NOT FOR QUOTATION
WITHOUT PERMISSION
OF THE AUTHOR

DESCRIPTION AND INVESTIGATION OF INVESTMENT PROCESSES IN MODELS OF ECONOMIC DYNAMICS

S.V. Chukanov
E.Yu. Khodjamirian

February 1986
CP-86-4

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

Many of today's most significant socioeconomic problems, such as slower economic growth, the decline of some established industries, and shifts in patterns of foreign trade, are inter- or transnational in nature in a variety of ways. Through analyses we attempt to identify the underlying processes of economic structural change and formulate useful hypotheses concerning future development, as some scholars argue that foreseen changes can not be precipitous. The understanding of these processes and future prospects provided the focus for the IIASA project on Comparative Analysis of Economic Structure and Growth.

This paper was mainly written during the stay of E.Yu. Khodjamirian at IIASA in the YSSP 1985. The authors present a model of the investment process and the results of its simulation under different assumptions on parameters, which characterize real problems of resource allocation over time and across industries in the construction sector.

A. Smyshlyaev

DESCRIPTION AND INVESTIGATION OF INVESTMENT PROCESSES IN MODELS OF ECONOMIC DYNAMICS

S.V. Chukanov and E.Yu. Khodjamirian

Committee for Systems Analysis, Presidum of the Academy of Sciences of the USSR, Prospect 60 Let Octyabria, 9 117312 Moscow, USSR

INTRODUCTION

One of the main distinctive features of the dynamic models of economics is the description of investment processes, i.e. of the mode of delay calculation between capital investments and commissioning of productive funds into the respective branches. The characteristic feature of some models is the task of determination of the dependence (usually linear) between capital investments and commissioning of funds. Many known modes of description of investment processes, that take an obvious delay into account, are in this case introduced in the so-called "normative" approach [1]. It is supposed that the construction of new funds in model branches is fulfilled by a given a priori fixed project.

The present paper deals with the description of the investment process as a controlled process, which means that there is a possibility to suspend the process of construction in the general sense of the word. As opposed to the traditional approach, the description of the investment processes as controlled processes makes it possible to formalize and investigate on a qualitative level the questions related with the problem of non-completed construction, efficiency of capital investment distribution, and freezing of construction in the branches.

DESCRIPTION OF INVESTMENT PROCESSES

The fund dynamics will be described in discrete time periods, and a year will be conventionally taken for the time unit. Let's suppose that all commissioning projects are characterized by the same (or sufficiently close) construction time τ, and the parameters γ_s, ρ_s $(s = \overline{1,\tau})$ that set the laws for investment entry into the construction and accretion of capacity volumes, respectively (see Figure 1).

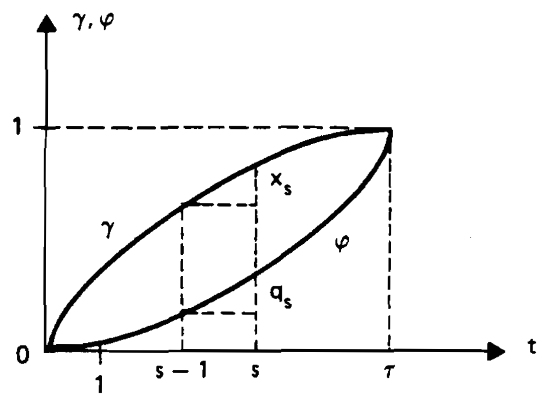


Figure 1. The laws for investments and commissioning of funds.

Then the capital investments K(t) during the year--period t--and capacity commissioning for this year $\Delta F(t)$ may be written in the following way (1):

$$K(t) = \sum_{s=1}^{\tau} x_s \, \mho(t-s+1) \quad ,$$

$$\Delta F(t) = \sum_{s=1}^{\tau} q_s \, \mho(t-s+1) \quad , \quad \mho(-t') = \text{fixe} \quad , \qquad (1)$$

$$t' = 1,2,\ldots,(\tau-1) \quad ,$$

where $\mathcal{V}(t)$ is the total value of the projects, the construction of which started at the beginning of the year t and is assumed to continue to the end of the year $t+\tau-1$; $x_s = \gamma_s - \gamma_{s-1}$ is the share of the complete cost of the projects requiring capital investments in a time period s-1 from the beginning of the construction; $q = \rho_s - \rho_{s-1}$ is the share of the project cost commissioned to the end of the year s-1 from the beginning of the construction. The values x and q, in their economic sense, should satisfy the following limitations:

$$\sum_{s=1}^{\tau} x_s = \sum_{s=1}^{\tau} q_s = 1 \quad ; \quad x_s, q_s \geq 0 \quad , \quad s = \overline{1,\tau}$$

$$\sum_{k=1}^{s} x_k \geq \sum_{k=1}^{s} q_s \quad ; \quad x_1 > 0 \quad , \quad q_\tau > 0 \quad . \tag{2}$$

Depending upon the forms of the graphs, and from the relations of equation (1) one can obtain other known models for the description of investment processes, where an obvious delay is taken into account (see Appendix 1).

To describe the controlled process of investment we shall have to link two periods of time with the construction process: the first one: calendar (current) time of construction t, and the second one: proper (active) time of construction s, i.e. the time span during which the project has actually been constructed. It is evident that $s \leq \tau$. The value of s as differing from the calendar time means the possibility of freezing (suspending) the construction and allows us to introduce additional control into the model.

Let's denote by $\psi_s(t)$ the volume of s-year capacities in the branch for the beginning of the year t. Let's also introduce the value $U_s(t)$ as the volume of capacities that have been in the process of active construction from the beginning of the year t (s-1 year) and are still in the construction process in year t.

Let's first consider the case when the process of construction proceeds strictly according to the project without freezing, i.e. the active time of construction coincides with the current one. In this case the construction process dynamics should be written in the following way:

$$\psi_1(t+1) = U_1(t)$$

$$\psi_s(t+1) = \psi_{s-1}(t) = U_s(t) \quad , \quad s = \overline{2,\tau-1} \tag{3}$$

$$\psi_\tau(t+1) = \psi_\tau(t) = U_\tau(t) \quad , \quad t = \overline{0,T} \quad ,$$

the value $U_1(t)$--the volume of the foundations laid for construction--being the only control here. It is clear that in this case $U_1(t)$ coincides with the value of $\mathcal{U}(t)$ introduced previously in the relation (1).

Now let us assume a possibility of freezing of the projects that are at various stages of completeness. In this case the volume of the construction frozen in the branch during the year t will show the difference $\psi_s(t) - U_{s+1}(t)$, $s = \overline{1,\tau-1}$. Then the following ratio of the dynamics of capacities, taking account of the construction process control, may be written:

$$\psi_s(t+1) = \psi_s(t) - U_{s+1}(t) + U_s(t) \quad , \quad s = \overline{1,\tau-1}$$

$$\psi_\tau(t+1) = \psi_\tau(t) + U_\tau(t) \quad . \tag{4}$$

The fulfillment of the following restrictions is also evident:

$$0 \leq U_s(t) \leq \psi_{s-1}(t) \quad , \quad s = \overline{2,\tau} \quad , \tag{5}$$

where the value of $U_1(t) \geq 0$ is the volume of the set-up constructions being non-restricted in principle by the above mentioned correlations (4) and (5).

BRANCH DEVELOPMENT PROBLEM WITH THE CONTROLLED PROCESS OF CONSTRUCTION

In order to estimate the efficiency of capital investment distribution as well as that of construction freezing within the framework of the proposed description, let us formulate the problem of branch development with the controlled process of construction. To facilitate the task of further discussion we will pass over to the vector designations:

$$\psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_\tau(t))^T ,$$

$$U(t) = (U_1(t), U_2(t), \ldots, U_\tau(t))^T$$

$$\psi^0(t) = (\psi_1^0, \psi_2^0, \ldots, \psi_\tau^0) , \quad K = (x_1, x_2, \ldots x_\tau) ,$$

$$\boldsymbol{\varphi} = (\rho_1, \rho_2, \ldots \rho_\tau)$$

$$E = \|\delta\|_{i,j=1}^{\tau} , \quad H = \|\delta_{ij}\|_{i,j-1=1}^{\tau}$$

where $$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Let the investments $v^\rho(t)$ during the whole plan period T be known. Then the following problem for optimal planning of the branch may be written:

$$\psi(t+1) = \psi(t) + (E-H)U(t) , \quad \psi(0) = \psi^0 , \quad t = \overline{0,T} \tag{6}$$

$$HU(t) \leq \psi(t) , \ U(t) \geq 0 \tag{7}$$

$$KU(t) \leq v^\rho(t) \tag{8}$$

$$\sum_{t=0}^{\tau} q(t)\ U(t) \rightarrow \max . \tag{9}$$

Here the correlations of capacity dynamics (6) and inequality (7) coincide with (4) and (5); inequality (8) shows the limitedness of the supplies of investments. The function of the problem has a rather general form since the values of $\psi(t)$ and $U(t)$ are simply bound by the difference equation (6). If we suppose that $q(t) = c(t)\Psi(E-H)$, then according to (1) the function acquires the sense of the total cost of capacity increment $\Delta F(t) = \Psi(E-H)U(t)$ calculated in a price variable with time $c(t)$.

The problem (6)-(9) is the problem of linear dynamic programming in discrete time and its optimal conditions are obtained directly from the theory of linear programming [2]. The main results of this problem investigation are given in Appendix 1.

The formulation of the optimum conditions--in terms of local time problems bound by dynamic correlations, e.g. in terms of the maximum principle--decomposes the problem in accordance with the specificity of the dynamic problem and proves convenient for qualitative analysis.

THE LOCAL (ONE-STEP) PROBLEM

The local problem of the maximum principle (see Appendix 1) is the problem of the investment distribution for capacity construction of various types of s, and if the $v^{\rho}(t)$ is of scalar quantity, it assumes an analytic solution. It should also be noted that in practice the plan solutions are frequently obtained on the basis of a one-step problem (6)-(9), which is entirely equivalent to the local problem.

The solution of the local (one-step) problem is given in Appendix 2. During the solving process we have seen that the efficiency of the investment distribution in construction is defined by the behavior of function $r_s = q_{s+1}/x_{s+1}$.

Provided that $\bar{s} = \arg\max_s r_s$, the distribution of the investments in the construction of starting projects (capacities) is optimal. In case of non-fulfillment of this condition, noncompleteness of starting projects proves to be advantageous, this increases the actual duration of the construction as well as the volumes of noncompleted construction in the current year.

ASYMPTOTIC PROPERTIES OF OPTIMAL TRAJECTORIES

The experience of solving the problems of economic dynamics shows that the structure of their optimal solutions is rather complicated. In this connection it is of great importance, in order to understand the peculiarities of the behavior of optimal trajectories and the construction of effective numerical methods of solution, to investigate the asymptotics of the solution at great intervals of planning. It is well known that for a wide class of dynamic models of the economy optimal trajectories are most of the time close to some outlined stationary trajectory that is called the turn-pike.

The investigation of stationary trajectories of the branch development problem is given in Appendix 3. We assumed that the economy develops at the α rate, i.e. $v^{\rho}(t) = \alpha^{t}v$, $q(t) = \alpha^{-t}q$. This suggests a hypothesis that the economy on the whole, as a unit consisting of a great number of branches, is of a stable nature and the processes that occur in a separate branch do not essentially influence this development.

During the process of solving the stationary problem we have seen that the optimal stationary trajectory is defined by the function $R_s = \sum_{k=1}^{\bar{s}} \alpha^{-k}q_k / \sum_{k=1}$. The solution of the problem corresponds to the uniform construction of capacities from the zero stage to the $\bar{s}$ stage. From the contextual point of view the function R_s provides efficiency conditions for uniform construction from the zero stage to the s stage: R_s equals the relation $\sum_{k=1}^{s} \alpha^{-k}q_k$ (the cost of capacity increment) to $\sum_{k=1}^{s} \alpha^{-k}x_k$ (the cost of expenditure calculated with due regard to discounted prices).

The simplest form of the turn-pike theory for the problem has been proved in [3], i.e. the turn-pike theorem in a weak form which ascertains the proximity of the optimal trajectory to the Neiman boundary. For the given problem the fulfillment of inequalities in (6)-(9) as equalities corresponds to the Neiman boundary. The proof of the strict form of the turn-pike theorem, i.e. of the proximity to the isolated stationary trajectory for linear problems, is based on the analysis of the behavior of trajectories within the Neiman boundary. The availability of the turn-pike qualities has been shown to their full extent by the results of numerical experiments that are discussed here.

THE RESULTS OF NUMERICAL EXPERIMENTS

It was also the aim of the numerical experiments carried out to detect the qualitative features of the problem of branch development with the controlled process of construction. The calculations were made on the basis of conventional information according to the problem (6)-(9) at the values T = 50 and 95 years. To make the comparison more vivid, the examples are brought out under one and the same conventional project of construction characterized by (see Figure 2)*:

$$\gamma_s = \sum_{k=1}^{s} x_k = (1;2;3;4;5) \quad ;$$

$$\rho_s = \sum_{k=1}^{s} q_s = (0,5;1,7;1,7;1,7;5) \quad ;$$

$$\tau = 5 \quad ; \quad s = \overline{1,\tau} \quad .$$

All the calculations have been conducted at $\psi^0 = 0$, $v = 3$ and different α and T (in the problem with the variable turn-pike α depends upon t). The values $\overline{U}$, $\overline{\Psi}$ correspond to the best stationary development defined for the stationary problem (Appendix 3, (A 19).

*The values of investment parameters γ and ρ are not standardized for a greater representation of the graphs given below.

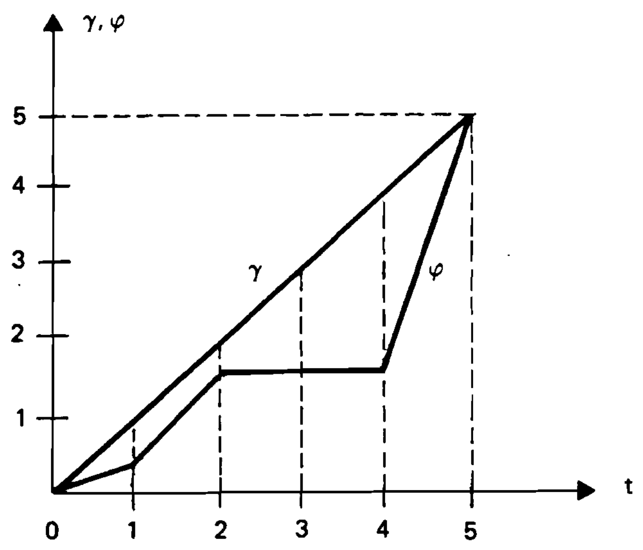


Figure 2. Conventional project of construction.

Example 1 (Figure 3):

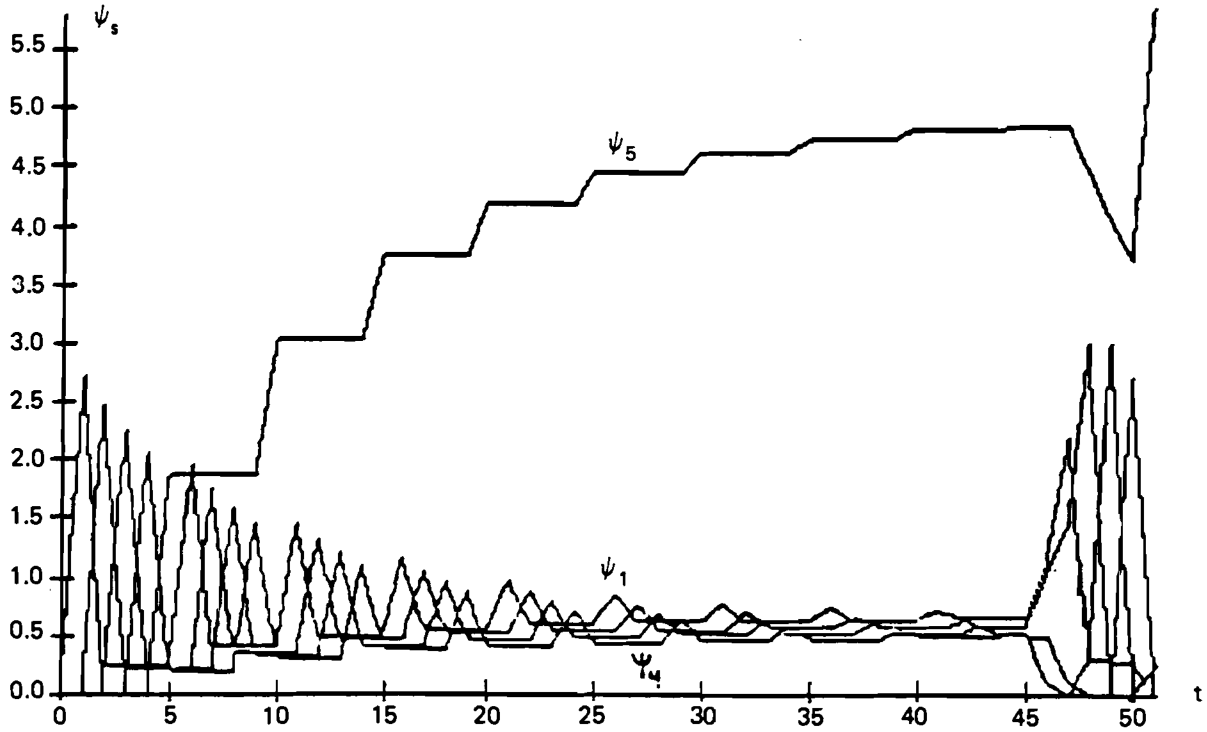
$T = 50;\ \alpha = 1,1$

$R_s = (0,5;\ 0,832;\ 0,582;\ 0,485;\ 0,919)$

$\bar{U} = (0,719;\ 0,653;\ 0,594;\ 0,54;\ 0,491)$

$\bar{\Psi} = (0,653;\ 0,594;\ 0,54;\ 0,491;\ -)$

In the given example $\bar{s} = \arg\max_s R_s = 5$. It is seem from Figure 3 that in this case a uniform construction of all types of capacities is observed; generally speaking, the value of ψ_s is non-defined since there is an accumulation of completed construction.



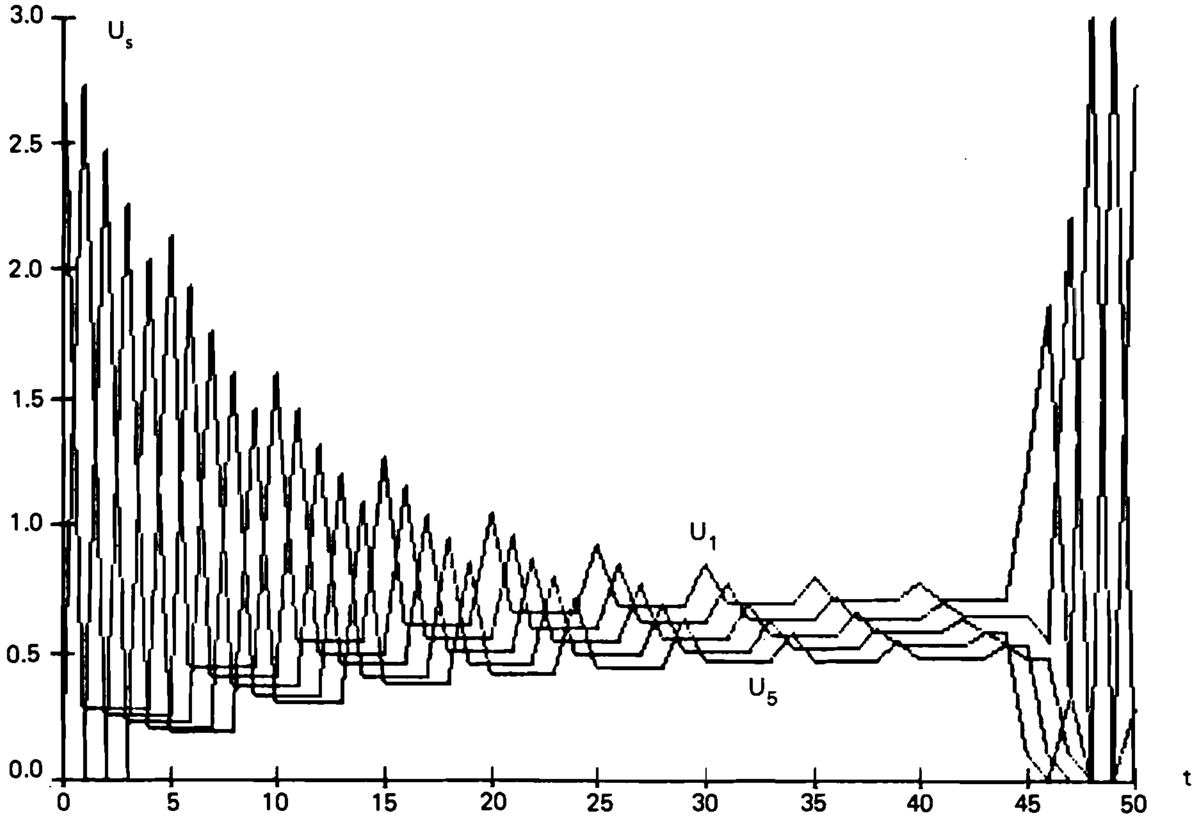


Figure 3. Optimal trajectories ψ and U to Example 1.

The turn-pike area of optimal trajectories is also seen in the figure.

Example 2 (Figure 4):

$T = 50; \alpha = 1,4$

$R_s = (0,5;\ 0,791;\ 0,61;\ 0,525;\ 0,776)$

$\bar{U} = (1,75;\ 1,25;\ 0;\ 0;\ 0)$

$\bar{\psi} = (1,25;\ 0;\ 0;\ 0;\ -)$.

Here $\bar{s} = 2$. In this case only the construction of one- and two-year projects takes place.

Example 3 (Figure 5):

$T = 95; \alpha(t) = 1 + \Delta t, \Delta = 0,005$

$\bar{s} = 5$ at $t \leq 67$

$\bar{s} = 2$ at $t > 67$

$\bar{U}(t) = 0,6(1+2\Delta t;\ 1+\Delta t;\ 1;\ 1-\Delta t;\ 1-2\Delta t)$ at $t \leq 67$

$\bar{U}(t) = 1,5(1+\frac{\Delta}{2}t;\ 1-\frac{\Delta}{2}t;\ 0;\ 0;\ 0)$ at $t > 67$.

The example refers to the case with the variable turn-pike trajectory, and $\bar{U}(t)$ correpsonds to the approximation of the calculated dependence of $\bar{U}$ upon t of the solution of problem (19). The function of R_s depends upon t (since α depends upon t) and its maximum changes with time. One can see in Figure 5 that in this case the construction of all types of projects takes place first and one- and two-year projects still remain; clearly seen is also the area of exit to the turn-pike as well as the transition area corresponding to the maximum shift $R_{\bar{s}}$, and again the turn-pike area for one- and two-year projects.

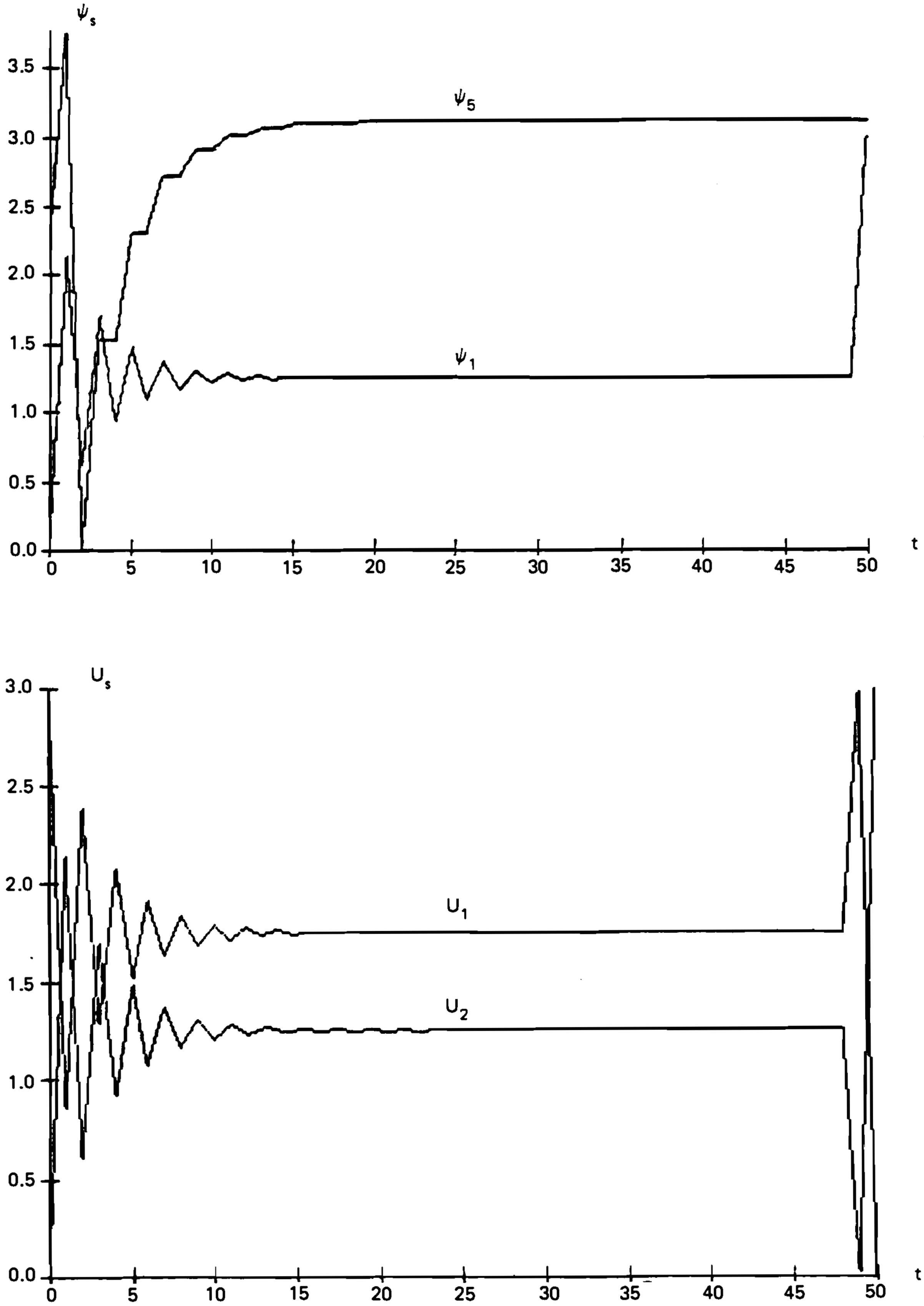


Figure 4. Optimal trajectories ψ and U to Example 2.

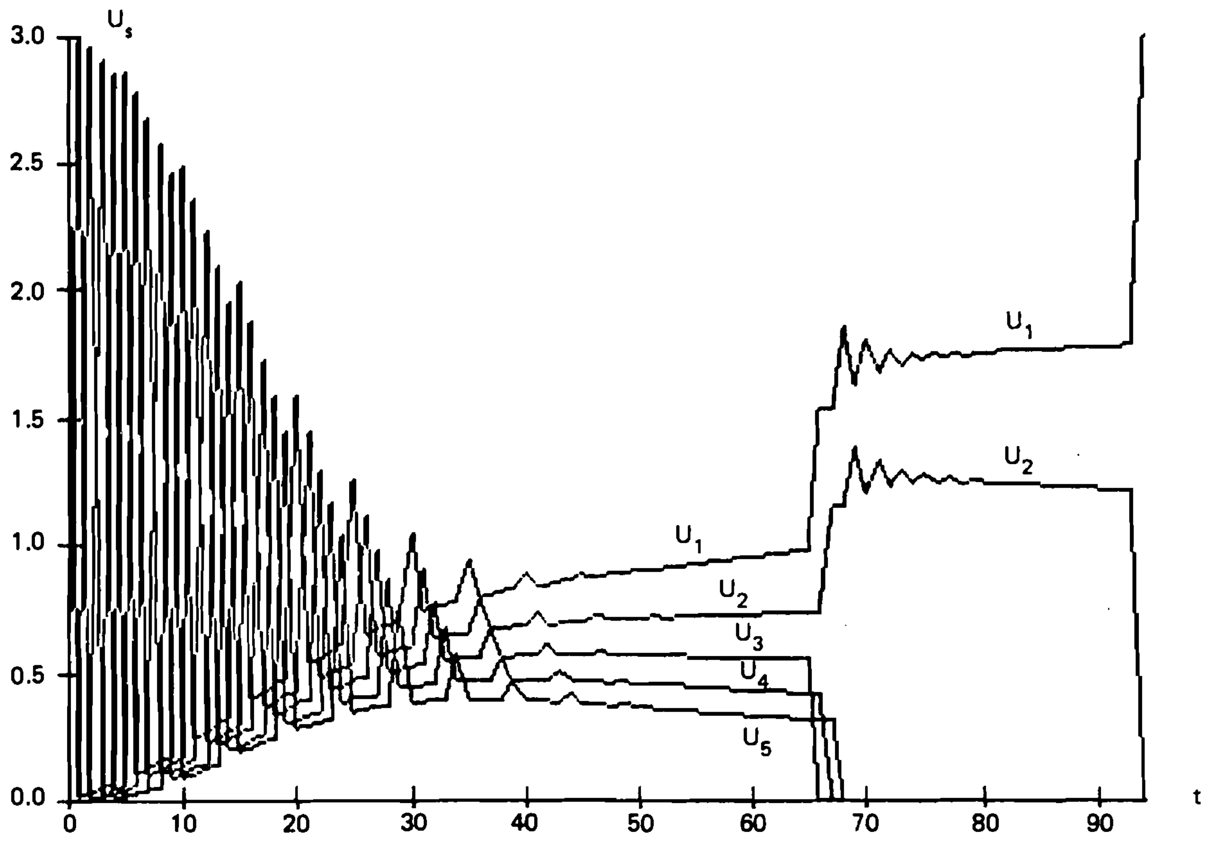


Figure 5. Optimal trajectories to Example 3.

Example 4 (Figure 6):

$T = 95;\ \alpha(t) = 1 + \Delta(100-t),\ \Delta = 0{,}005$

$\bar{s} = 2$ at $t \leq 33$

$\bar{s} = 5$ at $t > 33$

$\bar{U}(t) = 1{,}5(1+\frac{\Delta}{2}(100-t);\ 1-\frac{\Delta}{2}(100-t);\ 0;\ 0;\ 0)$ at $t \leq 33$

$\bar{U}(t) = 0{,}6(1+2\Delta(100-t);\ 1+\Delta(100-t);\ 1;\ 1-\Delta(100-t);$

$1-2\Delta(100-t))$ at $t > 33$.

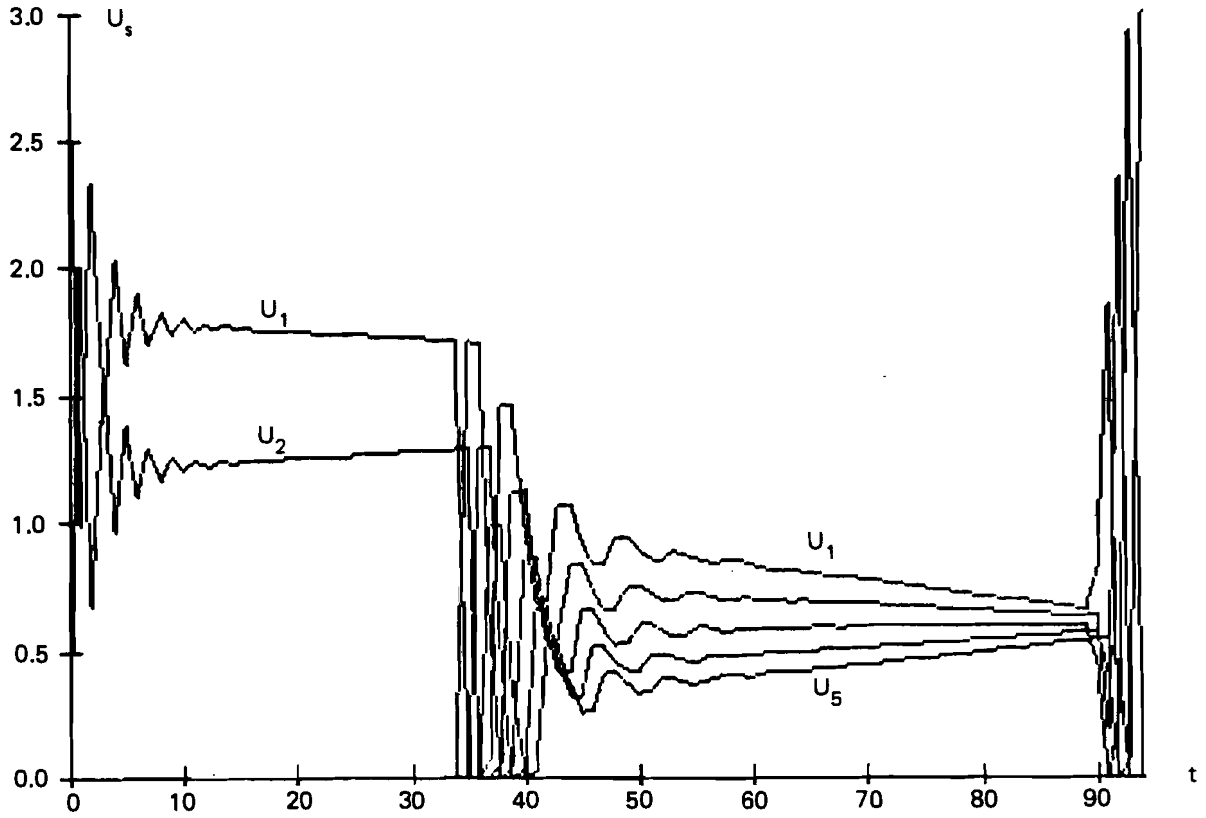


Figure 6. Optimal trajectories to Example 4.

A reverse picture is observed in the given example: here α decreases and, correspondingly, one- and two-year projects are constructed first and then the construction of all s types of projects is undertaken.

The given examples have shown that a long-term optimal dynamics of the problem has turn-pike properties: the optimal trajectory is most of the time close to the optimal stationary trajectory, which, as has been established in Appendix 3, is defined by the R_s function behavior.

CONCLUSIONS

Let's note again the most interesting and, from our point of view, profound moments of the investigation that are related with the controlled process of construction of new capacities.

If the number of steps in the dynamic problem is equal to one (a one-step problem), then the optimal distribution of capital investments into projects that are at various stages of construction is simply defined by some function of capital investment efficiency. Different types of requirements of the projects under construction in terms of investments are being satisfied with efficiency decrease.

A long-term optimal dynamics of the problem has asymptotic properties. The optimal trajectory is most of the time close to the best stationary trajectory. The latter is also defined by some efficiency function of capital investments, differing however from all the above-mentioned ones by the fact that the projects, where the investments are more effective in a one-step problem, turn out to be completely frozen in long-term optimal plans of the branch development. The investments on the indicated stationary trajectory are distributed in such a way as to provide to some extent uniform construction of projects from the initial stage up to the state of maximum efficiency.

APPENDIX 1. ANALYSIS OF THE BRANCH DEVELOPMENT PROBLEM

Let's briefly formulate the main results. Applying the standard Lagrange Function in direct and dual form

$$L = \sum_{t=0}^{T} q(t)U(t) + p(-1)(\psi^0 - \psi(0)) +$$

$$+ \sum_{t=0}^{T} [p(t)(\psi(t) + (E-H)\ U(t) - \psi(t+1)) +$$

$$+ \gamma(t)(\psi(t) - HU(t)) + w(t)(v^{\rho}(t) - KU(t))] =$$

$$= p(-1)\psi^0 + \sum_{t=0}^{T} w(t)v^{\rho}(t) - p(T)\psi(T+1) +$$

$$+ \sum_{t=0}^{T} [(p(t) - p(t-1) + \gamma(t))\psi(t) +$$

$$+ (q(t) + p(t)(E-H) - \gamma(t)H - w(t)K)U(t)] \quad ,$$

where $p(t)$, $w(t)$, $\gamma(t)$ are vector lines of the adequate dimensions, we obtain the dual problem to (6)-(9):

$$p(t-1) = p(t) + \gamma(t) \quad , \quad p(T) = 0 \quad , \quad t = \overline{0,T} \tag{A1}$$

$$\gamma(t)H + w(t)K \geq q(t) + p(t)(E-H) \quad ; \quad \gamma(t), w(t) \geq 0 \tag{A2}$$

$$p(-1)\psi^0 + \sum_{t=0}^{T} w(t)v^{\rho}(t) \rightarrow \min \quad , \tag{A3}$$

comprising the dynamic equation (A1) of dual phase variables $p(t)$, the limitations (A2) on dual controls $\gamma(t)$, $w(t)$, the function of the dual problem (A3) and the correlations of complementary non-rigidity:

$$\gamma(t)[\psi(t) - HU(t)] = w(t)[v^{\rho}(t) - KU(t)] =$$

$$= [q(t) + p(t)(E-H) - \gamma(t)H - \qquad \text{(A4)}$$

$$- w(t)K]U(t) = 0 \quad .$$

The existence of $\hat{p}(t)$, $\hat{\gamma}(t)$, $\hat{w}(t)$ satisfying (A1)-(A3), together with $\hat{\psi}(t)$, $\hat{U}(t)$ the correlations (A4), is the necessary and sufficient optimum condition of the functions $\hat{\psi}(t)$, $\hat{U}(t)$ in (6)-(9) [2].

It is obvious that the correlations (A4) are valid if and only if $\hat{U}(t)$ and $\hat{\gamma}(t)$, $\hat{w}(t)$ are the solutions of the pair of local dual tasks:

$$\begin{cases} q(t) + \hat{p}(t)(E-H)U \to \max \\ HU(t) \le \hat{\psi}(t),\ U \ge 0,\ KU \le v^{\rho}(t) \end{cases}$$

$$\qquad \text{(A5)}$$

$$\begin{cases} \gamma\hat{\psi}(t) + wv^{\rho}(t) \to \min \\ \gamma H + wK \ge \hat{p}(t)(E-H) + q(t);\ w,\gamma \ge 0 \end{cases} ,$$

or of a saddle point of the function

$$\bar{H}(t,\hat{p}(t),\hat{\psi}(t),U,\gamma,w = (q(t) + \hat{p}(t)(E-H))U +$$

$$+ \gamma(\hat{\psi}(t) - HU) + w(v^{\rho}(t) - KU)$$

on the set $U \ge 0;\ w,\gamma \ge 0$.

APPENDIX 2. SOLUTION OF THE LOCAL (ONE-STEP) PROBLEM

Thus, let's consider the local problem (A5) assuming t, $\hat{p}(t)$ and $\hat{\psi}(t)$ to be fixed. We shall first solve its duel problem that may be written as follows:

$$\min \{\gamma\hat{\psi}(t) + wv^{\rho}(t) : w,\gamma \geq 0;\ \gamma H + wK \geq g\} =$$

$$= \min_{w\geq 0} \{mv^{\rho}(t) + \min_{\substack{\gamma\geq 0 \\ \gamma H + wK \geq g}} \gamma\hat{\psi}(t)\} ,$$

where $g = \hat{p}(t)(E-H) + q(t)$. Taking into account the non-negativity of γ it is not difficult to calculate the interior minimum on the right-hand side of the equation:

$$\min_{\substack{\gamma\geq 0 \\ \gamma H + wK \geq g}} \gamma\hat{\psi}(t) = \sum_{s=1}^{\tau-1} \min_{\substack{\gamma_s \geq 0 \\ \gamma_s \geq g_{s+1} - wx_{x+1}}} \gamma_s\hat{\psi}_s(t) =$$

$$= \sum_{s=1}^{\tau-1} (g_{s+1} - wx_{s+1})_+ \hat{\psi}_s(t) ,$$

where

$$(a)_+ = \begin{cases} a & \text{at } a > 0 \\ 0 & \text{at } a \leq 0. \end{cases}$$

The value $\tilde{w}$ provides the minimum on the set $w \geq 0$ of the convex function

$$\mathfrak{F}(w) = wv^{\rho}(t) + \sum_{s=1}^{\tau-1} (g_{s+1} - wx_{s+1})_+ \hat{\psi}_s(t) ,$$

if and only if the derivatives $\mathfrak{F}(w)$ at the point $\tilde{w}$ in the directions assumed by the limitations $w \geq 0$ are non-negative. By calculating these derivatives one finds for $\tilde{w} = 0$

$$\mathcal{F}(\tilde{w},h) = h(v^{\rho}(t) - \sum_{s:g_{s+1}>0} x_{s+1}\,\hat{\psi}_s(t)) \geq 0 \qquad \forall h \geq 0 \qquad (A6)$$

and for $\tilde{w} > 0$

$$\mathcal{F}(\tilde{w},h) = h(v^{\rho}(t) - \sum_{s:(g_{s+1}-\tilde{w}x_{s+1})>0} x_{s+1}\,\hat{\psi}_s(t) \geq 0 \quad \forall h \geq 0$$

$$\mathcal{F}(\tilde{w},h) = h(v^{\rho}(t) - \sum_{s:(g_{s+1}-\tilde{w}x_{s+1})\geq 0} x_{s+1}\,\hat{\psi}_s(t) \geq 0 \quad \forall h \leq 0 \quad . \quad (A7)$$

It is seen from (A6) that the resource has a zero estimation $\tilde{w}$ if and only if it is sufficient for the continuation of all the constructions with positive values $g_{s+1} > 0$. Otherwise, $\tilde{w}$ is chosen from the correlations (A7) which, for convenience, may be rewritten in the following form:

$$\sum_{s:\tilde{w}<r_s} x_{s+1}\,\hat{\psi}_s(t) \leq v^{\rho}(t) \leq \sum_{s:\tilde{w}\leq r_s} x_{s+1}\,\hat{\psi}_s(t) \quad , \qquad (A8)$$

where

$$r_s = \frac{g_{s+1}}{x_{s+1}} \quad .$$

It is these correlations that define the structure of the solution: the needs of constructions with high values r_s are fully satisfied; the requirements of constructions with small r_s are not satisfied and only partially satisfied at $r_s = \tilde{w}$ (see Figure A1, where the shaded area is equal to v^{ρ}). In this case $\tilde{w} > 0$ is chosen such that the resource should be spent completely. Since in a one-step problem $g_{s+1} = q_{s+1} = c(\rho_{s+1} - \rho_s)$ is the capacity increment cost during the construction of s+1 unit volume, and x_{s+1} is the expenditure on this construction, the function r_s may be interpreted as efficiency of the fund-formation resource distribution in the

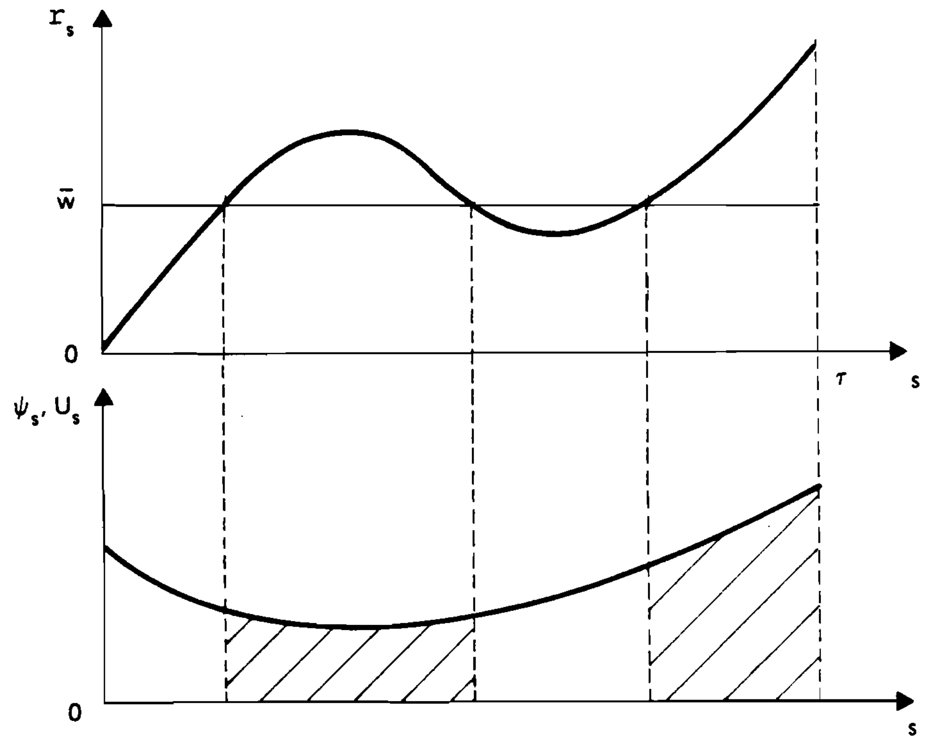


Figure A1. Distribution of the fund-formation resource in a one-step problem.

continuation of the construction from state s into state s+1. The function r_s is also in full agreement with the local problem of the maximum principle, but in current dual prices $\hat{g}(t) = \hat{q}(t) + \hat{p}(t)(E-H)$ for the construction volumes of various types s.

APPENDIX 3. ANALYSIS OF STATIONARY TRAJECTORIES

For further consideration we shall introduce new (discounted) variables:

$$\psi_N(t) = \alpha^{-t}\psi \quad , \quad U_N(t) = \alpha^{-t}U$$

$$p_N(t) = \alpha^{t}p \quad , \quad \gamma_N(t) = \alpha^{t}\gamma \quad , \quad w_N(t) = \alpha^{t}w \quad ,$$

and replace balance equalities by inequalities, which actually provides the possibility of destroying the constructed capacities. Formally the substitution of equalities for inequalities may enlarge the set of solutions, but because of the specific character of the problem, the new statement appears to be completely equivalent to the old one. Further, we will use only the new variables, omitting the "N" index. Besides, we shall assume, as before, the one-dimension of $v^{\rho}(t)$ that would enable us to carry out a complete analytical research of the problem. Then, with the new variables, the problem (6)-(9) and its dual problem will take the following form:

Direct problem:

$$\alpha\psi(t+1) \leq \psi(t) + (E-H)U(t) \quad , \quad \psi(0) \leq \psi^0$$

$$HU(t) \leq \psi(t) \quad , \quad U(t), \psi(t) \geq 0 \qquad \text{(A9)}$$

$$KU(t) \leq v$$

$$\sum_{t=0}^{T} qU(t) \rightarrow \max \quad .$$

Dual problem:

$$\alpha p(t-1) \geq p(t) + \gamma(t) \quad , \quad p(T) \geq 0$$

$$\gamma(t)H + w(t)K \geq q(t) + p(t)(E-H)$$

$$p(t),\ \gamma(t),\ w(t) \geq 0$$

$$\sum_{t=0}^{T} w(t)v + p(-1)\psi^0 \rightarrow \min \quad .$$

During the analysis of the asymptotics of the optimal solutions the stationary trajectories play an important role. By saying stationary here we mean:

$$U(t) = U \quad , \quad \psi(t+1) \geq \psi = \psi(t) \quad .$$

While passing on to t+1 one may formally discard $\psi(t+1) - \psi(t)$ and repeat the transition from t+1 to t+2. The inequality $\psi(t+1) \geq \psi(t)$ corresponds to the accumulation of the completed construction.

In terms of function (9) the best stationary trajectory in the plan interval [0,T] will be the trajectory with the maximum value of qU. Let's formulate the problem of the best stationary development in the branch:

$$\alpha\psi \leq \psi + (E-H)U \quad , \quad HU \leq \psi \quad , \quad KU \leq v \quad , \quad U,\psi \geq 0 \qquad \text{(A10)}$$

$$qU \rightarrow \max \quad ,$$

its dual

$$\alpha p - p - \gamma = \xi_\psi \geq 0$$

$$\gamma H + wK - q - p(E-H) = \xi_U \geq 0 \quad , \quad \gamma,\ p,\ w \geq 0 \qquad \text{(A11)}$$

$$wv \rightarrow \min \quad ,$$

and the correlations of complementary non-rigidity:

$$p[\psi + (E-H)U - \alpha\psi] = \gamma(\psi-HU) = w(v-KU) = \xi_\psi \psi = \\ = \xi_U U = 0 \quad . \tag{A12}$$

Let's solve the problem (A11). Consider the auxiliary problem

$$wK - q - P(E-\alpha H) = \xi \geq 0 \quad ; \quad p,w \geq 0 \quad , \\ wv \to \min \quad . \tag{A13}$$

Any assumed solution $(p,\gamma,w,\xi_\psi,\xi_U)$ of the problem (A11) generates an assumed solution $(p,w,\xi = \xi_U + H\xi_\psi)$ of the problem (A13). And vice versa, generally speaking, the solution of the problem (A13) is restored by any assumed solution (p,w,ξ) of the problem (A11), e.g. $(p,w,\gamma = (\alpha-1)p,\ \xi_U = \xi,\ \xi_\psi = 0)$. Therefore the values of the functions in these problems coincide, and by the optimal solution of the problem (A13) at least one optimal solution of the problem (A11) may be restored in a trivial way.

We shall rewrite (A13) in the form:

$$\bar{j} = \min \{wv : p_s, \xi_s, w \geq 0;\ p_s = \alpha p_{s-1} + wx_s - q - \xi_s, \\ s = \overline{1,\tau};\ p_0 \overset{df}{=} 0\} \quad .$$

By solving the difference equation for p_s, we find

$$j = \min \{wv : w, \xi_s \geq 0;\ p_s = \alpha^s [\sum_{k=1}^{s} \alpha^{-k}(wx_k - q_k - \xi_k)] \geq 0; \\ s = \overline{1,\tau}\} = \min \{wv : w, \xi_s \geq 0;\ w \sum_{k=1}^{s} \alpha^{-k} x_k \geq \\ \geq \sum_{k=1}^{s} \alpha^{-k}(q_k + \xi_k),\ s = \overline{1,\tau}\} \quad .$$

From the last expression we obtain the following properties of the optimal solution $(\bar{w},\bar{p},\bar{\xi})$:

$$\bar{w} + R_{\bar{s}} = \max_{s=\overline{1,\tau}} R_s \quad , \tag{A14}$$

where

$$R_s = \frac{\sum_{k=1}^{s} \alpha^{-k} q_k}{\sum_{k=1}^{s} \alpha^{-k} q_k} \quad , \quad \xi_s = 0 \text{ at } s \le \bar{s} \quad , \quad \bar{p}_{\bar{s}} = 0 \quad .$$

Further, suppose the uni-extremality of the function R_s. Then

$$\bar{p}_s = \alpha^s \left(\sum_{k=1}^{s} \alpha^{-k} x_k \right) (R_{\bar{s}} - R_s) > 0 \quad , \quad s < \bar{s} \quad , \tag{A15}$$

and among the optimal sets $(\bar{w},\bar{p},\bar{\xi})$ there are such as:

$$\xi_s > 0$$

and

$$\bar{p}_s = \alpha^s \left(\sum_{k=1}^{s} \alpha^{-k} x_k \right) \left[R_{\bar{s}} - R_s - \frac{\sum_{k=1}^{s} \alpha^{-k} \xi_k}{\sum_{k=1}^{s} \alpha^{-k} s_k} \right] > 0 \text{ at } s > \bar{s} \quad . \tag{A16}$$

By means of the obtained set $(\bar{p},\bar{w},\bar{\xi})$ we may restore the solution $(\bar{p},\bar{w},\bar{\gamma},\bar{\xi}_U,\bar{\xi}_\psi)$ of the problem (A11). For this purpose it is necessary to choose ξ_ψ so that

$$\bar{\xi}_\psi \ge 0; \; \bar{\gamma} = (\alpha-1)\bar{p} - \bar{\xi}_\psi \ge 0; \; \bar{\xi}_U = \bar{\xi}_U - \bar{\xi}_\psi H \ge 0 \quad .$$

Due to the inequalities (A16) among the solutions of the problem (A11) there are such that

$$\bar{\xi}_{\psi_s} > 0,\ \bar{\gamma}_s > 0,\ \bar{\xi}_{U_s} > 0 \text{ at } s > \bar{s}\ , \tag{A17}$$

and any solution according to (A14), (A15) satisfies the correlations

$$\bar{p}_s > 0;\ \bar{\gamma}_s = (\alpha-1)\bar{p}_s \text{ at } s < \bar{s}\ . \tag{A18}$$

Using the obtained solution of the problem (A11) and the correlations of complementary non-rigidity, we find that the solution of the problem (A10) should satisfy the system:

$$\sum_{s=1}^{\tau} x_s \bar{U}_s = v;\ \bar{U}_s = 0,\ \bar{\psi}_s = 0 \text{ at } s > \bar{s}\ ,$$

$$\alpha\bar{\psi}_s = (\bar{\psi}_s + \bar{U}_s - \bar{U}_{s+1});\ \bar{U}_{s+1} = \bar{\psi}_s \text{ at } s < \bar{s}\ .$$

Hence we find the complete description of the set of solutions (A10):

$$\bar{U}_{\bar{s}} = \frac{v}{\sum_{k=1}^{\bar{s}} \alpha^{(\bar{s}-k)} x_k}\ ,\ \bar{U}_s = \bar{U}_{\bar{s}}\,\alpha^{(\bar{s}-s)} \text{ at } s \leq \bar{s}\ ;$$

$$\bar{U}_s = 0 \text{ at } s > \bar{s}\ ;$$

$$\bar{\psi} = \bar{U}_1\alpha^{-s} \text{ at } s < \bar{s}\ ;\quad 0 \leq \bar{\psi}_{\bar{s}} \leq \bar{U}_1 \frac{\alpha^{(1-\bar{s})}}{(\alpha-1)}\ ; \tag{A19}$$

$$\bar{\psi}_s = 0 \text{ at } s > \bar{s}\ .$$

REFERENCES

[1] Ilyutovich A.E., Mokhov V.N. Normative approach to the description of fundformation processes in linear dynamic models of interbranch balance. Avtomatika i telemekhanika, 1978, N 12.

[2] Propoy A.I. Elements of optimal discrete processes. Moscow, "Nauka", 1973.

[3] Khodjamirian E.Yu., Chukanov S.V. The dynamic branch model with the controlled fundformation process. "Automated systems of planning and control", Yerevan, 1983.