# WORKING PAPER

LIMIT THEOREMS FOR PROPORTIONS OF BALLS IN A GENERALIZED URN SCHEME

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## Foreword

In this paper the authors continue to study the process of growth modeled by urn schemes containing balls of different colors. The rate of convergence for proportions of balls to the limit state is investigated. It is shown that Gaussian as well as non-Gaussian Markov random processes may describe the asymptotic behavior.

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# Limit Theorems for Proportions of Balls in a Generalized Urn Scheme

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## 1. Introduction

This paper is actually the second part of the paper [14] and for the sake of abridgment we adopt here all notations from [14]. We consider only the situation when the vector $X_n, n \to \infty$ of balls proportions converge with probability 1 to a non-random variable. This will be the case, in particular, when the set of fixed-points of urn functions is a singleton. The general case is studied in a forthcoming paper.

We investigate asymptotic normality as well as the law of iterated logarithm and the invariance principles. These results generalize results of the article [8]. The main idea of our approach originates in paper [3] and consists in an interpretation of related to the generalized urn scheme processes as a stochastic approximation type procedure.

## 2. Auxiliary Results

In subsequent theorems are important local properties of the process $X_n$ in a neighborhood of attracting point $\vartheta$. The following lemma establishes the necessary relations.

**Lemma** 1. If for any $\varepsilon > 0$ and all $x \in U^{N-1}(\vartheta, \varepsilon) \cap L_{N-1}$

1) There is $r \geq 3$ such that $sup_{t \geq 1} \sum_{i \in Z_+^N} ]i[^r q_t(i, x) \leq C_1$, then uniformly with respect to $x, \gamma$

a) $E(X_{\{\|\xi_t(X_t,\gamma_t)\| > R\}}\|\xi_t(X_t,\gamma_t)\|^2|X_t = x, \gamma_t = \gamma) \le C_2R^{2-r}$

where $\xi_t$ is the N-dimensional vector whose first $N-1$ coordinates coincide with corresponding coordinates of vector $\eta_t$ and the last coordinate is $]\beta_t(X_t)[ - r_t(X_t)$. Let also

2) there exist $q \ge 3$ and variables $q(i,x), x \in U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}, i \in Z_+^N$ such that $\sum_{i \in Z_+^N} q(i,x) = 1$, $\sum_{i \in Z_+^N} ]i[^q\, q(i,x) \le C_3$, $|q(i,x) - q_t(i,x)| \le \sigma_t \to 0$ with $t \to \infty$. Then

b) $r_t(x) = \rho(x) + r(t,x), E(\xi_t(X_t,\gamma_t)\xi_t(X_t,\gamma_t)^T|X_t$

$= x, \gamma_t = \gamma) = \sigma(x) + F(x) + M(t,x,\gamma) + R(t,x),$

where

$\rho(x) = \sum_{i \in Z_+^N} ]i[q(i,x), |r(t,x)| \le C_4\sigma_t^{H/N+H}, H =$

$= \min(r,q)-1, |M^{jk}(t,x,\gamma)| \le C_5\gamma^{-1}, |R^{jk}(t,x)| \le$

$\le C_6\sigma_t\frac{H-1}{N+H}, j,k = 1,2,\ldots,N; \sigma^{jk}(x) = \sum_{i \in Z_+^N}(i^j -$

$- x^i]i[)(i^k - x^k]i[)q(i,x), F^{jk}(x) = f^j(x)f^k(x), k = 1,2,\ldots,N-1, \sigma^{jN}(x) =$

$= \sum_{i \in Z_+^N} ]i[(i^j - x^j]i[)q(i,x), F^{jN}(x) = f^j(x)\rho(x), j = 1,2,\ldots,N-1, \sigma^{NN}(x) =$

$= \sum_{i \in Z_+^N} ]i[^2 q(i,x), F^{NN}(x) = -\rho(x)^2$ ,

and if, in addition,

3) functions $q(i,x), i \in Z_+^N$ are continuous on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$; then

c) $\sigma, f$ and $\rho$ are continuous on this set of functions. Furthermore, if conditions 2), 3) are satisfied and

4) partial derivatives $q^{(k)}(i,x) = \frac{\partial q(i,x)}{\partial x^k}$, $k = 1,2,\ldots,N-1$ exist and are continuous for all $i \in Z_+^N$, $x \in U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$, and series $\sum_{i \in Z_+^N} i^j q^{(k)}(i,x), j,k = 1,2,\ldots,N-1$ converge uniformly; then

d) $f$ and $\rho$ are continuously differentiable on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$ and

$$\Sigma_{j=1}^{N-1} \frac{\partial f^j(x)}{\partial x^k} = \rho(x) + (1 - \Sigma_{j=1}^{N-1} x^j) \frac{\partial \rho(x)}{\partial x^k}, \ k = 1,2,\dots,N-1.$$

If conditions 1) - 3) are satisfied and

5) $X_t \to \vartheta$ with probability 1 for $t \to \infty$, then

e) with probability 1 $\gamma_t t^{-1} \to \rho(\vartheta)$ for $t \to \infty$.

**Proof.** Furtheron we assume that $x \in U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$ and $\gamma$ is a natural number. Using estimate (7) from [1], condition 1) and Hölder inequality we obtain for $X_t = x, \gamma_t = \gamma$

$$\|\xi_t(X_t,\gamma_t)\|^2 = \|\eta_t(X_t,\gamma_t)\|^2 + []\beta_t(X_t)[ -$$

$$- r_t(X_t)]^2 \le 2\{\|\zeta_t(X_t,\gamma_t)\|^2 + \{E[\|\zeta_t(X_t,\gamma_t)\| |X_t,$$

$$\gamma_t]\}^2 + ]\beta_t(X_t)[^2 + r_t(X_t)^2\} \le 2\{4N\{]\beta_t(X_t)[^2 +$$

$$+ \{E[]\beta_t(X_t)[|X_t]\}^2\} + ]\beta_t(X_t)[^2 + r_t(X_t)^2\} =$$

$$= 2(4N+1)\left[]\beta_t(X_t)[^2 + r_t(X_t)^2\right] \le 2(4N+1)\left[]\beta_t(X_t)[^2 + C_1^{2/\tau}\right] .$$

If $R > 2C_1^{1/\tau}(4N+1)^{1/2}$ then

$$\{\|\xi_t(X_t,\gamma_t)\|^2 > R^2\} \subseteq \{2(4N+1)(]\beta_t(X_t)[^2 +$$

$$+ C_1^{2/\tau}] > R^2\} \subseteq \{]\beta_t(X_t)[^2 > \frac{1}{4}(4N+1)^{-1}R^2\}$$

and, taking into account inequality (12) from [1] with $\mu = \tau$, $\rho = 2$ and $\rho = 0$ we obtain

$$E[\chi_{\{\|\xi_t(X_t,\gamma_t)\| > R\}} \|\xi_t(X_t,\gamma_t)\|^2 |X_t,\gamma_t] \le$$

$$\le 2(4N+1)E\left\{\chi_{\{]\beta_t(X_t)[ > \frac{1}{2}(4N+1)^{-1/2}R\}} \left[]\beta_t(X_t)[^2 +\right.\right.$$

$$\left.\left. + C_1^{2/\tau}\right] |X_t\right\} \le 2^{\tau-1}(4N+1)^{\tau/2}C_1[1+4(4N+1)C_1^{2/\tau}R^{-2}]R^{2-\tau} <$$

$$< 3 \cdot 2^{\tau-1}(4N+1)^{\tau/2}C_1R^{2-\tau} = C_2R^{2-\tau} .$$

Thus, the assertion 2) is proved.

If the condition 2) is satisfied, then on the basis of estimate (8) from [1] and Hölder inequality

$$|f^j(x)| \le 2 \sum_{i \in Z_+^N} ]i[ q(i,x) = 2\rho(x) \le 2C_3^{1/q} \;,\; j = 1,2,\ldots,N-1 \tag{1}$$

Therefore due to assertions a), b) of Lemma 1 from [1]

$$\begin{aligned} & e[\zeta_t^j(X_t,\gamma_t)|X_t = x, \gamma_t = \gamma] E[\zeta_t^k(X_t,\gamma_t)|X_t = \\ & = x, \gamma_t = \gamma] = f^j(x) f^k(x) + M_1^{jk}(t,x,\gamma) + R_1^{jk}(t,x) \,, \end{aligned} \tag{2}$$

$$|M_1^{jk}(t,x,\gamma)| \le C_5^{(1)} \gamma^{-1}, |R_1(t,x)| \le C_6^{(1)} \sigma_t^{H/N+H} \;, \tag{3}$$

where $j, k = 1,2,\ldots,N-1$. It is easy to see that

$$\begin{aligned} & E\Big[\zeta_t^j(X_t,\gamma_t)\zeta_t^k(X_t,\gamma_t)|X_t = x, \gamma_t = \\ & = \gamma] = \sigma^{jk}(x) + M_2^{jk}(t,x,\gamma) + R_2^{jk}(t,x) \,, \end{aligned} \tag{4}$$

$$\begin{aligned} & M_2^{jk}(t,x,\gamma) = -\gamma^{-1} \sum_{i \in Z_+^N} \frac{(2+\gamma^{-1}]i[)]i[}{(1+\gamma^{-1}]i[)^2} (i^j - \\ & - x^j]i[)(i^k - x^k]i[) q_t(i,x) \,, \end{aligned}$$

$$R_2^{jk}(t,x) = \sum_{i \in Z_+^N} (i^j - x^j]i[)(i^k - x^k]i[)[q_t(i,x) - q(i,x)] \;.$$

Since $\frac{2+\tau}{1+\tau} \le 2$ with $\tau \ge 0$ then from condition 1), estimate (8) from [1] and the Hölder inequality

$$|M_2^{jk}(t,x,\gamma)| \le 8\gamma^{-1} \sum_{i \in Z_+^N} ]i[^3 q_t(i,x) \le 8C_1^{1/r} \gamma^{-1} \;. \tag{5}$$

As in proving assertion b) of Lemma 1 in [1], taking into account (8) and (12) with $\mu = \min(r,q), \rho = 2$ from [1] and conditions 1), 2)

$$\begin{aligned} & |R_2^{jk}(t,x)| \le 4 \sum_{i \in Z_+^N} ]i[^2 |q_t(i,x) - q(i,x)| \le \\ & \le 4(\sigma_t \sum_{i \in Z_+^N} ]i[^2 + \sum_{i \in Z_+^N \setminus Z_+^N(L)} ]i[^2 \Big[q_t(i,x) + q(i,x)\Big] \le \\ & \le 4\Big\{\sigma_t (L+1)^{N+1} + (L+1)^{1-H}\Big[C_1^{\min(1,q/r)} + C_3^{\min(1,r/q)}\Big]\Big\} \;. \end{aligned}$$

Assume that $L = L(t) = \sigma_t^{-1/N+H}$, then

$$|R_2^{jk}(t,x)| \leq C_6^{(2)} \sigma_t^{\frac{H-1}{N+H}} \quad . \tag{6}$$

Since

$$r(t,r) = \sum_{i \in Z_+^N} ]i[[q_t(i,x) + q(i,x)] \quad ,$$

then by reasoning in the same manner as when proving assertion of Lemma 1 from [1] we obtain

$$|r(t,x)| \leq C_4 \sigma_t^{H/N+H} \quad . \tag{7}$$

Based on conditions 1), 2)

$$E\left[]\beta_t(X_t)[^2 |X_t = x\right] = \sigma^{NN}(x) + R_1^{NN}(t,x) ,$$

$$R_1^{NN}(t,x) = \sum_{i \in Z_+^N} ]i[^2 \left[q_t(i,x) - q(i,x)\right] \quad , \tag{8}$$

therefore, due to (6)

$$|R_1^{NN}(t,x)| \leq C_6^{(2)} \sigma_t^{\frac{H-1}{N+H}} \quad . \tag{9}$$

Taking into account conditions 1), 2), assertions a), b) of Lemma 1 from [1] and (1), (7)

$$E\left[]\beta_t(X_t)[\xi_t^j(X_t, \gamma_t) |X_t = x, \gamma_t = \gamma\right] =$$

$$= \sigma^{Nj}(x) + F^{Nj}(x) + M_1^{Nj}(t,x,\gamma) + R_1^{Nj}(t,x) \quad , \tag{10}$$

$$|M_1^{Nj}(t,x,\gamma)| \leq C_5^{(1)} \gamma^{-1} \quad , \tag{11}$$

$$|R_1^{Nj}(t,x)| \leq C_6^{(1)} \sigma_t^{H/N+H} \quad , \tag{12}$$

where $j = 1,2,\ldots,N-1$. From definition $\xi_t$, relations (2) - (12) and the fact that by virtue of condition 2) $\sigma_t \to 0$, and therefore, $\sigma_t^{H/N+H} = 0\left(\sigma_t^{\frac{H-1}{N+H}}\right)$ for $t \to \infty$ the assertion b) follows.

From 2), estimates (8) and (12) with $\mu = q, \rho = 2$ or $\rho = 1$ from [1], series which define functions $\sigma, f$ and $\rho$ converge uniformly on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$. Then its sum is a continuos function [9, p. 431], and under condition 3) the assertion c) is valid. By differentiating formally the expression for $f^j$ we obtain

$$\frac{\partial f^j(x)}{\partial x^k} = -\delta_{jk}\rho(x) + \sum_{i \in Z_+^N} (i^j - x^j]i[)q^{(k)}(i,x) \quad , \tag{13}$$

where $\delta_{jk}$ is a Kronecker symbol. On the basis of assertion c) the first term here is a continuous function, and with account for condition 4) the series consists of continuous functions and convergences uniformly. Therefore [9, p. 431] $\frac{\partial f^j(x)}{\partial x^k}$, $j,k, = 1,2,\ldots,N-1$ exist and are continuous on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$ and $f$ is continuously differentiable on this set. From equality (13) it follows that

$$\sum_{j=1}^{N-1} \frac{\partial f^j(x)}{\partial x^k} = \rho(x) + (1 - \sum_{j=1}^{N-1} x^j)\frac{\partial \rho(x)}{\partial x^k}, j,k = 1,2,\ldots,N-1 \quad ,$$

i.e., with account for the foregoing, $\rho$ is continuously differentiable on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$ and the assertion d) is valid.

Based on relations (17), (18) from [1] to prove the assertion e) it suffices to show that with probability 1

$$n^{-1}\sum_{i=1}^{n-1} r_i(X_i) \longrightarrow \rho(\vartheta) \quad \text{for} \quad n \to \infty \quad .$$

This relation is valid by (7) and the continuity of the function $\rho$ (according to c)) on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$. Therefore, taking into account condition 5) $\rho(X_t) \to \rho(\vartheta)$ with probability 1 for $t \to \infty$. The lemma is proved.

**Remark 1.** If $N = 2$, functions $q(i,x), i \in Z_+^2$ are continuous on the set $(\vartheta - \varepsilon, \vartheta + \varepsilon) \cap R(0,1)$; series $\sum_{i \in Z_+^2} (i^1 + i^2)q(i,x)$ converges uniformly; on the set $(\vartheta - \varepsilon, \vartheta) \cap R(0,1)$ or $(\vartheta, \vartheta + \varepsilon) \cap R(0,1)$ functions $q^1(i,x)$ exist, are continuous and series $\sum_{i \in Z_+^2} i^j q^1(i,x), j = 1,2$ converge uniformly, then similar to proof of assertion d) it can be shown that the function $f$ is continuous on $(\vartheta - \varepsilon, \vartheta + \varepsilon) \cap R(0,1)$ and is continuously differentiable on $(\vartheta - \varepsilon, \vartheta) \cap R(0,1)$ or $(\vartheta, \vartheta + \varepsilon) \cap R(0,1)$ respectively.

The following facts are useful to study the asymptotic of the urn processes.

Let $M$ dimensional vectors $x_s, s \geq 1$ form a Markov process,

$$E\|x_s\|^r < \infty, s \geq 1 \quad , \tag{14}$$

$$x_s \to \vartheta \text{ with probability 1 for } s \to \infty \quad , \tag{15}$$

where $r > 0, \vartheta \in R^M$. Assume that there is $\varepsilon > 0$ such that for $x_s = x, x \in U^M(\vartheta,\varepsilon), s \geq 1$

$$E(x_{s+1}|x_s) = s^{-1}[g(x) + w_s(x)] \quad , \tag{16}$$

$$x_{s+1} - E(x_{s+1}|x_s) = s^{-1}z(s,x) \quad ,$$

$$E\chi_{\{\|z(s,x)\|>R\}}\|z(s,x)\|^2 \leq C_0R^{-1} \quad , \tag{17}$$

$$\lim \|D(s,x) - D\|_0 = 0 \quad , \tag{18}$$

$$\max(s^{-1},\|x-\vartheta\|) \to 0 \quad ,$$

where $g$ and $w_s$ are M-dimensional vector-functions; $D(s,x) = Ez(s,x)z(s,x)^T, D-$ is a symmetric non-negative definite matrix, $\|\cdot\|_0$ is a norm of $M \times M$ matrices. In all subsequent lemmas relations (14) - (18) are assumed to be satisfied.

Let $D^M[0,T]$ be a space of $M$-dimensional vector functions on $[0,T], T > 0$ without second order discontinuities with Skorokhod metric [10] (for $M = 1$ simply $D[0,T]$). In $D^M[0,T]$ and $D[0,T]$ for $n \geq 2$ we consider random processes

$$X_n(t) = (n+s)^{1/2}(x_{n+s} - \vartheta) \text{ for } \sum_{i=n}^{n+s} i^{-1} \leq t < \sum_{i=n}^{n+s+1} i^{-1} \quad ,$$

and

$$Y_n(t) = \left[\frac{n+1}{\ln(n+s)}\right]^{1/2}(x_{n+s} - \vartheta) \text{ for } \sum_{i=n}^{n+s}(i\, lni)^{-1} \leq t < \sum_{i=n}^{n+s+1}(i\, lni)^{-1} \quad .$$

**Lemma 2.** [5]. Assume that

1) in equality (16) function $g$ is differentiable at point $\vartheta$, i.e., for $x \to \vartheta, g(x) = G(x-\vartheta) + o(\|x-\vartheta\|)$;

2) matrix $G + \frac{1}{2}J_M$ is stable (i.e., real parts of its eigen-values are negative) where $J_M$ is a unit matrix in $R^M$;

3)

$$\lim_{s\to\infty} s^{1/2} \sup_{x\in U^M(\vartheta,\varepsilon)} \|w_s(x)\| = 0 \quad .$$

Then with $n \to \infty$ random processes $X_n$ weakly converge in $D^M[0,T]$ to a stationary gaussian Markov process $X$, which satisfies a stochastic differential equation of the following form

$$dX = (G + \frac{1}{2}J_M)X\,dt + D^{1/2}dw_M \quad ,$$

where $D^{1/2}$ is a non-negative square root of the symmetric metrix $D$, $w_M$ is a standard $M$-dimensional Wiener process (with $M = 1$, simply $w$).

**Lemma 3**. Let $M = 1$ and

1) with $x \to \vartheta$

$$g(x) = -\frac{1}{2}(x-\vartheta) + o(|x-\vartheta|),$$

2)

$$\lim_{s\to\infty} (s\ln s)^{1/2} \sup_{|x-\vartheta|<\varepsilon} |w_s(x)| = 0 \quad .$$

Then with $n \to \infty$ random processes $Y_n$ weakly converge in $D[0,T]$ to a stationary gaussian Markov process $Y$, satisfying the following stochastic differential equation

$$dY = -\frac{1}{2}Ydt + D^{1/2}dw \quad .$$

The proof of this lemma is based on limit theorems for random processes generated by series of weakly dependent random vector [11] and is similar to that in paper [5].

**Lemma 4**. [4]. Let $M = 1$,

1) for $x \to \vartheta$

$$g(x) = G(x-\vartheta) + o(|x - \vartheta|),\ G < -1/2\,;$$

2)

$$\lim_{s\to\infty} \left[\frac{s}{\ln\ln s}\right]^{1/2} \sup_{|x-\vartheta|<\varepsilon} |w_s(x)| = 0 \quad .$$

Then with probability 1

$$\overline{\lim_{n\to\infty}}\left[\frac{n}{ln ln n}\right]^{1/2}\left[\frac{-1-2G}{D}\right]^{1/2}(x_n-\vartheta)=1 \quad ,$$

$$\underline{\lim_{n\to\infty}}\left[\frac{n}{ln ln n}\right]^{1/2}\left[\frac{-1-2b}{D}\right]^{1/2}(x_n-\vartheta)=-1 \quad .$$

**Lemma 5** [6]. Suppose that $M = 2$,

1) with $x \to \vartheta$

$$g(x) = G(x^1)(x-\vartheta) + o(\|x-\vartheta\|) \quad ,$$

where $G(x^1) = G_1$ for $x^1 \geq \vartheta^1$ and $G(x^1) = G_2$ for $x^1 < \vartheta^1$;

2) matrices $G_i + \frac{1}{2}J_2$, $i = 1,2$ are stable; and condition 3) of Lemma 2 is satisfied.

Then with $n \to \infty$ random processes $X_n$ weakly converges in $D^2[0,T]$ to a stationary Markov process $X$ which satisfies a stochastic differential equation of the following form

$$dX = [G(X^1) + \frac{1}{2}J_2]Xdt + D^{1/2}dw_2 \quad .$$

Now we can prove some results on asymptotic behavior of balls proportions in the generalized urn scheme.

### 3. Limit Theorems

In $D^N[0,T]$, $T > 0$, we consider random processes

$$z_n(t) = \sqrt{n+s}\,(y_{n+s}-\tilde{\vartheta}) \text{ for } \sum_{i=n}^{n+s} i^{-1} \leq t < \sum_{i=n}^{n+s+1} i^{-1} \quad ,$$

where $n \geq 1$, $y_n, \tilde{\vartheta}$ are N-dimensional vectors whose $N-1$ coordinates are equal corresponding coordinates of the vectors $X_n, \vartheta$, and the last coordinates are $\mu_n$, $\rho(\vartheta)$, $\mu_n = \gamma_n / n$.

**Theorem 1.** Let

1) $\sup_{n\geq 1}\ \sup_{x\in L_{L-1}} \sum_{i\in Z_+^N} ]i[q_n(i,x) = C_1$;

2) $X_n \to \vartheta$ with probability for $n \to \infty$ and for some $\varepsilon > 0$ the following conditions

hold on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$

3) there exists $r \geq \varepsilon$ for which $\sup\limits_{n\geq 1} \sum\limits_{i \in Z_+^N} ]i[^r q_n(i,x) = C_2$;

4) there exists continuously differentiable functions $q(i,x), i \in Z_+^N$ such that $\sum\limits_{i \in Z_+^N} q(i,x) = 1$ for some $q \geq 3$, $\sum\limits_{i \in Z_+^N} ]i[^q q(i,x) \leq C_3$, $|q(i,x) - q_n(i,x)| \leq \sigma_n, n \geq 1$, series $\sum\limits_{i \in Z_+^k} i^j \frac{\partial q(i,x)}{\partial x^k}, j,k = 1,2,\ldots,N-1$, converge uniformly;

5) $\lim\limits_{n \to \infty} \sigma_n \frac{n^{N+H}}{2H} = 0$;

6) matrix $A + \frac{1}{2}J_N$ is stable, where $A^{jk} = \rho(\vartheta)^{-1}\frac{\partial f^i(\vartheta)}{\partial x^k}, A^{jN} = 0, j = 1,2,\ldots,N-1$, $A^{Nk} = \frac{\partial \rho(\vartheta)}{\partial x^k}, k = 1,2,\ldots,N-1, A^{NN} = -1$.

Then random processes $z_n$ weakly converge in $D^N[0,T]$ to a stationary gaussian Markov process $z$ which satisfies the following stochastic differential equation

$$dz = (A + \frac{1}{2}J_N)zdt + \Sigma(\vartheta)^{1/2}dw_N \quad ,$$

where

$$\Sigma^{ij}(\vartheta) = \rho(\vartheta)^{-2}\sigma^{ij}(\vartheta), i = 1,2,\ldots,N-1 \; ,$$

$$\Sigma^{Nj}(\vartheta) = \rho(\vartheta)^{-1}\sigma^{Nj}(\vartheta), j = 1,2,\ldots,N-1, \Sigma^{NN}(\vartheta) = \sigma^{NN}(\vartheta) + F^{NN}(\vartheta) \quad .$$

**Proof.** On the basis of equality (5) from [1] we have

$$\mu_{n+1} = \mu_n + \frac{1}{n+1}\Big[]\beta_n(X_n)[-\mu_n\Big] n \geq 1 \tag{19}$$

$$\mu_1 = \gamma_1 \; , \tag{20}$$

therefore under condition 1)

$$\begin{aligned} E|\mu_{n+1}| &\leq \left(1 + \frac{1}{n+1}\right)E|\mu_n| + \frac{1}{n+1}E\left\{E\Big[]\beta_n(X_n)[|X_n\Big]\right\} \leq \\ &\leq \left(1 + \frac{1}{n+1}\right)E|\mu_n| + \frac{1}{n+1}C_1, n \geq 1 \quad . \end{aligned} \tag{21}$$

Since $X_n^i \in (0,1), i = 1,2,\dots,N-1, n \geq 1$, then due to (20), (21)

$$E\|y_n\| < \infty, n \geq 1 \tag{22}$$

On the basis of assertion e) of Lemma 1 and conditions 2), 5) are satisfied we have

$$\mu_n \to \rho(\vartheta) \text{ with probability } 1, n \to \infty,$$

and, therefore

$$y_n \to \tilde{\vartheta} \text{ with probability } 1, n \to \infty, \tag{23}$$

Let $\varepsilon_0 = \min(\varepsilon, \rho(\vartheta)/2)$. Denote by $Y_n(\tilde{\vartheta}, \varepsilon_0)$ a set of points $y \in U^N(\tilde{\vartheta}, \varepsilon_0)$ whose first $N-1$ coordinates $x(y)$ belong to $U^{N-1}(\vartheta, \varepsilon) \cap L_{N-1}$ and the last $y^N$ is such that $ny^N$ is a natural number greater or equal $\gamma_1$. Using equalities (6) from [1], (19), assertions a), b) of Lemma 1 from [1], and assertion b) of Lemma 1 with conditions 3), 4) we have for $y_n = y, y \in Y_n(\tilde{\vartheta}, \varepsilon_0), n \geq 1$

$$E(y_{n+1}|y_n) = y + \frac{1}{n}[R(y) + w_n(y)], \tag{24}$$

$$y_{n+1} - E(y_{n+1}|y_n) = \frac{1}{n} z(n, y), \tag{25}$$

$$\|w_n(y)\| \leq C_4\left[\sigma_n^{H/N+H} + n^{-1}\right], \tag{26}$$

$$R^k(y) = (y^N)^{-1} f^k(x(y)), k = 1,2,\dots,N-1, R^N(y) = \rho(x(y)) - y^N, \tag{27}$$

where

$$w_n^k(y) = (y^N)^{-1}[\sigma_n^k(x(y)) + \delta^k(x(y), ny^N)],$$

$$z^k(n, y) = \eta_n^k(x(y), ny^N), k = 1,2,\dots,N-1, w_n^N(y) =$$

$$= (n+1)^{-1}[y^N + nr(n, x(y)) - \rho(x(y))], z^N(n, y) = \frac{n}{n+1}\left[]\beta_n(x(y))[ - r_n(x(y))\right].$$

Due to condition 4), assertion d) of Lemma 1 and equalities (27) the function $R$ is differentiable at point $\tilde{\vartheta}$. Since $f(\vartheta) = 0$, then

$$R(\tilde{y}_n) = A(\hat{y}_n - \tilde{\vartheta}) + o(\|\tilde{y}_n - \tilde{\vartheta}\|) \tag{28}$$

for $\tilde{y}_n \to \tilde{\vartheta}, \tilde{y}_n \in Y_n(\tilde{\vartheta}, \varepsilon_0)$. From estimate (1), conditions 3), 4) and assertions a), b) of Lemma 1 we have

$$E\chi_{\{\|z(n,\tilde{y}_n)\|>R\}}\|z(n, \tilde{y}_n)\|^2 \leq C_5 R^{2-\tau}, R > 0,$$

$$|Ez^k(n,\tilde{y}_n)z^j(n,\tilde{y}_n) - \Sigma^{kj}(x(y_n))| \le C_6[\sigma_n^{\frac{H-1}{N+H}} + n^{-1} + f(x(\tilde{y}_n))] \quad . \tag{29}$$

Since for $\tilde{y}_n \to \tilde{\vartheta}$ we have $x(\tilde{y}_n) \to \vartheta$ and the functions $\sigma, f, \rho$ are continuous on $U^{N-1}(\vartheta,\varepsilon) \cap L_{N-1}$ from condition 4) and assertion c) of Lemma 1, then due to condition 5) we obtain

$$\lim |Ez^k(n,\tilde{y}_n)z^j(n,\tilde{y}_n) - \Sigma^{kj}(\vartheta)| = 0 \quad ,$$

$$\max(n^{-1}, \|\tilde{y}_n - \tilde{\vartheta}\|) \to 0 \tag{30}$$

where

$$\tilde{y}_n \in Y_n(\tilde{\vartheta},\varepsilon_0),\, k,j = 1,2,\ldots,N \quad .$$

Relations (22)-(26), (28)-(30) and conditions 5), 6) make it possible to use lemma 2 which gives the required result. The theorem is proved.

Calculating directly the limit distribution of $z(t)$ with $t \to \infty$ we obtain the following assertion.

**Corollary 1.** Under conditions of Theorem 1

$$\sqrt{n}\begin{bmatrix} X_n & -\vartheta \\ \gamma_{n/n} & -\rho(\vartheta) \end{bmatrix} \longrightarrow N(0,B)\ , \quad n \to \infty$$

in probability, where $N(0,B)$ is a normal random vector of $N$ dimensionality with zero mean and variance matrix

$$B = \int_0^\infty e^{(A + \frac{1}{2}J_n)t}\, \Sigma(\vartheta)\, e^{(A^T + \frac{1}{2}J_N)t}\, dt \quad .$$

**Remark 2.** Let the number of balls added to the urn at each step be constant and equal to $V \ge 1$ (as, e.g., in [3], [7], [8]). i.e.

$$\sum_{\substack{i \in Z_+^N \\ |i| = \nu}} q_n(i,x) = 1,\ x \in L_{n-1} \quad .$$

Then conditions 1), 3) of Theorem 1 are satisfied, $\gamma_n = \gamma_1 + (n-1)\nu,\ n \ge 1$, i.e., this is not a random variable and instead of $z_n$ it is sufficient to consider the corresponding $N-1$-dimensional random process generated by $X_n - \vartheta,\ n \ge 1$.

From relations (22)-(26), (28)-(30), Lemma 3.4 and Remark 1.2 of the paper [1] we obtain the following.

**Theorem 2.** Let $N = 2$ and the number of balls added to the urn at each step is equal to $\nu$ constant $\nu \geq 1$. Suppose also that the following conditions are satisfied:

1) with probability 1 $X_n \to \vartheta, n \to \infty$;

2) for some $\varepsilon > 0, x \in (\vartheta-\varepsilon, \vartheta+\varepsilon) \cap R(0,1)$ there exist continuously differentiable functions $q((i,\nu-i)^T, x), 0 \leq i \leq \nu$, such that $|q((i,\nu-i)^T,x) - q_n((i,\nu-i)^T,x)| \leq \sigma_n, n \geq 1$ ;

3)

$$f'(\vartheta) = -1/2, \lim_{n\to\infty} (n\, lnn)^{1/2}\sigma_n = 0 \ .$$

Then random processes

$$v_n(t) = \left[\frac{n+s}{ln(n+s)}\right]^{1/2} (X_{n+s} - \vartheta) \text{ for } \sum_{i=n}^{n+s} (ilni)^{-1} \leq$$

$$\leq t < \sum_{i=n}^{n+s+1} (ilni)^{-1}, n \geq 2 \ ,$$

converge in $D[0,T]$ for $n \to \infty$ to a stationary gaussian Markov process $v$ of the following form

$$av = -\frac{1}{2}vdt + \sigma^{1/2}dw \ ,$$

where

$$\sigma = \sum_{i=0}^{\nu} (i-\nu\vartheta)^2 q((i,\nu-i)^T,\vartheta) \ .$$

**Corollary 2.** If conditions of Theorem 2 are satisfied, then

$$\sqrt{\frac{n}{lnn}}(X_n - \vartheta) \to N(0,\sigma) \ ,$$

in probability for $n \to \infty$.

**Theorem 3.** Let all conditions of Theorem 2 be satisfied except the 3) which is replaced by the following

3.1)

$$f'(\vartheta) < -1/2, \lim_{s\to\infty} \left[\frac{s}{lnlns}\right]^{1/2} \alpha_s = 0 .$$

Then

$$\overline{\lim_{n\to inf}} \sqrt{\frac{n}{lnlnn}} (X_n - \vartheta) = \sqrt{\frac{\sigma}{-1-2} f'(\vartheta)} ,$$

$$\underline{\lim_{n\to inf}} \sqrt{\frac{n}{lnlnn}} (X_n - \vartheta) = -\sqrt{\frac{\sigma}{-1-2f'(\vartheta)}} .$$

In Theorems 1,2 the limit random processes are gaussian, and the limit distribution of variables $X_n - \vartheta$ is normal. It appears that if we discard the requirement of differentiability of functions $q(i,\cdot), i \in Z_+^N$, at point $\vartheta$ then the limit random processes may not be gaussian, as well as the limit distribution of variables $X_n - \vartheta$ not be unfinite divisible. The theorem given below stipulates the appropriate result. The proof of this resul is based on relations (22)-(26), (28)-(30), Remark 1 and Lemma 5.

**Theorem 4.** Let $N = 2, \vartheta \in (0,1)$, conditions 1-3, 5 of Theorem 1 be satisfied as well as the following

4) there are continuously differentiable on $(\vartheta-\varepsilon, \vartheta) \cap R(0,1)$, $(\vartheta, \vartheta+\varepsilon) \cap R(0,1)$ functions $q(i,\cdot), i \in Z_+^2$ such that $\sum_{i \in Z_+^2} q(i,x) = 1$, for some $q \geq 3$, $\sum_{i \in Z_+^2} ]i[^q q(i,x) \leq C_3$, $|q(i,x) - q_n(i,x)| \leq \alpha_n$, $n \geq 1$,

5) matrices $A_i + \frac{1}{2} J_2$ are stable, where $i = 1,2$, $A_1^{11} = \rho(\vartheta)^{-1} f'(\vartheta+0)$, $A_2^{11} = \rho(\vartheta)^{-1} f'(\vartheta-0)$, $A_1^{12} = A_2^{12} = 0$, $A_1^{21} = \rho'(\vartheta+0)$, $A_2^{21} = \rho'(\vartheta-0)$, $A_1^{22} = A_2^{22} = -1$.

Then random processes $z_n$ weakly converge in $D^2[0,T]$ to a stationary Markov process $z$, satisfying the following stochastic differentiable equation

$$dz = [A(z^1) + \frac{1}{2} J_2] z \, dt + \Sigma(\vartheta)^{1/2} dw_2 ,$$

where $A(z^1) = A_1$ for $z^1 \geq 0, A(z^1) = A_2$ for $z^1 < 0$.

**Corollary 3.** Let $N = 2$, $\vartheta \in (0,1)$ and the number of balls added to the urn at each step be constant and equal to $\nu \geq 1$.

Suppose that

1) $x_n \to \vartheta$ with probability 1, $n \to \infty$;

2) for some $\varepsilon > 0$ there exist continuously differentiable on $(\vartheta-\varepsilon,\vartheta)$ and $(\vartheta,\vartheta+\varepsilon)$ functions $q((i,\nu-i)^T,x), 0 \leq i \leq \nu$ such that

$$|q((i,\nu-i)^T,x) - q_n((i,\nu-i)^T,x)| \leq \sigma_n,\ n \geq 1 ;$$

3)

$$\max(f'(\vartheta+0), f'(\vartheta-0)) < -1/2,\ \lim_{n\to\infty} n^{1/2}\sigma_n = 0 \quad .$$

Then the limit distribution of random variables $\sqrt{n}(X_n - \vartheta)$ has the density of the following form

$$p(x) = C\begin{cases} \exp\left\{\dfrac{ax^2}{2f'(\vartheta+0)+1}\right\}, & x \geq 0, \\ \exp\left\{\dfrac{ax^2}{2f'(\vartheta-0)+1}\right\}, & x < 0, \end{cases}$$

where $C$ is a constant such that $\int_{-\infty}^{\infty} p(x)dx = 1$.

Corollary 3 follows from Theorem 4, Remark 2 and the fact that the limit distribution $z(t), t \to \infty$ has the density $p$. The distribution with density $p$ is not unfinite divisible.

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