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AN INTERACTIVE FUZZY SATISFICING METHOD
FOR MULTIOBJECTIVE NONLINEAR PROGRAMMING
PROBLEMS WITH FUZZY PARAMETERS

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An Interactive Fuzzy Satisficing Method for Multiobjective
Nonlinear Programming Problems with Fuzzy Parameters

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Abstract

This paper presents an interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters. The fuzzy parameters in the objective functions and the constraints are characterized by the fuzzy numbers. On the basis of the α -level sets of the fuzzy numbers, the concept of α -multiobjective nonlinear programming and α -Pareto optimality is introduced. Through the interaction with the decision maker (DM), the fuzzy goals of the DM for each of the objective functions in α -multiobjective nonlinear programming are quantified by eliciting the corresponding membership functions. After determining the membership functions, in order to generate a candidate for the satisficing solution which is also α -Pareto optimal, if the DM specifies the degree α of the α -level sets and the reference membership values, the augmented minimax problem is solved and the DM is supplied with the corresponding α -Pareto optimal solution together with the trade-off rates among the values of the membership functions and the degree α . Then by considering the current values of the membership functions and as well as the trade-off rates, the DM responds by updating his reference membership values and/or the degree α . In this way the satisficing solution for the DM can be derived efficiently from among an α -Pareto optimal solution set. Based on the proposed method, a time-sharing computer program is written and an illustrative numerical example is demonstrated along with the corresponding computer outputs.

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I. INTRODUCTION

In most multiobjective nonlinear programming problems, multiple-objective functions usually conflict with each other in that any improvement of one objective function can be achieved only at the expense of another. Accordingly, the aim is to find the satisficing solution of the decision maker (DM) which is also Pareto optimal (e.g.[2],[24] etc.). However, when formulating the multiobjective nonlinear programming problem which closely describes and represents the real decision situation, various factors of the real system should be reflected in the description of the objective functions and the constraints. Naturally these objective functions and the constraints involve many parameters whose possible values may be assigned by the experts. In the conventional approach, such parameters are fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters.

In most practical situations, however, it is natural to consider that the possible values of these parameters are often only ambiguously known to the experts. In this case, it may be more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy subsets of the real line known as fuzzy numbers [4,5]. The resulting multiobjective nonlinear programming problem involving fuzzy parameters would be viewed as the more realistic version of the conventional one.

Recently, Tanaka and Asai [19,20] formulated the multiobjective linear programming problems with fuzzy parameters. Following the fuzzy decision or minimum operator proposed by Bellman and Zadeh [1] together with triangular membership functions for fuzzy parameters, they considered two types of fuzzy multiobjective linear programming problems; one is to

decide the nonfuzzy solution and the other is to decide the fuzzy solution.

More recently, Orlovski [12,13] formulated general multiobjective nonlinear programming problems with fuzzy parameters. He presented two approaches to the formulated problems by making systematic use of the extension principle of Zadeh [23] and demonstrated that there exist in some sense equivalent nonfuzzy formulations.

In this paper, in order to deal with the multiobjective nonlinear programming problems with fuzzy parameters characterized by fuzzy numbers, the concept of α -multiobjective nonlinear programming and α -Pareto optimality is introduced on the basis of the α -level sets of the fuzzy numbers. Then by assuming that the fuzzy goals of the DM for each of the objective functions in α -multiobjective nonlinear programming can be quantified by eliciting the corresponding membership functions, an interactive fuzzy satisficing method to derive the satisficing solution of the DM efficiently from among an α -Pareto optimal solution set is presented as a generalization of the results obtained in Sakawa et al.[14-17].

II. α -PARETO OPTIMALITY

In general, the multiobjective nonlinear programming (MONLP) problem is represented as the following vector-minimization problem:

$$\min f(x) \triangleq (f_1(x), f_2(x), \dots, f_k(x))^T \quad (1)$$

$$\text{subject to } x \in X = \{ x \in E^n \mid g_j(x) \leq 0, j=1, \dots, m \}$$

where x is an n -dimensional vector of decision variables, $f_1(x), \dots, f_k(x)$ are k distinct objective functions of the decision vector x , $g_1(x), \dots, g_m(x)$ are inequality constraints, and X is the feasible set of constrained decisions.

Fundamental to the MONLP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MONLP is one where any improvement of one objective function can be achieved only at the expense of another. Mathematically, a formal definition of a Pareto optimal solution to the MONLP is given below:

Definition 1. (Pareto optimal solution)

$x^* \in X$ is said to be a Pareto optimal solution to the MONLP, if and only if there does not exist another $x \in X$ such that $f_i(x) \leq f_i(x^*)$, $i=1, \dots, k$, with strict inequality holding for at least one i .

In practice, however, it would certainly be appropriate to consider that the possible values of the parameters in the description of the objective functions and the constraints usually involve the ambiguity of the experts' understanding of the real system. For this reason, in this paper, we consider the following multiobjective nonlinear programming problem with fuzzy parameters (MONLP-FP) :

$$\min f(x, \tilde{a}) \triangleq (f_1(x, \tilde{a}_1), f_2(x, \tilde{a}_2), \dots, f_k(x, \tilde{a}_k))^T \quad (2)$$

$$\text{subject to } x \in X(\tilde{b}) \triangleq \{ x \in E^n \mid g_j(x, \tilde{b}_j) \leq 0, j=1, \dots, m \}$$

where $\tilde{a}_i = (\tilde{a}_{i1}, \dots, \tilde{a}_{ip_i})$, $\tilde{b}_j = (\tilde{b}_{j1}, \dots, \tilde{b}_{jq_j})$ represent respectively a vector of fuzzy parameters involved in the objective function $f_i(x, \tilde{a}_i)$ and the constraint function $g_j(x, \tilde{b}_j)$.

These fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced by Dubois and Prade [3,4] . It is appropriate to review here that a real fuzzy number \tilde{p} is a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{p}}(p)$ is defined as:

- (1) A continuous mapping from E^1 to the closed interval $[0,1]$.
- (2) $\mu_{\tilde{p}}(p) = 0$ for all $p \in (-\infty, p_1]$
- (3) Strictly increasing on $[p_1, p_2]$
- (4) $\mu_{\tilde{p}}(p) = 1$ for all $p \in [p_2, p_3]$
- (5) Strictly decreasing on $[p_3, p_4]$
- (6) $\mu_{\tilde{p}}(p) = 0$ for all $p \in [p_4, +\infty]$.

Fig.1 illustrates the graph of the possible shape of the fuzzy number \tilde{p} .

We now assume that \tilde{a}_{ir} and \tilde{b}_{js} in the MONLP-FP are fuzzy numbers whose membership functions are $\mu_{\tilde{a}_{ir}}(a_{ir})$ and $\mu_{\tilde{b}_{js}}(b_{js})$ respectively. For simplicity in the notation, define the following vectors:

$$a_i = (a_{i1}, \dots, a_{ip_i}), \quad b_j = (b_{j1}, \dots, b_{jq_j})$$

$$a = (a_1, \dots, a_k), \quad \tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_k), \quad b = (b_1, \dots, b_m), \quad \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_m) .$$

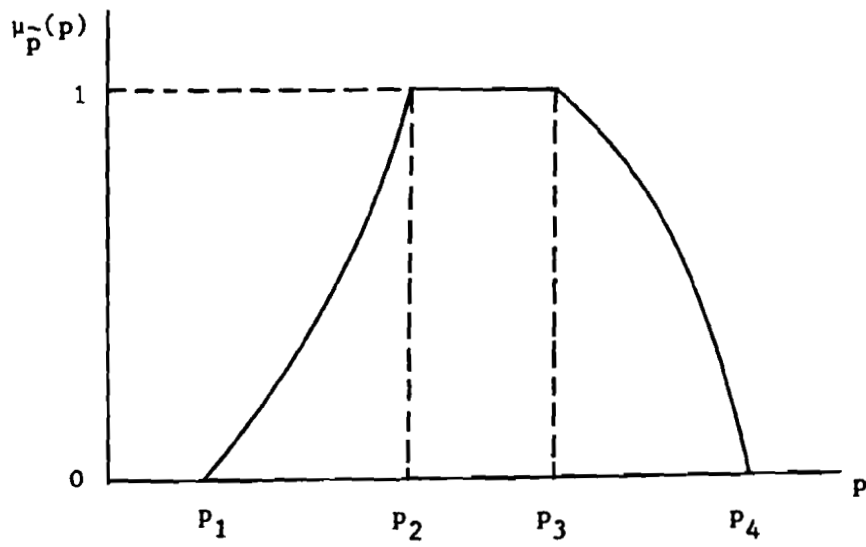


Fig. 1. Membership function of fuzzy \tilde{p}

Then we can introduce the following α -level set or α -cut [4] of the fuzzy numbers \tilde{a}_{ir} and \tilde{b}_{js} .

Definition 2. (α -level set)

The α -level set of the fuzzy numbers \tilde{a}_{ir} ($i=1, \dots, k$, $r=1, \dots, p_i$) and \tilde{b}_{js} ($j=1, \dots, m$, $s=1, \dots, q_j$) is defined as the ordinary set $L_\alpha(\tilde{a}, \tilde{b})$ for which the degree of their membership functions exceeds the level α :

$$L_\alpha(a, b) = \left\{ (a, b) \mid \begin{array}{l} \mu_{\tilde{a}_{ir}}(a_{ir}) \geq \alpha \quad (i=1, \dots, k, r=1, \dots, p_i), \\ \mu_{\tilde{b}_{js}}(b_{js}) \geq \alpha \quad (j=1, \dots, m, s=1, \dots, q_j) \end{array} \right\} \quad (3)$$

It is clear that the level sets have the following property:

$$\alpha_1 \leq \alpha_2 \quad \text{if and only if} \quad L_{\alpha_1}(\tilde{a}, \tilde{b}) \supset L_{\alpha_2}(\tilde{a}, \tilde{b})$$

For a certain degree α , the MONLP-FP (2) can be understood as the following nonfuzzy α -multiobjective nonlinear programming (α -MONLP) problem.

α -MONLP

$$\begin{aligned} \min \quad & f(x, a) \triangleq (f_1(x, a_1), f_2(x, a_2), \dots, f_k(x, a_k))^T \\ \text{subject to} \quad & x \in X(b) \triangleq \{x \in E^n \mid g_j(x, b_j) \leq 0, j=1, \dots, m\} \end{aligned} \quad (4)$$

$$(a, b) \in L_\alpha(\tilde{a}, \tilde{b})$$

It should be emphasized here that in the α -MONLP the parameters (a, b) are treated as decision variables rather than constants.

On the basis of the α -level sets of the fuzzy numbers, we introduce the concept of α -Pareto optimal solutions to the α -MONLP.

Definition 3. (α -Pareto optimal solution)

$x^* \in X(b)$ is said to be an α -Pareto optimal solution to the α -MONLP(4), if and only if there does not exist another $x \in X(b)$, $(a, b) \in L_{\alpha}(\tilde{a}, \tilde{b})$ such that $f_i(x, a_i) \leq f_i(x^*, a_i^*)$, $i=1, \dots, k$, with strict inequality holding for at least one i , where the corresponding values of parameters a^* and b^* are called α -level optimal parameters.

For practical purposes, however, since only local solutions are guaranteed in solving a scalar optimization problem by any standard optimization technique, unless the problem is convex, we deal with local α -Pareto optimal solutions instead of global α -Pareto optimal solutions.

Definition 4. (local α -Pareto optimal solution)

$x^* \in X(b)$ is said to be a local α -Pareto optimal solution to the α -MONLP (4), if and only if there does not exist another $x \in X(b) \cap N(x^*; r)$, $(a, b) \in L_{\alpha}(\tilde{a}, \tilde{b}) \cap N(a^*, b^*; r')$ such that $f_i(x, a_i) \leq f_i(x^*, a_i^*)$, $i=1, \dots, k$, with strict inequality holding for at least one i , where the corresponding values of parameters a^* and b^* are called α -level local optimal parameters and $N(x^*; r)$ denotes the set $\{x | x \in E^n, \|x - x^*\| < r\}$.

Usually, (local) α -Pareto optimal solutions consist of an infinite number of points, and some kinds of subjective judgement should be added to the quantitative analyses by the DM. The DM must select his (local) satisficing or compromise solution from among (local) α -Pareto optimal solutions.

In a minimization problem, a fuzzy goal stated by the DM may be to achieve "substantially less " than A . This type of statement can be quantified by eliciting a corresponding membership function.

In order to elicit a membership function $\mu_{f_i}(x, a_i)$ from the DM for each of the objective functions $f_i(x, a_i)$ in the α -MONLP (4), we first calculate the individual minimum f_i^{\min} and maximum f_i^{\max} of each objective function $f_i(x, a_i)$ under given constraints. By taking account of the calculated individual minimum and maximum of each objective function, the DM must determine his subjective membership function $\mu_{f_i}(x, a_i)$ which is a strictly monotone decreasing function with respect to $f_i(x, a_i)$. Fig. 2 illustrates the graph of the possible shape of the membership function representing the fuzzy goal to achieve substantially less than A .

So far we have considered a minimization problem and consequently assumed that the DM has a fuzzy goal such as " $f_i(x, a_i)$ should be substantially less than A ". In the fuzzy approaches, we can further treat a more general case where the DM has two types of fuzzy goals, namely fuzzy goals expressed in words such as " $f_i(x, a_i)$ should be in the vicinity of B " (called fuzzy equal) as well as " $f_i(x, a_i)$ should be substantially less than A " (called fuzzy min). Such a generalized α -MONLP ($G\alpha$ -MONLP) problem may now be expressed as:

$$\text{fuzzy min } f_i(x, a_i) \quad (i \in I)$$

$$\text{fuzzy equal } f_i(x, a_i) \quad (i \in \bar{I})$$

subject to $x \in X(b)$ (5)

$$(a,b) \in L_{\alpha}(\tilde{a},\tilde{b})$$

where $I \cup \bar{I} = \{1,2,\dots,k\}$.

In order to elicit a membership function from the DM for a fuzzy goal like " $f_i(x,a_i)$ should be in the vicinity of B", it is obvious that we can use different functions to the left and right sides of B .

Fig.3 illustrates the graph of the possible shape of the fuzzy equal membership function representing the fuzzy goal to be in the vicinity of B .

Now we introduce the concept of (local) M- α -Pareto optimal solutions which are defined in terms of membership functions instead of objective functions .

Definition 5. ((local) M- α -Pareto optimal solution)

$x^* \in X$ is said to be a (local) M- α -Pareto optimal solution to the G α -MONLP, if and only if there does not exist another $x \in X(b)$ ($\cap N(x^*;r)$), $(a,b) \in L_{\alpha}(\tilde{a},\tilde{b})$ ($\cap N(a^*,b^*;r')$) such that $\mu_{f_i}(x,a_i) \geq \mu_{f_i}(x^*,a_i^*)$, $i=1,\dots,k$, with strict inequality holding for at least one i .

Observe that the set of (local) α -Pareto optimal solutions is a subset of the set of (local) M- α -Pareto optimal solutions.

Having elicited the membership functions $\mu_{f_i}(x,a_i)$, $i=1,\dots,k$ from the DM for each of the objective functions $f_i(x,a_i)$, $i=1,\dots,k$, the α -MONLP (4) and/or the G α -MONLP (5) can be converted into the fuzzy α -MONLP (F α -MONLP) problem defined by:

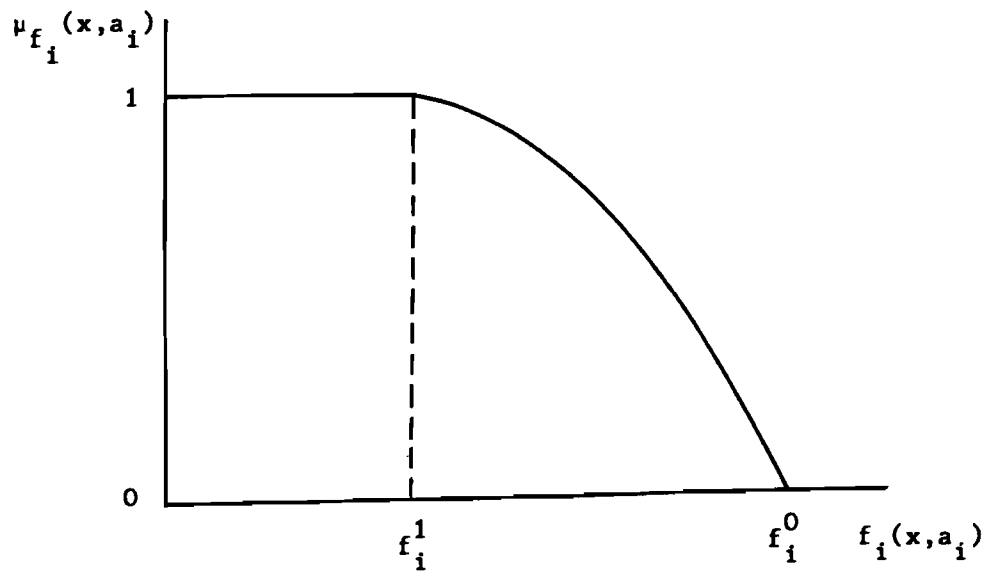


Fig. 2. An example of a fuzzy min membership function

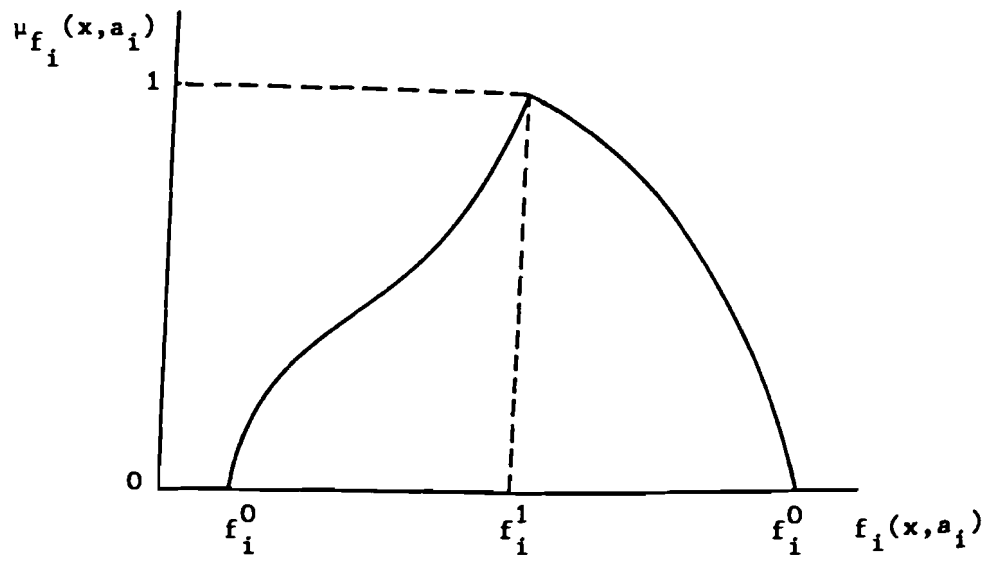


Fig. 3. An example of a fuzzy equal membership function

$$\begin{aligned} & \max_{\substack{x \in X(b) \\ (a,b) \in L_\alpha(\tilde{a}, \tilde{b})}} (\mu_{f_1}(x, a_1), \mu_{f_2}(x, a_2), \dots, \mu_{f_k}(x, a_k)) \end{aligned} \quad (6)$$

By introducing a general aggregation function

$$\mu_D(\mu_f(x, a)) = \mu_D(\mu_{f_1}(x, a_1), \mu_{f_2}(x, a_2), \dots, \mu_{f_k}(x, a_k)), \quad (7)$$

a general fuzzy α -multiobjective decision problem (F α -DMP) can be defined by:

$$\begin{aligned} & \max_{\substack{x \in X(b) \\ (a,b) \in L_\alpha(\tilde{a}, \tilde{b})}} \mu_D(\mu_f(x, a)) \end{aligned} \quad (8)$$

Observe that the value of $\mu_D(\mu_f(x))$ can be interpreted as the overall degree of satisfaction of the DM's fuzzy goals. The fuzzy decision or minimum-operator of Bellman and Zadeh [1]

$$\min_{\substack{1 \leq i \leq k \\ =}} (\mu_{f_1}(x, a_1), \mu_{f_2}(x, a_2), \dots, \mu_{f_k}(x, a_k)) \quad (9)$$

can be viewed only as one special example of $\mu_D(\mu_f(x, a))$.

In the conventional fuzzy approaches [25,26] it has been implicitly assumed that the minimum-operator is the proper representation of human decision makers' fuzzy preferences, and hence the F α -MDP (8) has been interpreted as follows:

$$\begin{aligned} & \max_{\substack{x \in X(b) \\ (a,b) \in L_\alpha(\tilde{a}, \tilde{b})}} \min_{1 \leq i \leq k} (\mu_{f_1}(x, a_1), \mu_{f_2}(x, a_2), \dots, \mu_{f_k}(x, a_k)) \end{aligned} \quad (10)$$

or equivalently

$$\begin{aligned}
 & \max \quad v \\
 & v, x \in X(b) \\
 & (a, b) \in L_{\alpha}(\tilde{a}, \tilde{b}) \\
 & \text{subject to} \quad v \leq \mu_{f_i}(x, a_i), \quad i=1, \dots, k.
 \end{aligned} \tag{11}$$

However, it should be emphasized here that this approach is preferable only when the DM feels that the minimum-operator is appropriate. In other words, in general decision situations, human decision maker do not always use the minimum-operator when they combine the fuzzy goals and/or constraints. Probably the most crucial problem in the F_{α} -MDP is the identification of an appropriate aggregation function which will represents the human decision makers' fuzzy preferences. If $\mu_D(\cdot)$ can be explicitly identified, then the F_{α} -MDP reduces to a standard mathematical programming problem. However, this rarely happens and as an alternative, it becomes evident that an interaction with the DM is necessary.

Throughout this paper we make the following assumptions.

Assumption 1.

The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions through the interaction with the DM.

Assumption 2.

$\mu_D(\cdot)$ exists and is known only implicitly to the DM, which means the DM cannot specify the entire form of $\mu_D(\cdot)$, but he can provide local information concerning his preference. Moreover, it is increasing and continuous.

Assumption 3.

All $f_i(x, \tilde{a}_i)$, $i=1, \dots, k$ and all $g_j(x, \tilde{b}_j)$, $j=1, \dots, m$ are continuously differentiable in their respective domains.

III. AUGMENTED MINIMAX PROBLEMS

Having determined the membership functions for each of the objective functions, in order to generate a candidate for the satisficing solution which is also (local) (M-) α -Pareto optimal, the DM is asked to specify the degree α of the α -level set and the reference levels of achievement of the membership functions, called the reference membership values. Observe that the idea of the reference membership values [16,17] can be viewed as an obvious extension of the idea of the reference point of Wierzbicki [21] .

For the DM's degree α and the reference membership values $\bar{\mu}_{f_i}$, $i=1, \dots, k$, the following augmented minimax problem is solved in order to generate the (local) (M-) α -Pareto optimal solution which is in a sense close to his requirement (or better, if the reference membership values are attainable).

$$\begin{aligned} \min_{\substack{x \in X(b) \\ (a,b) \in L_{\alpha}(\tilde{a}, \tilde{b})}} \quad & \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) \end{aligned} \quad (12)$$

or equivalently

$$\min_{x, v, a, b} \quad v + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) \quad (13)$$

$$\text{subject to} \quad \bar{\mu}_{f_i} - \mu_{f_i}(x, a_i) \leq v, \quad i=1, \dots, k \quad (14)$$

$$(a, b) \in L_{\alpha}(\tilde{a}, \tilde{b}) \quad (15)$$

$$x \in X(b) \quad (16)$$

The term augmented is adopted because the term $\rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i))$ is added to the usual minimax problems, where ρ is a sufficiently small positive scalar. Naturally, ρ should be a sufficiently small, but computationally significant, positive scalar. In most case, a computationally significant value of $\rho = 10^{-3} \sim 10^{-5}$ should suffice. Such an augmented minimax problem can be viewed as a modified fuzzy version of the augmented Tchebycheff norm problem of Steuer and Choo [18] or Choo and Atkins [3].

The relationships between the (local) optimal solutions of the augmented minimax problem and the (local) α -Pareto optimal concept of the α -MONLP can be characterized by the following theorems.

Theorem 1.

If (x^*, v^*, a^*, b^*) is a (local) optimal solution to the augmented minimax problem for some $\bar{\mu}_{f_i}$, $i=1, \dots, k$, with $0 < \mu_{f_i}(x^*, a_i^*) < 1$ holding for all i , then x^* is a (local) α -Pareto optimal solution and a^*, b^* are α -level (local) optimal parameters to the α -MONLP.

(Proof)

Assume that x^* is not a (local) α -Pareto optimal solution and a^*, b^* are not α -level (local) optimal parameters to the α -MONLP, then there exists $\bar{x} \in X(b)(\cap N(x^*, r))$, $(\bar{a}, \bar{b}) \in L_{\alpha}(\bar{a}, \bar{b})(\cap N(a^*, b^*; r'))$ such that $f(\bar{x}, \bar{a}) \leq f(x^*, a^*)$. This implies that $\mu_f(\bar{x}, \bar{a}) \leq \mu_f(x^*, a^*)$ or $\bar{\mu}_f - \mu_f(\bar{x}, \bar{a}) \leq \bar{\mu}_f - \mu_f(x^*, a^*)$, since by the hypothesis $0 < \mu_{f_i}(x^*, a_i^*) < 1$ for all i , where $\mu_f(x, a) = (\mu_{f_1}(x, a_1), \dots, \mu_{f_k}(x, a_k))$ and $\bar{\mu}_f = (\bar{\mu}_{f_1}, \dots, \bar{\mu}_{f_k})$. Then it holds that

$$\max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(\bar{x}, \bar{a}_i)) \leq \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(x^*, a_i^*))$$

This means that

$$\begin{aligned} & \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(\bar{x}, \bar{a}_i)) + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(\bar{x}, \bar{a}_i)) \\ < & \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(x^*, a_i^*)) + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x^*, a_i^*)) \end{aligned}$$

which contradicts the fact that (x^*, v^*, a^*, b^*) is a (local) optimal solution to the augmented minimax problem. Hence x^* is a (local) α -Pareto optimal solution and a^*, b^* are α -level (local) optimal parameters to the α -MONLP.

Theorem 2.

If x^* is a (local) α -Pareto optimal solution and a^*, b^* are α -level (local) optimal parameters to the α -MONLP with $0 < \mu_{f_i}(x^*, a_i^*) < 1$ holding for all i , then there exist $\bar{\mu}_{f_i}, i=1, \dots, k$ such that (x^*, v^*, a^*, b^*) is a (local) optimal solution to the augmented minimax problem.

(Proof)

Assume that (x^*, v^*, a^*, b^*) is not a (local) optimal solution to the augmented minimax problem for any $\bar{\mu}_{f_i}, i=1, \dots, k$, satisfying

$$\bar{\mu}_{f_1} - \mu_{f_1}(x^*, a_1^*) = \dots = \bar{\mu}_{f_k} - \mu_{f_k}(x^*, a_k^*).$$

Then there exists $\bar{x} \in X(\cap N(x^*, r))$ and $(\bar{a}, \bar{b}) \in L_\alpha(\bar{a}, \bar{b})(\cap N(a^*, b^*; r'))$ such that

$$\begin{aligned} & \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(x^*, a_i^*)) + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x^*, a_i^*)) \\ > & \max_{1 \leq i \leq k} (\bar{\mu}_{f_i} - \mu_{f_i}(\bar{x}, \bar{a}_i)) + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(\bar{x}, \bar{a}_i)) . \end{aligned}$$

This implies that

$$\max_{1 \leq i \leq k} (\mu_{f_i}(x^*, a_i^*) - \mu_{f_i}(\bar{x}, \bar{a}_i)) + \rho \sum_{i=1}^k (\mu_{f_i}(x^*, a_i^*) - \mu_{f_i}(\bar{x}, \bar{a}_i)) < 0$$

Now if either any $\mu_{f_i}(x^*, a_i^*) - \mu_{f_i}(\bar{x}, \bar{a}_i)$ is positive or all $\mu_{f_i}(x^*, a_i^*) - \mu_{f_i}(\bar{x}, \bar{a}_i)$, $i=1, \dots, k$, are zero, this inequality would be violated for sufficiently small positive ρ . Hence

$$\mu_{f_i}(x^*, a_i^*) - \mu_{f_i}(\bar{x}, \bar{a}_i) \leq 0, \quad i=1, \dots, k$$

must hold. Since by the assumption $0 < \mu_{f_i}(x^*, a_i^*) < 1$, we have $f(x^*, a_i^*) \geq f(x, a_i)$, which contradicts the fact that x^* is a (local) α -Pareto optimal solution and a_i^*, b_i^* are α -level (local) optimal parameters to the α -MONLP, and the theorem is proved.

Using the concept of (local) M - α -Pareto optimality, the following M - α -Pareto version of Theorem 1 and 2 can be obtained.

Theorem 3.

$x^* \in X(b)$ is a (local) M - α -Pareto optimal solution and a_i^*, b_i^* are α -level (local) optimal parameters to the $G\alpha$ -MONLP, if and only if there exist $\bar{\mu}_{f_i}$, $i=1, \dots, k$, such that (x^*, v^*, a_i^*, b_i^*) is a (local) optimal solution to the augmented minimax problem.

The proof of this theorem is much like that of Theorem 1 and 2 and thus is omitted.

It is significant to note here that from the property of the α -level set, the following relation holds for any two optimal solutions

(x^1, v^1, a^1, b^1) and (x^2, v^2, a^2, b^2) to the augmented minimax problems corresponding to α^1 and α^2 with the same reference membership values:

$$\alpha^1 \underset{=}{<} \alpha^2 \quad \text{if and only if} \quad \mu_{f_i}(x^1, a_i^1) \underset{=}{>} \mu_{f_i}(x^2, a_i^2) \quad i=1,2,\dots,k .$$

IV. TRADE-OFF RATES

Now given the (local)(M-) α -Pareto optimal solution for the degree α and the reference membership values specified by the DM by solving the corresponding augmented minimax problem, the DM must either be satisfied with the current (local)(M-) α -Pareto optimal solution, or update the reference membership values and/or the degree α . In order to help the DM express his degree of preference, trade-off information between a standing membership function and each of the other membership functions as well as between the degree α and the membership functions is very useful. Fortunately, such a trade-off information is easily obtainable since it is closely related to the strict positive Lagrange multipliers of the augmented minimax problem.

To derive the trade-off information, we first define the Lagrangian function L for the augmented minimax problem (13)-(16) as follows:

$$\begin{aligned}
 & L(x, v, a, b, \lambda^{\mu}, \lambda^a, \lambda^b, \lambda^g, \bar{\mu}_f, \alpha) \\
 = & v + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) + \sum_{i=1}^k \lambda_i^{\mu} (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i) - v) + \sum_{j=1}^m \lambda_j^g g_j(x, b_j) \\
 & + \sum_{i=1}^k \sum_{r=1}^{p_i} \lambda_{ir}^a (\alpha - \mu_{a_{ir}}) + \sum_{j=1}^m \sum_{s=1}^{q_j} \lambda_{js}^b (\alpha - \mu_{b_{js}}) \quad (11)
 \end{aligned}$$

In the following for notational convenience we denote the decision variable in the augmented minimax problem (13)-(16) by $y = (x, v, a, b)$ and let us assume that the augmented minimax problem has a unique local optimal solution y^* satisfying the following three assumptions.

Assumption 4.

y^* is a regular point of the constraints of the augmented minimax problem.

Assumption 5.

The second-order sufficiency conditions are satisfied at y^* .

Assumption 6.

There are no degenerate constraints at y^* .

Then the following existence theorem, which is based on the implicit function theorem [6] , holds.

Theorem 4.

Let $y^* = (x^*, v^*, a^*, b^*)$ be a unique local solution of the augmented minimax problem (13)-(16) satisfying the assumptions 4,5 and 6. Let $\lambda^* = (\lambda^{\mu^*}, \lambda^{a^*}, \lambda^{b^*}, \lambda^{g^*})$ denote the Lagrange multipliers corresponding to the constraints (14)-(16). Then there exist a continuously differentiable vector valued function $y(\cdot)$ and $\lambda(\cdot)$ defined on some neighborhood $N(\alpha^*)$ so that $y(\alpha^*) = y^*$, $\lambda(\alpha^*) = \lambda^*$, where $y(\alpha)$ is a unique local solution of the augmented minimax problem (13)-(16) for any $\alpha \in N(\alpha^*)$ satisfying the assumptions 4,5 and 6, and $\lambda(\alpha)$ is the Lagrange multiplier corresponding to the constraints (14)-(16).

In Theorem 4, $\inf_{x,v,a,b} \left\{ v + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) \mid \bar{\mu}_{f_i} - \mu_{f_i}(x, a_i) \leq v, (i=1, \dots, k), (a, b) \in L_{\alpha}(\tilde{a}, \tilde{b}), x \in X(b) \right\}$ can be viewed as the optimal value function of the augmented minimax problem (13)-(16) for any $\alpha \in N(\alpha^*)$. Therefore, the following theorem holds under the same assumptions in Theorem 4.

Theorem 5.

If all the assumptions in Theorem 4 are satisfied, then the following relations hold on some neighborhood $N(\alpha^*)$ of α^* .

$$\frac{\partial \left(v + \rho \sum_{i=1}^k (\bar{\mu}_{f_i} - \mu_{f_i}(x, a_i)) \right)}{\partial \alpha} = \frac{\partial L}{\partial \alpha} = \sum_{i=1}^k \sum_{r=1}^{p_i} \lambda_{ir}^a + \sum_{j=1}^m \sum_{s=1}^{q_j} \lambda_{js}^b \quad (18)$$

If all the constraints (14) of the augmented minimax problem are active, namely if $v(\alpha^*) = \bar{\mu}_{f_i} - \mu_{f_i}(x(\alpha^*), a_i(\alpha^*))$, then the following theorem holds.

Theorem 6.

Let all the assumptions in Theorem 4 are satisfied. Also assume that all the constraints (14) of the augmented minimax problem are active. Then it holds that

$$\left. \frac{\partial \mu_{f_i}(x, a_i)}{\partial \alpha} \right|_{\alpha=\alpha^*} = - \frac{1}{1+\rho k} \left(\sum_{i=1}^k \sum_{r=1}^{p_i} \lambda_{ir}^{a^*} + \sum_{j=1}^m \sum_{s=1}^{q_j} \lambda_{js}^{b^*} \right), \quad i=1, \dots, k. \quad (19)$$

Regarding a trade-off rate between $\mu_{f_1}(x)$ and $\mu_{f_i}(x)$ for each $i=2, \dots, k$, by extending the results in Haimes and Chankong [7], it can be proved that the following theorem holds [22].

Theorem 7.

Let all the assumptions in Theorem 4 are satisfied. Also assume that the constraints (14) are active. Then it holds that

$$\left. \frac{\partial \mu_{f_i}(x, a_i)}{\partial \mu_{f_1}(x, a_1)} \right|_{\alpha=\alpha^*} = - \frac{\lambda_1^{\mu^*}}{\lambda_i^{\mu^*}}, \quad i=2, \dots, k. \quad (20)$$

It should be noted here that in order to obtain the trade-off rate information from (19) and (20), all the constraints (14) of the augmented minimax problem must be active. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_{f_i}$ for inactive constraints by $\mu_{f_i}(x^*, a_i^*)$ and solve the corresponding augmented minimax problem for obtaining the Lagrange multipliers.

V. AN INTERACTIVE ALGORITHM

Following the above discussions, we can now construct the interactive algorithm in order to derive the (local) satisficing solution for the DM from among the (local)(M-) α -Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

Step 0 (Individual minimum and maximum)

Calculate the individual minimum f_i^{\min} and maximum f_i^{\max} of each objective function $f_i(x)$ under given constraints for $\alpha=1$.

Step 1* (Membership functions)

Elicit a membership function $\mu_{f_i}(x, a_i)$ from the DM for each of the objective functions.

Step 2* (Initialization)

Ask the DM to select the initial values of α ($0 < \alpha < 1$) and set the initial reference membership values $\bar{\mu}_{f_i}^{(1)} = 1, i=1, \dots, k$. Set the iteration index $r=1$.

Step 3 ((local) (M-) α -Pareto optimal solution)

Set $\bar{\mu}_{f_i} = \bar{\mu}_{f_i}^{(r)}$, $i=1, \dots, k$, solve the corresponding augmented minimax problem to obtain the (local)(M-) α -Pareto optimal solution $x^{(r)}$, $f(x^{(r)}, a^{(r)})$ and the membership function value $\mu_{f_i}(x^{(r)}, a^{(r)})$ together with the trade-off rate information between the membership functions and the degree α .

Step 4* (Termination or updating)

If the DM is satisfied with the current levels of $\mu_{f_i}(x^{(r)}, a^{(r)})$, $i=1, \dots, k$ of the (local) (M-) α -Pareto optimal solution, stop. Then the current (local) (M-) α -Pareto optimal solution $f(x^{(r)}, a^{(r)}) = (f_1(x^{(r)}, a^{(r)}), \dots, f_k(x^{(r)}, a^{(r)}))$ is the (local) satisficing solution of the DM.

Otherwise, ask the DM to update the current reference membership values $\bar{\mu}_{f_i}^{(r)}$ and/or the degree $\alpha^{(r)}$ to the new reference membership values $\bar{\mu}_{f_i}^{(r+1)}$ $i=1, \dots, k$ and/or the degree $\alpha^{(r+1)}$ by considering the current values of the membership functions together with the trade-off rates between the membership functions and the degree α . Set $r=r+1$ and return to Step 3.

Here it should be stressed for the DM that (1) any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions, and (2) the greater value of the degree α gives worse values of the membership functions for some fixed reference levels.

VI. AN INTERACTIVE COMPUTER PROGRAM AND AN ILLUSTRATIVE EXAMPLE

Interactive fuzzy satisficing processes for multiobjective nonlinear programming problems with fuzzy parameters include eliciting a membership function for each of the objective functions and reference membership values and/or degree α from the DM. Thus, interactive utilization of computer facilities is highly recommended. Based on the method described above, we have developed a new interactive computer program. Our new package includes graphical representations by which the DM can figure the shapes of his membership functions, and he can find incorrect assessments or inconsistent evaluations promptly, revise them immediately and proceed to the next stage more easily.

Our program is composed of one main program and several subroutines. The main program calls in and runs the subprograms with commands indicated by the user (DM). Here we give a brief explanation of the major commands prepared in our program.

- (1) MINMAX: Displays the calculated (local) individual minimum and maximum of each of the objective functions under the given constraints for $\alpha=1$.
- (2) MF: Elicit a membership function from the DM for each of the objective functions.
- (3) GRAPH: Depicts graphically the shape of the membership function for each of the objective functions.
- (4) GO: Derives the (local) satisficing solution for the DM from among the (local) (M-) α -Pareto optimal solution set by updating the reference membership values and/or the degree α .
- (5) STOP: Exists from the program.

- (6) SAVE: Saves all the necessary information, which has been put in, in a file.
- (7) READ: Restores the information which was saved in the file.

In our computer program, the DM can select his membership function in a subjective manner from among the following five types of functions; linear [25], exponential, hyperbolic [11], hyperbolic inverse and piecewise linear [8] functions. Then the parameter values are determined through the interaction with the DM. Here, it is assumed that $\mu_{f_i}(x) = 0$ if $f_i(x) \geq f_i^0$ and $\mu_{f_i}(x) = 1$ if $f_i(x) \leq f_i^1$, where f_i^0 is an unacceptable level for $f_i(x)$ and f_i^1 is a totally desirable level for $f_i(x)$.

(1) Linear membership function:

$$\mu_{f_i}(x) = (f_i(x) - f_i^0) / (f_i^1 - f_i^0)$$

The linear membership function can be determined by asking the DM to specify the two points f_i^0 and f_i^1 within f_i^{\min} and f_i^{\max} , where f_i^a represents the value of $f_i(x)$ such that the degree of membership function $\mu_{f_i}(x)$ is a.

(2) Exponential membership function:

$$\mu_{f_i}(x) = a_i (1 - \exp (-b_i (f_i(x) - f_i^0) / (f_i^1 - f_i^0)))$$

The exponential membership function can be determined by asking the DM to specify the three points f_i^0 , $f_i^{0.5}$ and f_i^1 within f_i^{\max} and f_i^{\min} .

(3) Hyperbolic membership function:

$$\mu_{f_i}(x) = a_i \tanh \left((f_i(x) - b_i) \alpha_i \right) + c_i$$

The hyperbolic membership function can be determined by asking the DM to specify the four points $f_i^0, f_i^{0.5}, f_i^1$ and b_i within f_i^{\max} and f_i^{\min} .

(4) Hyperbolic inverse membership function:

$$\mu_{f_i}(x) = a_i \tanh^{-1} \left((f_i(x) - b_i) \alpha_i \right) + c_i$$

The hyperbolic inverse membership function can be determined by asking the DM to specify the four points $f_i^0, f_i^{0.5}, f_i^1$ and b_i within f_i^{\max} and f_i^{\min} .

(5) Piecewise linear membership function:

$$\mu_{f_i}(x) = \sum_{j=1}^{N_i} \alpha_{ij} |f_i(x) - g_{ij}| + \beta_i f_i(x) + \gamma_i$$

Here, it is assumed that $\mu_{f_i}(x) = t_{ir} f_i(x) + s_{ir}$ for each segment $g_{ir-1} \leq f_i(x) \leq g_{ir}$. The piecewise linear membership function can be determined by asking the DM to specify the degree of membership in each of several values of objective functions within f_i^{\max} and f_i^{\min} .

We now demonstrate the interaction processes using our computer program by means of an illustrative example which is designed to test the program.

Consider the following three objective nonlinear programming problem with fuzzy parameters.

$$\text{fuzzy min } f_1(x, \tilde{a}_1) = (x_1 + 5)^2 + \tilde{a}_{11}x_2^2 + 2(x_3 - \tilde{a}_{12})^2$$

$$\text{fuzzy min } f_2(x, \tilde{a}_2) = \tilde{a}_{21}(x_1 - 45)^2 + (x_2 + 15)^2 + 3(x_3 + \tilde{a}_{22})^2$$

$$\text{fuzzy equal } f_3(x, \tilde{a}_3) = \tilde{a}_{31}(x_1 + 20)^2 + \tilde{a}_{32}(x_2 - 45)^2 + (x_3 + 15)^2$$

$$\text{subject to } g_1(x, b_1) = b_{11}x_1^2 + b_{12}x_2^2 + b_{13}x_3^2 \leq 100$$

$$0 \leq x_i \leq 10, \quad i = 1, 2, 3.$$

The membership functions for the fuzzy numbers \tilde{a}_1 , \tilde{a}_2 , \tilde{a}_3 and \tilde{b}_1 in this example are explained in Table 1. where L and E represent respectively linear and exponential membership functions.

In applying our computer program to this problem, suppose that the interaction with the hypothetical DM establishes the following membership functions and corresponding assessment values for the three objective functions.

$$f_1 : \text{ exponential, } (f_1^0, f_1^{0.5}, f_1^1) = (5400, 5000, 3300)$$

$$f_2 : \text{ hyperbolic, } (f_2^0, f_2^{0.5}, f_2^1, b_2) = (6900, 4600, 3900, 4400)$$

$$f_3 \left\{ \begin{array}{l} \text{left : exponential, } (f_3^0, f_3^{0.5}, f_3^1) = (7800, 8200, 10000) \\ \text{right : hyperbolic inverse, } (f_3^0, f_3^{0.5}, f_3^1, b_3) = (13300, 11000, 10000, 12000) \end{array} \right.$$

Table 1. Fuzzy numbers

\tilde{p}	(p , p , p , p)	TYPE	
		left	right
\tilde{a}_{11}	(3.8, 4.0, 4.0, 4.3)	L	E
\tilde{a}_{12}	(48.5, 50.0, 50.0, 52.0)	E	E
\tilde{a}_{21}	(1.85, 2.0, 2.0, 2.2)	E	L
\tilde{a}_{22}	(18.2, 20.0, 20.0, 22.5)	L	E
\tilde{a}_{31}	(2.9, 3.0, 3.0, 3.15)	E	L
\tilde{a}_{32}	(4.7, 5.0, 5.0, 5.35)	L	L
\tilde{b}_{11}	(0.9, 1.0, 1.0, 1.1)	E	E
\tilde{b}_{12}	(0.8, 1.0, 1.0, 1.2)	E	E
\tilde{b}_{13}	(0.85, 1.0, 1.0, 1.15)	E	L

In Fig. 4, the interaction processes using the time-sharing computer program under TSS of ACOS-1000 digital computer in the computer center of Kobe University in Japan are explained especially for the first iteration through the aid of some of the computer outputs. $M-\alpha$ -Pareto optimal solutions are obtained by solving the augmented minimax problem using the revised version of the generalized reduced gradient (GRG) [9] program called GRG2[10].

In this example, at the 4th iteration, the satisficing solution of the DM is derived and the values of the objectives and decision variables are shown in Fig. 5. The whole interactive processes are summarized in Table 2. CPU time required in this interaction process was 3.713 seconds and the example session takes about 10 minutes.

COMMAND:
=GO

INPUT SUFFICIENTLY SMALL POSITIVE SCALAR FOR AUGEMENTED TERM:
=0.001

INPUT THE DEGREE ALFA OF THE ALFA LEVEL SETS
FOR THE FUZZY PARAMETERS:
=0.4

-----< ITERATION 1 >-----

INITIATES AN INTERACTION WITH ALL THE INITIAL REFERENCE
MEMBERSHIP VALUES ARE 1

(KUHN-TUCKER CONDITIONS SATISFIED)

M-ALFA-PARETO OPTIMAL SOLUTION
TO THE AUGEMENTED MINIMAX PROBLEM
FOR INITIAL REFERENCE MEMBERSHIP VALUES

MEMBERSHIP		I	OBJECTIVE FUNCTION	
M(F1) =	0.7274	I	F(1) =	4663.3827
M(F2) =	0.7274	I	F(2) =	4302.5119
M(F3) =	0.7274	I	F(3) =	10323.4849
X(1) =	8.4836	X(2) =	5.8945	
X(3) =	2.1681			

TRADE-OFFS AMONG MEMBERSHIP FUNCTIONS

-DM(F2)/DM(F1) = 1.3260
-DM(F3)/DM(F1) = 2.6925

TRADE-OFFS BETWEEN ALFA AND MEMBERSHIPS

-DM(F)/DALFA = 0.1560

ARE YOU SATISFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION ?
=NO

-----< ITERATION 2 >-----

CONSIDER THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION TOGETHER WITH
THE TRADE-OFFS AMONG THE MEMBERSHIP FUNCTIONS.

THEN INPUT YOUR REFERENCE MEMBERSHIP VALUES
FOR EACH OF THE MEMBERSHIP FUNCTIONS:
=0.6 0.8 0.7

INPUT THE DEGREE ALFA OF THE ALFA LEVEL SETS
FOR THE FUZZY PARAMETERS:
=0.45

Fig. 4. Interactive fuzzy satisficing processes

ARE YOU SATISFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION ?
=YES

THE FOLLOWING VALUES ARE YOUR SATISFICING SOLUTION :

MEMBERSHIP		I	OBJECTIVE FUNCTION	
M(F1) =	0.6705	I	F(1) =	4766.6664
M(F2) =	0.7705	I	F(2) =	4245.5291
M(F3) =	0.6905	I	F(3) =	10401.5823
X(1) =	8.6080	X(2) =	5.7819	
X(3) =	1.6575			

COMMAND:
=STOP

*** [CPU-TIME = 3.713 SEC.] ***

Fig. 5. Satisficing solution of the DM

Table 2. Interactive processes

Iteration	1	2	3	4
$\hat{\mu}_1$	1	0.6	0.65	0.67
$\hat{\mu}_2$	1	0.8	0.78	0.77
$\hat{\mu}_3$	1	0.7	0.7	0.69
α	0.4	0.45	0.45	0.5
μ_{f_1}	0.69	0.57	0.61	0.62
μ_{f_2}	0.69	0.77	0.74	0.72
μ_{f_3}	0.69	0.67	0.66	0.64
f_1	4740.64	4912.43	4856.50	4846.98
f_2	4356.56	4243.79	4282.59	4313.28
f_3	10412.46	10445.48	10468.40	10529.64
x_1	8.52	8.71	8.68	8.67
x_2	7.53	5.65	5.62	5.54
x_3	2.30	1.47	1.76	1.85
$\partial\mu_{f_2}/\partial\mu_{f_1}$	-1.19	-0.79	-0.91	-0.95
$\partial\mu_{f_3}/\partial\mu_{f_1}$	-2.10	-1.42	-1.55	-1.48
$\partial\mu_f/\partial\alpha$	-0.25	-0.26	-0.25	-0.24

VII. CONCLUSION

In this paper, we have proposed an interactive fuzzy satisficing method in order to deal with the imprecise nature of the DM's judgement in multiobjective nonlinear programming problems with fuzzy parameters characterized by fuzzy numbers. Through the use of the concept of the α -level sets of the fuzzy numbers, a new solution concept called the α -Pareto optimality has been introduced. In our interactive scheme, after determining the membership functions, the (local) satisficing solution of the DM can be derived by updating the reference membership values and/or the degree based on the current values of the membership functions and α together with the trade-off rates between the membership functions and the degree α . Furthermore, (local) (M-) α -Pareto optimality of the generated solution in each iteration is guaranteed. Based on the proposed method, the time-sharing computer program has been written to facilitate the interaction processes. An illustrative numerical example demonstrated the feasibility and efficiency of both the proposed method and its interactive computer program by simulating the responses of the hypothetical DM. However, further applications must be carried out in cooperation with a person actually involved in decision-making. From such experiences the proposed method and its computer program must be revised.

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