Progressive Taxation and the Equal Sacrifice Principle*

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June 1986
CP-86-18

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Foreword

A traditional justification for progressive taxation is that it imposes equal sacrifice on all taxpayers in loss of utility. Nevertheless, many plausible utility functions yield strictly *regressive* taxes when all taxpayers sacrifice equally. If tax rates are required to be progressive, nonnegative, and independent of scale, then there is a unique family of positive, increasing, continuous utility functions that is consistent with equal rate of sacrifice. They determine a unique family of tax schedules that seem not to have been studied before except in isolated cases.

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1. Introduction

"I have not been able to discover who was the first to proclaim the thesis that because the degree of utility of income decreases when income increases, it follows that equality of sacrifice entails progressive taxation". ([2], p. 48). So begins a learned paper by A.J. Cohen Stuart (1889) on the utilitarian foundations of progressive taxation. Cohen Stuart points out that 'equal sacrifice' may be interpreted in two ways. Equal *absolute* sacrifice means that in paying taxes everyone gives up the same amount of utility relative to his initial position. Equal *rate* of sacrifice means that everyone gives up the same percentage in utility. Assuming that the marginal utility of income falls as income rises, it is clear that the richer must pay more in tax than the poorer if all are to sacrifice equally by either criterion. It is not so clear, however, that the richer must pay a *higher percentage* of their incomes in taxes to achieve equality of sacrifice. In fact, for many plausible utility functions this is not the case. By itself, then, the equal sacrifice doctrine does not imply progressive taxation.

Instead of attempting to justify progressive taxes by equal sacrifice, we may turn the argument around: if a progressive (or flat) tax is deemed to be the appropriate outcome of an equal sacrifice model, what implications does this have for the form of the utility function? More generally, might it be possible to deduce pertinent information about the utility function by examining the properties of the tax schedules that result from the equal sacrifice principle? The answer is affirmative. We shall show that three simple properties of the tax system—

* This work was supported by the National Science Foundation under Grant SES-8319530 at the University of Maryland.

nonnegativity, progressivity, and scale-invariance—are enough to completely determine the form of the utility functions, whether equal absolute or equal rate of sacrifice is the criterion. In both cases, the implied family of tax schedules is the same.

The study of formal properties of tax schedules has a long and distinguished tradition going back to the work of Cohen Stuart [2], Edgeworth [3]. and others around the turn of the last century. This axiomatic point of view has recently been revived in the work of W.F. Richter [9], [10], Eichhorn and Funke [4], Eichhorn, Funke and Richter [5], and Pfaehler [7], among others. It differs substantially from the optimal tax literature (e.g., Mirrlees [6]) in that tax systems are derived from elementary principles of equity and consistency, rather than from criteria of economic optimality.

2. The equal sacrifice principle

Let x represent an individual's taxable income, and $u(x)$ the utility of income level x. For purposes of this discussion, every individual is assumed to have the same utility function for income. We make the following regularity assumptions: $u(x)$ is defined for all $x > 0$. $u(x)$ may or may not be bounded above or below.

A *tax schedule* is a function $t = f(x)$ defined for all $x > 0$, where t is the tax paid on income x and $0 \le f(x) < x$. That is, taxes are nonnegative and nonconfiscatory. f is *progressive* if, for all x, $f(x)/x$ is monotone nondecreasing in x. f is *strictly progressive* if $f(x)/x$ is strictly monotone increasing in x. Thus the flat tax [$f(x) = rx$ where $0 \le r < 1$] is progressive but not strictly so.

Say that $f(x)$ *induces equal sacrifice* if

$$u(x) - u(x) - f(x)) = c \text{ for some constant } c \text{ and all } x > 0 \quad . \qquad (1)$$

$f(x)$ *induces an equal rate of sacrifice* if

$$\frac{u(x) - u(x - f(x))}{u(x)} = r \text{ for some constant } r \text{ and all } x > 0 \quad . \qquad (2)$$

Note that, since $0 \le f(x) < x$ and $u(x)$ is strictly increasing, we must have $c \ge 0$ in (1). Moreover, $c > 0$ unless $f(x)$ is identically zero. For (2) to be well-defined, $u(x) \ne 0$, and hence by continuity either $u(x) > 0$ for all $x > 0$, or $u(x) < 0$ for all $x > 0$. The former case is the more natural, and implies that $0 \le r < 1$ in (2).

Our goal is to gain more information about the form of the utility functions in (1) and (2) by studying the properties of the tax schedules that they engender. We have already required that taxes be nonnegative, i.e., that there be no minimum income guarantee or other form of direct subsidization through the tax system. This is a significant assumption, but accords with much current practice. Second, it seems intuitively reasonable that a tax schedule which metes out equal sacrifice will be either progressive or flat, but not strictly regressive. In other words, if a proposed measure of utility yields nonprogressive taxation, then we may suspect that it attributes too great a marginal value of income to the rich.

Third, it seems reasonable that tax *rates* should depend only on the relative distribution of incomes, not on the absolute monetary units in which incomes are measured. In other words, if incomes all increase by a fixed percentage, so that the relative distribution of incomes remains unchanged, and if the total tax burden increases by the same percentage, then the relative distribution of taxes should remain unchanged. Such a tax system is ***scale-invariant***. Scale-invariance can be interpreted as a simple principle of distributive equity: if taxes *and* incomes all increase or decrease in some fixed proportion, then equity is preserved. It can also be regarded as merely a pragmatic principle that allows the tax system to be re-indexed without changing the relative distribution of the tax burden.

3. Utility functions and tax systems consistent with absolute equality of sacrifice

Let $f(x)$ be a tax schedule that induces equal absolute sacrifice under the utility function $u(x)$. Thus f and u satisfy (1), or equivalently

$$f(x) = x - u^{-1}[u(x) - c] \text{ for some } c \geq 0 \text{ and all } x > 0 \quad . \tag{3}$$

Note that the inverse of u exists because u is assumed to be strictly increasing. Expression (3) shows that f can be regarded as a function of two variables: the income level x and the amount of sacrifice c. As c varies, $f(x,c)$ defines a *family* of tax schedules that allows different amounts of tax to be raised relative to any fixed distribution of incomes. $f(x,c)$ is called a ***parametric tax schedule***. Parametric families crop up in a wide variety of fair division problems [11].

Scale-invariance of the tax system based on $f(x,c)$ amounts to saying that, for every scale factor $\lambda > 0$, and for every constant $c \geq 0$, there exists a constant c' (depending on λ) such that

$$t = f(x,c) \quad \text{iff} \quad \lambda t = f(\lambda x, c') \quad \text{for all} \quad x > 0 \ . \tag{4}$$

In other words, the tax schedule can be *re-indexed* by changing c to c', and the proportion of tax paid to income received will remain fixed for all persons.

If f imposes absolute sacrifice and is scale-invariant then several important properties of the utility function may be deduced. In particular, differences in utility levels will be preserved (in an ordinal sense) under a uniform change of scale. Consider two individuals, one with initial income x and after-tax income y, $0 < y \leq x$, and another with initial income x' and after tax income y', $0 < y' \leq x'$. Equal absolute sacrifice implies that $u(x) - u(y) = u(x') - u(y') = c \geq 0$. By definition of $f, y = x - f(x,c)$ and $y' = x' - f(x',c)$. Since f is scale-invariant, for every $\lambda > 0$ there is a $c' \geq 0$ such that

$$\lambda y = \lambda x - f(\lambda x, c') \text{ and } \lambda y' = \lambda x' - f(\lambda x', c') \ .$$

Hence by (3)

$$u(\lambda x) - u(\lambda y) = u(\lambda x') - u(\lambda y') = c' \ .$$

That is, for every $x, x', y, y' > 0$ and every $\lambda > 0$,

$$u(x) - u(y) = u(x') - u(y') \ \ iff \ \ u(\lambda x) - u(\lambda y) = u(\lambda x') - u(\lambda y') \ . \tag{5}$$

Such a function u is said to be *homogeneous in differences.*

Lemma. If $u(x)$ is continuous, nondecreasing, and homogeneous in differences, then $u(x)$ is of form

(i) $u(x) = b$, or

(ii) $u(x) = a \ln x + b$, $a > 0$, or

(iii) $u(x) = a x^p + b$, $ap > 0$.

Proof.* If $u(x)$ is homogeneous in differences, then there is a single-valued function F such that for all $0 < y \leq x$ and all $\lambda > 0$,

$$u(\lambda x) - u(\lambda y) = F(u(x) - u(y), \lambda) \ .$$

Since u is continuous, F is too. Let $D = \{u(x) - u(y) : 0 < y \leq x\}$. By continuity, D is an interval of R_+ of form $[0,m)$ or $[0,m]$, where $m > 0$ unless $u(x)$ is constant, which is case (i) of the lemma. Suppose then that $m > 0$. If $z, z' \geq 0$ and $z + z' < m$, then $z + z' = u(x) - u(y)$ for some x and y. Since $0 \leq z \leq u(x) - u(y)$, we have

$$u(x) \geq u(x) - z \geq u(y) \ .$$

Thus by the continuity of $u(x)$ there exists some w between x and y such that $u(w) = u(x) - z$; that is $z = u(x) - u(w)$. Hence also $z' = u(w) - u(y)$. This argument shows that for every $\lambda > 0$,

$$F(z + z', \lambda) = F(z, \lambda) + F(z', \lambda) \text{ provided } z, z' \geq 0 \text{ and } z + z' < m \ . \quad (6)$$

Since F is continuous, F may be continuously extended so that (6) also holds whenever $z, z' \geq 0$ and $z + z' = m$. It follows from [1] (Section 2.1.4, Theorem 3) that

$$F(z, \lambda) = c(\lambda) z \text{ for all } 0 \leq z \leq m, \text{ all } \lambda > 0 \ .$$

Since u is nondecreasing,

$$u(\lambda x) - u(\lambda y) = c(\lambda)[u(x) - u(y)] \text{ for all } x, y > 0 \ .$$

Fix $y^* > 0$, and let $d(\lambda) = u(\lambda y^*) - c(\lambda) u(y^*)$. Then

$$u(\lambda x) = c(\lambda) u(x) + d(\lambda) \text{ for all } x, \lambda > 0.$$

For all real numbers r define $u^*(r) = u(e^r)$, $c^*(r) = c(e^r)$, and $d^*(r) = d(e^r)$. Making the substitutions $s = \ln \lambda$ and $t = \ln x$, the preceding becomes

$$u^*(s + t) = c^*(s) u^*(t) + d^*(s) \ .$$

By [1] (Section 3.1.3, Theorem 1) this, together with the continuity and monotonicity of u^*, implies that

$$u^*(r) = b \ ,$$

*The proof of this result originated in collaboration with Janos Aczel during a pleasant drive together across southern Germany in July, 1985. His contribution and helpful comments on the manuscript are gratefully acknowledged.

or

$$u^*(r) = ar + b \ , \quad a > 0 \ ,$$

or

$$u^*(r) = ae^{pr} + b \ , \quad ap > 0 \ .$$

Therefore,

$$u(x) = b \ ,$$

or

$$u(x) = a \ln x + b \ , \quad a > 0 \ ,$$

or

$$u(x) = ax^p + b \ , \quad ap > 0 \ . \quad \blacksquare$$

It should be remarked that, if $u(x)$ is assumed to be continuous, homogeneous in differences, and *nonconstant* (instead of nondecreasing), then the above argument shows that u must be of form (i) or (ii) with $a \neq 0$, or (iii) with $ap \neq 0$.

Theorem. Let the utility of income $u(x)$ be continuous and strictly increasing for all incomes $x > 0$, and let $f(x,c)$ be a scale-invariant, progressive family of tax schedules that induces an equal level of sacrifice $c \geq 0$ for all positive income levels. Then for some fixed $a,p > 0$ and all $x > 0$,

$$f(x,c) = (1 - e^{-c/a})x \quad \text{and} \quad u(x) = a \ln x + b, \ a > 0; \tag{i}$$

or

$$f(x,c) = x - \frac{x}{[1+cx^p/a]^{1/p}} \quad \text{and} \ u(x) = -ax^{-p} + b, \ \ a,p > 0 \ . \tag{ii}$$

Proof: By the lemma, $u(x)$ must be of the form $a \ln x + b$ for some $a > 0$ or $ax^p + b$ for $ap > 0$. The case of constant $u(x)$ is ruled out since it is assumed here that $u(x)$ is strictly increasing. $f(x)$ yields a constant level of sacrifice $c \geq 0$ if and only if

$$f(x,c) = x - u^{-1}[u(x) - c] \ . \tag{7}$$

If $u(x) = a \ln a + b$, (7) implies that

$$f(x,c) = x - e^{\ln x - c/a} = (1 - e^{-c/a})x \quad ,$$

which is case (i) of the theorem. If $u(x) = ax^p + b$ we have two possibilities:

Case 1: $u(x) = ax^p + b,\ a > 0,\ p > 0$.

Then $u^{-1}(y) = [(y - b)/a]^{1/p}$, and (7) becomes $f(x,c) = x - [x^p - c/a]^{1/p}$. This is not progressive for $c > 0$, since $f(x,c)/x = 1 - [1 - c/x^p a]^{1/p}$ is strictly decreasing in x.

Case 2: $u(x) = ax^p + b,\ a < 0,\ p < 0$.

Replace a by $-a$ and p by $-p$ and write $u(x) = -ax^{-p} + b$, where $a > 0$, $p > 0$, and b is unrestricted. Then $u^{-1}(y) = [a/(b - y)]^{1/p}$ and (7) becomes

$$f(x,c) = x - \frac{x}{[1+cx^p/a]^{1/p}} \quad .$$

These schedules are progressive for all $c \geq 0$, and strictly so for $c > 0$. ß

Set $a = 1$ and for every $p > 0$ define $f_p(x,c) = x - x/(1 + cx^p)^{1/p}$. Also, define $f_0(x,c) = (1 - e^{-c})x$ for all $x > 0$. These tax schedules form a single family with two parameters: p essentially determines the progressivity of the schedule and for each fixed p, c determines the amount of tax to be levied.

4. Utility functions and tax systems consistent with an equal rate of sacrifice

Consider now a utility function that induces an equal *rate* of sacrifice over all income levels. For this notion to be well-defined, we must have $u(x) \neq 0$ for all $x > 0$. That is, the utility of income must be everywhere positive or everywhere negative. The more natural case is the former, and there is no real loss of generality in assuming it, for if $u(x)$ is negative, increasing, and an equal rate of sacrifice regime prevails, then it also prevails under $1/|u(x)|$, which is positive and increasing.

Let $u(x)$ be positive, continuous, and strictly increasing, and let $f(x)$ induce a constant rate of sacrifice r, $0 \leq r < 1$. By definition,

$$1 - \frac{u(x - f(x))}{u(x)} = r \qquad \text{for all} \quad x > 0 \ . \tag{8}$$

Thus

$$ln\ u(x) - ln\ u(x - f(x)) = -ln(1 - r) \quad \text{for all} \quad x > 0\ .$$

Therefore the arguments of the preceding section apply, since $f(x)$ induces equal absolute sacrifice relative to the transformed utility function $ln\ u(x)$. It follows that the only progressive and scale-invariant tax schedules that are consistent with an equal rate of sacrifice are essentially the same as those that are consistent with equal absolute sacrifice. The difference between the two cases is the required form of the utility function. There are two possibilities. One is that $ln\ u(x) = a\ ln\ x + b,\ a > 0$, in which case $u(x) = \beta x^a$ where $a > 0$ and $\beta = e^b > 0$. Then an equal rate of sacrifice at rate r results from imposing the flat tax $t = \tilde{f}(x,r) = [1 - (1-r)^{1/a}]x$.

The second possibility is that $ln\ u(x) = -ax^{-p} + b$, where $a, p > 0$. Then $u(x)$ is of the form

$$u(x) = \beta\, e^{-ax^{-p}}\ , \qquad a, \beta, p > 0\ .$$

An equal rate of sacrifice at rate r results from imposing the tax $t = \tilde{f}(x,r) = x - x/(1 - [ln(1-r)]\ x^p/{}^{1/p}$. These results are summarized below.

Corollary. Let utility of income $u(x)$ be positive, continuous, and strictly increasing for all incomes $x > 0$, and let $\tilde{f}(x,r)$ be a scale-invariant, progressive family of tax schedules that induces an equal rate of sacrifice $0 \leq r < 1$ on all positive incomes. Then

$$\tilde{f}(x,r) = [1 - (1-r)^{1/a}]x \quad \text{and} \quad u(x) = \beta x^a,\ a, \beta > 0\ , \tag{i}$$

or

$$\tilde{f}(x,r) = x - \frac{x}{[1 - [ln(1-r)]x^p/a]^{1/p}} \quad \text{and} \quad u(x) = \beta e^{-ax^{-p}}, a, \beta, p > 0\ . \tag{ii}$$

Cohen Stuart, who originated the formal approach to equal sacrifice in taxation, advocated the use of a utility function of form $u(x) = a ln\ x + b,\ a > 0$, together with equal *rate* of sacrifice as the criterion. This gives rise to tax schedules of the form

$$t = x - e^{-rb/a} x^{(1-r)}\ ,$$

where r, $0 \leq r < 1$ is the rate of sacrifice. These functions are known as *Cohen Stuart taxes*. They were also mentioned as a possibility by Edgeworth, who proposed that they might be appropriate as a surtax yoked onto some other tax [3]. Let $x^* = e^{-b/a}$, which may be interpreted as the "minimum subsistence" level of income. Then Cohen Stuart's utility and tax functions may be written as $u(x) = a ln(x/x^*)$ and $t = x - x^*(x/x^*)^{1-r}$.

Cohen Stuart taxes do not satisfy the conditions of the foregoing theorem on several counts. First, the rate of sacrifice is undefined when $x = x^*$, i.e., when the zero level of utility is reached. Second, when $x < x^*$, taxes become negative. Third, and perhaps most disturbing, the tax rates paid by individuals depend not only on the relative distribution of incomes in society, but also on the absolute level x^* that is supposed to have zero utility. In other words, a precise definition of what constitutes the subsistence level is crucial to the validity of the scheme in equal sacrifice terms.

5. Conclusion

The equal sacrifice approach to progressive taxation is appealing in principle. But it is far from clear *a priori* what form the utility function should take, or whether equal absolute or equal rate of sacrifice is the most appropriate criterion. One can gain insight into these questions by imposing elementary conditions on the tax functions that arise from an equal sacrifice model. As we have seen, equal sacrifice plus nonnegativity, progressivity, and scale-invariance is consistent with only a very special class of utility functions.

The flat tax implies equal absolute sacrifice if and only if utility is represented by $u(x) = a ln\ x + b$ for some $a > 0$, and implies equal rate of sacrifice if and only if utility is represented by $u(x) = \beta x^{a}$, $a, \beta > 0$. A strictly progressive, nonnegative, and scale-invariant tax implies equal absolute sacrifice if and only if utility is represented by $u(x) = -ax^{-p} + b$, where $a, p > 0$. It implies equal rate of sacrifice if and only if utility is represented by $u(x) = be^{-ax^{-p}}$, where $a, b, p > 0$. The class $-ax^{-p} + b$ is bounded above and unbounded below, whereas the class $be^{-ax^{-p}}$ is positive, bounded both above and below, and $u(x) \to 0$ as $x \to 0$. Since a positive utility of income to be more sensible intuitively than one that is unbounded below, it appears that equal *rate* of sacrifice seems to be the more natural criterion when we examine both the form of the utility functions and of the tax schedules that are consistent with it.

Finally, there is only one family of tax schedules that is consistent with equal sacrifice ***under either definition***, and is nonnegative, progressive, and scale-invariant. This same family has another desirable feature: namely, a tax can be levied in successive installments with the same outcome as if it had been levied all at once [12].

Schedules of the above type can be fit quite closely to recent tax schedules in the U.S. and in West Germany [12]. This fact suggests that one could use actual tax data to estimate the value of p in formula (ii) of the Theorem (or the Corollary). For recent US and West German data, the estimated values of p are in the vicinity of .65 to .75. If one hypothesizes that tax schedules to some degree reflect what a given society perceives to be equal sacrifice, then such estimates might be treated in turn as estimates of the parameters in the utility functions themselves.

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