WORKING PAPER

DIFFERENTIAL CALCULUS OF SET-VALUED MAPS.
AN UPDATE

Jean-Pierre Aubin

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International Institute
for Applied Systems Analysis
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IIASA

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FOREWORD

IIASA has played a crucial role in the development of the "graphical approach" to the differential calculus of set-valued maps, around J.-P. Aubin, H. Frankowska, R.T. Rockafellar and allowed to make contacts with Soviet and eastern European mathematicians (C. Olech, B. Pschenichnyiy, E. Polovinkin, V. Tihomirov, ...) who were following analogous approaches. Since 1981, they and their collaborators developed this calculus and applied it to a variety of problems, in mathematical programming (Kuhn-Tucker rules, sensitivity of solutions and Lagrange multipliers), in nonsmooth analysis (Inverse Functions Theorems, local uniqueness), in control theory (controllability of systems with feedbacks, Pontryagin's Maximum Principle, Hamilton-Jacobi-Bellman equations, observability and other issues), in viability theory (regulation of systems, heavy trajectories), ...

The first version of this survey appeared at IIASA in 1982, and constituted the seventh chapter of the book APPLIED NONLINEAR ANALYSIS published in 1984 by I. Ekeland and the author. Since then, many other results have been motivated by the successful applications of this calculus, and, may be unfortunately, other concepts (such the concept of intermediate tangent cone and derivatives introduced and used by H. Frankowska). Infinite-dimensional problems such as control problems or the more classical problems of calculus of variations require the use of adequate adaptations of the same main idea, as well as more technical assumptions.

The time and the place (IIASA) were ripe to update the exposition of this differential calculus. The Russian translation of APPLIED NONLINEAR ANALYSIS triggered this revised version.

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Introduction

There are few ideas in mathematics, but so many ways to implement them .... At each stage of the development of a branch, when the technical improvements require too much technical skill from the mathematicians, it is time to return to basic ideas.

This is what happened with the basic idea of differential calculus, when, despite a strong reluctance for using set-valued maps, the pressure of the many contemporary problems arising in systems theory (optimization, regulation, viability and control of evolution systems) forced many applied mathematicians to use set-valued maps as naturally as the familiar single-valued maps.

Natural inertia (or is it conservatism ?) led most of us to consider set-valued maps as ... a maps, and not as a graphs (or representative curves), as it should be[1], and as it was at the very origin of analytical geometry, when our ancestors used representative curves before the concept of function.

Fortunately, facts are there to force us to return to long forgotten tracks. During the last decades, the "graphical" side of set-valued maps took some preeminence among mathematicians using maximal monotone operators, graphical and epigraphical convergence, and the graphical derivatives we are about to describe. All the tool were there, though, but ignored. The road was paved by French and Polish mathematicians, Baire, Bouligand, Choquet, Kuratowski, Painlevé and many other, but abandoned for some reasons.

There is no doubt that mathematical programming and control theory provided in the sixties the motivation to study again set-valued maps. Let us mention the pioneer roles of C. Berge, A.F. Filippov and T. Ważewski during this crucial period. But at that time, the set-valued maps were mostly regarded as maps.

The beginning of the eighties saw the emergence of the concept of "graphical derivative", which goes back to Pierre de Fermat[2]. The idea

[1] at least in the instances of interest for us.

[2] Fermat was one of the most important originator in the history of mathematics. Even Newton did recognize explicitly that he received the hint of the differential calculus from Fermat's method of building tangents devised half a century earlier. Fermat was also

behind the construction of a differential calculus of set-valued maps is simple and is still the one to which all of us have been first acquainted during our teens. It starts with the concept of tangent to the graph of a function: the derivative is the slope of the tangent to the curve. We should say, now, that the tangent space to the graph of the curve is the graph of the differential. This is this statement that we take as a basis for adapting to the set-valued case the concept of derivative.

Consider a set-valued map $F : X \rightsquigarrow Y$, which is characterized by its graph (the subset of pairs (x, y) such that y belongs to $F(x)$).

We need first an appropriate notion of tangent cone to a set in a Banach space at a given point, which coincides with the tangent space when the set is an embedded differentiable manifold and with the tangent cone of convex analysis when the set is convex. At the time, experience shows that three tangent cones seem to be useful:

1. Bouligand's contingent cone[3]

2. Adjacent tangent cone[4]

3. Clarke's tangent cone[5]

They correspond to different regularity requirements. The tangent cone of Clarke is always convex. There already exists a sufficiently detailed calculus of these cones, which is exposed below.

Once a concept of tangent cone is chosen, we can associate with it a notion of derivative of a set-valued map F at a point (x, y) of its graph:

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the one who discovered that the derivative of a (polynomial) function vanishes when it reaches an extremum (Euler-Lagrange equations, Pontryagin's maximum principle are just implementations to infinite-dimensional problems of what should be called the FERMAT RULE). He also was the first to discover the "principle of least time" in optics, the prototype of the variational principles governing so many physical and mechanical laws. He shared with Descartes the independent invention of analytic geometry and with Pascal the creation of the mathematical theory of probability. He was on top of that a poet, a linguist, a lawyer and, if it has to be recalled, the author of the Fermat Theorems, ... consequences of a revolutionary treatment of number theory

[3]introduced in the thirties

[4]used by H. Frankowska under the name of intermediate tangent cone

[5]introduced in 1975

it is a set-valued map $F'(x, y)$ the graph of which is equal to the tangent cone to the graph of F at the point (x, y).

In this way, we associate with the contingent cone, the adjacent and the Clarke tangent cones the following concepts of derivatives:

1. contingent derivative, corresponding to the Gâteaux derivative,
2. adjacent derivative, corresponding to the Fréchet derivative,
3. circatangent derivative, corresponding to the continuous Fréchet derivative.

For instance, if dx is a direction in the space X, a direction dy in the space Y belongs to the **contingent derivative** $DF(x, y)(dx)$ of F at the point (x, y) in the direction dx if and only if the pair (dx, dy) belongs to the contingent cone to the graph of F at (x, y).

These derivatives keep enough properties of the derivatives of smooth functions to be quite efficient. They enjoy a pretty rich calculus, and such basic theorems of analysis as the inverse function theorem can be extended to the multivalued case.

Derivatives of set-valued maps (and also of nonsmooth single-valued maps) are set-valued maps, which are positively homogeneous. They are convex (in the sense that their graph is convex) when they depend in a "continuous" way of (x, y). Such maps, , are the set-valued analogues of continuous linear operators.

The chain rule is in particular an example of a property which remains (almost) true.

But what about Newton and Leibnitz, who introduced the derivatives as limits of differential quotients?

Our first duty is to characterize the various graphical limits as adequate limits of differential quotients. Unfortunately, the formulas become very often quite ugly, and nobody in a right frame of mind would have invented them from scratch if they were not derived from the graphical approach.

But all these limits are pointwise limits, which classify all these generalized derivatives in a class different from the class of distributional derivatives introduced by L. Schwartz and S. Sobolev in the fifties, for solving partial differential equations: Their objective was to keep the linearity of the differential operators, by allowing convergence of the differential quotients in weaker and weaker topologies, the price to be paid being that derivatives may no longer be functions, but distributions.

This survey presents only the definitions, the main properties and the calculus of the graphical derivatives of set-valued maps and epigraphical derivatives of extended real-valued functions, useful whenever the order relation of the real line plays a role, as in mathematical programming or Lyapunov style stability theory of dynamical systems.

The applications to optimization, control theory and viability theory are not described here.

We just provide a small bibliographical complement to the list of references of APPLIED NONLINEAR ANALYSIS

1 Tangent Cones

We devote this section to the definitions of some (and may be, too many) of the tangent cones which have been used in applications.

It is difficult to strike the right balance between simplicity (use only the contingent cones) and the needs of more results motivated by further studies.

We have chosen to postpone to the end of this presentation the dual concepts (normal cones, codifferential, generalized gradients) since their properties can be derived from the properties of the tangent cones.

We shall also provide the calculus in infinite-dimensional spaces, since it is required in the framework of control problems and of the calculus of variations, despite ugliness of the technical assumptions which, for the time, have not been simplified.

Definition 1.1 (Tangent cones) *Let $K \subset X$ be a subset of a topological vector space X and $x \in \overline{K}$ belong to the closure of K. We denote by*

(1) $$S_K(x) \ := \ \bigcup_{h>0} \frac{K-x}{h}$$

the cone spanned by $K - x$.

We introduce the three following tangent cones

1. *the* **contingent**[6] **cone** $T_K(x) := T_K^\natural(x)$, *defined by*

   (2) $$T_K(x) \ := \ \{ \ v \mid \liminf_{h\to 0+} d_K(x+hv)/h = 0 \ \}$$

2. *the* **adjacent**[7] **cone** $T_K^\flat(x)$, *defined by*

   (3) $$T_K^\flat(x) \ := \ \{ \ v \mid \lim_{h\to 0+} d_K(x+hv)/h = 0 \ \}$$

3. *the* **Clarke**[8] **tangent cone** $C_K(x)$, *defined by*

   (4) $$C_K(x) \ := \ \{ \ v \mid \lim_{h\to 0+, K\ni x'\to x} d_K(x'+hv)/h = 0 \ \}$$

[6]from Latin *contingere*, to touch on all sides, introduced by G. Bouligand

[7]from Latin *adjacere*, to lie near, recently introduced and applied under the name **intermediate cone** by H. Frankowska and the name of **derivable** cone by R. T. Rockafellar

[8]from Canadian *Frank H. Clarke*; we shall use the adjective **circatangent** to mention properties derived from this tangent cone, for instance, circatangent derivatives and epiderivatives

We see at once that these three tangent cones are closed, that these tangent cones to K and the closure $\overline{K}$ of K do coincide, that

$$(5) \qquad C_K(x) \subset T_K^\flat(x) \subset T_K(x) \subset \overline{S_K(x)}$$

and that

$$(6) \qquad \text{if } x \in \operatorname{Int}(K), \text{ then } C_K(x) = X$$

It is very convenient to use the following characterization of these cones in terms of sequences.

Proposition 1.1 *Let x belong to K.*

$$\begin{cases} i) & v \in T_K(x) \textit{ if and only if } \exists\, h_n \to 0+, \\ & \exists\, v_n \to v \textit{ such that } \forall\, n,\ x + h_n v_n \in K \\ ii) & v \in T_K^\flat(x) \textit{ if and only if } \forall\ h_n \to 0+, \\ & \exists\, v_n \to v \textit{ such that } \forall\, n,\ x + h_n v_n \in K \\ iii) & v \in C_K(x) \textit{ if and only if } \forall\ h_n \to 0+,\ \forall\, x_n \to x, \\ & (x_n \in K),\ \exists\, v_n \to v \textit{ such that } \forall\, n,\ x_n + h_n v_n \in K \end{cases}$$

Remark These tangent cones can be defined in terms of Kuratowski upper and lower limits of $\frac{K-x}{h}$, as the following statement shows:

Proposition 1.2 *Let x belong to K. The following equalities*

$$\begin{cases} i) & T_K(x) = \limsup_{h \to 0+} \frac{K-x}{h} \\ ii) & T_K^\flat(x) = \liminf_{h \to 0+} \frac{K-x}{h} \\ iii) & C_K(x) = \liminf_{h \to 0+,\, K \ni x' \to x} \frac{K-x'}{h} \end{cases}$$

hold true. □

Let us begin by proving an astonishing fact: the Clarke tangent cone $C_K(x)$ is always a closed convex cone.

Proposition 1.3 *The Clarke tangent cone $C_K(x)$ is a closed convex cone satisfying the following properties*

$$\begin{cases} i) & C_K(x) + T_K(x) \subset T_K(x) \\ ii) & C_K(x) + T_K^\flat(x) \subset T_K^\flat(x) \end{cases}$$

Proof

1. Let v_1 and v_2 belong to $C_K(x)$. For proving that $v_1 + v_2$ belongs to this cone, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $x_n \in K$ converging to x. There exists a sequence of elements v_{1n} converging to v_1 such that the elements $x_{1n} := x_n + h_n v_{1n}$ do belong to K for all n. But since x_{1n} does also converge to x in K, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall n, \ x_{1n} + h_n v_{2n} = x_n + h_n(v_{1n} + v_{2n}) \in K$$

This implies that $v_1 + v_2$ belongs to $C_K(x)$ because the sequence of elements $v_{1n} + v_{2n}$ converges to $v_1 + v_2$.

2. Now, let v_1 belong to $T_K(x)$ and v_2 belong to $C_K(x)$. There exists a sequence of elements $h_n > 0$ converging to 0 and v_{1n} converging to v_1 such that the elements $x_{1n} := x + h_n v_{1n}$ do belong to K for all n. But since x_{1n} does also converge to x in K, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall n, \ x_{1n} + h_n v_{2n} = x + h_n(v_{1n} + v_{2n}) \in K$$

This implies that $v_1 + v_2$ belongs to $T_K(x)$

3. The proof is analogous for the cone $T_K^\flat(x)$. □

Remark We can interpret the above inclusions by saying that the Clarke tangent cone is contained in the Minkowski difference (or the asymptotic cone, the convex kernel) of the adjacent and contingent cones. Let us recall that the **Minkowski difference** $K \ominus L$ of two subsets K and L is the subset $\bigcap_{x \in L}(K - x)$ of elements z such that

$$L + z \subset K$$

When P is a closed cone, the Minkowski difference $P \ominus P$ is always a convex cone. □

Unfortunately, the price to pay for enjoying this convexity property of the Clarke tangent cones is that they may often be reduced to the trivial cone $\{0\}$.

But we shall show in just a moment that the Clarke tangent cone and that the contingent cone do coincide at those points x where the set-valued map $x \leadsto T_K(x)$ is lower semicontinuous.

Definition 1.2 (Sleek Subsets) *We shall say that a subset $K \subset X$ is* **sleek** *at $x \in K$ if the set-valued map*

$$K \ni x' \leadsto T_K(x') \text{ is lower semicontinuous at } x$$

We shall say that it is **sleek** *if and only if it is sleek at every point of K.*

We shall prove later that smooth manifolds and convex subsets of finite dimensional vector-spaces are sleek.

But for the time, we just deduce from Theorem 1.1 below this quite important regularity property:

Theorem 1.1 (Tangent Cones of Sleek Subsets) *Let K be a weakly closed subset of a reflexive Banach space. If K is sleek at $x \in K$, then the contingent and Clarke tangent cones do coincide, and consequently, are convex.*

For that purpose, when X is a normed space, it is quite useful to introduce the following notations:

$$(7) \quad \begin{cases} i) & D_\uparrow d_K(x)(v) \\ & := \liminf_{h\to 0+}(d_K(x+hv) - d_K(x))/h \\ ii) & D_\uparrow^\flat d_K(x)(v) \\ & := \limsup_{h\to 0+}(d_K(x+hv) - d_K(x))/h \\ iii) & C_\uparrow d_K(x)(v) \\ & := \limsup_{h\to 0+,\, x'\to x}(d_K(x'+hv) - d_K(x'))/h \end{cases}$$

which will be justified later[9].

We need the estimates we provide below to prove our theorem as well as other consequences.

[9] they are the contingent, adjacent and circatangent epiderivatives of the distance functions d_K.

Theorem 1.2 *Let K be a weakly closed subset of a reflexive Banach space and $\pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\|y - z\| = d_K(y)$. Then we have the following inequalities:*

$$(8) \quad \begin{cases} i) & D_\uparrow d_K(y)(v) \leq d(v, T_K(\pi_K(y))) \\ ii) & D_\uparrow^\flat d_K(y)(v) \leq d(v, T_K^\flat(\pi_K(y))) \\ iii) & C_\uparrow d_K(y)(v) \leq d(v, C_K(\pi_K(y))) \end{cases}$$

Proof

1. We begin by proving these inequalities when y belongs to K. Indeed, for all $w \in X$, inequality $d_K(y+hv) \leq d_K(y+hw) + h\|v - w\|$ implies that

$$\begin{cases} i) & \liminf_{h\to 0+} d_K(y+hv)/h \leq \|v - w\| \text{ when } w \in T_K(y) \\ ii) & \limsup_{h\to 0+} d_K(y+hv)/h \leq \|v - w\| \text{ when } w \in T_K^\flat(y) \end{cases}$$

2. Assume now that $y \notin K$ and choose $z \in \pi_K(y)$. Then

$$\begin{cases} (d_K(y+hv) - d_K(y))/h \\ \leq (\|y - z\| + d_K(z+hv) - d_K(y))/h \\ = d_K(z+hv)/h \end{cases}$$

Since z belongs to K, inequalities (8) with $y = z$ imply that

$$\begin{cases} i) & D_\uparrow d_K(y)(v) \leq d(v, T_K(z)) \\ ii) & D_\uparrow^\flat d_K(y)(v) \leq d(v, T_K^\flat(z)) \end{cases}$$

For proving inequality

$$C_\uparrow d_K(y)(v) \leq d(v, C_K(z)) = \inf_{w\in C_K(x)} \|v - w\|$$

we first observe that when $y \notin K$,

$$\forall z \in \pi_K(y), \ \forall x \in K, \ \|z - x\| \leq 2\|y - x\|$$

Hence

$$\sup_{\substack{h\leq\alpha \\ \|y-x\|\leq\beta}} (d_K(y+hv) - d_K(y))/h \leq \sup_{\substack{h\leq\alpha \\ \|z-x\|\leq 2\beta}} d_K(z+hw)/h + \|v - w\|$$

so that our claim is established. We end the proof by taking the infimum when z ranges over $\pi_K(y)$. □

We shall need actually the following

Corollary 1.1 *Let K be a weakly closed subset of a reflexive Banach space. Then*

$$d_K(x+tv) - d_K(x) \;\leq\; \int_0^t d(v, T_K(\pi_K(x+\tau)v))d\tau$$

Proof We set $g(t) := d_K(x+tv)$. Since $g(\cdot)$ is locally lipschitzean, it is almost everywhere differentiable. Theorem 1.2 implies that $g'(\tau) \leq d(v, T_K(\pi_K(x+\tau)v))$. We then integrate from 0 to t. □

Theorem 1.3 *Let K be a weakly closed subset of a reflexive Banach space. Let us consider a set-valued map $F : K \rightsquigarrow X$ satisfying*

$$(9) \qquad \begin{cases} i) & F \text{ is lower semicontinuous} \\ ii) & \forall\, x \in K,\;\; F(x) \subset T_K(x) \end{cases}$$

Then,

$$(10) \qquad \forall\, x \in K, \;\; F(x) \subset C_K(x)$$

Proof Let us take $x \in K$ and $v \in F(x)$. Since

$$\forall\, z \in \pi_K(y+tv),\; \forall\, x \in K,\; \|z-x\| \;\leq\; 2\|y+tv-x\| \;\leq 2\|x-y\| + 2t\|v\|$$

we infer that, for all $\epsilon > 0$, $y \in K$ close to x and τ small enough, the lower semicontinuity of F at x implies that

$$\begin{cases} d(v, T_K(\pi_K(y+\tau v))) & \leq \;\; d_K(v, F(\pi_K(y+\tau v))) \\ & \leq \;\; d(v, F(x)) + \epsilon \;\; = \;\; \epsilon \end{cases}$$

because v belongs to $F(x)$ by assumption. Corollary 1.1 thus implies that, for all $y \in K$ close to x and for all $t \in]0, h]$ for some positive h,

$$d_K(y+tv) \;\leq\; \int_0^t d(v, T_K(\pi_K(y+\tau v)))d\tau \;\;\leq\;\; t\epsilon$$

We have proved that v belongs to $C_K(x)$. □

Remark In particular, we deduce the following characterization of the tangent cones:

$$(11) \qquad \begin{cases} i) & T_K(x) & := & \{ v \mid D_\uparrow d_K(x)(v) \;\; \leq 0 \} \\ ii) & T_K^\flat(x) & := & \{ v \mid D_\uparrow^\flat d_K(x)(v) \;\; \leq 0 \} \\ iii) & C_K(x) & := & \{ v \mid C_\uparrow d_K(x)(v) \;\; \leq 0 \} \end{cases}$$

These equalities (11) suggest to extend the definition of these tangent cones to elements which are outside K.

Definition 1.3 *Let K be a subset of a normed space X and x belong to X. We extend the notions of contingent and adjacent cones to K at points outside K in the following way:*

$$(12) \quad \begin{cases} i) & T_K(x) := \{ v \mid D_\uparrow d_K(x)(v) \leq 0 \} \\ ii) & T_K^\flat(x) := \{ v \mid D_\uparrow^\flat d_K(x)(v) \leq 0 \} \\ iii) & C_K(x) := \{ v \mid C_\uparrow d_K(x)(v) \leq 0 \} \end{cases}$$

We deduce at once from Theorem 1.2 the following corollary:

Corollary 1.2 *The tangent cones at points outside K are related to the tangent cones at their projections in the following way:*

$$\begin{cases} i) & \forall\, y \in \pi_K(x), \; T_K(y) \subset T_K(x) \\ ii) & \forall\, y \in \pi_K(x), \; T_K^\flat(y) \subset T_K^\flat(x) \\ iii) & \forall\, y \in \pi_K(x), \; C_K(y) \subset C_K(x) \end{cases} \quad \square$$

It will be convenient to name the points x of a subset K where two of the above tangent cones do coincide.

Definition 1.4 *We shall say that a subset $K \subset X$*

1. *is* **pseudo-convex** *at $x \in K$ if and only if either one of the equivalent properties*

$$(13) \quad \begin{cases} a/ & T_K(x) = \overline{S_K(x)} =: \overline{\bigcup_{h>0} \frac{K-x}{h}} \\ b/ & K \subset x + T_K(x) \end{cases}$$

holds true.

2. *is* **derivable** *at $x \in K$ if and only if*

$$(14) \quad T_K^\flat(x) = T_K(x)$$

Remark We shall justify later why we are led to introduce this ménagerie of tangent cones. Each of them corresponds to a classical regularity requirement. We shall see that the contingent cone is related to

Gàteaux derivatives, the adjacent cone to the Fréchet derivative and the Clarke tangent cone to the continuous Fréchet derivative.

If $L \subset K$ is a subset of K, Bouligand has also introduced the **paratingent**[10] **cone** $P_K^L(x)$ to K relative to L at $x \in L$, defined by

$$(15) \qquad P_K^L(x) := \{ v \mid \limsup_{h \to 0+, L \ni x' \to x} d_K(x' + hv)/h = 0 \}$$

and we observe that

$$(16) \qquad T_K(x) - T_K^\flat(x) \subset P_K^K(x) \quad \square$$

We can also introduce open tangent cones. Let us mention the two following ones:

Definition 1.5 *Let x belong to K.*

1. *The cone $D_K(x)$ defined by*

$$(17) \qquad \begin{cases} v \in D_K(x) \text{ if and only if } \forall\ h_n \to 0+, \\ \forall\, v_n \to v, \text{ we have } \forall\, n,\ x + h_n v_n \in K \end{cases}$$

is called the **Dubovicki-Miliutin tangent cone**

2. *The cone $H_K(x)$ defined by*

$$(18) \qquad \begin{cases} v \in H_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \forall\, x_n \to x, \\ (x_n \in L),\ \forall\, v_n \to v, \text{ we have } \forall\, n,\ x_n + h_n v_n \in K \end{cases}$$

is called the **hypertangent cone.**

We see at once that

$$(19) \qquad \begin{cases} i) & H_K(x) \subset C_K(x) \cap D_K(x) \\ ii) & D_K(x) \subset T_K^\flat(x) \\ iii) & H_K(x) + T_K^\flat(x) \subset D_K(x) \quad \square \end{cases}$$

[10] Shi Shuzhong showed that when K is the closure of its interior, the contingent cones and the paratingent cones (relative to the boundary) are generically equal (they coincide on a G_δ dense of the boundary), a consequence of Choquet's theorem.

2 Tangent Cones to Convex sets

For convex subsets K, the situation is dramatically simplified by the fact that the Clarke tangent cones, the adjacent and the contingent cones coincide with the closed cone spanned by $K - x$.

Proposition 2.1 (Tangent Cones to Convex Sets) *Let us assume that K is convex. Then the contingent cone $T_K(x)$to K at x is convex and*

$$C_K(x) \quad = \quad T_K^\flat(x) \quad = \quad T_K(x) \quad = \quad \overline{S_K(x)}$$

In particular,

$$N_K(x) \; = \; N_K^\circ(x) \; = \; \{p \in X^\star \mid \max_{y in K} < p, y > = < p, x >\}$$

Remark We shall denote by $T_K(x)$ the common value of these cones, and call it the **tangent cone** to the convex subset K at x. □

Proof We begin by stating the following consequence of convexity:

(20) $\quad \forall\, v \in S_K(x),\ \exists\, h > 0,\ \text{ such that } \forall\, t \in [0, h],\ x + tv \ \in \ K$

since we can write that

$$x + tv \quad = \quad (1 - \frac{t}{h})x + \frac{t}{h}(x + hv)$$

is a convex combination of elements of K.

It is enough to prove that $S_K(x)$ is contained in the Clarke tangent cone. Let $v := (y - x)/h$ belong to $S_K(x)$ (where $y \in K$ and $h > 0$) and let us consider sequences of elements $h_n > 0$ and $x_n \in K$ converging to 0 and x respectively. We see that $v_n := (y - x_n)/h$ converges to v and that

$$x_n + h_n v_n \quad = \quad (1 - \frac{h_n}{h})x_n + \frac{h_n}{h}y \ \in \ K$$

since it is a convex combination of elements of K. □

Actually, convex subsets of finite dimensional vector-spaces are sleek:

Theorem 2.1 *Let K be a closed convex subset of a finite dimensional vector-space X. Then K is sleek.*

Proof It is equivalent to prove that the graph of the set-valued map $K \ni x \rightsquigarrow N_K(x)$ is closed[11].

But this is obviously the case: let us consider sequences of elements $x_n \in K$ and $p_n \in N_K(x_n)$ converging to x and p respectively. Then inequalities

$$\forall y \in K, \ < p_n, y > \ \leq \ < p_n, x_n >$$

implies by passing to the limit inequalities

$$\forall y \in K, \ < p, y > \ \leq \ < p, x >$$

which state that p belongs to $N_K(x)$. Hence the graph is closed, so that the set-valued map $T_K(\cdot)$ is lower semicontinuous, since the dimension of X is finite. □

We observe easily that the normal cones are contained in the barrier cone of a convex subset K:

Proposition 2.2 *Let K be convex. Then, for all $x \in K$,*

$$\begin{cases} i) & N_K(x) \ \subset \ b(K) \\ ii) & \text{the asymptotic cone } b(K)^- \ \subset \ T_K(x) \end{cases}$$

It may be useful to characterize the interior of the tangent cone to a convex subset.

Proposition 2.3 (Interior of a Tangent Cone) *Assume that the interior of $K \subset X$ is not empty. Then*

$$\forall x \in K, \quad \mathrm{Int}(T_K(x)) \ = \ \bigcup_{h>0} (\frac{\mathrm{Int}(K) - x}{h}$$

Furthermore, the graph of the set-valued map $K \ni x \rightsquigarrow \mathrm{Int}(T_K(x))$ is open.

Proof

1. The union of the interiors of $(K - x)/h$ being open, it is contained in the interior of the tangent cone. Since K is convex, so are the

[11] see [5, Proposition 3.1.18., p.117].

cones $S_K(x)$, and thus, the closures of $S_K(x)$ and of their interiors do coincide. Then it is enough to prove that if v belongs to the interior of $S_K(x)$, it is interior to one of the $(K-x)/h$.

Let $\eta > 0$ such that $v + \eta B \subset S_K(x)$. If $x + v$ belongs to the interior of K, the proof is completed. If not, let us choose $x_0 \in \operatorname{Int}(K)$ and set $v_0 := x_0 - x$. Hence $v - \eta v_0/\|v_0\|$ belongs to $S_K(x)$, and thus, there exists some $h > 0$ such that $x + h(v - \eta v_0/\|v_0\|)$ belongs to K. By setting

$$\lambda := h\eta/(h\eta + \|v_0\|)$$

we deduce that

$$x + (1-\lambda)hv \;=\; \lambda x_0 + (1-\lambda)(x + h(v - \eta v_0/\|v_0\|))$$

Since x_0 belongs to the interior of K, $x + h(v - \eta v_0/\|v_0\|)$ belongs to K and λ is smaller than 1, we deduce from the convexity of K that $x + (1-\lambda)hv$ belongs to the interior of K, i.e., that v belongs to the interior of $(K-x)/(1-\lambda)h$.

2. Let us take a pair (x_0, v_0) in the interior of the graph of $T_K(\cdot)$. Then, by the above statement, there exists $h > 0$ such that

$$v_0 \in (\operatorname{Int}(K) - x_0)/h$$

Hence there exists $\eta > 0$ such that

$$x_0 + hv_0 + \eta B \;=\; x_0 + h(v + \frac{\eta}{h}B) \;\subset\; \operatorname{Int}(K)$$

Therefore

$$(x_0 + \frac{\eta}{2}B) \times (v_0 + \frac{\eta}{2h}B) \;\subset \operatorname{Graph}(T_K(\cdot)) \quad \square$$

Remark Convex subsets are star-shaped around each of their elements and thus, share with them some properties.

Definition 2.1 (Star-Shaped Subsets) *A subset K is said to be* **star-shaped around** $x \in K$ *if*

$$\forall\, y \in K,\;\; \forall\, \lambda \in [0,1],\;\; x + \lambda(y - x) \in K$$

We observe the following

Lemma 2.1 *If $K \subset X$ is star-shaped around $x \in K$, then it is pseudo-convex and derivable at this point.*

For the convenience of the reader, we list below some useful calculus of tangent cones to convex subsets (see [5, Section 4.1.]). The subsets K, K_i, L, M, ... are assumed to be convex.

Properties of Tangent and Normal Cones (1)

1. If $K \subset L$, then

$$T_K(x) \subset T_L(x) \;\; \& \;\; N_L(x) \subset N_K(x) \tag{21}$$

2. If $K_i \subset X_i$, $\quad (i = 1, \cdots, n)$, then

$$\begin{cases} i) & T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n T_{K_i}(x_i) \\ ii) & N_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n N_{K_i}(x_i) \end{cases} \tag{22}$$

3. If K_1 and K_2 are contained in X, then

$$\begin{cases} i) & T_{K_1+K_2}(x_1 + x_2) = \overline{T_{K_1}(x_1) + T_{K_2}(x_2)} \\ ii) & N_{K_1+K_2}(x_1 + x_2) = N_{K_1}(x_1) \cap N_{K_2}(x_2) \end{cases} \tag{23}$$

In particular, if P is a closed vector subspace, then

$$\begin{cases} i) & T_{K+P}(x_1 + x_2) = \overline{T_{K_1}(x_1) + P} \\ ii) & N_{K_1+P}(x_1 + x_2) = N_{K_1}(x_1) \cap P^{\perp} \end{cases} \tag{24}$$

4. If $B \in \mathcal{L}(X, Y)$, then

$$\begin{cases} i) & T_{B(K)}(x) = \overline{B(T_K(x))} \\ ii) & N_{B(K)}(x) = B^{\star^{-1}} N_K(x) \end{cases} \tag{25}$$

5. If $L \subset X$ and $M \subset Y$ are ***closed convex*** subsets and $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that the ***qualification constraint condition***

$$0 \in \mathrm{Int}(M - A(L)) \tag{26}$$

holds true, then

$$\begin{cases} i) & T_{L \cap A^{-1}(M))} = T_L(x) \cap A^{-1} T_M(Ax) \\ ii) & N_{L \cap A^{-1}(M))} = N_L(x) + A^{\star} N_M(Ax) \end{cases} \tag{27}$$

Properties of Tangent and Normal Cones (2)

6. If $M \subset Y$ is a ***closed convex*** subset and if $A \in \mathcal{L}(X,Y)$ is a continuous linear operator such that

$$(28) \qquad 0 \in \mathrm{Int}(\mathrm{Im}(A) - M)$$

then

$$(29) \qquad \begin{cases} i) & T_{A^{-1}(M)}(x) = A^{-1}T_M(Ax) \\ ii) & N_{A^{-1}(M)}(x) = A^{\star}N_M(Ax) \end{cases}$$

7. If K_1 and K_2 are closed convex subsets contained in X and satisfy[12]

$$(30) \qquad 0 \in \mathrm{Int}(K_1 - K_2)$$

then

$$(31) \qquad \begin{cases} i) & T_{K_1 \cap K_2}(x) = T_{K_1}(x) \cap T_{K_2}(x) \\ ii) & N_{K_1 \cap K_2}(x) = N_{K_1}(x) + N_{K_2}(x) \end{cases}$$

8. If $K_i \subset X$, $(i = 1, \ldots, n)$, are closed and convex and if there exists $\gamma > 0$ such that

$$(32) \qquad \forall x_i \mid \|x_i\| \leq \gamma, \ \bigcap_{i=1}^{n}(K_i - x_i) \neq \emptyset$$

Then,

$$(33) \qquad \begin{cases} i) & T_{\bigcap_{i=1}^n K_i}(x) = \bigcap_{i=1}^n T_{K_i}(x) \\ ii) & N_{\bigcap_{i=1}^n K_i}(x) = \sum_{i=1}^n N_{K_i}(x) \end{cases}$$

[12]This property is false when assumption (30)is not satisfied. Take for instance two balls K_1 and K_2 tangent at a point x. The tangent cone to the intersection $\{x\}$ is reduced to $\{0\}$, whereas the intersection of the tangent cones is a hyperplane. This shows that we cannot dispense of the **constraint qualification** assumptions in the calculus of tangent cones to inverse images and intersections

3 Inverse Function Theorems

We derive from the basic Inverse stability Theorem[13] a series of equivalent results which extend in several ways the Liusternik Inverse Fuction Theorem. We refer to [?] for more powerful results based on the concept of "variations" of set-valued map defined on any metric space, and which are related to images of the unit ball by derivatives of set-valued maps when the definition space is normed.

[13] See [4, Theorem 3.1]:

Theorem 3.1 (Inverse Stability Theorem) *Let X and Y be two Banach spaces. We introduce a sequence of continuous linear operators $A_n \in \mathcal{L}(X,Y)$, a sequence of closed subsets $K_n \subset X$.*

Let us consider elements $x_n^\star$ of the subsets K_n such that both $x_n^\star$ converges to $x_0^\star$ and $A_n x_n$ converges to y_0.

We posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(34)\quad \begin{cases} \forall x_n \in K_n \cap B(x_0,\eta), \\ A_n S_{K_n}(x_n) \cap B_Y \subset A_n\left(T_{K_n}(x_n) \cap cB_X\right) + \alpha B_Y \end{cases}$$

Let us set $l := c/(1-\alpha)$, $\rho < \eta/3l$ and consider elements y_n and x_{0n} satisfying:

$$(35)\quad \begin{cases} i) & x_{0n} \in K_n \cap B(x_0,\eta/3),\ A_n x_{0n} \in B(y_0,\rho) \\ ii) & y_n \in A_n(K_n) \cap B(y_0,\eta/3) \end{cases}$$

Then, for any $l' > l$ and $n > 0$, there exist solutions $\widehat{x_n}$ satisfying

$$(36)\quad \begin{cases} i) & \widehat{x_n} \in K_n \ \&\ A_n\widehat{x_n} = y_n \\ ii) & \|\widehat{x_n} - x_{0n}\| \leq l'\|y_n - A_n x_{0n}\| \end{cases}$$

so that

$$(37)\quad \begin{cases} d(x_0, K_n \cap A_n^{-1}(y_n)) \leq l\|y_n - A_n x_{0n}\| \\ \leq \|x_0 - x_{0n}\| + l\|y_n - y_0\| + l\|y_0 - A_n x_{0n}\| \end{cases}$$

converges to 0 when x_{0n} converges to x_0 and both $A_n x_{0n}$ and $y_n \in A_n K_n$ converge to y_0.

Theorem 3.2 (Criterion of Pseudo-Lipschitzeanity) *Let K be a closed subset of a Banach space X and $A \in \mathcal{L}(X,Y)$ be a continuous linear operator from X to another Banach space Y. Let us assume that for some $x_0 \in K$, there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(38) \qquad \begin{cases} \forall\, x \in K \cap B(x_0,\eta), \\ AS_K(x) \cap B_Y \;\subset\; A(T_K(x) \cap cB_X) + \alpha B_Y \end{cases}$$

Then the set-valued map

$$(39) \qquad A(K) \;\ni\; y \rightsquigarrow A^{-1}(y) \cap K$$

is **pseudo-lipschitzean** *around (Ax_0, x_0): For any x_1 close to x_0 and $y \in K$ close to Ax_0,*

$$(40) \qquad d(x_1, A^{-1}(y) \cap K) \;\le\; l\|y - Ax_1\|$$

Remark Assumption (4.3) can be written in the form

$$\sup_{x \in B_K(x_0,\eta)} \; \sup_{u \in S_K(x)} \; \inf_{v \in T_K(x), Av \in Au + \alpha\|u\|B} \frac{\|v\|}{\|u\|} \;\le\; c$$

Observe that when $\alpha = 0$, assumption (4.3) implies that $A(K)$ is **pseudo-convex** on a neighborhood of $Ax_0 \in K$ since for all x in this neighborhood,

$$S_{A(K)}(Ax) = A(S_K(x)) \subset A(T_K(x)) \subset T_{A(K)}(Ax)$$

Hence, we can regard stability assumption (4.3) as a weakened local pseudo-convexity. □

In particular, we obtain the following inverse mapping theorem:

Theorem 3.3 (Linear Inverse Function Theorem) *Let K be a closed subset of a Banach space X and $A \in \mathcal{L}(X,Y)$ be a continuous linear operator from X to another Banach space Y. Let us assume that for some $x_0 \in K$, there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(41) \qquad \begin{cases} \forall\, x \in K \cap B(x_0,\eta), \\ B_Y \;\subset\; A(T_K(x) \cap cB_X) + \alpha B_Y \end{cases}$$

Then Ax_0 belongs to the interior of $A(K)$ and the set-valued map $y \rightsquigarrow A^{-1}(y) \cap K$ is **pseudo-lipschitzean** *around (Ax_0, x_0).*

It implies the following apparently more general statement:

Theorem 3.4 (Set-Valued Inverse Function Theorem) *Let us consider a closed set-valued map $F : X \rightsquigarrow Y$, an element (x_0, y_0) of its graph and let us assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall\, (x,y) \in \mathrm{Graph}(F) \cap B((x_0,y_0),\eta), \\ \forall\, v \in Y,\ \exists\, u \in X,\ \exists\, w \in Y \quad \textit{such that}\ v\ \in DF(x,y)(u) + w \\ \textit{and}\ \|u\| \ \leq\ c\|v\| \quad \& \quad \|w\| \ \leq\ \alpha\|v\| \end{cases}$$

Then y_0 belongs to the interior of the image of F and F^{-1} is pseudo-lipschitzean around (x_0, y_0).

which, actually, is equivalent, because, by taking for F the restriction of A to K, or, even more generally, the restriction of a differentiable single-valued map, we infer that

Theorem 3.5 (Constrained Inverse Function Theorem) *Let X and Y be two Banach spaces. We introduce a (single-valued) continuous map $f : X \mapsto Y$, a closed subset $K \subset X$ and an element x_0 of K.*

We assume that f is differentiable on a neighborhood of x_0 and we posit the following **stability assumption***: there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(42) \qquad \begin{cases} \forall\, x \in K \cap B(x_0,\eta), \\ B_Y \ \subset\ f'(x)(T_K(x) \cap cB_X) + \alpha B_Y \end{cases}$$

Then $f(x_0)$ belongs to the interior of $f(K)$ and the set-valued map $y \rightsquigarrow f^{-1}(y) \cap K$ is **pseudo-lipschitzean** *around $(f(x_0), x_0)$.*

We obtain as a consequence the Liusternik Inverse Function Theorem:

Corollary 3.1 (Liusternik Theorem) *Let X and Y be two Banach spaces. We introduce a (single-valued) continuous map f from X to Y. We assume that f is continuously differentiable on a neighborhood of x_0 and we posit the following* **surjectivity assumption**

$$f'(x_0) \text{ is surjective}$$

Then the set-valued map $y \rightsquigarrow f^{-1}(y)$ is **pseudo-lipschitzean** *around $(f(x_0), x_0)$.*

Proof Since the continuous linear operator $f'(x_0)$ is surjective, we deduce from the Banach Theorem that there exists a constant c such that

$$\forall\, v \in Y,\ \exists\, u \in X \mid f'(x_0)u = v \ \ \& \ \ \|u\| \leq c\|v\|$$

Since $x \mapsto f'(x)$ is continuous at x_0, we infer that

$$\begin{array}{l} \forall\, v \in Y,\ \exists\, u \in X \quad \text{such that} \\ f'(x)u = v + w \ \ \& \ \ \|u\| \leq c\|v\|,\ \|w\| \leq \|f'(x) - f'(x_0)\|\|u\| \end{array}$$

so that $\|w\| \leq \alpha\|v\|$ when x is close to x_0. □

We can extend this theorem to the case of set-valued maps by introducing and adequate definition of strongly sleek map.

Definition 3.1 (Strongly Sleek Sets and Maps) *We shall say that a closed subset K is* **strongly sleek** *at $x_0 \in K$ if the cone-valued map $K \ni x \rightsquigarrow T_K(x)$ is* **strongly lower semicontinuous** *at x_0 in the sense that*

$$\lim_{x \to x_0} \sup_{u \in T_K(x_0) \cap B_X} d(u, T_K(x)) = 0$$

We shall say that F is **strongly sleek** *at a point (x_0, y_0) of its graph if its graph is strongly sleek at this point*[14].

With this definition, we can state a natural set-valued version of Liusternik's Theorem

[14]Namely, if

$$\sup_{(u,v) \in \mathrm{Graph}(DF(x_0,y_0)),\ \max_{\|u\|,\|v\|} \leq 1} d((u,v), \mathrm{Graph}(DF(x,y)))$$

converges to 0 when (x,y) converges to (x_0, y_0).

Theorem 3.6 (Set-Valued Liusternik Theorem) *Let us consider reflexive Banach spaces X and Y, a closed set-valued map $F : X \rightsquigarrow Y$ and an element (x_0, y_0) of its graph. Let us assume that F is strongly sleek at (x_0, y_0). If*

$$DF(x_0, y_0) \text{ is surjective}$$

then y_0 belongs to the interior of the image of F and F^{-1} is pseudo-lipschitzean around (x_0, y_0). If the dimension of Y is finite, it is sufficient to assume that F is sleek at (x_0, y_0).

Actually, this results follows (and thus, is equivalent) to its "constrained linear" version.

Theorem 3.7 (Pointwise Inverse Function Theorem) *Let X and Y be reflexive Banach spaces, K be a weakly closed subset of X and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator. If K is strongly sleek at x_0 and if*

$$AT_K(x_0) \quad = \quad Y$$

then Ax_0 belongs to the interior of $A(K)$ and the set-valued map $y \rightsquigarrow A^{-1}(y) \cap K$ is **pseudo- lipschitzean** *around (Ax_0, x_0). If the dimenssion of Y is finite, it is sufficient to asume that K is sleek at x_0.*

Proof We have to prove that in both cases, the stability assumption is satisfied. The proof of the first case is easy. There exists a constant $c > 0$ such that, for all v in the unit sphere S_Y, there exists a solution u_0 to the equation $Au = v$ such that $\|u_0\| \leq c\|v\|\|$, thanks to Robinson-Ursescu's Theorem, because $T_K(x_0)$ is a closed convex cone, K being sleek at x_0.

Since K is actually strongly sleek at x_0, we can associate with any $\epsilon > 0$ an $\eta > 0$ such that, for all $u_0 \in X$ and all $x \in B_K(x_0, \eta)$, there exists $u \in T_K(x)$ such that $\|u - u_0\| \leq \epsilon\|u_0\|$.

Hence any $v \in S_Y$ can be written $v = Au + w$ where $\|u\| \leq (1 + \epsilon)c\|v\|$ and $\|w\| \leq \|A\|\|u_0 - u\| \leq \alpha\|v\|$ when $\epsilon \leq \alpha/\|A\|$.

When the dimension of Y is finite, the unit sphere S_Y is compact. We know that for any $v_i \in S_Y$, there exits a solution u_{0i} to the equation $Au = v_i$ such that $\|u_{0i}\| \leq c\|v_i\|\|$. Hence for any $\epsilon > 0$ and v_i, there exist $\eta_i > 0$ such that, for all $x \in B_K(x_0, \eta)$, there exists $u \in T_K(x)$ such that $\|u_i - u_{0i}\| \leq \epsilon\|u_{0i}\|/2\|A\|$.

We can cover S_Y by p balls $B(v_i, \epsilon/2)$ so that, by taking $\eta := \min_{i=1,\ldots,p} \eta_i$, we obtain that for any $v \in S_Y$ and any $x \in B_K(x_0, \eta)$, there exist $u_i \in T_K(x)$ and $w_i \in Y$ related by the equation $v = Au_i + w_i$ where $\|u_i\| \leq c\|v_i\| = c$ and where $\|w_i\| \leq \|v - Au_{0i}\| + \|A\|\|u_{0i} - u_i\| \leq \epsilon$. □

We provide now theorems on local uniqueness and injectivity of set-valued maps.

Definition 3.2 *Let $F : X \rightsquigarrow Y$ be a set-valued map. We shall say that it enjoys* **local inverse univocity** *around an element (x^*, y^*) of its graph if and only if there exists a neighborhood $N(x^*)$ such that*

$$\{ \, x \mid \text{such that } y^* \in F(x) \, \} \ \cap \ N(x^*) \ = \ \{x^*\}$$

If the neighborhood $N(x^*)$ coincides with the domain of F, F is said to have **(global) inverse univocity**.

We shall say that it is **locally injective** *around x^* if and only if there exists a neighborhood $N(x^*)$ such that, for all $x_1 \neq x_2 \in N(x^*)$, we have $F(x_1) \cap F(x_2) = \emptyset$. It is said to be* **(globally) injective** *if we can take for neighborhood $N(x^*)$ the whole domain of F.*

Since $0 \in DF(x^*, y^*)(0)$, we observe that to say that the "linearized system" $DF(x^*, y^*)$ enjoys the inverse univocity amounts to saying that the inverse image $DF(x^*, y^*)^{-1}(0)$ contains only one element, i.e., that its kernel $\mathrm{Ker}DF(x^*, y^*)$ is equal to 0, where the kernel is naturally defined by

$$\mathrm{Ker}DF(x^*, y^*) \ := \ DF(x^*, y^*)^{-1}(0)$$

Theorem 3.8 *Let F be a set-valued map from a finite dimensional vector-space X to a Banach space Y and (x^*, y^*) belong to its graph.*

If the kernel of the contingent derivative $DF(x^\star, y^\star)$ of F at $(x^\star, y^\star)$ is equal to $\{ 0 \}$, then there exists a neighborhood $N(x^\star)$ such that

$$(43) \qquad \{ x \text{ such that } y^\star \in F(x) \} \cap N(x^\star) = \{x^\star\}$$

Let us assume that there exits $\gamma > 0$ such that $F(x^\star + \gamma B)$ is relatively compact and that F has a closed graph. If for all $y \in F(x^\star)$ the kernels of the paratingent[15] derivatives $PF(x^\star, y)$ of F at $(x^\star, y)$ are equal to $\{0\}$, then F is locally injective around $x^\star$.

Proof

1. Assume that the conclusion (43) is false . Then there exists a sequence of elements $x_n \neq x^\star$ converging to $x^\star$ satisfying

$$\forall n \geq 0, \quad y^\star \in F(x_n)$$

Let us set $h_n := \|x_n - x^\star\|$, which converges to 0, and

$$u_n := (x_n - x^\star)/h_n$$

The elements u_n do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) u_n does converge to some u different from 0. Since the above equation can be written

$$\forall n \geq 0, \quad y^\star + h_n 0 \in F(x^\star + h_n u_n)$$

we deduce that

$$0 \in DF(x^\star, y^\star)(u)$$

Hence we have proved the existence of a non zero element of the kernel of $DF(x^\star, y^\star)$, which is a contradiction.

2. Assume that F is not locally injective. Then there exists a sequence of elements $x_n^1, x_n^2 \in N(x^\star)$, $x_n^1 \neq x_n^2$, converging to $x^\star$ and y_n satisfying

$$\forall n \geq 0, \quad y_n \in F(x_n^1) \cap F(x_n^2)$$

Let us set $h_n := \|x_n^1 - x_n^2\|$, which converges to 0, and

$$u_n := (x_n^1 - x_n^2)/h_n$$

The elements u_n do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) u_n does converge to some u different from 0.

[15] by definition, the graph of the paratingent derivative $PF(x.y)$ of F at (x, y) is the paratingent cone to the graph of F at (x, y).

Then for all large n

$$y_n \in F(x_n^1) \cap F(x_n^2) \ := \ F(x_n^2 + h_n u_n) \cap F(x_n^2) \ \subset \ F(x^\star + \gamma B)$$

we deduce that a subsequence (again denoted) y_n converges to some $y \in F(x^\star)$ (because Graph(F) is closed).

Since the above equation implies that

$$\forall\, n \geq 0, \quad y_n + h_n 0 \ \in \ F(x_n^2 + h_n u_n)$$

and we deduce that

$$0 \ \in \ PF(x^\star, y)(u)$$

Hence we have proved the existence of a non zero element of the kernel of $PF(x^\star, y)$, which is a contradiction. □

4 Calculus of Tangent Cones

We shall present now a calculus of tangent cones, from which we shall deduce a calculus of derivatives of set-valued maps and a calculus of epi-derivatives.

4.1 Subsets and Products

If $K \subset L$, then

$$(44) \qquad T_K(x) \subset T_L(x) \ \& \ T_K^\flat(x) \subset T_L^\flat(x)$$

If $K_i \subset X$, $(i = 1, \cdots, n)$, then

$$(45) \qquad \begin{cases} i) & T_{\bigcup_{i=1}^n K_i}(x) = \bigcup_{i=1}^n T_{K_i}(x) \\ ii) & T^\flat_{\bigcup_{i=1}^n K_i}(x) = \bigcup_{i=1}^n T^\flat_{K_i}(x) \end{cases}$$

If $K_i \subset X_i$, $(i = 1, \cdots, n)$, then

$$(46) \qquad \begin{cases} i) & T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) \subset \prod_{i=1}^n T_{K_i}(x_i) \\ ii) & T^\flat_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n T^\flat_{K_i}(x_i) \\ iii) & C_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n C_{K_i}(x_i) \end{cases}$$

4.2 Inverse Images

Let us consider now two topological vector spaces X and Y, subsets $L \subset X$ and $M \subset Y$ and a differentiable single-valued map f from X to Y. The following statement is obvious:

Proposition 4.1 *For any $x \in L \cap f^{-1}(M)$, we always have*

$$\begin{cases} i) & T_{L \cap f^{-1}(M)}(x) \subset T_L(x) \cap f'(x)^{-1} T_M(f(x)) \\ ii) & T^{\flat}_{L \cap f^{-1}(M)}(x) \subset T^{\flat}_L(x) \cap f'(x)^{-1} T^{\flat}_M(f(x)) \end{cases}$$

We shall deduce from the Constrained Inverse Function Theorem[16] converse inclusions.

---

[16]Let us recall this statement

Theorem 4.1 (Constrained Inverse Function Theorem) *Let X and Y be two Banach spaces. We introduce a (single-valued) continuous map $f : X \mapsto Y$, a closed subset $K \subset X$ and an element x_0 of K.*

We assume that f is differentiable on a neighborhood of x_0 and we posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall x \in K \cap B(x_0, \eta), \\ B_Y \subset f'(x)(T_K(x) \cap cB_X) + \alpha B_Y \end{cases}$$

Then $f(x_0)$ belongs to the interior of $f(K)$ and the set-valued map $y \rightsquigarrow f^{-1}(y) \cap K$ is **pseudo-lipschitzean** *around $(f(x_0), x_0)$.*

Theorem 4.2 *Let X and Y be Banach spaces, $L \subset X$ and $M \subset Y$ be* **closed** *subsets, f be a continuously differentiable map around an element $x_0 \in L \cap f^{-1}(M)$. If X and Y are finite dimensional vector-spaces, we posit the pointwise transversality condition*

$$(47) \qquad f'(x_0)(C_L(x_0)) - C_M(f(x_0)) = Y$$

Then

$$\left\{\begin{array}{ll} i) & T_L^\flat(x_0) \cap f'(x_0)^{-1} T_M(f(x_0)) \subset T_{L\cap f^{-1}(M)}(f(x_0)) \\ ii) & T_L^\flat(x_0) \cap f'(x_0)^{-1} T_M^\flat(f(x_0)) = T_{L\cap f^{-1}(M)}^\flat(f(x_0)) \\ iii) & C_L(x_0) \cap f'(x_0)^{-1} C_M(f(x_0)) \subset C_{L\cap f^{-1}(M)}(f(x_0)) \end{array}\right.$$

Otherwise, we have to replace the pointwise transversality condition by the local transversality condition condition *: there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$(48) \qquad \left\{\begin{array}{l} \forall\, x \in L \cap B(x_0, \eta),\ \ y \in M \cap B(f(x_0), \eta) \\ B_Y \ \subset \ f'(x)(T_L^\flat(x) \cap cB_X) - T_M(y) + \alpha B_Y \end{array}\right.$$

As a consequence, we infer that $L \cap f^{-1}(M)$ is sleek (respectively derivable) whenever L and M are sleek (respectively derivable).

Proof Let us prove for instance the inclusion for the Clarke tangent cones. Take any sequence of elements $x_n \in L \cap f^{-1}(M)$ which converges to x. Let us take any $u \in C_L(x)$ such that $f'(x_0)u \in C_M(f(x))$. Hence for any sequence $h_n > 0$, there exist sequences u_n and v_n converging to u and $f'(x_0)u$ respectively such that, for all $n \geq 0$,

$$x_n + h_n u_n \in L \ \ \& \ \ f(x_n) + h_n v_n \in M$$

We apply now the Constrained Inverse Function Theorem 3.5 to the subset $L \times M$ of $X \times Y$ and the continuous map $f \ominus \mathbf{1}$ associating to any (x, y) the element $f(x) - y$, since we can write

$$K \ := \ L \cap f^{-1}(M) \ = \ (f \ominus \mathbf{1})^{-1}(0) \cap (L \times M)$$

It is obvious that the transversality condition (48) implies the stability assumption of the Constrained Inverse Function Theorem. The pair $(x_n + h_n u_n, f(x_n) + h_n v_n)$ belongs to $L \times M$ and

$$(f \ominus \mathbf{1})(x_n + h_n u_n, f(x_n) + h_n v_n) \ \text{ converges to } \ 0$$

because f is continuously differentiable at x_0.

Therefore, by the Constrained Inverse Function Theorem 3.5, there exits a solution $(\widehat{x_n}, \widehat{y_n}) \in L \times M$ to the equation $(f \ominus \mathbf{1})(\widehat{x_n}, \widehat{y_n}) = 0$ (i.e., $\widehat{y_n} = f(\widehat{x_n})$) such that

$$\|x_n + h_n u_n - \widehat{x_n}\| + \|f(x_n) + h_n v_n - \widehat{y_n}\| \leq l\|f(x_n + h_n u_n) - f(x_n) - h_n v_n - 0\|$$

Hence $\widehat{u_n} := (x_n - \widehat{x_n})/h_n$ converges to u, and for all $n \geq 0$, we know that $x_n + h_n u_n$ belongs to $L \cap f^{-1}(M)$ because $x_n + h_n \widehat{u_n} = \widehat{x_n}$ belongs to L and $f(x_n + h_n \widehat{u_n}) = \widehat{y_n}$ belongs to M. □

We list now three useful corollaries of this theorem:

Corollary 4.1 (Tangent Cones to Inverse Images) *Assume that $M \subset Y$ is a closed subset and that f is a continuously differentiable map around an element $x_0 \in f^{-1}(M)$.*

When the dimension of X and Y is finite, we suppose that

$$\operatorname{Im}(f'(x_0)) \; + \; C_M(f(x_0)) \; = \; Y$$

Then

$$\begin{cases} i) & T_{f^{-1}(M)}(x_0) \; = \; f'(x_0)^{-1} T_M(f(x_0)) \\ ii) & T^{\flat}_{f^{-1}(M)}(x_0) \; = \; f'(x_0)^{-1} T^{\flat}_M(f(x_0)) \\ iii) & C_{f^{-1}(M)}(x_0) \; \subset \; f'(x_0)^{-1} C_M(f(x_0)) \end{cases}$$

Otherwise, we assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$\begin{cases} \forall\, x \in B(x_0, \eta), \;\; y \in B(f(x_0), \eta) \cap M \\ B_Y \;\; \subset \;\; \operatorname{Im}(f'(x)) \cap cB_X) + T_M(y) + \alpha B_Y \end{cases}$$

Corollary 4.2 (Tangent Cone to an Intersection) *If K_1 and K_2 are closed subsets contained in X. If X is a finite dimensional vector-space, we assume that*

$$C_{K_1}(x) \; - \; C_{K_2}(x) \; = \; X$$

Then

$$\begin{cases} i) & T^{\flat}_{K_1}(x) \cap T_{K_2}(x) \; \subset \; T_{K_1 \cap K_2}(x) \\ ii) & T^{\flat}_{K_1}(x) \cap T^{\flat}_{K_2}(x) \; = \; T^{\flat}_{K_1 \cap K_2}(x) \\ iii) & C_{K_1}(x) \cap C_{K_2}(x) \; \subset \; C_{K_1 \cap K_2}(x) \end{cases}$$

Otherwise, we suppose that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$\begin{cases} \forall\, x \in K_1 \cap B(x_0,\eta), \;\; y \in B(x_0,\eta) \cap K_2 \\ B_Y \;\; \subset \;\; AT^{\flat}_{K_1}(x) \cap cB_X) - T_{K_2}(y) + \alpha B_Y \end{cases}$$

Finally, for a finite intersection, we can state:

Corollary 4.3 (Tangent Cone to a Finite Intersection) *Let us consider n closed subsets $K_i \subset X$. When the dimension of X is finite, we assume that*

$$\forall\, v_1, \ldots, v_n \in X, \quad \bigcap_{i=1}^{n} (C_{K_i}(x_0) - v_i) \;\; \neq \;\; \emptyset$$

Then

$$\begin{cases} i) & \bigcap_{i=1}^{n} T^{\flat}_{K_i}(x_0) \; = \; T^{\flat}_{\bigcap_{i=1}^{n} K_i}(x_0) \\ ii) & \bigcap_{i=1}^{n} C_{K_i}(x_0) \; \subset \; C_{\bigcap_{i=1}^{n} K_i}(x_0) \end{cases}$$

If the dimension of X is infinite, we assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$\begin{cases} \forall\, x_i \in K_i \cap B(x_0,\eta), \; \forall\, v_i \in X, \\ \exists\, w_i \in X \;\; , \exists\, u \in \bigcap_{i=1}^{n} (T_{K_i}(x_i) - v_i - w_i) \\ \text{such that } \|u\| \;\; \leq \;\; c \max_{i=1,\ldots,n} \|v_i\| \;\; \& \;\; \|w_i\| \;\; \leq \;\; \alpha \max_{i=1,\ldots,n} \|v_i\| \end{cases}$$

4.3 Direct Images

Let us consider now two topological vector spaces X and Y, a subset $K \subset X$ and a differentiable single-valued map f from X to Y. The following statement is obvious:

Proposition 4.2 *For any $y \in f(K)$, we have*

$$\begin{cases} i) & \overline{\bigcup_{x\in K\cap f^{-1}(y)} f'(x)(T_K(x))} \subset T_{f(K)}(y) \\ ii) & \overline{\bigcup_{x\in K\cap f^{-1}(y)} f'(x)(T_K^\flat(x))} \subset T_{f(K)}^\flat(y) \end{cases}$$

It is not that easy to find elegant sufficient conditions implying the equality

$$(49) \qquad \overline{\bigcup_{x\in K\cap f^{-1}(y)} f'(x)(T_K(x))} = T_{f(K)}(y)$$

Let us review some simple ones:

Proposition 4.3 *If $A \in \mathcal{L}(X,Y)$ is a continuous linear operator and if K is pseudo-convex at some point x of $K \cap A^{-1}(y)$, then*

$$\overline{AT_K(x)} = T_{A(K)}(y)$$

In particular, when K is convex, we have

$$\bigcap_{x\in K\cap A^{-1}(y)} \overline{A(T_K(x))} = T_{A(K)}(y)$$

The Criterion of Pseudo-Lipschitzeanity[17] provides a more general sufficient condition:

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[17]Let us recall this statement

Theorem 4.8 (Criterion of Pseudo-Lipschitzeanity) *Let K be a closed subset of a Banach space X and $A \in \mathcal{L}(X,Y)$ be a continuous linear operator from X to another Banach space Y. Let us assume that for some $x_0 \in K$, there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall x \in K \cap B(x_0,\eta), \\ AS_K(x)\cap B_Y \subset A(T_K(x)\cap cB_X) + \alpha B_Y \end{cases}$$

Then the set-valued map

$$A(K) \ni y \rightsquigarrow A^{-1}(y)\cap K$$

is **pseudo-lipschitzean** *around (Ax_0, x_0): For any x_1 close to x_0 and $y \in K$ close to Ax_0,*

$$d(x_1, A^{-1}(y)\cap K) \le l\|y - Ax_1\|$$

Theorem 4.4 *Let K be a closed subset of a reflexive Banach space X and $A \in \mathcal{L}(X,Y)$ be a continuous linear operator from X to another Banach space Y. Let us assume that for some $x_0 \in K \cap A^{-1}(y_0)$, there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall\, x \in K \cap B(x_0, \eta), \\ AS_K(x) \cap B_Y \;\subset\; A(T_K(x) \cap cB_X) + \alpha B_Y \end{cases}$$

Then, if X is supplied with the weak topology, we obtain the equality

$$\overline{AT_K(x_0)} \;=\; T_{A(K)}(y_0) \tag{50}$$

Proof Let v belong to $T_{A(K)}(y_0)$. Then there exist sequences of elements $h_n > 0$ and v_n converging to 0 and v respectively such that

$$y_0 + h_n v_n \;=\; Ax_n \;\in\; A(K) \tag{51}$$

The point is to choose solutions $x_n \in K$ to the above equation (51) and a solution $x_0 \in K$ to the equation $Ax_0 = y_0$ such that

a subsequence of $u_n := (x_n - x_0)/h_n$ converges to some u

Such an element u belongs to the contingent cone $T_K(x_0)$ and is a solution to the equation $Au = v$.

Since the set-valued map $A(K) \ni y \rightsquigarrow K \in A^{-1}(y)$ is pseudo-lipschitzean around (y_0, x_0) by the Criterion of Pseudo-Lipschitzeanity, there exist a constant l' and solutions $x_n \in K$ to the equation (51) such that

$$\|x_0 - x_n\| \;\le\; \|y_0 - y_0 - h_n v_n\| \;=\; h_n \|v_n\|$$

Therefore, the sequence of elements u_n is bounded, so that a subsequence (again denoted) u_n converges (weakly when the dimension of X is infinite) to some u. □

Remark Any sufficient condition implying that for some $x_0 \in K \cap A^{-1}(y_0)$, the set-valued map $A(K) \ni y \rightsquigarrow K \cap A^{-1}(y)$ is pseudo-lipschitzean will automatically imply the above equality (50) between the contingent cone to the image and the closure of the image of the contingent cone. □

The sequence of elements $u_n := (x_n - x_0)/h_n$ satisfies

$$\text{(52)} \qquad \left\{ \begin{array}{lrcl} i) & u_n & \in & S_K(x_0) \\ ii) & Au_n & = & v_n \end{array} \right.$$

Therefore, any properness criterion[18] of the map A from the closed cone $\overline{S_K(x_0)}$ spanned by $K - x_0$ to Y implies equality (50).

In particular, the Closed Range Theorem provides such a simple criterion. Since the barrier cone of $S_K(x_0)$ is its polar cone, which is the **subnormal cone** $N_K^0(x_0)$ to K at x_0, we then obtain the following statement:

Theorem 4.5 *Let K be a closed subset of a reflexive Banach space X and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator from X to another Banach space Y. Let us assume that for some $x_0 \in K \cap A^{-1}(y_0)$,*

$$\text{(53)} \qquad \mathrm{Im}(A^\star) \ + \ N_K^0(x_0) \ = \ X^\star$$

Then, if X is supplied with the weak topology, we obtain the equality (50).

Remark Criterion (53), which is easy to use, is much too strong, since it requires implicitly that the inverse $y \rightsquigarrow K \cap A^{-1}(y)$ of the restriction of A to K is actually locally single-valued. Indeed, by taking the polar cones in equality (53), we obtain

$$\ker(A) \cap \overline{co}(S_K(x_0)) \ = \ \{0\}$$

Actually, when the dimension of X is finite, we have a stronger criterion:

[18]Banach's Closed Graph Theorem allows to assume that A is surjective: It is sufficient to decompose A as the product $\tilde{A} \circ \phi$ of the canonical surjection ϕ from X onto its factor space $X/\ker(A)$ and the associated bijective map $\tilde{A}$, which is an isomorphism. Then the properness of A is equivalent to the properness of ϕ.

Proposition 4.4 *If the dimension of X is finite and if for some $x_0 \in K \cap A^{-1}(y_0)$, then condition*

$$\ker(A) \cap T_K(x_0) \;=\; \{0\}$$

implies equality (50).

Proof We have to prove that the sequence of solutions u_n to (51) are bounded. If not, $\|u_n\|$ will go to ∞. Hence a subsequence of elements $\widehat{u_n} := u_n/\|u_n\|$ of the unit sphere, which is compact, converges to some $u \neq 0$. Since the sequence $A\widehat{u_n} = v_n/\|u_n\|$ converges to 0, we infer that u belongs to the kernel of A. It belongs also to the contingent cone $T_K(x_0)$, since we can write

$$x_0 + h_n\|u_n\|\widehat{u_n} \;\in\; K$$

and since $h_n\|u_n\|$ converges to 0. □

5 Tangent Cones in Lebesgue Spaces

Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and X be a Banach space. Let us consider a sequence of measurable set-valued maps

$$K : \omega \in \Omega \rightsquigarrow K(\omega) \subset X.$$

We associate to it the subsets $\mathcal{K}$ of $L^p(\Omega, X)$ defined by

$$\mathcal{K} \;:=\; \{\; x(\cdot) \in L^p(\Omega, X) \;\mid\; \text{for almost all } \; \omega \in \Omega, \;\; x(\omega) \in K(\omega) \;\}$$

We shall characterize the adjacent and contingent cones to such a subset in termes of the tangent cones to the subsets $K(\omega)$.

Theorem 5.1 *Let us assume that the set-valued map K is measurable and that the subset $\mathcal{K}$ is not empty. Then*

$$\begin{cases} \{v(\cdot) \in L^p(\Omega, X) \mid \text{for almost all } \; \omega, \;\; v(\omega) \in T^\flat_{K(\omega)}(x(\omega))\} \\ \subset \;\; T^\flat_{\mathcal{K}}(x(\cdot)) \;\; \subset \;\; T_{\mathcal{K}}(x(\cdot)) \\ \subset \;\; \{v(\cdot) \in L^p(\Omega, X) \mid \text{for almost all } \omega, \;\; v(\omega) \in T_{K(\omega)}(x(\omega))\} \end{cases}$$

Proof

1. Let $v(\cdot)$ belong to the first subset. We have to prove that that when $h > 0$ goes to 0, there exists functions $v_h(\cdot) \in L^p(\Omega, X)$ converging to $v(\cdot)$ such that

   $$\text{for almost all } \omega \in \Omega, \;\; x(\omega) + hv_h(\omega) \;\; \in \;\; K(\omega)$$

   Let us set:

   $$a_h(\omega) \;\; := \;\; d(v(\omega), (K(\omega) - x(\omega))/h)$$

   The function a_h is measurable and converge to 0 almost everywhere because for almost all $\omega \in \Omega$, $v(\omega)$ belongs to the adjacent cone to $K(\omega)$ at $x(\omega)$.

Since

$$\text{for almost all } \omega, \ a(\omega) \ \leq \ \|v(\omega)\|$$

and since the right-hand side of this inequality belongs to $L^p(\Omega)$, we deduce from Lebesgue's Theorem that the functions $a_h(\cdot)$ do converge to 0 in $L^p(\Omega)$. Let us introduce now the subsets $L_h(\omega)$ defined by

$$L_h(\omega) \ := \ \{z \in \overline{K(\omega)} \ | \ d(v(\omega), (z - x(\omega))/h) = a_h(\omega)\}$$

It is clear that the set-valued map $L_h(\cdot)$ is also measurable. The Measurable Selection Theorem allows us to choose a measurable selection $z_h(\cdot)$ of the set-valued map $L_h(\cdot)$.

We define now the functions $v_h(\cdot)$ by

$$v_h(\omega) \ := \ (z_h(\omega) - x(\omega))/h)$$

They are measurable, satisfy

$$\|v_h(\omega) - v(\omega)\| \ = \ a_h(\omega)$$

and thus, converges to $v(\cdot)$ in $L^p(\Omega; X)$ since $a_h(\cdot)$ converges to 0 in $L^p(\Omega)$. We infer that $v(\cdot)$ belongs to $T_K^\flat(x(\cdot))$ because

$$\text{for almost all } \Omega \in \Omega, \ x(\omega) + hv_h(\omega) \ \in \ K(\omega)$$

2. Let us choose now some $v(\cdot)$ in the contingent cone to the subset K. Then there exist subsequences $h_n > 0$ and $v_n(\cdot)$ converging respectively to 0 and to $x(\cdot)$ in $L^p(\Omega; X)$ and satisfying

$$\text{for almost all } \omega \in \Omega, \ x(\omega) + h_n v_n(\omega) \ \in \ K(\omega)$$

Then a subsequence (again denoted) $v_n(\cdot)$ converges almost everywhere to $v(\cdot)$ and consequently, for almost all ω, $v(\omega)$ belongs to the contingent cone to the subset $K(\omega)$ at $x(\omega)$. □

Naturally, we infer that

Corollary 5.1 *Let us assume that the set-valued map K is measurable and that the subset K is not empty. If the subsets $K(\omega)$ are derivable, so is K and*

$$T_K(x(\cdot)) \ = \ \{v(\cdot) \in L^p(\Omega, X) \ | \ \text{for almost all } \omega, \ v(\omega) \in T_{K(\omega)}(x(\omega))\}$$

6 Derivatives of Set-Valued maps

We shall derive from each concept of tangent cone to a subset an associated concept of **graphical derivative** of a set-valued map F from a topological vector space X to another Y.

The idea is very simple, and goes back to the protohistory of the differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function.

The tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \mapsto f'(x)u$.

It is possible to implement this idea for any set-valued map F since we have introduced (unfortunately, several) ways to implement the concept of tangency for any subset of a topological vector space. Therefore, in the framework of a given problem, we can choose the adequate concept of tangent cone, and thus, regard this tangent cone to the graph of the set-valued map F at some point (x, y) of its graph as the graph of the associated "graphical" derivative of F at this point (x, y).

Since the tangent cones are at least ... cones, all these derivatives are at least **positively homogeneous** set-valued maps (also called **processes**). This is what remains of the familiar, but luxurious, requirement of linearity.

However, they are closed convex processes, i.e., set-valued analogues of continuous linear operators, when the tangent cones happen to be closed and convex (this is the case when we use the Clarke tangent cone or the Minkowski differences of the contingent or adjacent cones).

Hence, we start with some definitions and notations.

Definition 6.1 *Let $F : X \rightsquigarrow Y$ be a set-valued map from a topological vector space X to another Y.*

We introduce the three following graphical derivatives

1. *the* **contingent derivative** $DF(x,y) := D^{\sharp}F(x,y)$, *defined by*

$$\text{(54)} \qquad \mathrm{Graph}(DF(x,y)) \;:=\; T_{\mathrm{Graph}(F)}(x,y)$$

2. *the* **adjacent derivative** $D^{\flat}F(x,y)$ *defined by*

$$\text{(55)} \qquad \mathrm{Graph}(D^{\flat}F(x,y)) \;:=\; T^{\flat}_{\mathrm{Graph}(F)}(x,y)$$

3. *the* **circatangent derivative** $CF(x,y)$ *defined by*

$$\text{(56)} \qquad \mathrm{Graph}(CF(x,y)) \;:=\; C_{\mathrm{Graph}(F)}(x,y)$$

We shall say that F *is* **sleek** *at* $(x,y) \in \mathrm{Graph}(F)$ *if and only if*

$$(x',y') \rightsquigarrow \mathrm{Graph}(DF)(x',y') \ \textit{is lower semicontinuous at} \ (x,y)$$

and it is **sleek** *if it is sleek at every point of its graph.*

We shall say that F *is* **derivable** *at* $(x,y) \in \mathrm{Graph}(F)$ *if and only if the contingent and adjacent derivatives coincide:*

$$DF(x,y) \;:=\; D^{\flat}F(x,y)$$

and that it is **derivable** *if it is derivable at every point of its graph.*

Finally, we shall say that F *is* **pseudo-convex** *at* $(x,y) \in \mathrm{Graph}(F)$ *if and only if*

$$\forall\, x' \in \mathrm{Dom}(F), \;\; F(x') \subset DF(x,y)(x'-x) + y$$

We see at once that these three graphical derivatives are closed processes and that

$$\text{(57)} \qquad \forall\, u, \;\; CF(x,y)(u) \subset D^{\flat}F(x,y)(u) \subset DF(x,y)(u)$$

Naturally, the circatangent derivative is always a closed convex process, and the contingent derivative is a closed convex process whenever F is sleek at (x,y).

When $F := f$ is single-valued, we set

$$Df(x) := Df(x, f(x)), \quad D^{\flat} f(x) := D^{\flat} f(x, f(x)), \quad Cf(x) := Cf(x, f(x))$$

We see easily from Propositions 6.2 and 6.3 below that

$$(58) \quad \begin{cases} i) & Df(x))(u) = f'(x)u & \text{if } f \text{ is Gâteaux differentiable} \\ ii) & D^{\flat} f(x) = f'(x)u & \text{if } f \text{ is Fréchet differentiable} \\ iii) & Cf(x)(u) = f'(x)u & \text{if } f \text{ is continuously differentiable} \end{cases}$$

Restrictions $F := f|_K$ of single-valued maps f to subsets $K \subset X$ provide a wide class of set-valued maps defined by

$$f|_K(x) := \begin{cases} f(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

for which we obtain the following formulas: if f is continuously differentiable around a point $x \in K$, then **the derivative of the restriction is the restriction of the derivative to the corresponding tangent cone:**

$$(59) \quad \begin{cases} i) & D(f|_K)(x) := D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)} \\ ii) & D^{\flat}(f|_K)(x) := D^{\flat}(f|_K)(x, f(x)) = f'(x)|_{T^{\flat}_K(x)} \\ iii) & C(f|_K)(x) := C(f|_K)(x, f(x)) = f'(x)|_{C_K(x)} \end{cases}$$

Actually, this follows from the useful

Proposition 6.1 *Let X and Y be normed spaces, f a continuously differentiable operator from an open subset $\Omega \subset X$ to Y, $L \subset X$ and $M \subset Y$ closed subsets. Let $F : X \rightsquigarrow Y$ be the set-valued map defined by:*

$$F(x) := \begin{cases} f(x) - M & \text{when } x \in L \\ \emptyset & \text{when } x \notin L \end{cases}$$

Let (x, y) belong to the graph of F. Then its adjacent derivative is equal to

$$D^{\flat} F(x, y)(u) := \begin{cases} f'(x)u - T^{\flat}_M(f(x) - y) & \text{when } u \in T^{\flat}_L(x) \\ \emptyset & \text{when } u \notin T^{\flat}_L(x) \end{cases}$$

The same formula holds true for the circatangent derivative and the Clarke tangent cones.

Proof

Let v belong to $DF(x,y)(u)$ and let us prove that it belongs to $f'(x)u - T_M(f(x)-y)$. We know that for all $h_n > 0$ converging to 0, there exist sequences u_n and v_n converging to u and v respectively such that $x + h_n u_n$ belongs to L and $y + h_n v_n$ belongs to $f(x + h_n u) - M$ for all n. This implies that u belongs to to $T_L^\flat(x)$ and, since we can write $f(x + h_n u_n) = f(x) + h_n(f'(x)u + o(h_n))$, that $f(x) - y + h_n(f'(x)u - v_n + o(h_n))$ belongs to M. Hence $f'(x)u - v$ does belongs to $T_M^\flat(f(x) - y)$.

Conversely, assume that u belongs to $T_L^\flat(x)$ and that $f'(x)u - v$ belongs to $T_M^\flat(f(x)-y)$. Hence, for all sequence $h_n > 0$ converging to 0, there exist sequence u_n amd w_n converging to u and $f'(x)u - v$ such that $x + h_n u_n$ and $f(x) - y + h_n w_n$ belongs to L and M respectively. Then $v_n := f'(x)u + o(h_n) - w_n$ converges to v and satisfies $y + h_n v_n \in f(x) + h_n u_n - M$. □

Remark For contingent derivatives, we can only prove that

$$DF(x,y)(u) \quad \subset \quad \begin{cases} f'(x)u - T_M(f(x)-y) & \text{when} \quad u \in_L (x) \\ \emptyset & \text{when} \quad u \notin T_L(x) \end{cases}$$

and that

$$DF(x,y)(u) \quad \supset \quad \begin{cases} f'(x)u - T_M^\flat(f(x)-y) & \text{when} \quad u \in_L (x) \\ \emptyset & \text{when} \quad u \notin T_L(x) \end{cases} \quad \square$$

Another familiar instance of set-valued maps is the inverse of a set-valued map F (or even of a non injective single-valued map). We can easily compute any of their graphical derivative because **a graphical derivative of the inverse of a set-valued map F is the inverse of the derivative**:

$$(60) \qquad \begin{cases} i) & D(F)^{-1}(y,x) & = & DF(x,y)^{-1} \\ ii) & D^\flat(F)^{-1}(y,x) & = & D^\flat F(x,y)^{-1} \\ iii) & C(F)^{-1}(y,x) & = & CF(x,y)^{-1} \end{cases}$$

The first task is to characterize these derivatives by adequate limits of difference quotients. We begin with the case of contingent derivatives.

Proposition 6.2 *Let $(x, y) \in \mathrm{Graph}(F)$ belong to the graph of a set-valued map $F : X \rightsquigarrow Y$ from a normed space X to another Y. Then*

$$(61) \quad \begin{cases} v \text{ belongs to } DF(x,y)(u) \text{ if and only if} \\ \liminf_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{cases}$$

If F is lipschitzean around an element $x_0 \in \mathrm{Int}(\mathrm{Dom}(F))$, then

$$(62) \quad \begin{cases} v \text{ belongs to } DF(x,y)(u) \text{ if and only if} \\ \liminf_{h\to 0+,} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0 \end{cases}$$

Furthermore, if the dimension of Y is finite, or if Y is a reflexive Banach space supplied with the weak topology, then

$$(63) \quad \mathrm{Dom}(DF(x,y)) \;=\; X$$

Proof The first two statements being obvious, let us check the last one. Let u belong to X. Then, for all $h > 0$ small enough and $y \in F(x)$,

$$y \;\in\; F(x) \;\subset F(x+hu) + \;\; lh\|u\|B$$

Hence there exits $y_h \in F(x+hu)$ such that $v_h := (y_h - y)/h$ belongs to $l\|u\|B$, which is compact. Hence a subsequence (again denoted) of v_h converges (weakly if the dimension of Y is infinite) to some v, which belongs to $DF(x,y)(u)$.

In order to characterize adjacent and circatangent derivatives in terms of limits of difference quotients, we need to introduce the concept of "**lim sup inf**" of functions of two variables.

Definition 6.2 (Lim sup inf) *Let L and M be two metric spaces and $\phi : L \times M \mapsto \mathbf{R}$ be a function. We set*

$$\limsup_{x'\to x} \inf_{y'\to y} \phi(x',y') := \sup_{\epsilon>0} \inf_{\eta>0} \sup_{x'\in B(x,\eta)} \inf_{y'\in B(y,\epsilon)} \phi(x',y')$$

Hence, by translating the definition of the adjacent and the Clarke tangent cones, we obtain the following characterizations:

Proposition 6.3 *Let* $(x,y) \in \text{Graph}(F)$ *belong to the graph of a set-valued map* $F : X \leadsto Y$ *from a normed space* X *to another* Y. *Then*

$$(64) \quad \begin{cases} v \textit{ belongs to } D^\flat F(x,y)(u) \quad \textit{if and only if} \\ \limsup_{h\to 0+} \inf_{u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{cases}$$

and

$$(65) \quad \begin{cases} v \textit{ belongs to } CF(x,y)(u) \quad \textit{if and only if} \\ \limsup_{h\to 0+,(x',y')\to(x,y)} \inf_{u'\to u} d\left(v, \frac{F(x'+hu')-y'}{h}\right) = 0 \end{cases}$$

If F *is lipschitzean around an element* $x \in \text{Int}(\text{Dom}(F))$, *then the formulas become much simpler:*

$$(66) \quad \begin{cases} v \textit{ belongs to } D^\flat F(x,y)(u) \quad \textit{if and only if} \\ \lim_{h\to 0+} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{cases}$$

and

$$(67) \quad \begin{cases} v \textit{ belongs to } CF(x,y)(u) \quad \textit{if and only if} \\ \lim_{h\to 0+,(x',y')\to(x,y)} d\left(v, \frac{F(x'+hu')-y'}{h}\right) = 0 \end{cases}$$

Let us mention the following property:

Proposition 6.4 *Let us assume that the images of* F *are convex and that* F *is lipschitzean around* x. *Then the images of the adjacent derivative* $DF^\flat(x,y)$ *are convex.*

Proof Let v^1 and v^2 belong to $DF(x,y)(u)$. Then, for any sequence $h_n > 0$, there exist sequences u_n^1 and u_n^2 converging to u and sequences v_n^1 and v_n^2 converging to v^1 and v^2 respectively such that

$$\forall\, n, \;\; y + h_n v_n^i \;\in\; F(x + h_n u_n^i) \quad (i = 1,2)$$

Since F is lipschitzean around x,

$$y + h_n v_n^2 \;\in\; F(x + h_n u_n^1) + l h_n \|u_n^2 - u_n^1\|$$

so that there exists another sequence $\overline{v_n^2}$ converging to v^2 such that

$$y + h_n \overline{v_n^2} \;\subset\; F(x + h_n u_n^1)$$

Now, $F(x + h_n u_n^1)$ being convex, we deduce that for all $\lambda \in [0,1]$,

$$y + h_n(\lambda v_n^1 + (1-\lambda)\overline{v_n^2}) \in F(x + h_n u_n^1)$$

Since $\lambda v_n^1 + (1-\lambda)\overline{v_n^2}$ converges to $\lambda v^1 + (1-\lambda) v^2$, we deduce that this element belongs to $DF^\flat(x,y)(u)$. □

Remark: Kernel of the Derivative The kernels of the various derivatives characterize the associated tangent cones to the inverse image.

Proposition 6.5 *Let $F : X \rightsquigarrow Y$ be a set-valued map and (x, y) belong to its graph. Then*

$$\begin{cases} i) & T_{F^{-1}(y)}(x) \subset \ker DF(x,y) := DF(x,y)^{-1}(0) \\ ii) & T^{\flat}_{F^{-1}(y)}(x) \subset \ker D^{\flat}F(x,y) \end{cases}$$

If F^{-1} is pseudo-lipschitzean around (y, x), we have

$$\begin{cases} i) & \ker DF(x,y) = T_{F^{-1}(y)}(x) \\ ii) & \ker D^{\flat}F(x,y) = T^{\flat}_{F^{-1}(y)}(x) \\ iii) & \ker CF(x,y) \subset C_{F^{-1}(y)}(x) \end{cases}$$

Proof The first inclusions are obvious. To prove the converse inclusions, let u belong to the kernel of $CF(x, y)$ for instance: for all sequence $x_n \in F^{-1}(y)$ converging to x and sequence $h_n > 0$ converging to 0, there exist sequences u_n and v_n converging respectively to u and 0 satisfying

$$\forall n, \quad y + h_n v_n \in F(x_n + h_n u_n)$$

Since F^{-1} is pseudo-lipschitzean around (y, x), there exists an element $x_n^1 \in F^{-1}(y)$ such that

$$\|x_n^1 - (x_n + h_n u_n)\| \leq l \|y_n - (y_n + h_n v_n)\|$$

Hence, by setting $u_n^1 := (x_n^1 - x_n)/h_n$, we see that $x_n + h_n u_n^1 = x_n^1$ belongs to $F^{-1}(y)$ and that u_n^1 converges to u because $\|u_n^1 - u_n\| \leq l \|v_n\|$ and because v_n converges to 0.

Therefore we have proved that u belongs to the Clarke tangent cone to $F^{-1}(y)$ at x. The proofs for the other tangent cones are the same. □

Example: Derivatives of monotone operators Let X be a Hilbert space (identified with its dual). We recall that a set-valued map $F : X \rightsquigarrow X$ is monotone if and only if

$$(68) \qquad \forall\,(x,p),(y,q) \in \text{Graph}(F), \; < p-q \, , \, x-y > \; \geq \; 0$$

Subdifferentials of convex functions are monotone[19] maps. We also recall that when F is monotone, its **resolvent** $J := (1+A)^{-1}$ is single-valued and lipschitzean (with constant equal to 1) on its domain. Therefore, we can easily compute derivatives of F in terms of the derivatives of its resolvent.

Proposition 6.6 *Let X be a Hilbert space. We identify its dual with X and we supply it with the weak toplogy. If F is a monotone set-valued map, then its adjacent derivative $D^{\flat}F(x,p)$ at some pair (x,p) of its graph is semi positive-definite in the sense that*

$$\forall\,(u,r) \in \text{Graph}(D^{\flat}F(x,p)), \; < r \, , \, u > \geq \; 0$$

The same is true for the circatangent derivative. Furthermore, the following statements are equivalent:

$$\begin{cases} a) & r \; \in \; DF(x,p)(u) \\ b) & r \; \in \; DJ(x+p)(r+u) \end{cases}$$

This last statement remains true for the adjacent and circatangent derivatives.

Proof The first statement is obvious, since $< r, u >$ is the limit of the sequence $< r_n.u_n >$ (because u_n converges to u strongly and r_n converges to v weakly) and since

$$< r_n \, , \, u_n > \; = \; < x + h_n u_n - x \, , \, p + h_n r_n - p > \; \geq \; 0$$

For proving the second statement, we observe that p belongs to $F(x)$ if and only if $x = J(x+p)$. Since J is the inverse of $(1+A)$, we deduce that

[19] we refer to [5, Sections 6.6 & 6.7] for an introduction to monotone and maximal monotone maps.

$$s \in D(\mathbf{1} + A)(x,q)(u) \iff u \in DJ(q)(s)$$

Hence, by setting $q := x + p$ and $s := r + u$, we obtain the formula we were looking for. □

Since the cone-valued map N_K associating with any $x \in K$ the normal cone $N_K(x)$ to a closed convex subset is maximal monotone (because the normal cone is the subdifferential of the indicator of K), and since its resolvaent is the best approximation projector, we deduce the following corollary:

Corollary 6.1 *Let K be a closed convex subset of a Hilbert space, and let p belong to the normal cone $N_K(x)$ to K at some $x \in K$. Let π_K denote the best approximation projector onto K. Then, the two following satatements are equivalent:*

$$\begin{cases} i) & q \in D^\flat N_K(x,p)(u) \\ ii) & u \in D^\flat \pi_K(x+p)(u+q) \end{cases}$$

7 Calculus of Derivatives

We derive from the calculus of tangent cones the associated calculus of derivatives of set-valued maps. We begin naturally by the chain rule for computing the composition product of a set-valued map $G : X \leadsto Y$ and a set-valued map $H : Y \leadsto Z$.

One can conceive two dual ways for defining composition products of set-valued maps (which coincide when G is single-valued):

Definition 7.1 *Let X, Y, Z be Banach spaces and $G : X \leadsto Y$, $H : Y \leadsto Z$ be set-valued maps:*

1. the usual composition product (called simply the **product***) $H \circ G : X \leadsto Z$ of H and G at x is defined by*

$$(H \circ G)(x) \;:=\; \bigcup_{y \in G(x)} H(y)$$

2. the **square product**[20] *$H \square G : X \leadsto Z$ of H and G at x is defined by*

$$(H \square G)(x) \;:=\; \bigcap_{y \in G(x)} H(y)$$

[20]Observe that square products are implicitely involved in the factorization of maps. Let X be a subset, $\mathcal{R}$ be an equivalence relation on X and ϕ denote the canonical surjection from X onto the factor space $X/\mathcal{R}$. If f is a single-valued map from X to Y, its factorization $\tilde{f} : X/\mathcal{R} \mapsto Y$ is defined by

$$\tilde{f}(\xi) \;:=\; (f \square \phi^{-1})(\xi)$$

It is non trivial if and only if f is consistent with the equivalence relation $\mathcal{R}$, i.e., if and only if $f(x) = f(y)$ whenever $\phi(x) = \phi(y)$. When $F : X \leadsto Y$ is a set-valued map, we can define its factorization $\tilde{F} : X/\mathcal{R} \leadsto Y$ by

$$\tilde{F}(\xi) \;:=\; (F \square \phi^{-1})(\xi) \quad \square$$

Let us recall that there are two manners to define the inverse image[21] by a set-valued map G of a subset M:

$$\begin{cases} a) & G^-(M) := \{ x \mid G(x) \cap M \neq \emptyset \} \\ b) & G^+(M) := \{ x \mid G(x) \subset M \} \end{cases}$$

We deduce the following formulas

$$(69) \qquad \begin{cases} i) & \text{Graph}(F \circ G) = (G \times \mathbf{1})^- \text{Graph}(H) \\ ii) & \text{Graph}(F \square G) = (G \times \mathbf{1})^+ \text{Graph}(H) \end{cases}$$

as well as the formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$\begin{cases} i) & (H \circ G)^{-1}(y) = G^-(H^{-1}(y)) \\ ii) & (H \square G)^{-1}(y) = G^+(H^{-1}(y)) \end{cases}$$

We also point aout the following relation:

$$(70) \qquad \text{Graph}(H \circ G) = (\mathbf{1} \times H)\text{Graph}(G)$$

We shall need the following result:

Proposition 7.1 *Let F be a set-valued map from X to Y and K be a subset of X. Assume that F is lipschitzean around some $x \in K$. Then, for any $y \in F(x)$, we have*

$$(71) \qquad D^\flat F(x,y) T_K(x) \subset T_{F(K)}(y)$$

As a consequence, we deduce that if M is a subset of Y, then

$$(72) \qquad T_{F^+(M)}(x) \subset D^\flat(x,y)^+ T_M(y)$$

Proof

[21] We recall also that a set-valued map G is upper semicontinuous if and only if the inverse images G^- of open subsets are open and that it is lower semicontinuous if and only if the inverse images G^+ of open subsets are open.

Take u in $T_K(x)$ and $v \in D^\flat F(x,y)(u)$. Then there exist sequences $h_n > 0$ converging to 0, u_n^1 and u_n^2 conveging to u and v_n conveging to v such that

$$x_n + h_n u_n^1 \in K \ \ \& \ \ y + h_n v_n \in F(x + h_n u_n^2)$$

Since F is lipschitzean around x, we deduce that

$$y + h_n v_n \in F(x + h_n u_n^1) + l h_n \|u_n^1 - u_n^2\|$$

so that there exists another sequence $v_n^\star$ converging to v such that

$$y + h_n v_n^\star \in F(x + h_n u_n^1) \subset F(K)$$

This implies that v belongs to the contingent cone to $F(K)$ at y.

Consider now $K := F^+(M)$. Since $F(F^+(M))$ is contained in M, we deduce that

$$D^\flat F(x,y) T_{F^+(M)}(x) \subset T_{F(F^+(M))}(y) \subset T_M(y)$$

from which formula (72) follows. □

Remark Naturally, the formula

$$DF(x,y) T_K^\flat(x) \ \subset \ T_{F(K)}(y)$$

is also true. □

We begin by this simple result:

Theorem 7.1 *Let us consider a set-valued map $G : X \rightsquigarrow Y$ and a set-valued map $H : Y \rightsquigarrow Z$.*

Let us assume that H is lipschitzean around y where y belongs to $G(x)$. Then, for any $z \in H(y)$, we have

$$(73) \qquad D^\flat H(y,z) \circ DG(x,y) \ \subset \ D(H \circ G)(x,z)$$

Let us assume that G is lipschitzean around x. Then, for all $y \in G(x)$ and $z \in (H \square G)(x)$, we have

$$(74) \qquad D(H \square G)(x,z) \ \subset \ DH(y,z) \square D^\flat G(x,y)$$

In particular, if $G := g$ is single-valued and lipschitzean around x, we obtain

$$D(Hg)(x,z)(u) \subset DH(g(x),z)(g'(x)u)$$

and the equality holds true when H is lipschitzean around $g(x)$.

Proof We apply Proposition 7.1 to formulas (70) and (69)ii) respectively for proving the two first formulas. When $G := g$ is single-valued, we use the fact that both composition products coincide. □

We state now a more powerful result:

Theorem 7.2 *Let us consider a set-valued map $G : X \rightsquigarrow Y$ and a set-valued map $H : Y \rightsquigarrow Z$.*

If the dimension of Y is finite, we supose that

$$\mathrm{Im}(CG(x_0,y_0)) - \mathrm{Dom}(CH(y_0,z_0)) = Y$$

Then

$$\begin{cases} i) & D^\flat H(y_0,z_0) \circ DG(x_0,y_0) \subset D(H \circ G)(x_0,z_0) \\ ii) & D^\flat H(y_0,z_0) \circ D^\flat G(x_0,y_0) = D^\flat(H \circ G)(x_0,z_0) \\ iii) & CH(y_0,z_0) \circ CG(x_0,y_0) \subset C(H \circ G)(x_0,z_0) \end{cases}$$

If Y is any Banach space, we assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$(75) \quad \begin{cases} & \forall\ (x,y_1) \in \mathrm{Graph}(G) \cap B((x_0,y_0),\eta), \\ & \forall\ (y_2,z) \in \mathrm{Graph}(H) \cap B((y_0,z_0),\eta), \\ i) & B_Y \subset \mathrm{Im}(D^\flat G(x,y_1)) \cap cB_X - \mathrm{Dom}(DH(y_2,z)) + \alpha B_Y \\ ii) & \|DG^\flat(x,y_1)\| \le c \\ iii) & \|DH^{-1}(z,y_2)\| \le c \end{cases}$$

Proof If we denote by ω the continuous linear operator from $X \times Y \times Y \times Z$ to Y associating to (x,y_1,y_2,z) the element $y_1 - y_2$ and by $\pi_{X\times Z}$ the canonical projection from $X \times Y \times Y \times Z$ onto $X \times Z$, we observe that

$$(76) \quad \mathrm{Graph}(H \circ G) = \pi_{X\times Z}((\mathrm{Graph}(G) \times \mathrm{Graph}(H)) \cap \omega^{-1}(0))$$

Therefore, we apply Theorem 4.2 with $A = \omega$, $L = \mathrm{Graph}(G) \times \mathrm{Graph}(H)$ and $M = \{0\}$.

Assumption (75) implies the transversality condition. Indeed, for any $v \in Y$, we can find $v_1 \in \text{Im}(D^\flat G(x,y_1)), v_2 \in \text{Dom}(DH(y_2,z))$ and $w \in Y$ such that $v = v_1 - v_2 + e$ with $\|v_1\| \leq c\|v\|$ and $\|e\| \leq \alpha\|v\|$. Hence $\|v_2\| \leq (c+1+\alpha)\|v\|$ and there exist $u \in DG^\flat(x,y_1)^{-1}(v_1)$ and $w \in DH(y_2,z)(v_2)$ such that $\|u\| \leq c\|v_1\|$ and $\|w\| \leq c\|v_2\|$.

Therefore, $v = \omega(u,v_1,v_2,w) + e$ where (u,v_1,v_2,w) belongs to the contingent cone to the product of the graphs of G and H and $e \in \alpha B_Y$.

Consequently, we deduce that, for instance,

$$\begin{cases} (\text{Graph}(D^\flat G(x_0,y_0)) \times \text{Graph}(D^\flat H(y_0,z_0))) \cap \omega^{-1}(0) \\ = (T^\flat_{\text{Graph}(G)}(x_0,y_0) \times T^\flat_{\text{Graph}(H)}(y_0,z_0)) \cap \omega^{-1}(0) \\ = T^\flat_{\text{Graph}(G)\times\text{Graph}(H)}(x_0,y_0,y_0,z_0) \cap \omega^{-1}(0) \\ = T^\flat_{\text{Graph}(G)\times\text{Graph}(H)\cap\omega^{-1}(0)}(x_0,y_0,y_0,z_0) \end{cases}$$

By applying the projection $\pi_{X\times Z}$ to both sides of these equalities, we deduce that

$$\begin{cases} \text{Graph}(D^\flat H(y_0,z_0) \circ D^\flat G(x_0,y_0)) \\ = \pi_{X\times Z}((\text{Graph}(D^\flat G(x_0,y_0)) \times \text{Graph}(D^\flat H(y_0,z_0))) \cap \omega^{-1}(0)) \\ = \pi_{X\times Z}(T^\flat_{\text{Graph}(G)\times\text{Graph}(H)\cap\omega^{-1}(0)}(x_0,y_0,y_0,z_0)) \\ \subset T^\flat_{\pi_{X\times Z}(\text{Graph}(G)\times\text{Graph}(H)\cap\omega^{-1}(0))}(x_0,z_0) \\ = T^\flat_{\text{Graph}(H\circ G)}(x_0,y_0) \\ = \text{Graph}(D^\flat(H\circ G)(x_0,z_0)) \quad \square \end{cases}$$

We provide now some examples of situations where we have equality in chain rule formulas.

We denote by $\overline{F}$ the set-valued map whose graph is the closure of the graph of F.

Proposition 7.2 *Let us consider a set-valued map $G : X \rightsquigarrow Y$ and a set-valued map $H : Y \rightsquigarrow Z$. The following inclusion*

$$D(H \circ G)(x_0, z_0) \quad \subset \quad \overline{DH(y_0, z_0) \circ DG(x_0, y_0)}$$

holds true under one of the following assumptions:

1. *G is pseudo-convex at $(x_0, y_0) \in \mathrm{Graph}(G)$ and H is pseudo-convex at $(y_0, z_0) \in \mathrm{Graph}(H)$, (and in particular, G and H are convex),*
2. *The dimension of X and Y is finite and*

$$(77) \qquad \ker(DH(x_0, y_0)) \cap DG(x_0, y_0)(0) \quad = \quad \{0\}$$

Proof We deduce these statements from the criteria implying that the contingent cones of the images by $\pi_{X \times Z}$ are the closures of the images of the contingent cones:

$$\left\{ \begin{array}{l} \mathrm{Graph}(D(H \circ G)(x_0, z_0)) \\ = T_{\mathrm{Graph}(H \circ G)}(x_0, y_0) \\ = T_{\pi_{X \times Z}(\mathrm{Graph}(G) \times \mathrm{Graph}(H) \cap \omega^{-1}(0))}(x_0, z_0) \\ = \overline{pi_{X \times Z}(T_{\mathrm{Graph}(G) \times \mathrm{Graph}(H) \cap \omega^{-1}(0)}(x_0, y_0, y_0, z_0))} \\ \subset \overline{\pi_{X \times Z}((\mathrm{Graph}(DG(x_0, y_0)) \times \mathrm{Graph}(DH(y_0, z_0))) \cap \omega^{-1}(0))} \\ = \overline{\mathrm{Graph}(DH(y_0, z_0) \circ DG(x_0, y_0))} \end{array} \right.$$

When the graph of G is pseudo-convex at (x, y) and the graph of H is pseudo-convex at (y, z), we derive the above property from Proposition 4.3.

The second case follows from Proposition 4.4 because condition (77) implies obviously that

$$\ker \pi_{X \times Z} \cap T_{\mathrm{Graph}(G) \times \mathrm{Graph}(H) \cap \omega^{-1}(0)}(x_0, y_0, y_0, z_0)) = 0 \quad \square$$

Remark It is quite useful to relate the tangent cones to the domain (or the image) of a set-valued map to the domain (or the image) of its derivative.

Proposition 7.3 *Let us consider two topological vector spaces X and Y, a set-valued map $F : X \rightsquigarrow Y$ from X to Y and a point (x_0, y_0) of its graph. We always have*

$$(78) \qquad \overline{\mathrm{Dom}(DF(x_0, y))} \subset T_{\mathrm{Dom}(F)}(x_0)$$

Equality

$$(79) \qquad \overline{\mathrm{Dom}(DF(x_0, y))} = T_{\mathrm{Dom}(F)}(x_0)$$

holds true under one of the following assumptions:

1. *F is pseudo-convex at $(x_0, y_0) \in \mathrm{Graph}(F)$, (and in particular, F is convex),*
2. *X and are reflexive Banach spaces and the cosubdifferential of F at (x_0, y_0) is surjective:*

$$(80) \qquad \mathrm{Im}(DF(x_0, y_0)^{0\star}) = Y^\star$$

3. *The dimension of X and Y is finite and*

$$(81) \qquad \ker(DF(x_0, y_0)) = \{0\}$$

Proof We deduce these statements from the criteria (Proposition 4.3, Proposition 4.5 and Proposition 4.4) implying that the contingent cones of the images are the closures of the images of the contingent cones, since

$$(82) \qquad \mathrm{Dom}(F) = \pi_X \mathrm{Graph}(F)$$

where π_X is the projection from $X \times Y$ onto X. We thus deduce that

$$(83) \qquad \begin{cases} \overline{\mathrm{Dom}(DF((x_0, y_0)))} = \overline{\pi_X \mathrm{Graph}(DF)((x_0, y_0))} \\ = \overline{\pi_X T_{\mathrm{Graph}(F)}((x_0, y_0))} \subset T_{\pi_X(\mathrm{Graph}(F))}(x_0) \\ = T_{\mathrm{Dom}(F)}(x_0) \quad \square \end{cases}$$

Remark By using the paratingent derivatives[22], we obtain upper estimates of the usual composition product.

[22]by definition, the graph of the paratingent derivative $PF(x.y)$ of F at (x, y) is the paratingent cone to the graph of F at (x, y).

We can also define the lop-sided paratingent derivatives $P_x F(x, y)$ and $P_y F(x, y)$ in the following way:

$$\mathrm{Graph}(P_1 F(x, y)) := P^{\mathrm{Dom}(F)}_{\mathrm{Graph}(F)}(x, y) \ \& \ \mathrm{Graph}(P_2 F(x, y)) := P^{\mathrm{Im}(F)}_{\mathrm{Graph}(F)}(x, y)$$

Theorem 7.3 *Assume that G is lipschitzean around x. Then*

1. Y is a finite dimensional vector-space and $G(x)$ is bounded, then

$$D(H \circ G)(x,z) \subset \bigcup_{y\in G(x)} P_1H(y,z) \circ P_2G(x,y)$$

2. and

$$P(H \square G)(x,z) \subset \bigcap_{y\in G(x)} PH(y,z) \square CG(x,y)$$

Proof Let w belong to $D(H \circ G)(x,z)(u)$: there exist sequences $h_n > 0$, u_n and v_n converging to 0, u and v such that

$$\forall\, n,\ \ z + h_n v_n \in H(y_n) \text{ where } y_n \in G(x + h_n u_n)$$

Since G is lipschitzean around x, there exist elements $y_n^0 \in G(x)$ such that

$$v_n := \frac{y_n - y_n^0}{h_n} \text{ satisfies } \|v_n\| \leq\ l\|u_n\|$$

Furthermore, $G(x)$ being relatively compact, a subsequence (again denoted) y_n^0 converges to some y. We can also extract a subsequence (again denoted) v_n which converges to some v, since this sequence is bounded and the dimension of Y is finite.

Since

$$y_n^0 + h_n v_n \in G(x + h_n u_n)\ \ \&\ \ z + h_n w_n \in H(y_n^0 + h_n v_n)$$

we infer that w belongs to $PH(y,z)(v)$ and v belongs to $PG(x,y)(u)$

Let $u \in \text{Dom } CG(x,y)$ and w belong to $P(H \square G)(x,z)(u)$. Hence there exist a sequence $h_n > 0$ converging to 0 and sequences of elements $(x_n, z_n) \in \text{Graph}(H \square G)$, u_n and w_n converging to (x,z), u and w respectively such that

$$\forall\, n \geq 0,\ \ z_n + h_n w_n \ \in \bigcap_{y\in G(x_n + h_n u_n)} H(y)$$

The set-valued map G being lipschitzean, there exists a sequence of elements $y_n \in G(x_n)$ converging to y. By definition of the square product, we know that $z_n \in H(y_n)$ (because $z_n \in (H \square G)(x_n)$).

Take now any v in $CG(x,y)(u)$. Since G is lipschitzean around x, there exists a sequence of elements v_n converging to v such that

$$\forall\, n \geq 0,\ \ y_n + h_n v_n \ \in \ G(x_n + h_n u_n)$$

Therefore,

$$\forall\, n \geq 0,\ \ z_n + h_n w_n \ \in \ H(y_n + h_n v_n)$$

so that we infer that

$$w \ \in \ PH(y,z)(v)$$

Since this is true for every element v of $CG(x,y)(u)$, we deduce that

$$w \in \bigcap_{v \in CG(x,y)(u)} PH(y,z)(v) = PH(y,z)\square CG(x,y)(u) \qquad \square$$

8 Epiderivatives

Let us consider an extended real-valued function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ whose domain

$$\text{Dom}(V) \;:=\; \{ x \in X \mid V(x) < +\infty \}$$

is not empty. (Such a function is said to be **proper** in convex and non smooth analysis. We shall rather say that it is **nontrivial** for avoiding confusion with proper maps).

We can naturally regard it as the set-valued map $\mathbf{V} : X \rightsquigarrow \mathbf{R}$ defined by

$$\mathbf{V}(x) \;:=\; \begin{cases} V(x) & \text{if } x \in \text{Dom}(V) \\ \emptyset & \text{if } x \notin \text{Dom}(V) \end{cases}$$

so that we can define in the usual way contingent, adjacent and circatangent derivatives of V at $x \in \text{Dom}(V)$. We shall set:

$$\begin{cases} i) & DV(x)(u) \;:=\; DV(x, V(x))(u) \\ & = \{v \mid \liminf_{h\to 0+, u'\to u} \|V(x+hu') - V(x) - hv\|/h = 0\} \\ ii) & D^\flat V(x)(u) \;:=\; D^\flat V(x, V(x))(u) \\ & = \{v \mid \limsup_{h\to 0+} \inf_{u'\to u} \|V(x+hu') - V(x) - hv\|/h = 0\} \\ iii) & CV(x)(u) \;:=\; CV(x, V(x))(u) \\ & = \{v \mid \limsup_{h\to 0+, x'\to x} \inf_{u'\to u} \|V(x+hu') - V(x) - hv\|/h = 0\} \end{cases}$$

However, minimization problems and Lyapunov functions involve obviously the order relation of $\mathbf{R}$. Hence, when dealing with such problems, we associate with the extended real-valued function V two new set-valued maps $\mathbf{V}_\uparrow$ and $\mathbf{V}_\downarrow$ defined in the following way:

$$(84) \qquad \begin{cases} i) & \mathbf{V}_\uparrow \;:=\; \begin{cases} V(x) + \mathbf{R}_+ & \text{if } x \in \text{Dom}(V) \\ \emptyset & \text{if } x \notin \text{Dom}(V) \end{cases} \\ ii) & \mathbf{V}_\downarrow \;:=\; \begin{cases} V(x) - \mathbf{R}_+ & \text{if } x \in \text{Dom}(V) \\ \emptyset & \text{if } x \notin \text{Dom}(V) \end{cases} \end{cases}$$

We see at once that

$$(85) \qquad \begin{cases} i) & \text{Graph}(\mathbf{V}_\uparrow) \;=\; \text{Ep}(V) \\ ii) & \text{Graph}(\mathbf{V}_\downarrow) \;=\; \text{Hp}(V) \end{cases}$$

Therefore, we are led naturally to associate with these two set-valued maps $\mathbf{V}_\uparrow$ and $\mathbf{V}_\downarrow$ their contingent, adjacent and circatangent derivatives at points $(x, V(x))$ and thus, unfortunately, to introduce still new definitions.

Definition 8.1 (Epiderivatives) *Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended real-valued function and x belong to its domain. We shall say that the functions $D_\uparrow(V)(x)$, $D_\uparrow^\flat(V)(x)$ and $C_\uparrow(V)(x)$ from X to $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$ defined respectively by*

$$\left\{\begin{array}{ll} i) & D_\uparrow(V)(x)(u) \\ & := \liminf_{h\to 0+, u'\to u}(V(x+hu') - V(x))/h \\ ii) & D_\uparrow^\flat(V)(x)(u) \\ & := \limsup_{h\to 0+} \inf_{u'\to u}(V(x+hu') - V(x))/h \\ iii) & C_\uparrow(V)(x)(u) \\ & := \limsup_{h\to 0+, x'\to x, V(x')\leq\lambda'\to V(x)} \inf_{u'\to u}(V(x+hu') - \lambda')/h \end{array}\right.$$

are the **contingent, adjacent and circatangent epiderivatives** *of V at x in the direction u.*

We define in a symmetric way the **contingent, adjacent and circatangent hypoderivatives** *$D_\downarrow(V)(x)$, $D_\downarrow^\flat(V)(x)$ and $C_\downarrow(V)(x)$ from X to $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$*

$$\left\{\begin{array}{ll} i) & D_\downarrow(V)(x)(u) \;=\; -(D_\uparrow(-V)(x_0)(u)) \\ & := \limsup_{h\to 0+, u'\to u}(V(x+hu') - V(x))/h \\ ii) & D_\downarrow^\flat(V)(x)(u) \;=\; -(D_\uparrow^\flat(-V)(x_0)(u)) \\ & := \liminf_{h\to 0+} \sup_{u'\to u}(V(x+hu') - V(x))/h \\ iii) & C_\downarrow(V)(x)(u) \;=\; -(C_\uparrow(-V)(x_0)(u)) \\ & := \liminf_{h\to 0+, x'\to x, V(x')\geq\lambda'\to V(x)} \sup_{u'\to u}(V(x+hu') - \lambda')/h \end{array}\right.$$

Such an epi or hypo derivative is said to be **nontrivial** *if and only if it never takes the value $+\infty$ and has at least one finite value.*

Naturally, they coincide with the directional derivatives $< V'(x), u >$ when V is respectively Gâteaux, Fréchet and continuously differentiable.

Remark Definition (8.1) provide another interpretation of the epiderivatives in terms of epilimits of the difference quotients

$$\nabla_h V(x) := u \mapsto (V(x+hu) - V(x))/h$$

1. the contingent epiderivative is the epi-lower limit of the difference quotients $\nabla_h V(x)$ when $h \to 0+$ 2. the adjacent epiderivative is the epi-upper limit of the difference quotients $\nabla_h V(x)$ when $h \to 0+$ 3. the circatangent epiderivative is the epi-upper limit of the difference quotients $u \mapsto (V(x'+hu) - \lambda')/h$ when $h \to 0+$ and $(x', \lambda') \in \mathrm{Ep}(V) \to (x, V(x))$ □

If V is continuously differentiable around a point $x \in K$, then the **epiderivative of the restriction is the restriction of the epiderivative to the corresponding tangent cone:**

$$\begin{cases} i) & D_{\uparrow}(V|_K)(x)(u) := \begin{cases} < V'(x), u > & \text{if } u \in T_K(x) \\ +\infty & \text{if if not} \end{cases} \\ ii) & D_{\uparrow}^{\flat}(V|_K)(x)(u) := \begin{cases} < V'(x), u > & \text{if } u \in T_K^{\flat}(x) \\ +\infty & \text{if if not} \end{cases} \\ iii) & C_{\uparrow}(V|_K)(x)(u) := \begin{cases} < V'(x), u > & \text{if } u \in C_K(x) \\ +\infty & \text{if if not} \end{cases} \end{cases}$$

The formulas become much more simpler when V is lipschitzean.

Proposition 8.1 *Let us assume that $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ is lipschitzean around a point x of its domain. Then*

$$(86) \quad \begin{cases} i) & D_{\uparrow}(V)(x)(u) \\ & := \liminf_{h \to 0+}(V(x+hu') - V(x))/h \text{ and} \\ ii) & D_{\uparrow}^{\flat}(V)(x)(u) \\ & := \limsup_{h \to 0+}(V(x+hu') - V(x))/h \\ & \text{are Dini derivatives} \\ iii) & C_{\uparrow}(V)(x)(u) =: V^{\circ}(x,u) \\ & := \limsup_{h \to 0+, x' \to x}(V(x+hu') - V(x'))/h \\ & \text{is the Clarke directional derivative.} \end{cases}$$

Furthermore,

$$\begin{cases} i) & (x,u) \in X \times \text{Int}(\text{Dom}(V)) \mapsto C_{\uparrow}V(x)(u) \\ & \text{is upper semicontinuous} \\ ii) & u \mapsto C_{\uparrow}V(x)(u) \text{ is lipschitzean} \\ iii) & C_{\uparrow}(-V)(x)(u) := C_{\uparrow}V(x)(-u) \end{cases}$$

Proposition 8.2 *When the function V is convex, all these epiderivatives are equal and the formula becomes:*

$$D_\uparrow(V)(x)(u) \;=\; \liminf_{u'\to u}\left(\inf_{h>0}\frac{V(x+hu')-V(x)}{h}\right)$$

Furthermore, when x belongs to the interior of the domain of V, i.e., when V is lipschitzean around x, we obtain

$$D_\uparrow(V)(x)(u) \;=\; \inf_{h>0}\frac{V(x+hu')-V(x)}{h}$$

We first observe that

Proposition 8.3 *Let $V : X \mapsto \mathbf{R}\cup\{+\infty\}$ be a nontrivial extended real-valued function and x belong to its domain.*

Then the tangent cones to the epigraph of V at $(x, V(x))$ are epigraphs of the corresponding epiderivatives of V at x:

$$\begin{cases} i) & \mathrm{Ep}D_\uparrow V(x) = T_{\mathrm{Ep}(V)}(x, V(x)) \\ ii) & \mathrm{Ep}D_\uparrow^\flat V(x) = T^\flat_{\mathrm{Ep}(V)}(x, V(x)) \\ iii) & \mathrm{Ep}C_\uparrow V(x) = C_{\mathrm{Ep}(V)}(x, V(x)) \end{cases}$$

Then the circatangent epiderivative is always lower semicontinuous convex and positively homogeneous.

Proof We shall check this fact only for the contingent case, the proof being similar for the adjacent and circatangent cases.

Actually, we shall prove that for all $w \geq V(x)$, we have

$$\mathrm{Ep}D_\uparrow V(x) = T_{\mathrm{Ep}(V)}(x, V(x)) \;\subset T_{\mathrm{Ep}(V)}(x, w) \tag{87}$$

Let $v \geq D_\uparrow V(x)(u)$ where u belongs to the domain of the contingent epiderivative of V at x. Then there exist sequences u_n, v_n and $h_n > 0$ converging to u, v and 0 such that $h_n v_n \geq V(x + h_n u_n) - V(x)$. Since $w - V(x) \geq 0$, we deduce that $w - V(x) + h_n v_n \geq V(x + h_n u_n) - V(x)$, i.e., that the pair (u, v) belongs to $T_{\mathrm{Ep}(V)}(x, w)$.

Let us assume that (u,v) belongs to $T_{\mathrm{Ep}(V)}(x,w)$. We infer that there exists sequences u_n, v_n and $h_n > 0$ converging to u, v and 0 such that

$$w + h_n v_n \;\geq\; V(x + h_n u_n) \tag{88}$$

If $w = V(x)$, we deduce that $v \geq D_\uparrow(V)(x)(u)$. Hence the equality between the contingent cone to the epigraph of V at $(x, V(x))$ and the epigraph of the contingent epiderivative holds true. □

This result leads us to introduce the following definitions.

Definition 8.2 *We shall say that an extended function V is* **pseudo-convex** *at x if*

$$\forall\, y \in X,\;\; D_\uparrow V(x)(y-x) \;\leq\; V(y) - V(x)$$

We shall say that it is **epi-derivable** *at x if the contingent and adjacent epiderivatives do coincide and that it is* **epi-sleek** *at x if the epigraph of V is sleek at (x, Vx).*

Proposition 8.4 *Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended real-valued function and x belong to its domain. Then*

$$\mathrm{Dom}(D_\uparrow(V)(x)) \times \mathbf{R} \subset T_{\mathrm{Ep}(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

If the restriction of V to its domain is upper semicontinuous, then, for all $w > V(x)$,

$$T_{\mathrm{Ep}(V)}(x,w) \;=\; T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

Proof If u belongs to the domain of the contingent epiderivative of V at x, if $w \geq V(x)$ and if u is any real number, we check that (u,v) belongs to the epigraph of $D_\uparrow(V)(x)$.

Indeed, there exist sequences of elements $h_n > 0$, u_n and v_n converging to 0, u, and $D_\uparrow(V)(x)$ respectively such that

$$(x + h_n u_n, V(x) + h_n v_n) \;\in\; \mathrm{Ep}(V)$$

But we can write

$$(x + h_n u_n, w + h_n v) = (x + h_n u_n, V(x) + h_n v_n) + (0, w - V(x) + h_n(v - v_n))$$

Since $w - V(x) + h_n(v - v_n)$ is strictly positive when $w > V(x)$ and h_n is small enough, we deduce that $(x + h_n u_n, w + h_n v)$ belongs to the epigraph of V, i.e.,that (u,v) belongs to the epigraph of $D_\uparrow(V)(x)$.

If (u,v) belongs to $T_{\mathrm{Ep}(V)}(x,w)$, we deduce from (88) that $x + h_n u_n$ belongs to the domain of V for all n, i.e., that u belongs to the contingent cone to the domain of V at x.

Let w be strictly larger than x and u belong to $T_{\mathrm{Dom}(V)}(x)$. Then there exists sequences u_n and $h_n > 0$ converging to u and 0 such that $V(x + h_n u_n) < +\infty$ for all n.

Since V is upper semicontinuous on its domain, for all $\epsilon \in]0, \frac{w-V(x)}{2}[$, there exists $\eta > 0$ such that, for all $h_n \|u_n\| < \eta$, we obtain

$$V(x + h_n u_n) \;\leq\; V(x) + \epsilon \;<\; w - \epsilon$$

Let v given arbitrarily in $\mathbf{R}$. Then, for any $h_n > 0$ when $v \geq 0$ or for any $h_n \in]0, \frac{\epsilon}{-v}[$ when $v < 0$, inequality $w - \epsilon \leq w + h_n v$ implies that $V(x + h_n u_n) \leq w + h_n v$, i.e., that the pair (u,v) belongs to $T_{\mathrm{Ep}(V)}(x,w)$.
□

Proposition 8.5 *Let* $V \; :\mapsto \; \mathbf{R} \cup \{+\infty\}$ *be an extended function and* x *belong to its domain. Then*

$$\begin{cases} \{D_\uparrow V(x)(u), D_\downarrow V(x)(u)\} \\ \subset DV(x)(u) \\ \subset [D_\uparrow V(x)(u), D_\downarrow V(x)(u)] \end{cases}$$

These subsets are equal when the images of the contingent derivative are connected.

Proof Since

$$\mathrm{Graph}(V) \subset \mathrm{Ep}(V) \cap \mathrm{Hp}(V)$$

we deduce that the inclusions

$$T_{\mathrm{Graph}(V)}(x, V(x)) \subset T_{\mathrm{Ep}(V)}(x, V(x)) \cap T_{\mathrm{Hp}(V)}(x, V(x))$$

can be translated in the following inclusions:

$$\mathrm{Graph}(DV(x)) \subset \mathrm{Ep} D_\uparrow V(x) \cap \mathrm{Hp} D_\downarrow V(x)$$

from which inclusion $DV(x)(u) \subset [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$ follows.

Since the contingent epiderivative of V at x in the direction u is equal to

$$D_\uparrow V(x)(u) := \liminf_{h\to 0+, u'\to u} \frac{V(x+hu') - V(x)}{h}$$

we see that $D_\uparrow V(x)(u)$ is the limit of a subsequence of $\frac{V(x+hu')-V(x)}{h}$, and thus, the pair $(u, D_\uparrow V(x)(u))$ belongs to the contingent cone to the graph of V at $(x, V(x))$. The same is true with the contingent hypoderivative. □

Remark There is another intimate connection between tangent cones and their corresponding epiderivatives than Proposition 11 linking the tangent cones to the lower sections of the corresponding epiderivatives of the function $d_K(\cdot)$ and Proposition 8.3 linking the tangent cone to the epigraph to the epigraph of the epiderivative.

Let ψ_K be the **indicator** of a subset K, defined by

$$\psi_K(x) := \begin{cases} 0 & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

It is easy to observe that

$$(89) \quad \begin{cases} i) & D_\uparrow(\psi_K)(x) = \psi_{T_K(x)} \\ ii) & D_\uparrow^\flat(\psi_K)(x) = \psi_{T_K^\flat(x)} \\ iii) & C_\uparrow(\psi_K)(x) = \psi_{C_K(x)} \end{cases}$$

Hence we can either derive properties of the epiderivatives from properties of the tangent cones or take the opposite approach by using the above formulas. □

9 Calculus of Epiderivatives

We present below some useful formulas concerning the epiderivatives.

Theorem 9.1 *Let us consider two Banach spaces X and Y, a continuous single-valued map $f : X \mapsto Y$, and two extended real-valued functions V and W from X and Y to $\mathbf{R} \cup \{+\infty\}$ respectively. Let x_0 belong to the Kuratowski lower limit of the domains of the functions $U := V + W \circ f$. We asume that f is continuously differentiable around x_0. 1. We always have:*

$$D_\uparrow V(x_0)(u) + D_\uparrow W(f(x_0))(f'(x_0)u) \leq D_\uparrow(V + W \circ f)(x_0)(u)$$

2. If X and Y are finite dimensional, we suppose that

$$\mathrm{Dom}(C_\uparrow W(f(x_0))) - f'(x_0)\mathrm{Dom}(C_\uparrow V(f(x_0))) = Y$$

Then, the epiderivatives of the function $U := V+W\circ f$ satisfy the estimates:

$$(90)\begin{cases} i) & D_\uparrow(U)(x_0)(u) \leq D_\uparrow^\flat(V)(x_0)(u) + D_\uparrow(W)(f(x_0))(f'(x_0)u) \\ ii) & D_\uparrow^\flat(U)(x_0)(u) \leq D_\uparrow^\flat(V)(x_0)(u) + D_\uparrow^\flat(W)(f(x_0))(f'(x_0)u) \\ iii) & C_\uparrow(U)(x_0)(u) \leq C_\uparrow(V)(x_0)(u) + C_\uparrow(W)(f(x_0))(f'(x_0)u) \end{cases}$$

3. If X or Y is a Banach space, we posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that, for all n,*

$$(91)\begin{cases} i) & \forall\, x \in \mathrm{Dom}(V) \cap B(x_0, \eta), \ \forall\, y \in \mathrm{Dom}(W) \cap B(f(x_0), \eta) \\ & B_Y \subset f'(x_0)\left(\mathrm{Dom}(D_\uparrow^\flat(V)(x)) \cap cB_X\right) \\ & -\mathrm{Dom}(D_\uparrow(W)(y)) + \alpha B_Y \\ ii) & \sup_{u\in\mathrm{Dom}(D_\uparrow^\flat(V)(x))} |D_\uparrow^\flat(V)(x)(u)|/\|u\| \leq c \\ iii) & \sup_{v\in\mathrm{Dom}(D_\uparrow(W)(y))} |D_\uparrow(W)(y)(v)|/\|v\| \leq c \end{cases}$$

Proof We shall prove the formula only in the case of circatangent epiderivatives.

If we set

$$\begin{cases} i) & K := \mathrm{Ep}(V) \times \mathrm{Ep}(W) \times \mathbf{R} \subset X \times \mathbf{R} \times Y \times \mathbf{R} \times \mathbf{R} \\ ii) & G(x,a,y,b,c) := (f(x) - y, a + b - c) \\ iii) & H(x,a,y,b,c) := (x,c) \end{cases}$$

we can write

$$\mathrm{Ep}(U) = H(K \cap G^{-1}(0,0))$$

Therefore, we shall us Theorem 4.2 for estimating tangent cones to $K \cap G^{-1}(0,0)$. We first observe that the transversality conditions of our theorem imply the corresponding stability assumptions of Theorem 4.2.

This is obvious when X and Y are finite dimensional spaces. Otherwise, we have to check that there exists a constant $c' > 0$ such that, for all n, for all

$$(x,a,y,b,c) \in K \text{ close to } (x_0, V_\uparrow(x_0), y_0, W_\uparrow(f(x_0)), 0)$$

for all $(z,\lambda) \in X \times \mathbf{R}$, there exist (u,μ,v,ν,δ) and e such that

$$\begin{cases} i) & z = f'(x)u - v + e \ \& \ \lambda = \mu + \nu - \delta \\ ii) & \|e\| \leq \alpha(\|z\| + |\lambda|) \\ iii) & \|u\| + \|v\| + |\mu| + |\nu| + |\delta| \leq c'(\|z\| + |\lambda|) \end{cases}$$

Assumptions (91)i) & ii) imply right away that

$$\begin{cases} i) & z = f'(x)u - v + e, \\ ii) & \|e\| \leq \alpha\|z\| \ \& \ \|u\| \leq c\|z\| \\ iii) & \|v\| + \ \leq \leq (1 + \alpha + c\|f'(x)\|)\|z\| \end{cases}$$

Let us take now $\mu := c\|u\|$, $\nu := c\|v\|$ and $\delta := c(\|u\| + \|v\|) - \lambda$. We deduce from (91)iii) that (u,μ) belongs to $\mathrm{Ep}(D_\uparrow^\flat V)(x)$, that (v,ν) belongs to $\mathrm{Ep}(D_\uparrow W)(y)$ and that

$$D_\uparrow^\flat(V)(x)(u) + D_\uparrow(W)(y)(v) \leq c(\|u\| + \|v\|) \leq \mu + \nu = \lambda + \delta$$

and that $|\delta| \leq (|\lambda| + c(\|u\| + \|v\|)) \leq c'(\|z\| + |\lambda|)$.

Let us set $z_0 = (x_0, V_\uparrow(x_0), f(x_0), W_\uparrow(f(x_0)), U_\uparrow(x_0))$ Hence, we deduce that

$$C_K(z_0) \cap G'(z_0)^{-1}(0,0) \subset \quad C_{K \cap G^{-1}(0)}(z_0)$$

It remains to show that this inclusion implies inequality (90).

Let us set $\lambda = C_\uparrow(V)(x_0)(u)$, $\mu = C_\uparrow(W)(f(x_0))(f'(x_0)u)$ and $\nu := \lambda + \mu$.

Hence the element $(u, \lambda, f'(x_0)u, \mu, \nu)$ belongs to $C_K(z_0) \cap G'(z_0)^{-1}(0,0)$ and thus, to the Clarke tangent cone to the subset $K \cap G^{-1}(0,0)$ at z_0. Then, for all sequence $h_n > 0$, there exist elements (u_n, λ_n, μ_n) converging to (u, λ, μ) such that, for all $n \geq N$,

$$(x_n + h_n u_n, a_n + h_n \lambda_n, f(x_n) + h_n f'(x_0) u_n, b_n + h_n \mu_n, a_n + b_n + h_n \nu_n) \in K$$

Therefore, the pairs $(x_n + h_n u_n, a_n + b_n + h_n(\lambda_n + \mu_n))$ belong to the epigraph of U. Since $(u_n, \lambda_n + \mu_n)$ converges to (u, ν), we deduce that

$$C_\uparrow(U)(x_0)(u) \quad \leq \quad \nu \quad = \quad C_\uparrow(V)(x_0)(u) + C_\uparrow(W)(f(x_0))(f'(x_0)u) \quad \square$$

Let us state explicitely useful formulas of the epiderivatives of the restriction of a function V to a closed subset, in the finite-dimensional case for sinmplicity.

Corollary 9.1 *Let X be a finite-dimensional space, V be an extended function defined on X and K be a closed subset of X. Let x_0 belong to $K \cap \mathrm{Dom}(V)$.*

We always have:

$$\forall\, u \in T_K(x_0), \quad D_\uparrow V(x_0)(u) \leq D_\uparrow(V|_K)(x_0)(u)$$

If we assume that

$$\mathrm{Dom}(C_\uparrow(V(x_0))) - C_K(x_0) \;=\; X$$

then

$$\begin{cases} i) & \forall\, u \in T_K^\flat(x_0), \quad D_\uparrow(V|_K)(x_0) \leq D_\uparrow V(x_0)(u) \\ ii) & \forall\, u \in T_K^\flat(x_0), \quad D_\uparrow^\flat(V|_K)(x_0) \leq D_\uparrow^\flat V(x_0)(u) \\ iii) & \forall\, u \in C_K(x_0), \quad C_\uparrow(V|_K)(x_0) \leq C_\uparrow V(x_0)(u) \end{cases}$$

Let us consider now a family of functions $V_i : X \mapsto \mathbf{R} \cup \{+\infty\}$, $(i \in I)$ and let us associate with it the function U defined by

$$U(x) \ := \ \max_{i\in I} V_i(x)$$

We set $I(x) := \{i \in I \mid V_i(x) = U(x)\}$. The following estimate is obvious:

(92) $$\forall\, u \in X, \ \sup_{i\in I(x)} D_\uparrow V_i(x)(u) \ \leq \ D_\uparrow U(x)(u)$$

because, for all $i \in I(x)$,

$$(V_i(x+hu) - V_i(x))/h \ \leq \ (U(x+hu) - U(x))/h$$

Conversely, we can obtain the following result:

Proposition 9.1 *Let us consider n extended real-valued functions $V_i : X \mapsto \mathbf{R} \cup \{+\infty\}$ and the function U defined by*

$$U(x) \ := \ \max_{i=1,\dots,n} V_i(x)$$

If the dimension of X is finite and if we posit the transversality assumption

$$\forall\, u_i \in X, \ \bigcap_{i=1}^{n}(\mathrm{Dom}(C_\uparrow V_i(x_0)) - u_i) \neq \emptyset$$

then

$$\begin{cases} i) & \forall\, u \in \bigcap_{i=1}^{n} \mathrm{Dom}(D_\uparrow^\flat V_i(x_0)), \ D^\flat U(x_0)(u) \leq \max_{i\in I(x_0)} D_\uparrow^\flat V_i(x_0)(u) \\ ii) & \forall\, u \in \bigcap_{i=1}^{n} \mathrm{Dom}(C_\uparrow V_i(x_0)), \ CU(x_0)(u) \leq \max_{i\in I(x_0)} C_\uparrow V_i(x_0)(u) \end{cases}$$

Proof Since the epigraph of U is the intersection of the epigraphs of the n functions V_i, we shall us Corollary 4.3, stating that under convenient assumptions we shall check in a moment,

$$T^\flat_{\mathrm{Ep}(U)}(x_0, U(x_0)) \ \supset \ \bigcap_{i=1}^{n} T^\flat_{\mathrm{Ep}(V_i)}(x_0, U(x_0))$$

The left-hand side of this formula is the epigraph of the adjacent epiderivative of U at x_0. For the right-hand side, either i belongs to $I(x_0)$, and thus the adjacent tangent cone to the epigraph of V_i at $U(x_0)$ is equal to the epigraph of the adjacent derivative of V_i at x_0, or $V_i(x_0) < U(x_0)$, and we deduce from Proposition 8.4 that the adjacent tangent cone contains $\mathrm{Dom}(D_\uparrow^\flat V_i(x_0)) \times \mathbf{R}$.

Then, we deduce from the above relation that

$$\forall u \in \bigcap_{i=1}^{n} \mathrm{Dom}(D_{\uparrow}^{\flat}V_i(x_0)),\ D^{\flat}U(x_0)(u) \leq \max_{i\in I(x_0)} D_{\uparrow}^{\flat}V_i(x_0)(u)$$

Since the dimension of X is finite, we have to check that for all pairs (u_i, λ_i),

$$\bigcap_{i=1}^{n}(C_{\mathrm{Ep}(V_i)}(x_0, U(x_0)) - (u_i, \lambda_i)) \neq \emptyset$$

It is clear that this property follows from

$$\bigcap_{i=1}^{n}(\mathrm{Dom}(C_{\uparrow}V_i(x_0)) - u_i) \neq \emptyset \quad \square$$

Let us consider two topological vector spaces X and Y and an extended real-valued function $U : X \times Y \mapsto \mathbf{R} \cup \{+\infty\}$.

We associate with it the marginal function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ defined by

$$V(x) \ := \ \inf_{y\in Y} U(x, y)$$

Let π denote the projection from $X \times Y \times \mathbf{R}$ to $X \times \mathbf{R}$.

We observe that:

$$\pi\mathrm{Ep}(U) \ \subset \ \mathrm{Ep}(V) \ \subset \ \overline{\pi\mathrm{Ep}(U)}$$

The first inclusion is obvious. The very definition of the infimum implies that for every $\epsilon > 0$ and every $(x, \lambda) \in \mathrm{Ep}(V)$, there exists $y_\epsilon \in Y$ such that $(x, y_\epsilon, \lambda + \epsilon)$ belongs to $\mathrm{Ep}(U)$.

Proposition 9.2 *Let us consider two topological vector spaces X and Y, an extended real-valued function $U : X \times Y \mapsto \mathbf{R} \cup \{+\infty\}$, and its marginal function V. Suppose that there exists $y_0 \in Y$ which achieves the minimum of $U(x_0, \cdot)$ on Y:*

$$V(x_0) \ = U(x_0, y_0)$$

The inclusion

$$\forall u \in X,\ \ D_{\uparrow}(V)(x_0)(u) \leq \liminf_{u'\to u}(\inf_{v\in Y} D_{\uparrow}(U)(x_0, y_0)(u', v))$$

is always true.

Equality

$$\forall\, u \in X,\quad D_\uparrow(V)(x_0)(u) = \liminf_{u' \to u}(\inf_{v\in Y} D_\uparrow(U)(x_0,y_0)(u',v))$$

holds true under one of the following assumptions:

1. *U is pseudo-convex at $(x_0,y_0) \in \mathrm{Dom}(U)$, (and in particular, U is convex),*
2. *X and Y are reflexive Banach space and the subdifferential of U at (x_0,y_0) satisfies*

   (93) $\forall (p,q) \in X^\star \times Y^\star,\ \exists p_0 \in X^\star$ *such that* $(p-p_0,q) \in \partial^0 U(x,y)$

3. *The dimension of X and Y is finite and*

   (94) $$\{\ v \ \mid\ D_\uparrow(U)(x_0,y_0)(0,v)\ \le\ 0\ \} := \{0\}$$

Proof We deduce these statements from the criteria implying that the contingent cones of the images are the closures of the images of the contingent cones since

$$\begin{cases} \overline{(\pi \mathrm{Ep} D_\uparrow(U)(x_0,y_0))} & = & \overline{(\pi T_{\mathrm{Ep}(U)}(x_0,y_0,U(x_0,y_0)))} \\ \subset T_{\pi(\mathrm{Ep}U)}(x_0,V(x_0)) & = & T_{\mathrm{Ep}(V)}(x_0,V(x_0)) \\ = \mathrm{Ep}(D_\uparrow(V)(x_0)) & & \end{cases}$$

which can be easily translated into the first inequality.

The equality is obtained when U is pseudo-convex because its epigraph is then pseudo-convex, (see Proposition 4.3), when condition (93) holds true because it implies that

$$\mathrm{Im}(\pi^\star) + N^0_{\mathrm{Ep}(U)}(x_0,y_0,U(x_0,y_0)) \;=\; X^\star \times Y^\star \times \mathbf{R}$$

(see Proposition 4.5), and when condition (94) is satisfied because it is equivalent to

$$\ker(\pi) \cap T_{\mathrm{Ep}(U)}(x_0,y_0,U(x_0,y_0)) \;=\; \{0\}$$

thanks to Proposition 4.4.

10 Normal Cones and Generalized Gradients

We devote this section to dual concepts of tangent cones, derivatives of set-valued maps and epiderivatives of extended real-valued functions. There are three reasons to do so. The first one is familiarity with more classical concepts. For usual functions on Hilbert spaces, there is a canonical identification between, say, a derivative of a differentiable function and its gradient, and it became traditional to formulate many results in terms of gradients, transposes of derivatives and normal cones. The second reason is that the first attempts to generalize the concept of gradients was by limiting procedures. Since it seems easier to take limits of elements (the gradients, for instance) than functionals (the associated directional derivatives, for example), many generalizations of concept of gradient dealt with set of limits of cluster points taken in a variety of ways. The third, and, from our view point, the most important justification for dealing with dual concepts, is the availability of the one to one correspondences between closed convex cones and their polar cones, continuous linear operators and their transposes, lower semicontinuous convex functions and their conjugates. This is why we should use only those concepts which can be "dualized". Unfortunately, this is just a paradisiac wish, since many problems which are not smooth by nature, force us to use naturally concepts as contingent cones, contingent derivatives and contingent epiderivatives. The price to pay in terms of loss of information for playing with duality just to be able to conserve some familiar classical formulation is indeed too high in many situations. Therefore, the dual concepts we are about to present are recommended only in convex, or more generally, **sleek** situations.

Since the Clarke tangent cone is convex, it can be characterized by its polar cone, which, by analogy with the case of smooth manifolds, will be regarded as the normal cone. On the other hand, we wish to adapt to the nonsmooth case the concept of a normal to a set at a given point, which is orthogonal to all vectors starting from this point and pointing into this set. Except the convex and (more generally), the sleek case, these two concepts are different.

Definition 10.1 *Let x belongs to $K \subset X$. We shall say that the (negative) polar cone*

(95) $$N_K(x) := C_K(x)^- = \{p \in X^\star \mid \forall v \in T_K(x), < p, v > \leq 0\}$$

is the normal cone *to K at x. We also say that the polar cone*

(96) $$N_K^\circ(x) := S_K(x)^- = \{ p \in X^\star \mid \max_{y \in K} < p, y > = < p, x > \}$$

to the cone spanned by $K - x$ is the **subnormal cone to** K *at x.*

The normal cone is pretty big since it contains the polar cones of the adjacent and contingent cones and the subnormal cone:

(97) $$N_K^\circ(x) \subset T_K(x)^- \subset T_K^\flat(x)^- \subset N_K(x)$$

and is equal to the whole space whenever the Clarke tangent cone is reduced to 0.

Let us point out the following property:

Proposition 10.1 *Let K be a subset of a Hilbert space. Then*

$$\forall y \notin K, \ \forall x \in \pi_K(y), \ y - x \in T_K(x)^- \subset N_K(x)$$

Proof Let v belong to the contingent cone $T_K(x)$: there exists a sequence $h_n > 0$ converging to 0 and a sequence v_n converging to v such that $x + h_n v_n$ belongs to K for all n. Since $\|y - x\| \leq \|y - x - h_n v_n\|$, we deduce that $< x - y, v > \geq 0$ for all $v \in T_K(x)$. □

Remark When X is a Banach space, we may consider the subdifferential $J(x)$ of its norm at x. then the above proposition can be extended to

(98) $$0 \in J(x-y) + N_K(x) \quad \square$$

The transpose of the derivatives of differentiable maps are often used, in writing chain rule formulas, just to quote an instance. The circatangent derivative being a closed convex process can be transposed. This brings us to introduce the following definitions

Definition 10.2 *Let $F : X \rightsquigarrow Y$ be a set-valued map from a topological vector space X to another Y. The transpose $CF(x,y)^*$ of the circatangent derivative $CF(x,y)$ of F at a $(x,y) \in \mathrm{Graph}(F)$ defined by*

(99) $$\begin{cases} p \in CF(x,y)^*(q) & \text{if and only if} \\ \forall\, u \in X, \ \forall\, v \in CF(x,y)(u), & <p,u> \ \leq \ <p,v> \end{cases}$$

is the **codifferential** *of F at (x,y). We shall say that the closed convex process from Y^* to X^* defined by*

(100) $$\begin{cases} p \in DF(x,y)^{\circ *}(q) & \text{if and only if} \\ \forall\, (x',y') \in \mathrm{Graph}(F), & <p,x'-x> \ \leq \ <q,y'-y> \end{cases}$$

is the **cosubdifferential** *of F at (x,y).*

When a real-valued function V is continuously differentiable at x, its gradient $V'(x)$ being a continuous linear functional, is therefore an element $V'(x) \in X^*$ of the dual of X.

Since the circatangent epiderivative of a nontrivial extended real-valued function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ at a point x of its domain is always lower semicontinuous convex and positively homogeneous, it is the support function of a closed convex subset, which is the generalized gradient of V at x. In the same time, we shall deal with the subdifferentials introduced by Moreau and Rockafellar for convex functions.

Definition 10.3 *Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended real-valued function and x belong to its domain. We shall say that the*

(101) $$\partial V(x) \ := \ \{\, p \in X^* \mid \forall\, u \in X, \ <p,u> \ \leq \ C_\uparrow V(x)(u) \,\}$$

is the **generalized gradient** *of V at x.*

We shall say that the closed convex subset of the dual X^* of X defined by

(102) $\partial^0 V(x) := \{p \in X^* \mid \forall\, y \in X,\ < p, y - x > \leq V(y) - V(x)\}$

is the **subdiferential** *of V at x.*

Naturally, when V is continuously differentiable at x, the circatangent epiderivative coincides with the gradient $V'(x)$, so that the generalized gradient is reduced to the only gradient:

$$\partial V(x) \ = \ \{V'(x)\} \text{ when } V \text{ is continuously differentiable at } x$$

We observe that

$$\partial^0 V(x) \ \subset \ \partial V(x)$$

When V is convex, both the generalized gradient and the subdiferential are equal:

$$\partial V(x) = \partial^0 V(x) \text{ when } V \text{ is convex}$$

More generally, this also happens when V is sleek and pseudo-convex at x.

If V is continuously differentiable around a point $x \in K$, then the **generalized gradient of the restriction is the sum of the gradient and the normal cone**:

$$\partial(V|_K)(x) \ = \ V'(x) + N_K(x)$$

We also note that the generalized gradient of the indicator of a subset is the normal cone:

$$\partial\psi_K(x) \ = \ N_K(x)$$

Remark When a real-valued function V is continuously differentiable at x, its gradient $V'(x)$ being a continuous linear functional, it is both an element $V'(x) \in X^*$ of the dual and the image $V'(x)^*(+1)$ of $+1$ by the transpose V'^* of $V'(x)$, a linear operator from $\mathbf{R}$ to X^*.

When V is no longer continuously differentiable, the generalized gradient remains intimately connected with the transpose of the circatangent derivative $C\mathbf{V}_{\uparrow}(x, V(x))$, (which, shall we recall, is the codifferential $CV_{\uparrow}(x, V(x))^{\star}$) of the set-valued map $\mathbf{V}_{\uparrow}$ at $(x, V(x))$.

> **the generalized gradient is the value at 1 of the codifferential of $\mathbf{V}_{\uparrow}$ at $(x, V(x))$.**

Indeed, the codifferential is a closed convex process from $\mathbf{R}$ to $X^{\star}$ which, being positively homogeneous, needs to be defined only at the points -1, 0 and $+1$.

We obtain:

$$(103) \quad \begin{cases} i) & CV_{\uparrow}(x, V(x))^{\star}(-1) \; = \; \emptyset \\ ii) & CV_{\uparrow}(x, V(x))^{\star}(0) \; = \; (\mathrm{Dom}(C_{\uparrow}V)(x))^{-} \\ iii) & CV_{\uparrow}(x, V(x))^{\star}(+1) \; = \; \partial V(x) \; \Box \end{cases}$$

The table of formulas on support functions allows to translate the properties of the circatangent epiderivatives into corresponding properties of the generalized gradient and vice-versa.

For instance, Proposition 8.1 can be restated in the following form.

Proposition 10.2 *When V is locally lipschitzean on the interior of its domain, then the generalized gradient satisfies:*

$$\begin{cases} i) & (x, u) \in X \times \mathrm{Int}(\mathrm{Dom}(V)) \mapsto \sigma(\partial V(x), u) \\ & \textit{is upper semicontinuous,} \\ & \textit{and thus, } \partial V(\cdot) \textit{ is upper hemicontinuous} \\ ii) & \partial V(x) \textit{ are nonempty bounded closed convex} \\ iii) & \partial V(x) \; = \; -\partial(-V(x)) \end{cases}$$

In the same way, the Fermat and Ekeland rules can be presented in the following fashion:

Theorem 10.1 (Fermat and Ekeland Rules) *Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended real-valued function.*

1. *Let $x \in \mathrm{Dom}(V)$ achieve the minimum of V on X.*

Then x is a solution to the inclusion:

$$0 \in \partial V(x)$$

the converse being true when V is convex or. more generally, pseudo-convex at x.

2. *Let X be a Banach space, $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ be a nontrivial non-negative extended real-valued function and $x_0 \in \mathrm{Dom}(V)$ be a given point of its domain. Then, for any $\epsilon > 0$, there exists a solution $x_\epsilon \in \mathrm{Dom}(V)$ to:*

$$(104) \quad \begin{cases} i) & V(x_\epsilon) + \epsilon\|x_\epsilon - x_0\| \leq V(x_0) \\ ii) & \forall\, u \in X,\ 0 \in \partial V(x_\epsilon) + \epsilon B \end{cases}$$

Remark

When the functions are not sleek, the use of generalized gradients and normal cones involves some loss of information since we have to replace the contingent epiderivative by the larger, but **convex**, circatangent epiderivative.

We could save part of the information using subsets of the form

$$\{\, p \in X^\star \mid \forall\, v \in X,\ < p, u > \leq D_\uparrow V(x)(u) \,\}$$

and the polar of the contingent cones.

This will lead us to increase the population of our ménagerie with species doomed to disappear through Darwinian evolution, since, their use do not allow to recover the original information for lack of duality. The use of generalized gradients for functions is then recommended for functions or cones which are sleek. □

For the convenience of the reader, we list below a summary of some formulas dealing with the adjacent[23] and normal cones to subsets. The subsets K, K_i, L, M, ... are assumed to be nonempty.

Properties of Adjacent and Normal Cones (1)

1. If $K \subset L$, then

$$T_K^\flat(x) \subset T_L(x)$$

2. If $K_i \subset X_i$, $(i = 1, \cdots, n)$, then

$$\left\{ \begin{array}{lrcl} i) & T^\flat_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) & = & \prod_{i=1}^n T^\flat_{K_i}(x_i) \\ ii) & N_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) & = & \prod_{i=1}^n N_{K_i}(x_i) \end{array} \right.$$

3. If K_1 and K_2 are contained in X, then

$$T_{K_1+K_2}(x_1 + x_2) \supset \overline{T_{K_1}(x_1) + T_{K_2}(x_2)}$$

4. If $f : X :\mapsto Y$, is differentiable at x, then

$$T_{f(K)}(x) \supset \overline{f'(x)(T_K(x))}$$

5. If X and Y are finite dimensional vector-spaces, if $L \subset X$ and $M \subset Y$ are *closed* subsets, if $f : X \mapsto Y$ is continuously differentiable at x and satisfy the ***transversality condition***

$$f'(x)C_L(x) - C_M(f(x)) = Y$$

then

$$\left\{ \begin{array}{lrcl} i) & T^\flat_{L\cap A^{-1}(M))} & = & T^\flat_L(x) \cap A^{-1}T^\flat_M(Ax) \\ ii) & N_{L\cap A^{-1}(M))} & \subset & N_L(x) + A^\star N_M(Ax) \end{array} \right.$$

[23] we chose the adjacent cone rather than the contingent or the Clarke tangent cones because they enjoy more often equalities in the formulas. Formulas for the Clarke tangent cones can be deduced from polarity from the formulas on normal cones.

Properties of Adjacent and Normal Cones (2)

6. If X and Y are finite dimensional vector-spaces, if $M \subset Y$ is a *closed* subset and if $f : X \mapsto Y$ is continuously differentiable at x such that the ***transversality condition***

$$\mathrm{Im}(f'(x)) - C_M(x) = Y$$

holds true, then

$$\begin{cases} i) & T^\flat_{A^{-1}(M)}(x) = A^{-1}T^\flat_M(Ax) \\ ii) & N_{A^{-1}(M)}(x) \leq A^\star N_M(Ax) \end{cases}$$

7. If X is a finite dimensional vector-space, if K_1 and K_2 are closed subsets contained in X and satisfy

$$C_{K_1}(x) - C_{K_2}(x) = X$$

then

$$\begin{cases} i) & T^\flat_{K_1 \cap K_2}(x) = T^\flat_{K_1}(x) \cap T^\flat_{K_2}(x) \\ ii) & N_{K_1 \cap K_2}(x) \subset N_{K_1}(x) + N_{K_2}(x) \end{cases}$$

8. If X is a finite dimensional vector-space, if $K_i \subset X$, $(i = 1, \ldots, n)$, are closed and if

$$\forall\, v_i \in X, \quad \bigcap_{i=1}^{n} (C_{K_i}(x) - v_i) \neq \emptyset$$

then,

$$\begin{cases} i) & T^\flat_{\bigcap_{i=1}^n K_i}(x) = \bigcap_{i=1}^n T^\flat_{K_i}(x) \\ ii) & N_{\bigcap_{i=1}^n K_i}(x) \subset \sum_{i=1}^n N_{K_i}(x) \end{cases}$$

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