Hence there exists a sequence $\{\hat{y}^k\}$ in $Y(\hat{u})$ such that

$$|\hat{y} + h_k(y^k - \frac{d^k}{h_k}) - \hat{y}^k| \leqq M\,|\hat{u} + h_k u^k - \hat{u}|$$

i.e.

$$\left| \frac{1}{h_k}(\hat{y} - \hat{y}^k) + y^k - \frac{d^k}{h_k} \right| \leq M \left| u^k \right|$$

for all k sufficiently large. Since $u^k \to u$, the right-hand side of the above inequality converges to $M|u|$. Therefore, the sequence $\{\frac{1}{h_k}(\hat{y} - y^k) + y^k - \frac{d^k}{h_k}\}$ is bounded. Since $\frac{|d^k|}{h_k} \to +\infty$, the sequence

$$\{\frac{h_k}{|d^k|}(\frac{1}{h_k}(\hat{y} - \hat{y}_k) + y^k - \frac{d^k}{h_k})\}$$

$$= \{-\frac{\hat{y}^k - \hat{y}}{|d^k|} + \frac{h_k}{|d^k|} y^k - \frac{d^k}{|d^k|}\}$$

converges to the zero vector in R^p. Since $y^k \to y$, the second term converges to the zero vector. We may assume that $\frac{d^k}{|d^k|} \to \bar{d}$ for some $\bar{d} \in P$ with $|\bar{d}| = 1$. Hence $\frac{\hat{y}^k - \hat{y}}{|d^k|} \to -d$. However, this implies that $-d \in [\, cl \bigcup_{\alpha>0} \alpha(Y(\hat{u}) - \hat{y})] \cap (-P)$, which contradicts the assumption of the proper P-minimality of $\hat{y}$. This completes the proof of the proposition. ■

Corollary 3.1. If $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$ and if Y is upper locally Lipschitz at $\hat{u}$, then

$$Min_P DY(\hat{u}, \hat{y})(u) = Min_P D(Y + P)(\hat{u}, \hat{y})(u) \quad \text{for} \quad \forall u \in U \quad . \qquad (3.9)$$

(Proof) In view of Proposition 3.1, by using Proposition 3.1.2 of Sawaragi *et al.* [6], we can prove that

$$Min_P DY(\hat{u}, \hat{y})(u) = Min_P(DY(\hat{u}, \hat{y})(u) + P) = Min_P D(Y + P)(\hat{u}, \hat{y})(u) \quad .$$ ■

By combining Theorem 2.1 and Corollary 3.1, we have the following theorem.

Theorem 3.2. If Y is P-minicomplete near $\hat{u}$ and upper locally Lipschitz at $\hat{u}$, and if $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$, then

$$Min_P DY(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u) \quad \text{for} \quad \forall u \in U \quad .$$

Example 3.1 shows that the minicompleteness of Y is essential for the above theorem. The following two examples illustrate the importance of the other two conditions in Theorem 3.1, namely the Lipschitz property of Y and the proper minimality of $\hat{y}$.

Example 3.3. (Y is not upper locally Lipschitz at $\hat{u}$). Let $U = R$, $p = 1$, $P = R_+$ and Y be defined by

$$Y(u) = \begin{cases} \{0\} & \text{if } u \leqq 0 \\ \{0, -\sqrt{u}\} & \text{if } u > 0 \end{cases} .$$

Then

$$W(u) = \begin{cases} \{0\} & \text{if } u \leqq 0 \\ \{-\sqrt{u}\} & \text{if } u > 0 \end{cases} .$$

Let $\hat{u} = 0$ and $\hat{y} = 0$. Then

$$DY(0,0)(u) = \begin{cases} \{0\} & \text{if } u \neq 0 \\ \{y \mid y \leqq 0\} & \text{if } u = 0 \end{cases}$$

$$Min_P DY(0,0)(u) = \begin{cases} \{0\} & \text{if } u \neq 0 \\ \phi & \text{if } u = 0 \end{cases}$$

$$D(Y + P)(0,0)(u) = \begin{cases} \{y \mid y \geqq 0\} & \text{if } u < 0 \\ \mathbb{R} & \text{if } u \geqq 0 \end{cases}$$

$$Min_P D(Y + P)(0,0)(u) = \begin{cases} \{0\} & \text{if } u < 0 \\ \phi & \text{if } u \geqq 0 \end{cases}$$

$$DW(0,0)(u) = \begin{cases} \{0\} & \text{if } u \leqq 0 \\ \phi & \text{if } u > 0 \end{cases} .$$

Hence

$$\{0\} = Min_P DY(0,0)(u) \not\subset DW(0,0)(u) = \phi \quad \text{for} \quad u > 0 \quad .$$

Example 3.4. ($\hat{y}$ is not properly P-minimal). Let $U = \mathbb{R}$, $p = 2$, $P = R^2_+$ and Y be defined by

$$Y(u) = \{y \mid Y_1 + y_2 = 0,\ y_1 \leqq u\} \cup \{y \mid y_1 + y_2 + 1 = 0,\ y_1 > 0\} \quad .$$

Then

$$W(u) = \{y \mid y_1 + y_2 = 0,$$
$$y_1 \leqq \min(0,u)\} \cup \{y \mid y_1 + y_2 + 1 = 0,\ y_1 > 0\}$$

Let $\hat{u} = 0$ and $\hat{y} = (0,0)$. Then

$$DY(\hat{u},\hat{y})(u) = Min_P DY(\hat{u},\hat{y})(u)$$

$$= \{ y \mid y_1 + y_2 = 0,\ y_1 \leqq u \} \quad ,$$

$$D(Y+P)(\hat{u},\hat{y})(u) = \{ y \mid y_1 + y_2 \geqq 0,$$

$$y_2 \geqq -u \} \cup \{ y \mid y_1 \geqq 0 \} \quad ,$$

$$Min_P D(Y+P)(\hat{u},\hat{y})(u) = \begin{cases} \{ y \mid y_1 + y_2 = 0,\ y_1 \leqq u \} & \text{if } u < 0 \\ \{ y \mid y_1 + y_2 = 0,\ y_1 < 0 \} & \text{if } u \geqq 0 \end{cases}$$

$$DW(\hat{u},\hat{y})(u) = \{ y \mid y_1 + y_2 = 0,\ y_1 \leqq \min(0,u) \} \quad .$$

Hence

$$(1,-1) \notin DW(\hat{u},\hat{y})(1), \text{ while } (1,-1) \in Min_P DY(\hat{u},\hat{y})(1) \quad .$$

4. SENSITIVITY ANALYSIS IN GENERAL MULTIOBJECTIVE OPTIMIZATION

In this section we deal with a general multiobjective optimization problem in which the feasible set $Y(u)$ is given by the composition of the feasible decision set $X(u)$ and the objective function $f(x,u)$. Namely, let X be a set-valued map from R^m to R^n, f be a continuously differentiable function from $R^n \times R^m$ into R^p and Y be defined by

$$Y(u) = f(X(u),u) = \{ y \mid y = f(x,u),\ x \in X(u) \} \text{ for } \forall u \in R^m \quad . \tag{4.1}$$

First, we investigate a relationship between the contingent derivatives of X and Y. Let $\hat{u} \in R^m$, $\hat{x} \in X(\hat{u})$ and $\hat{y} = f(\hat{x},\hat{u}) \in Y(\hat{u})$.

Proposition 4.1. For any $u \in R^m$,

$$\nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u \subset DY(\hat{u},\hat{y})(u) \tag{4.2}$$

where $\nabla_x f(\hat{x},\hat{u})$ (or $\nabla_u f(\hat{x},\hat{u})$) is the $p \times n$ (or $p \times m$) matrix whose (i,j) component is $\dfrac{\partial f_i(\hat{x},\hat{u})}{\partial x_j}$ (or $\dfrac{\partial f_i(\hat{x},\hat{u})}{\partial u_j}$). Moreover, let

$$\tilde{X}(u,y) = \{ x \mid x \in X(u),\ f(x,u) = y \} \quad . \tag{4.3}$$

If $\tilde{X}$ is upper locally Lipschitz at $(\hat{u},\hat{y})$ and $\tilde{X}(\hat{u},\hat{y}) = \{\hat{x}\}$, then the converse inclusion of (4.2) is also valid, i.e.,

$$\nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u = DY(\hat{u},\hat{y})(u) \text{ for } \forall u \in R^m \tag{4.4}$$

(Proof). First we prove (4.2). Let $x \in DX(\hat{u},\hat{x})(u)$. Then there exist sequences

$\{h_k\} \subset \overset{\circ}{R}_+, \{u^k\} \subset R^m$ and $\{x^k\} \subset R^n$ such that $h_k \to 0+$, $u^k \to u$, $x^k \to x$ and

$$\hat{x} + h_k x^k \in X(\hat{u} + h^k) \quad \text{for} \quad \forall k \quad .$$

Then

$$f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) \in Y(\hat{u} + h_k u^k) \quad \text{for} \quad \forall k$$

i.e.

$$\hat{y} + h_k \frac{f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u})}{h_k} \in Y(\hat{u} + h_k u^k) \quad \text{for} \quad \forall k \quad .$$

Since $h_k \to 0+$, $u^k \to u$ and $x^k \to x$,

$$\lim_{k \to \infty} f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u}) = \nabla_x f(\hat{x}, \hat{u}) \cdot x + \nabla_u f(\hat{x}, \hat{u}) \cdot u \quad .$$

Hence

$$\nabla_x f(\hat{x}, \hat{u}) \cdot x + \nabla_u f(\hat{x}, \hat{u}) \cdot u \in DY(\hat{u}, \hat{y})(u) \quad .$$

Thus (4.2) has been established. Next we prove (4.4). Let $y \in DY(\hat{u}, \hat{y})(u)$ along with sequences $\{h_k\} \subset \overset{\circ}{R}_+, \{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$ such that $h_k \to 0+$, $u^k \to u$, $y^k \to y$ and $\hat{y} + h_k y^k \in Y(\hat{u} + h_k u^k)$. Then there exists another sequence $\{x^k\} \subset R^n$ such that

$$\hat{x} + h_k x^k \in \tilde{X}(\hat{u} + h_k u^k, \hat{y} + h_k y^k) \quad .$$

Since $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$ and $\tilde{X}(\hat{u}, \hat{y}) = \{\hat{x}\}$, there exists a positive number M such that

$$|\hat{x} + h_k x^k - \hat{x}| \leqq M |(\hat{u} + h_k u^k, \hat{y} + h_k y^k) - (\hat{u}, \hat{y})|$$

i.e.

$$|x^k| \leqq M |(u^k, y^k)|$$

for all k sufficiently large. Since the right-hand side of the above inequality converges to $M|(u, y)|$ as $k \to \infty$, we may assume without loss of generality that x^k converges to some x. Then clearly $x \in DX(\hat{u}, \hat{x})(u)$. Moreover,

$$y = \lim_{k \to \infty} y^k = \lim_{k \to \infty} \frac{f(\hat{x} + h_k x^k, \hat{u} + h_k u^k) - f(\hat{x}, \hat{u})}{h_k}$$

$$= \nabla_x f(\hat{x}, \hat{u}) \cdot x + \nabla_u f(\hat{x}, \hat{u}) \cdot u \quad .$$

Therefore

$$y \in \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u$$

and the proof of the proposition is completed. ∎

The following two examples show that the additional conditions in Proposition 4.1 are essential for (4.4).

Example 4.1. ($\tilde{X}(\hat{u},\hat{y}) \neq \{\hat{x}\}$). Let

$$X(u) = \{x \in R \mid 0 \leqq x \leqq \max(1,1+u)\} \quad \text{for} \quad u \in R,$$

$f(x,u) = x(x-1)$, $\hat{u} = 0$, $\hat{x} = 0$ and $\hat{y} = f(\hat{x},\hat{u}) = 0$. Then $\tilde{X}(\hat{u},\hat{y}) = \{0,1\}$ and

$$Y(u) = \begin{cases} \{y \mid -\frac{1}{4} \leqq y \leqq 0\} & \text{if } u \leqq 0 \\ \{y \mid -\frac{1}{4} \leqq y \leqq u(1+u)\} & \text{if } u > 0 \end{cases} .$$

Hence, by taking $h_k = \frac{1}{k}$, $u^k = 1$ and $y^k = 1$, we can prove that

$$1 \in DY(\hat{u},\hat{y})(1) \quad .$$

On the other hand, $DX(\hat{u},\hat{y})(1) = R_+$, $\nabla_x f(\hat{x},\hat{y}) = -1$ and $\nabla_u f(\hat{x},\hat{u}) = 0$. Therefore

$$1 \notin \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(1) + \nabla_u f(\hat{x},\hat{u}) \cdot 1$$

and (4.4) is not true.

Example 4.2. ($\tilde{X}$ is not upper locally Lipschitz at $(\hat{u},\hat{y})$). Replace $X(u)$ by

$$X(u) = \{x \in R \mid 0 \leqq x < \max(1,1+u)\}$$

in Example 4.1. In this case $\tilde{X}(\hat{u},\hat{y}) = \{0\}$, but $\tilde{X}$ is not upper locally Lipschitz at $(\hat{u},\hat{y})$. We can analogously prove that

$$1 \in DY(\hat{u},\hat{y})(1) \quad \text{but}$$

$$1 \notin \nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot 1 \quad .$$

Example 4.3. ($\tilde{X}$ is not upper locally Lipschitz at $(\hat{u},\hat{y})$). Let $X(u) = [0,1] \subset R$ for every $u \in R$, $f(x,u) = x^2$, $\hat{u} = 0$, $\hat{x} = 0$ and $\hat{y} = 0$. Then $Y(u) = [0,1]$ and

$$DY(\hat{u},\hat{y})(u) = R_+ \quad \text{for} \quad \forall u \in R \quad .$$

However, $\nabla_x f(\hat{x},\hat{u}) \cdot DX(\hat{u},\hat{x})(u) + \nabla_u f(\hat{x},\hat{u}) \cdot u = \{0\}$. In this case $\tilde{X}(u,y) = \sqrt{y}$ for $y \geqq 0$ and any u, which is not upper locally Lipschitz at (0,0).

Finally we should note sufficient conditions for the Lipschitz continuity of Y at $\hat{u}$.

Lemma 4.1. If X is upper locally Lipschitz at $\hat{u}$ and if $X(\hat{u})$ is bounded, then Y is upper locally Lipschitz at $\hat{u}$.

(Proof). Since X is upper locally Lipschitz at $\hat{u}$, there exist some neighborhood N of $\hat{u}$ and a positive number M_1 such that

$$X(u) \subset X(\hat{u}) + M_1|u - \hat{u}|B \quad \text{for} \quad \forall u \in N \quad .$$

Since f is continuously differentiable,

$$M_2 = \max\{\|\nabla_{x,u} f(x,u)\| \mid (x,u) \in cl\,(X(\hat{u}) \times N)\} < +\infty \ .$$

For any $u \in N$ and $y \in Y(u)$, there exists $x \in X(u)$ such that $f(x,u) = y$. Then there exists $\bar{x} \in X(\hat{u})$ such that $|x - \bar{x}| \leqq M_1|u - \hat{u}|$. Hence

$$\begin{aligned} |f(x,u) - f(\bar{x},\hat{u})| &\leqq M_2|(x,u) - (\bar{x},\hat{u})| \\ &\leqq M_2(|x - \bar{x}| + |u - \hat{u}|) \\ &\leqq M_2(1 + M_1)|u - \hat{u}| \ . \end{aligned}$$

Putting $M = (1 + M_1)M_2$, we have

$$y \in Y(\hat{u}) + M|u - \hat{u}|B$$

Namely Y is upper locally Lipschitz at $\hat{u}$. This completes the proof of the lemma. ■

The following example shows that the condition of the boundedness of $X(\hat{u})$ is essential in Lemma 4.1.

Example 4.4. ($X(\hat{u})$ is not bounded). Let

$$X(u) = \{x \in R^2 \mid x_1 = u\} \qquad \text{for} \quad u \in R$$

and $f(x,u) = x_1 x_2$. Then

$$Y(u) = \{y \in R \mid y = u x_2\} \qquad \text{for} \quad u \in R \ .$$

Clearly Y is not upper locally Lipschitz at $\hat{u} = 0$.

Finally, we have the following theorem. Note that Y is P-minicomplete near $\hat{u}$ if $X(u)$ is compact for each u near $\hat{u}$.

Theorem 4.1. Assume the following conditions:

(i) X is upper locally Lipschitz at $\hat{u}$;

(ii) X is compact for each u near $\hat{u}$;

(iii) $\hat{y}$ is a properly P-minimal point of $Y(\hat{u})$;

(iv) $\tilde{X}(\hat{u},\hat{y}) = \{\hat{x}\}$;

(v) $\tilde{X}$ is upper locally Lipschitz at $(\hat{u},\hat{y})$.

Then, for any $u \in R^m$,

$$Min_P \{\nabla_x f(\hat{x},\hat{u}) \cdot x + \nabla_u f(\hat{x},\hat{u}) \cdot u \mid x \in DX(\hat{u},\hat{x})(u)\}$$
$$\subset DW(\hat{u},\hat{y})(u) \quad . \tag{4.5}$$

5. SENSITIVITY ANALYSIS IN MULTIOBJECTIVE PROGRAMMING

In this section we apply the results obtained in the preceding section to a usual multiobjective programming problem:

$$P\text{-minimize} \quad f(x) = (f_1(x),\dots,f_p(x)) \tag{5.1}$$
$$\text{subject to} \quad g(x) = (g_1(x),\dots,g_m(x)) \leqq 0 \ , \quad x \in R^n$$

and discuss the sensitivity in connection with the Lagrange multipliers. Recall that in usual nonlinear programming, the sensitivity of the perturbation function with respect to the parameter on the right-hand side of each inequality constraint is given by $-\lambda_j (j = 1,\dots,m)$, where λ_j is the corresponding Lagrange multiplier. Our final result will be an extension of this fact. Throughout this section, each function $f_i (i = 1,\dots,p)$ and $g_j (j = 1,\dots,m)$ is assumed to be continuously differentiable.

Let X be the set-valued map from R^m to R^n defined by

$$X(u) = \{x \in R^n \mid g(x) \leqq u\} \qquad \text{for} \quad u \in R^m \tag{5.2}$$

Hence, in this case, the feasible set-map Y from R^m to R^p is defined by

$$Y(u) = f(X(u)) = \{y \in R^p \mid y = f(x),\ x \in X(u)\} \tag{5.3}$$
$$= \{y \in R^p \mid y = f(x),\ g(x) \leqq u\}$$

Of course, the nominal value of the parameter vector u is 0 in R^m. Take a point

$\hat{x} \in X(0)$ and denote the index set of the active constraints at $\hat{x}$ by $J(\hat{x})$, i.e.

$$J(\hat{x}) = \{j \mid g_j(\hat{x}) = 0\} \tag{5.4}$$

First, we consider the contingent derivative of the set-valued map X.

Lemma 5.1. The contingent derivative of X at $(0,\hat{x})$ is given as follows:

$$DX(0,\hat{x})(u) = \{x \mid < \nabla g_j(\hat{x}), x > \leqq u_j \quad \text{for} \quad \forall j \in J(\hat{x})\} \tag{5.5}$$

where $< \cdot, \cdot >$ denotes the inner product in the Euclidean space.

(Proof) Note that

$$graphX = \{(u,x) \mid g_j(x) - u_j \leqq 0, \; j = 1,\ldots,m\}$$

is specified by m inequality constraints. The gradient vector of the jth constraint at $(0,\hat{x})$ with respect to (u,x) is $(-e^j, \nabla g_j(\hat{x}))$, where e^j is the j th basic unit vector in R^m, i.e. $e^j_k = 0$ if $k \neq j$ and $e^j_j = 1$. Hence these gradient vectors are linearly independent and so the tangent cone to graph X is given by

$$T_{graphX}(0,\hat{x}) = \{(u,x) \mid <(-e^j, \nabla g_j(\hat{x})), (u,x) > \leqq 0 \quad \text{for} \quad j \in J(\hat{x})\}$$

$$= \{(u,x) \mid < \nabla g_j(\hat{x}), x > \leqq u_j \quad \text{for} \quad j \in J(\hat{x})\} \; .$$

This completes the proof of the lemma. ■

In this case $X(u)$ is a closed set for every u, since g is continuous. The next lemma provides sufficient conditions for the Lipschitz continuity of X around $\hat{u} = 0$.

Lemma 5.2. If there exists a vector $\bar{u} > 0$ such that $X(\bar{u})$ is bounded, $X(0) \neq \phi$ and if the Cottle constraint qualification is satisfied at every $\bar{x} \in X(0)$, i.e.,

$$\sum_{j \in J(\bar{x})} \lambda_j \nabla g_j(\bar{x}) = 0 \text{ and } \lambda_j \geqq 0 \text{ for } j \in J(\bar{x})$$

$$\text{imply that } \lambda_j = 0 \text{ for } \forall j \in J(\bar{x}) \; , \tag{5.6}$$

then X is compact-valued and Lipschitz in a neighborhood of $\hat{u} = 0$.

(Proof) This lemma is due to Rockafellar [5] (combine Theorem 2.1 and Corollary 3.3 in [5]).

Of course, if X is Lipschitz in a neighborhood of $\hat{u}$, then it is upper locally Lipschitz at $\hat{u}$. Analogously we have the following lemma concerning the set-valued map

$$\tilde{X}(u,y) = \{x \mid f(x) = y,\ g(x) \leqq u\} \tag{5.7}$$

Lemma 5.3. If $\tilde{X}$ is locally bounded around $(0,\hat{y})$, $\tilde{X}(0,\hat{y}) \neq \phi$ and the Mangasarian-Fromovitz constraint qualification is satisfied at every $\hat{x} \in \tilde{X}(0,\hat{y})$, i.e.

$$\sum_{j=1}^{p} \mu_i \nabla f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \lambda_j \nabla g_j(\hat{x}) = 0 \text{ and}$$

$$\lambda_j \geqq 0 \text{ for } j \in J(\hat{x})$$

imply that

$$\mu_i = 0 \text{ for all } i = 1,\dots,p \text{ and } \lambda_j = 0 \text{ for all } j \in J(\hat{x}) , \tag{5.8}$$

then $\tilde{X}$ is compact-valued and Lipschitz in a neighborhood of $(0,\hat{y})$.

We will proceed with the discussion under the following assumptions:

Assumption 5.1.

(i) There exists $\bar{u} > 0$ such that $X(\bar{u})$ is bounded.

(ii) The Cottle constraint qualification (5.6) is satisfied at each $\bar{x} \in X(0)$.

(iii) $\tilde{X}(0,\hat{y}) = \{x \mid f(x) = \hat{y},\ g(x) \leqq 0\} = \{\hat{x}\}$.

(iv) The Mangasarian-Fromovitz constraint qualification (5.8) is satisfied at $\hat{x}$.

In addition to Assumption 5.1, we also assume that $\hat{y}$ is a properly P-minimal point of $Y(0)$†. Then we can apply Theorem 4.1 to obtain the relationship

$$Min_P \nabla f(\hat{x}) \cdot DX(0,\hat{x})(u) \subset DW(0,\hat{y})(u) \quad \text{for} \quad \forall u \in R^m . \tag{5.9}$$

In view of (5.5)

$$\nabla f(\hat{x}) \cdot DX(0,\hat{x})(u) = \{y \mid y_i = \langle \nabla f_i(\hat{x}),x \rangle \text{ for } i = 1,\dots,p \ ;$$
$$\langle \nabla g_j(\hat{x}),x \rangle \leqq u_j \text{ for } \forall j \in J(\hat{x})\} .$$

Hence the left-hand side of (5.9) consists of all the P-minimal values of the linear multiobjective programming problem:

$$\begin{cases} P\text{-minimize} & \langle \nabla f_i(\hat{x}),x \rangle, \quad i = 1,\dots,p \\ \text{subject to} & \langle \nabla g_j(\hat{x}),x \rangle \leqq u_j, \quad j \in J(\hat{x}) . \end{cases}$$

†In this case we call $\hat{x}$ a properly P-minimal solution to the problem (5.1).

The necessary and sufficient P-minimality conditions for the above problems are that there exist a multiplier vector $(\mu,\lambda) \in R^p \times R^m$ such that

$$\sum_{i=1}^{p} \mu_i \nabla f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \lambda_j \nabla g_j(\hat{x}) = 0 \tag{5.10}$$

$$\mu \in \operatorname{int} P^+ = \{\nu \in R^p \mid < \nu, d > > 0 \quad \text{for} \quad \forall d \neq 0 \in P\} \tag{5.11}$$

$$\lambda_j \geqq 0 \qquad \text{for} \quad j \in J(\hat{x}) \tag{5.12}$$

$$\lambda_j(< \nabla g_j(\hat{x}), x > - u_j) = 0 \qquad \text{for} \quad j \in J(\hat{x}) \quad . \tag{5.13}$$

Since $\hat{x}$ is a properly P-minimal solution to the problem (5.1), there exists a vector $(\mu,\lambda) \in R^m \times R^p$ satisfying (5.10) - (5.12). Hence, if $x \in R^n$ satisfies

$$\begin{cases} < \nabla g_j(\hat{x}), x > \leqq u_j & \text{for} \quad \forall j \in J(\hat{x}) \ \text{ such that } \ \lambda_j = 0 \\ < \nabla g_j(\hat{x}), x > = u_j & \text{for} \quad \forall j \in J(\hat{x}) \ \text{ such that } \ \lambda_j > 0 \ , \end{cases} \tag{5.14}$$

then $\nabla f(\hat{x}) \cdot x \in Min_P DY(0,\hat{y})(u)$. Moreover

$$\sum_{i=1}^{p} \mu_i < \nabla f_i(\hat{x}), x > + \sum_{j=1}^{m} \lambda_j u_j = 0 \quad .$$

Thus we have proved the following theorem.

Theorem 5.1. Suppose that $\hat{x}$ is a properly P-minimal solution to the multiobjective programming problem (5.1) and Assumption 5.1 is satisfied. Let (μ,λ) be the corresponding multiplier vector. Then, for each $x \in R^n$ satisfying (5.14),

$$\nabla f(\hat{x}) \cdot x \in DW(0,\hat{y})(u) \quad .$$

Moreover,

$$\sum_{i=1}^{p} \mu_i < \nabla f_i(\hat{x}), x > + \sum_{j=1}^{m} \lambda_j u_j = 0 \quad .$$

6. CONCLUSION

In this paper we have studied sensitivity analysis in multiobjective optimization. The essential result we have proved is that every cone minimal vector of the contingent derivative of the feasible set map in a direction is also the element of the contingent derivative of the perturbation map in that direction under some conditions (Theorem 3.2). We have also obtained the relationship between the contingent derivative of the perturbation map and the Lagrange multipliers for multiobjective programming problems (Theorem 5.1).

However, there remain several open problems which should be solved in the future. Some of them are the following. First, the contingent derivative of the perturbation map is not completely characterized. In other words, sufficient conditions for the converse inclusion of Theorem 3.2 have not been obtained yet. Secondly, the Lipschitz continuity of the perturbation map is not studied here. Thirdly, some more refined results may be obtained in the case of multiobjective programming. Finally, we should clarify effects of the convexity or linearity assumption.

REFERENCES

[1] J.P. Aubin, "Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions", in *Advances in Mathematics Supplementary Studies*, L. Nachbin (ed.), Academic Press, New York, pp. 160-232 (1981).

[2] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York (1984).

[3] A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, Academic Press, New York (1983).

[4] R.T. Rockafellar, "Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming", *Mathematical Programming Study* 17, pp. 28-66 (1982).

[5] R.T. Rockafellar, "Lipschitzian properties of multifunctions", *Nonlinear Analysis, TMA*, Vol. 9, No. 8, pp. 867-885 (1985).

[6] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, New York (1985).

[7] T. Tanino and Y. Sawaragi, "Stability of nondominated solutions in multicriteria decision-making", *Journal of Optimization Theory and Applications*, Vol. 30, No. 2, pp. 229-253 (1980).