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SMALL NOISE ANALYSIS FOR PIECEWISE LINEAR STOCHASTIC CONTROL PROBLEMS

Giovanni B. Di Masi
Wolfgang J. Runggaldier

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

FOREWORD

In line with the increasing attention that various researchers - including the authors - have recently devoted to the study of stochastic dynamical systems with piecewise linear coefficients, this paper deals with a stochastic control problem relative to a model of this type. In particular, it is shown that, for vanishing noise, such control problems can be approximated by suitably chosen linear adaptive control problems.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

ABSTRACT

A discrete-time stochastic control problem is considered for a dynamical model with piecewise linear coefficients and not necessarily Gaussian disturbances. The cost criteria and the class of admissible controls include piecewise polynomial costs and piecewise linear controls respectively. It is shown that relevant asymptotic (for vanishing noise) properties of this problem coincide with the corresponding properties of a suitably chosen adaptive control problem with linear dynamics. In particular, it turns out that the optimal values of the two problems tend to coincide and that almost optimal controls for one problem are almost optimal also for the other.

CONTENTS

SMALL NOISE ANALYSIS FOR PIECEWISE LINEAR STOCHASTIC CONTROL PROBLEMS

Giovanni B. Di Masi and Wolfgang J. Runggaldier

1. INTRODUCTION

This paper is concerned with an asymptotic analysis (for vanishing noise) of a discrete-time, nonlinear stochastic control problem (P) described as follows:

The state x_t, which for convenience of notation and without loss of generality we assume to be scalar, evolves according to

$$x_{t+1} = a_t(x_t) + u_t + \beta_{t+1}; \qquad (t = 0, 1, \ldots, T-1) \tag{1}$$

where $a_t(x)$ is piecewise linear in x, namely

$$a_t(x) = \sum_{i=1}^{N} [A_t(i)x + B_t(i)] I_{\Pi_i}(x) \tag{2}$$

with $\{\Pi_i\}_{i=1,\ldots,N}$ a finite partition of $\mathbf{R}$; furthermore, $a_t(x)$ is assumed to be continuous so that it is Lipschitz and of linear growth, more precisely, defining

$$A := \max_{i \leq N, t < T} \{|A_t(i)|\} \tag{3.a}$$

$$B := \max_{i \leq N, t < T} \{|B_t(i)|\} \tag{3.b}$$

we have

$$|a_t(x)| \leq A|x| + B \tag{4.a}$$

$$|a_t(x) - a_t(y)| \leq A|x - y| \tag{4.b}$$

The initial condition x_0 and the disturbances β_t are assumed to be distributed according to finite mixtures of normal densities; more precisely, we have ($\sim$ means "is distributed according to" and $g(x; \mu, \sigma^2)$ denotes the normal density with mean μ and variance σ^2)

$$x_0 \sim \sum_{i=1}^{N_0} \alpha_{0i} g(x; \mu_{0i}, \sigma_{0i}^2) \tag{5.a}$$

$$\beta_t \sim \sum_{i=1}^{N_t} \alpha_{ti} g(x;\, \mu_{ti},\, \sigma_{ti}^2) \tag{5.b}$$

with $\sigma_{0i}^2 > 0$; $\sigma_{ti}^2 > 0$. A possible representation for these random variables can be obtained in the following way. Consider a discrete random variable θ_0 which takes the finite number of values $\mu_{0i} (i = 1, \ldots, N_0)$ with probabilities $P\{\theta_0 = \mu_{0i}\} = \alpha_{0i}$ and define the mapping $Q_0 : \{\mu_{0i}\} \to \mathbf{R}$ by

$$Q_0(\mu_{0i}) = \sigma_{0i} \tag{6}$$

Assuming $v_0 \sim g(x;\, 0,\, 1)$, we have that the random variable

$$x_0 = \theta_0 + Q_0(\theta_0) v_0 \tag{7.a}$$

satisfies (5.a). With an analogous procedure we can obtain representations for β_t of the form

$$\beta_t = \theta_t + Q_t(\theta_t) v_t \tag{7.b}$$

where $\{v_t\}_{t=1,\ldots,T}$ is a standard Gaussian white noise, independent of v_0, θ_t are discrete random variables taking values in $\{\mu_{ti}\}$ with probabilities $P\{\theta_t = \mu_{ti}\} = \alpha_{ti}$ and $Q_t(\cdot)$ satisfies

$$Q_t(\mu_{ti}) = \sigma_{ti} \tag{8}$$

With such representations for x_0 and $\{\beta_t\}$, the mutual dependence of $x_0, \beta_1, \ldots, \beta_T$ will be related to the joint a-priori probability $p(\theta_0, \theta_1, \ldots, \theta_T)$ and a suitable choice of the latter allows a considerable flexibility as far as the possible dependence patterns are concerned. In what follows we let

$$\theta' := [\theta_0, \theta_1, \ldots, \theta_T] \tag{9}$$

$$\epsilon := \max_{0 \le t \le T, \theta_t} \{Q_t(\theta_t)\} \tag{10}$$

Furthermore, we shall denote by E_u^ϵ integration with respect to the measure induced by model (1), (7) for given $u_0, u_1, \ldots, u_T$ and given $p(\theta) = p(\theta_0, \theta_1, \ldots, \theta_T)$.

The class of admissible controls consists of feedback controls $u_t = u_t(x_t)$ such that there exist positive constants K_1, K_2, K_3 (independent of u) for which

$$|u_t(x)| \le K_1|x| + K_2 \tag{11.a}$$

$$|u_t(x) - u_t(y)| \le K_3|x - y| \tag{11.b}$$

In the following, when convenient, we shall denote by u an admissible strategy $(u_0, u_1, \ldots, u_{T-1})$.

As objective function to be minimized we consider

$$V^\epsilon(u) := E_u^\epsilon\left\{\sum_{t=0}^{T} f_t(x_t;\, \theta,\, u)\right\} \tag{12}$$

where f_t satisfies the relation

$$|f_t(x;\, \theta,\, u) - f_t(y;\, \theta,\, u)| \leq P_t(|x|, |y|)|x - y| \tag{13}$$

with P_t a polynomial independent of u and θ; furthermore, f_t is bounded from below. Notice that, because of (11), any f_t which is a polynomial in x_t and $u_t(x_t)$ satisfies (13) so that the given objective function generalizes the commonly used quadratic cost criterion.

In analogy to a previous paper on piecewise linear filtering [1] the aim here is to show that, asymptotically when ϵ in (10) tends to zero, the optimal value (and the δ-optimal controls) of our nonlinear problem (P) coincides with the optimal value (and the δ-optimal controls) of an adaptive linear stochastic control problem $(\hat{P})$. For these reasons such adaptive problem $(\hat{P})$ can be considered as an approximation to the original nonlinear problem (P).

In order to provide a more precise definition of $(\hat{P})$ we introduce further random processes related to the linear behaviors of $a_t(x)$ in (1), (2). To this end, given an admissible control sequence u, define the processes ξ_t^u and η_t^u by

$$\xi_{t+1}^u = a_t(\xi_t^u) + u_t(\xi_t^u) + \beta_{t+1};\; \xi_0^u = \theta_0 \tag{14}$$

$$\eta_t^u := \sum_{i=1}^{N} i\, I_{\Pi_i}(\xi_t^u) \tag{15}$$

Consider now a process $\{\hat{x}_t\}$ satisfying the following model

$$\hat{x}_{t+1} = A_t(\eta_t^u)\hat{x}_t + B_t(\eta_t^u) + u_t + \beta_{t+1} \tag{16}$$

where $A_t(\cdot)$, $B_t(\cdot)$ are the quantities appearing in (2); the initial condition and the disturbances are as in (7).

Notice that, for a given admissible control u and asymptotically for $\epsilon \downarrow 0$), the process η_t^u "tracks" the linear behavior of $a_t(x_t)$ in the sense that a.s.

$$\lim_{\epsilon \downarrow 0} \{A_t(\eta_t)x_t + B_t(\eta_t) - a_t(x_t)\} = 0 \tag{17}$$

Taking into account that η_t^u depends only on u and θ, $A_t(\eta_t)$ and $B_t(\eta_t)$ can be rewritten, with obvious abuse of notation, as $A_t(\theta, u)$ and $B_t(\theta, u)$ respectively so that, writing explicitly β_{t+1} as in (7.b), model (16) becomes

$$\hat{x}_{t+1} = \hat{a}_t(\hat{x}_t;\, \theta,\, u) + u_t + \theta_{t+1} + Q_{t+1}(\theta_{t+1})v_{t+1} \tag{18}$$

where

$$\hat{a}_t(\hat{x}_t;\, \theta,\, u) := A_t(\theta,\, u)\hat{x}_t + B_t(\theta,\, u) \tag{19}$$

In this case, corresponding to (17) we have a.s.

$$\lim_{\epsilon \downarrow 0} \{\hat{a}_t(\hat{x}_t;\, \theta,\, u) - a_t(x_t)\} = 0 \tag{20}$$

so that, for given u, besides (14) we also have

$$\xi_{t+1}^u = \hat{a}_t(\xi_t^u;\, \theta,\, u) + u_t(\xi_t^u) + \beta_{t+1};\ \xi_0^u = \theta_0 \tag{21}$$

Notice that $\hat{a}_t(\hat{x}_t, \theta, u)$ satisfy condition (4) uniformly in u and θ, namely

$$|\hat{a}_t(x;\, \theta,\, u)| \le A|x| + B \tag{22.a}$$

$$|\hat{a}_t(x;\, \theta,\, u) - \hat{a}_t(y;\, \theta,\, u)| \le A|x - y| \tag{22.b}$$

We can now describe problem $(\hat{P})$ as the discrete-time stochastic control problem with state evolving according to (18), with initial condition as in (7) and with the same a-priori $p(\theta) = p(\theta_0, \ldots, \theta_T)$ (let $\hat{E}_u^\epsilon$ denote integration with respect to the measure thus induced). The class of admissible controls remains the same as for (P), while the objective function $\hat{V}^\epsilon(u)$, which is to be minimized, is given by relation (12) with $\hat{E}_u^\epsilon$ replacing E_u^ϵ.

The main result of this paper is Theorem 2.1 below, whose immediate consequence (Corollary 2.1) is that, for vanishing ϵ, $|V^\epsilon(u) - \hat{V}^\epsilon(u)|$ converges to zero uniformly in u. This in turn implies (Corollary 2.2) that the optimal values V^ϵ and $\hat{V}^\epsilon$ of problems (P) and $(\hat{P})$ respectively, defined as

$$V^\epsilon := \inf_u V^\epsilon(u) \tag{23.a}$$

$$\hat{V}^\epsilon := \inf_u \hat{V}^\epsilon(u) \tag{23.b}$$

coincide for vanishing ϵ and, furthermore, that almost optimal controls for one problem are almost optimal also for the other.

2. ASYMPTOTIC ANALYSIS

Given any admissible strategy u, let $p_u^\epsilon(x_0,\dots,x_t;\,\theta)$ and $\hat p_u^\epsilon(x_0,\dots,x_t;\,\theta)$ denote the joint distributions of $x_0,\dots,x_t$ and θ corresponding to model (1) and (18) respectively. From (1), (7), (18) we have for p_u^ϵ and $\hat p_u^\epsilon$ the following recursive relations

$$p_u^\epsilon(x_0,\dots,x_{t+1};\,\theta) = g(x_{t+1};\, a_t(x_t) + u_t(x_t) + \theta_{t+1},\, Q_{t+1}^2(\theta_{t+1}))\cdot \qquad (24.a)$$

$$\cdot\, p_u^\epsilon(x_0,..,x_t;\,\theta)$$

$$\hat p_u^\epsilon(x_0,\dots,x_{t+1};\,\theta) = g(x_{t+1};\, \hat a_t(x_t;\,\theta,\,u) + u_t(x_t) + \theta_{t+1},\, Q_{t+1}^2(\theta_{t+1}))\cdot \qquad (24.b)$$

$$\cdot \hat p_u^\epsilon(x_0,\dots,x_t;\,\theta)$$

with initial condition

$$p_u^\epsilon(x_0;\,\theta) = \hat p_u^\epsilon(x_0;\,\theta) = g(x_0;\,\theta_0,\,Q_0^2(\theta_0))p(\theta) \qquad (24.c)$$

LEMMA 2.1 *For any $\epsilon_0 > 0$ and positive integer q there exists an $M > 0$ such that for all admissible u, all θ and all ϵ with $0 < \epsilon < \epsilon_0$ we have*

$$\int |x_t|^q p_u^\epsilon(x_0,\dots,x_t;\,\theta)\,\mathrm{d}x_0\dots\mathrm{d}x_t < M \qquad (25.a)$$

$$\int |x_t|^q \hat p_u^\epsilon(x_0,\dots,x_t;\,\theta)\,\mathrm{d}x_0\dots\mathrm{d}x_t < M \qquad (25.b)$$

PROOF We shall first prove (25.a) proceeding by induction. For $t = 0$ we have

$$\int |x_0|^q p_u^\epsilon(x_0;\,\theta)\,\mathrm{d}x_0 = \int |x_0|^q g(x_0;\,\theta_0,\,Q_0^2(\theta_0))p(\theta)\,\mathrm{d}x_0$$

Since the expression on the right is a polynomial in θ_0 and $Q_0^2(\theta_0)$, and recalling that θ takes only a finite number of possible values, we have (25.a) for $t = 0$.

Assuming (25.a) true for $t - 1$ and using (24.a) we have

$$\int |x_t|^q p_u^\epsilon(x_0,\dots,x_t;\,\theta)\,\mathrm{d}x_0\dots\mathrm{d}x_t =$$

$$= \int \left[\int |x_t|^q g(x_t;\, a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t,\, Q_t^2(\theta_t))\,\mathrm{d}x_t\right]\cdot$$

$$\cdot p_u^\epsilon(x_0,\dots,x_{t-1};\,\theta)\,\mathrm{d}x_0\dots\,dx_{t-1} \le$$

$$\le \int P(|x_{t-1}|,\,|\theta|,\,Q_t^2(\theta_t))p_u^\epsilon(x_0,\dots,x_{t-1};\,\theta)\,\mathrm{d}x_0\dots\mathrm{d}x_{t-1}$$

where P is a polynomial. The induction hypothesis and the fact that θ takes only a finite number of values then provides (25.a).

By similar arguments it is possible to prove (25.b). □

LEMMA 2.2 *For $f_t(x_t; \theta, u)$ satisfying (13) we have*

$$\lim_{\epsilon \downarrow 0} \sup_u | \int p_u^\epsilon(x_0, \ldots, x_{t-1}; \theta)\, dx_0 \ldots dx_{t-1} \cdot$$

$$\cdot [\int f_t(x_t; \theta, u) g(x_t; a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t, Q_t^2(\theta_t))\, dx_t -$$

$$- f_t(a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t; \theta, u)]| = 0 \tag{26.a}$$

$$\lim_{\epsilon \downarrow 0} \sup_u | \int \hat{p}_u^\epsilon(x_0, \ldots, x_{t-1}; \theta)\, dx_0 \ldots dx_{t-1} \cdot$$

$$\cdot [\int f_t(x_t; \theta, u) g(x_t; \hat{a}_{t-1}(x_{t-1}; \theta, u) + u_{t-1}(x_{t-1}) + \theta_t, Q_t^2(\theta_t))\, dx_t -$$

$$- f_t(\hat{a}_{t-1}(x_{t-1}; \theta, u) + u_{t-1}(x_{t-1}) + \theta_t; \theta, u)]| = 0 \tag{26.b}$$

where the "sup" is over all admissible controls.

PROOF We shall first prove (26.a). From (13) we have that an upper bound for the absolute value in (26.a) is given by

$$\int p_u^\epsilon(x_0, \ldots, x_{t-1}; \theta)\, dx_0 \ldots dx_{t-1} \cdot$$

$$\cdot \int P_t(|x_t|, |a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t|) \cdot$$

$$\cdot |x_t - a_{t-1}(x_{t-1}) - u_{t-1}(x_{t-1}) - \theta_t| \cdot$$

$$\cdot g(x_t; a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t, Q_t^2(\theta_t))\, dx_t =$$

$$= |Q_t(\theta_t)| \int P_t(|Q_t(\theta_t) z + a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t| \;,$$

$$|a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t|) \cdot$$

$$\cdot |z| g(z; 0, 1) p_u^\epsilon(x_0, \ldots, x_{t-1}; \theta)\, dz dx_0 \ldots dx_{t-1} \tag{27}$$

Using (4.a), (11.a), the fact that θ takes a finite number of values and Lemma 2.1, it is easily seen that the integral in the rightmost member of (27) is bounded uniformly in u so that (26.a) holds. The proof of (26.b) proceeds in an analogous way. □

Given $f_t(x; \theta, u)$ satisfying (13) let

$$\varphi_{t-1}(x; \theta, u) := f_t(a_{t-1}(x) + u_{t-1}(x) + \theta_t; \theta, u) \tag{28.a}$$

$$\hat{\varphi}_{t-1}(x; \theta, u) := f_t(\hat{a}_{t-1}(x; \theta, u) + u_{t-1}(x) + \theta_t; \theta, u) \tag{28.b}$$

We have

LEMMA 2.3 *The functions* $\varphi_{t-1}(x_{t-1}; \theta, u)$ *and* $\hat{\varphi}_{t-1}(x_{t-1}; \theta, u)$ *defined in (28) satisfy condition (13).*

PROOF Using the fact that $f_t(x_t; u, \theta)$ satisfies (13), we have for φ_{t-1}

$$|\varphi_{t-1}(x; \theta, u) - \varphi_{t-1}(y; \theta, u)| =$$

$$= |f_t(a_{t-1}(x) + u_{t-1}(x) + \theta_t; \theta, u) - f_t(a_{t-1}(y) + u_{t-1}(y) + \theta_t; \theta, u)| \leq$$

$$\leq P_t(|a_{t-1}(x) + u_{t-1}(x) + \theta_t|, |a_{t-1}(y) + u_{t-1}(y) + \theta_t|) \cdot$$

$$\cdot[|a_{t-1}(x) - a_{t-1}(y)| + |u_{t-1}(x) - u_{t-1}(y)|] \leq$$

$$\leq \bar{P}_{t-1}(|x|, |y|)|x - y|$$

where $\bar{P}_{t-1}$ is a suitable polynomial and where for the last inequality (4) and (11) have been used.

The proof for $\hat{\varphi}_{t-1}$ proceeds in an analogous way. □

THEOREM 2.1 *For* $f_t(x_t; \theta, u)$ *satisfying (13) we have*

$$\lim_{\epsilon \downarrow 0} \sup_u |\int f_t(x_t; \theta, u) p_u^\epsilon(x_0, \ldots, x_t; \theta)\, dx_0 \ldots dx_t -$$

$$- f_t(\xi_t^u; \theta, u) p(\theta)| = 0 \tag{29.a}$$

$$\lim_{\epsilon \downarrow 0} \sup_u |\int f_t(x_t; \theta, u) \hat{p}_u^\epsilon(x_0, \ldots, x_t; \theta)\, dx_0 \ldots dx_t -$$

$$- f_t(\xi_t^u; \theta, u) p(\theta)| = 0 \tag{29.b}$$

where the "sup" is over all admissible controls.

PROOF We shall first prove (29.a) proceeding by induction. For $t = 0$ the statement reduces to

$$\lim_{\epsilon \downarrow 0} \sup_u |\int f_0(x_0; \theta, u) g(x_0; \theta_0, Q_0^2(\theta_0)) p(\theta)\, dx_0 - f_0(\theta_0; \theta, u) p(\theta)| =$$

$$= p(\theta) \lim_{\epsilon \downarrow 0} \sup_u |\int f_0(x_0; \theta, u) g(x_0; \theta_0, Q_0^2(\theta_0))\, dx_0 - f_0(\theta_0; \theta, u)| = 0$$

whose proof is analogous to that of Lemma 2.2. Assume now (29.a) true for $t - 1$, then, using (28.a) we have

$$|\int f_t(x_t;\,\theta,\,u)p_u^\epsilon(x_0,\ldots,x_t;\,\theta)\,\mathrm{d}x_0\ldots\mathrm{d}x_t - f_t(\xi_t^u;\,\theta,\,u)p(\theta)| \leq$$

$$\leq |\int p_u^\epsilon(x_0,\ldots,x_{t-1};\,\theta)\Big[\int f_t(x_t;\,\theta,\,u)g(x_t;\,a_{t-1}(x_{t-1}) +$$

$$+ u_{t-1}(x_{t-1}) + \theta_t;\,Q_t^2(\theta_t))\,\mathrm{d}x_t -$$

$$- f_t(a_{t-1}(x_{t-1}) + u_{t-1}(x_{t-1}) + \theta_t;\,\theta,\,u)\Big]\mathrm{d}x_0\ldots\mathrm{d}x_{t-1}| +$$

$$+ |\int\varphi_{t-1}(x_{t-1};\,\theta,\,u)p_u^\epsilon(x_0,\ldots,x_{t-1};\,\theta)\,\mathrm{d}x_0\ldots\mathrm{d}x_{t-1} - \varphi_{t-1}(\xi_{t-1}^u;\,\theta,\,u)| \qquad (30)$$

By Lemma 2.2, the induction hypothesis and Lemma 2.3, the right hand side of (30) is infinitesimal with ϵ, uniformly in u, thereby completing the proof of (29.a). The proof of (29.b) proceeds in an analogous way, noticing that the process ξ_t^u satisfies not only (14) but also (21). □

As an immediate consequence of Theorem 2.1 we have the following

COROLLARY 2.1 *For the objective function $V^\epsilon(u)$ and $\hat{V}^\epsilon(u)$ relative to problem (P) and $(\hat{P})$ respectively we have*

$$\lim_{\epsilon\downarrow 0}\sup_u |V^\epsilon(u) - \hat{V}^\epsilon(u)| = 0$$ □

As mentioned in the Introduction, a consequence of Corollary 2.1 is that, asymptotically, the optimal values of problems (P) and $(\hat{P})$ coincide and that almost optimal controls for $(\hat{P})$ are almost optimal also for (P) and vice versa. This will be shown in the following Corollary 2.2.

COROLLARY 2.2 *For the optimal values V^ϵ and $\hat{V}^\epsilon$ defined in (23) we have*

$$\lim_{\epsilon\downarrow 0}|V^\epsilon - \hat{V}^\epsilon| = 0 \qquad (31)$$

Furthermore, let u and $\hat{u}$ be γ-optimal controls for (P) and $(\hat{P})$ respectively; then, for $\delta > 0$ given, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have that u and $\hat{u}$ are $(2\gamma + 2\delta)$-optimal for $(\hat{P})$ and (P) respectively.

PROOF From Corollary 2.1, for fixed $\delta > 0$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and all u we have $|V^\epsilon(u) - \hat{V}^\epsilon(u)| < \delta$. For given $\gamma > 0$ let now u and $\hat{u}$ be such that

$$V^\epsilon(u) \leq V^\epsilon + \gamma$$

$$\hat{V}^\epsilon(\hat{u}) \leq \hat{V}^\epsilon + \gamma$$

then

$$V^\epsilon \leq V^\epsilon(\hat{u}) \leq \hat{V}^\epsilon(\hat{u}) + \delta \leq \hat{V}^\epsilon + \gamma + \delta$$

$$\hat{V}^\epsilon \leq \hat{V}^\epsilon(u) \leq V^\epsilon(u) + \delta \leq V^\epsilon + \gamma + \delta$$

Therefore

$$| V^\epsilon - \hat{V}^\epsilon | \leq \gamma + \delta$$

which proves (31). Furthermore

$$V^\epsilon(\hat{u}) \leq \hat{V}^\epsilon + \gamma + \delta \leq V^\epsilon + 2\gamma + 2\delta$$

$$\hat{V}^\epsilon(u) \leq V^\epsilon + \gamma + \delta \leq \hat{V}^\epsilon + 2\gamma + 2\delta$$

which proves the second assertion of the Corollary. □

REFERENCE

[1] Di Masi, G.B. and W.J. Runggaldier: "Asymptotic analysis for piecewise linear filtering". IIASA Working Paper WP-87-53, June 1987.

Authors' permanent addresses:

Giovanni B. Di Masi, CNR-Ladseb, Corso Stati Uniti 4, I-35020 Padova, Italy

Wolfgang J. Runggaldier, Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, I-35131 Padova, Italy