WORKING PAPER

OBSERVABILITY OF SYSTEMS UNDER UNCERTAINTY

Jean-Pierre Aubin
and
Halina Frankowska

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Halina Frankowska*

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*CEREMADE, Université de Paris-Dauphine,
Paris, France

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

FOREWORD

The authors observe the evolution $t \in [0,T] \mapsto x(t) \in X$ of the state $x(\cdot)$ of a **system under uncertainty** governed by a differential inclusion

$$\text{for almost all } t \in [0,T], \quad x'(t) \in F(t, x(t))$$

through an observation map H

$$\forall\, t \in [0,T], \quad y(t) = h(x(t)) + \epsilon(t), \quad \epsilon(t) \in Q(t)$$

The set-valued character due to the uncertainty leads them to introduce

Sharp Input-Output map which is the (usual) product

$$\forall\, x_0 \in X, \quad I_-(x_0) := (H \circ S)(x_0) := \bigcup_{x(\cdot) \in S(x_0)} H(x(\cdot))$$

Hazy Input-Output map which is the **square product**

$$\forall\, x_0 \in X, \quad I_+(x_0) := (H \,\square\, S)(x_0) := \bigcap_{x(\cdot) \in S(x_0)} H(x(\cdot))$$

Recovering the input x_0 from the outputs $I_-(x_0)$ or $I_+(x_0)$ means that these Input-Output maps are "injective" in the sense that, locally,

$$\text{whenever } x_1 \neq x_2, \text{ then } I(x_1) \cap I(x_2) = \emptyset$$

They provide criteria for both sharp and hazy local observability in terms of (global) sharp and hazy observabiliy of the variational inclusion

$$w'(t) \in DF(t, \overline{x}(t), \overline{x}'(t))(w(t))$$

which is a "linearization" of the differential inclusion along a solution $\overline{x}(\cdot)$, where for almost all t, $DF(t,x,y)(u)$ denotes the contingent derivative of the set-valued map $F(t,\cdot,\cdot)$ at a point (x,y) of its graph. They reach these conclusions by implementing the following strategy:

1. Provide a general principle of local injectivity and observability of a set-valued map I, which derives these properties from the fact that the kernel of an adequate derivative of I is equal to 0.
2. Supply chain rule formulas which allow to compute the derivatives of the usual product I_- and the square product I_+ from the derivatives of the observation map H and the solution map $\mathcal{S}$.
3. Characterize the various derivatives of the solution map $\mathcal{S}$ in terms of the solution maps of the associated variational inclusions.
4. Piece together these results for deriving local sharp and hazy observability of the origial system from sharp and hazy observability of the variational inclusions.
5. Study global sharp and hazy observability of the variational inclusions.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

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Observability of Systems under Uncertainty

JEAN-PIERRE AUBIN & HALINA FRANKOWSKA

1 Introduction

We describe the evolution $t \in [0,T] \mapsto x(t) \in X$ of the state $x(\cdot)$ of a system under uncertainty by a differential inclusion

$$(1) \qquad \text{for almost all } t \in [0,T], \quad x'(t) \in F(t, x(t))$$

where the set-valued map character takes into account disturbances and/or perturbations of the system[1]. This system is observed through an observation map H, which is generally a set-valued map from the state space X to some observation space Y, which associates with each solution to the differential inclusion (1) an observation[2] $y(\cdot)$ satisfying

$$(2) \qquad \forall t \in [0,T], \quad y(t) \in H(x(t))$$

Observability concepts deal with the possibility of recovering the initial state $x_0 = x(0)$ of the system knowing only the evolution of an observation $t \in [0,T] \mapsto y(t)$ during the interval $[0,T]$, and naturally, knowing the laws (1) and (2). Once we get the initial state x_0, we may, by studying the differential inclusion, gather information about the solutions starting from x_0, using the many results provided by the theory of differential inclusions[3]. Let $S := S_F$

[1] A familiar representation of uncertainty is represented in parametrized form

$$\text{for almost all } t \in [0,T], \quad x'(t) = f(t, x(t)) + g(t, d(t)), \quad d(t) \in D(t)$$

[2] generally, given in a parametrized form

$$\forall t \in [0,T], \quad y(t) = h(x(t)) + \epsilon(t), \quad \epsilon(t) \in Q(t)$$

We assume for simplicity that H does not depend of the time t, but we shall provide in the appropriate remarks the extensions to the time-dependent case.

[3] For instance, under an adequate Lipschitz property, we know that for every $\overline{x}(\cdot) \in S(x_0)$,

$$S(x_0) \in \overline{x}(\cdot) + M \int_0^T \text{diam} F(t, \overline{x}(t)) dt B$$

where M is a constant independent of $\overline{x}(\cdot)$ and B denotes the closed unit ball in the Sobolev space $W^{1,1}(0,T)$.

from X to $\mathcal{C}(0,T;X)$ denote the solution map associating with every initial state $x_0 \in X$ the (possibly empty) set $\mathcal{S}(x_0)$ of solutions to the differential inclusion (1) starting at x_0 at the initial time $t = 0$.

In other words, we have introduced an Input-Output system where the

1. **inputs** , are the **initial states** x_0
2. **outputs**, are the **observations** $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through H

$$\begin{array}{ccccc}
\textbf{Inputs} & \overset{\mathcal{S}}{\rightsquigarrow} & \textbf{States} & \overset{H}{\mapsto} & \textbf{Outputs} \\
\downarrow & & \downarrow & & \downarrow \\
X \ni x_0 & \rightsquigarrow & x(\cdot) \in \mathcal{S}(x_0) & \mapsto & y(\cdot) \in H(x(\cdot)) \\
\uparrow & & \uparrow & & \uparrow \\
\text{Initial States} & & \left\{ \begin{array}{l} x'(t) \in F(t, x(t)) \\ x(0) = x_0 \end{array} \right. & & \text{Observations}
\end{array}$$

It remains to define an Input-Output map. But, because of the set-valued character (the presence of uncertainty), one can conceive two dual ways for defining composition products of the set-valued maps $\mathcal{S}$ from X to the space $\mathcal{C}(0,T;X)$ and H from $\mathcal{C}(0,T;X)$ to $\mathcal{C}(0,T;Y)$. So, for systems under uncertainty, we have to deal with **two Input-Output maps** from X to $\mathcal{C}(0,T;Y)$: the

Sharp Input-Output map which is the (usual) product

$$\forall\, x_0 \in X,\ \ I_-(x_0) := (H \circ \mathcal{S})(x_0) := \bigcup_{x(\cdot)\in \mathcal{S}(x_0)} H(x(\cdot))$$

Hazy Input-Output map which is the **square product**

$$\forall\, x_0 \in X,\ \ I_+(x_0) := (H \,\square\, \mathcal{S})(x_0) := \bigcap_{x(\cdot)\in \mathcal{S}(x_0)} H(x(\cdot))$$

The **sharp** Input-Output map tracks **at least** the evolution of a state starting at some initial state x_0 whereas the **hazy** Input-Output map tracks **all** such solutions.

Opinions may differ about which would be the "right" Input-Output map, just because they depend upon the context in which a given problem is stated. So, we shall study observability properties of **both** the sharp and hazy Input-Output maps.

Recovering the input x_0 from the outputs $I_-(x_0)$ or $I_+(x_0)$ means that the set-valued maps are "injective" in some sense.

When H and $\mathcal{S}$ are single-valued maps, the input-output map is called observable whenever the product $I := H \circ \mathcal{S}$ is injective, i.e.,

$$(3) \qquad H\mathcal{S}(x_1) = H\mathcal{S}(x_2) \implies x_1 = x_2$$

When we adapt this definition to the set-valued case, we come up with two possibilities: If I stands now for either I_- or I_+, we can require either the property

$$I(x_1) = I(x_2) \implies x_1 = x_2$$

or the stronger condition

$$I(x_1) \cap I(x_2) \neq \emptyset \implies x_1 = x_2$$

The first way would not be, in general, useful in the framework of uncertain systems since we often observe just one output $y(\cdot) \in H\mathcal{S}(x_0)$ and not the whole set of possible outputs $H\mathcal{S}(x_0)$. That is why we shall the second point of view, by saying that the sharp or hazy Input-Output map I is "observabale" **around** x_0 if

$$(4) \qquad \text{whenever } x_1 \neq x_2, \text{ then } I(x_1) \cap I(x_2) = \emptyset$$

If this property holds only on a neighborhood of some x_0, we shall say that I is "locally observable around x_0.

This a very pleasant concept, which we shall study for hazy Input-Output maps.

However, it is a bit too strong for sharp observability, and we shall be content with the weaker condition that the inverse image $I^{-1}(y_0)$ of some observation y_0 contains **at most** one input x_0:

$$(5) \qquad \text{whenever } x_1 \neq x_0, \text{ then } y_0 \notin I(x_1)$$

If this is the case, we shall say that the Input-Output map I is "observable" **at** x_0 , and "locally observable" **at** x_0 if it holds only on a neighborhood of x_0 (instead of "around" x_0).

In other words, sharp observability at x_0 means that whenever y_0 is an observation of some solution $x^\star(\cdot)$, i.e., $y_0 \in H(x^\star(\cdot))$, then $x^\star(0) = x_0$. Local sharp observability means that the above holds true only for those $x^\star$'s not too far from x_0.

Hazy observability at x_0 of y_0 means that y_0 can be a "common" observation only for one input x_0. In other words, if we (hopefully) observe an output y. which is a common observation of all solutions $x(\cdot) \in \mathcal{S}(\overline{x_0})$, then $\overline{x_0} = x_0$.

Actually, the purpose of this paper is to derive local observability of both the sharp and hazy Input-Output maps from the global sharp and hazy observability at 0 of "variational inclusions" through a linearization[4] of the Input-Output map.

Here, variational inclusions are "linearizations" of the differential inclusion (1) along a solution $\overline{x}(\cdot) \in \mathcal{S}(x_0)$ of the form

$$(6) \qquad w'(t) \in DF(t, \overline{x}(t), \overline{x}'(t))(w(t))$$

where for almost all t, $DF(t,x,y)(u)$ denotes an adequate concept of derivative (the contingent derivative, defined below) of the set-valued map $F(t,\cdot,\cdot)$ at a point (x,y) of its graph. Let us just say for the time that they are set-valued analogues of continuous linear operators.

(These linearized differential inclusions are called **variational inclusions** because they extend (in various ways) the classical variational equations of ordinary differential equations: their solutions starting at some u provide the directional derivative of the solution to the initial system in the direction u.)

To say that the variational inclusion is **hazily** (respectively **sharply**) observable at 0 amounts to saying that whenever **all** (respectively **at least one**) solutions $w(\cdot)$ to the variational inclusion (6) starting at u satisfy

$$(7) \qquad \forall\, t \in [0,T], \quad H'(\overline{x}(t))w(t) = 0$$

then $u = 0$.

To reach such conclusions, we shall choose the following strategy:

1. Provide a general principle of local injectivity and observability of a set-valued map I, which derives these properties from the fact that the kernel of an adequate derivative of I is equal to 0.

2. Supply chain rule formulas which allow to compute the derivatives of the usual product I_- and the square product I_+ from the derivatives of the observation map H and the solution map $\mathcal{S}$.

[4] The linearization techniques based on the differential calculus and inverse function theorems for set-valued maps has been succesfully used in the study of local controllability of differential inclusions and control systems with feedbacks. (See [13,10,11,12,20].)

3. Characterize the various derivatives of the solution map $\mathcal{S}$ in terms of the solution maps of the associated variational inclusions.

4. Piece together these results for deriving local sharp and hazy observability of the origial system from sharp and hazy observability of the variational inclusions.

5. Study global sharp and hazy observability of the variational inclusions[5]

But, before implementing this program, we have to avoid the trivial case when the hazy Input-Output map I_+ takes (locally) empty values.

For doing that, we " project" the differential inclusion (1) onto a differential inclusion

$$\text{(8)} \qquad \text{for almost all } t \in [0,T], \quad y'(t) \in G(t, y(t))$$

in such a way that the following property

$$\text{(9)} \quad \begin{cases} \forall\, (x_0, y_0) \in \text{Graph}(H) \ \text{ all solutions } x(\cdot) \text{ to (1) and } y(\cdot) \text{ to (8)} \\ \text{satisfy } \forall\, t \in [0,T], \quad y(t) \in H(x(t)) \end{cases}$$

holds true. If such is the case, then the hazy Input-Output map I_+ is well defined.

To proceed further, we need to introduce the concept of "contingent[6] derivative" of a set-valued map H from a Banach space X to a Banach space Y at a point (x, y) of its graph: It is the set-valued map $DH(x, y) : X \rightsquigarrow Y$ which associates with any direction u the set $DH(x, y)(u)$ of directions v satisfying

$$\text{(10)} \qquad \liminf_{h \to 0+, u' \to u} d\left(v, \frac{H(x + hu') - y}{h}\right) = 0$$

[5]This has already been done in [4,5] for time-independant closed convex processes, where it was shown that sharp observability is a dual concept of controllability and where various characterizations were provided. See below the comments on the oservability of a system around an equilibrium.

[6]The choice of this particular derivative is motivated by the fact that its graph is the **contingent cone** to the graph of H at (x, y), where the contingent cone $T_K(x)$ to $K \subset X$ at $x \in K$ is the set of directions $v \in X$ such that

$$\liminf_{h \to 0} d(x + hv, K)/h = 0$$

For our purpose, the contingent cone plays a major role compared to other tangent cones. However, we shall need other tangent cones and associated derivatives.

It is said "derivable" if for every (x, y) in the graph of F, v belongs to $DF(x, y)(u)$ if and only if

$$\lim_{h \to 0+} d\left(v, \frac{H(t, x + hu) - y}{h}\right) = 0$$

We extend the concept of C^1-function by saying that H is "sleek" if and only if[7]

$$\text{Graph}(H) \ni (x, y) \quad \rightsquigarrow \quad \text{Graph}(DH(x, y)) \text{ is lower semicontinuous}$$

(See the Appendix for more details on the differential calculus of set-valued maps).

Returning to the projection problem, we shall say that a set-valued map $G : [0, T] \times Y \rightsquigarrow Y$ is a " lipschitzean[8] square projection" of the set-valued map $F : [0, T] \times X \rightsquigarrow X$ by H if and only if

$$(11) \quad \begin{cases} i) & F \times G \text{ is lipschitzean around } [0, T] \times \text{Graph}(H) \\ ii) & \forall\, (x, y) \in \text{Graph}(H), \;\; G(t, y) \subset \bigcap_{v \in F(t,x)} DH(x, y)(v) \end{cases}$$

We shall prove that *if there exists a lipschitzean square projection of F by H, then the hazy Input-Output map $I_+ := H \square S$ has non empty values for any initial value $y_0 \in H(x_0)$.*

We state now the observability properties of the hazy Input-Output map around a solution $\bar{x}(\cdot)$ to the differential inclusion (1). We assume that F satisfies the following assumptions:

$$(12) \quad \begin{cases} i) & \forall x \in X \text{ the set-valued map } F(\cdot, x) \text{ is measurable} \\ ii) & \forall\, t \in [0, T],\ \forall\, x \in X,\ \ F(t, x) \text{ is a closed nonempty set} \\ iii) & \exists\, k(\cdot) \in L^1(0, T) \text{ such that for almost all } t \in [0, T] \\ & \text{the map } F(t, \cdot) \text{ is } k(t) - \text{Lipschitzean} \end{cases}$$

Theorem 1.1 *Let us assume that H is continuously differentiable, that F satisfies assumptions (12), that it has linear growth[9] and that it has a lipschitzean square projection G by H.*

[7] In this case, the graph of $DF(t, x, y)$ is a closed convex cone. Maps whose graphs are closed convex cones, called closed convex processes, are the set-valued analogues of continuous linear operators, and enjoy most of their properties.

[8] This means that

$$F(t, x) \subset F(t, y) + k(t)\|x - y\| B$$

[9]

$$\exists\, c > 0 \quad \text{such that} \quad \|F(t, x)\| \leq c(\|x\| + 1)$$

1. *If F is derivable and if for some $\overline{x}(\cdot) \in S(x_0)$ the contingent variational inclusion*

   (13) for almost all $t \in [0,T]$, $w'(t) \in DF(t, \overline{x}(t), \overline{x}'(t))(w(t))$

   is globally hazily observable through $H'(\overline{x}(\cdot))$ at 0, then the system (1) is locally hazily observable through H **at** x_0.

2. *If F is sleek and if for every solution $x(\cdot)$ to the differential inclusion (1) starting at x_0, the contingent variational inclusions*

   (14) for almost all $t \in [0,T]$, $w'(t) \in DF(t, x(t), x'(t))(w(t))$

   is globally hazily observable through $H'(x(\cdot))$ at 0, then the system (1) is locally hazily observable through H **around** x_0.

Observability properties of sharp Input-Output maps require stronger assumptions. We state first the result for a more simple, convex case.

Theorem 1.2 *Let us assume that H is linear and that the graphs of the set-valued maps $F(t,\cdot) : X \rightsquigarrow X$ are closed and convex. If for some $\overline{x}(\cdot) \in S(x_0)$ the contingent variational inclusion (13) is globally sharply observable through H at 0, then the system (1) is globally sharply observable through H at x_0.*

A more general case requires some additional assumptions.

Theorem 1.3 *Assume that F has closed convex images, is continuous, derivable, Lipschitz in the second variable with a constant independent of t and that the growth of F is linear with respect to the state. Let H be a twice continuously differentiable function from X to another finite dimensional vector-space Y. Consider an observation $y^* \in I_-(x_0)$ and assume that for every solution $\overline{x}(\cdot)$ to the differential inclusion (1) satisfying $y^*(\cdot) = H(\overline{x}(\cdot))$ and for all $t \in [0,T]$ we have*

$$\mathrm{Ker}\, H'(\overline{x}(t)) \cap (F(t,\overline{x}(t)) - F(t,\overline{x}(t)))^{\perp} = \{0\}$$

If for all $\overline{x}(\cdot)$ as above the contingent variational inclusion (13) is globally sharply observable through $H'(\overline{x}(t))$ around 0, then the system (1) is locally sharply observable through H at (x_0, y^*).

2 Hazy and Sharp Input-Output Systems

Let us consider a set-valued input-output system of the following form built through a differential inclusion

$$(15) \qquad \text{for almost all } t \in [0,T], \quad x'(t) \;\in\; F(t, x(t))$$

whose dynamics are described by a set-valued map F from $[0,T] \times X$ to X, where X is a finite dimensional vector-space (the **state space**) and $0 < T \leq \infty$. It governs the (uncertain) evolution of the state $x(\cdot)$ of the system. The **inputs** are the **initial states** x_0 and the **outputs** are the **observations** $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through a single-valued (or set-valued) map H from X to an **observation space** Y.

Let $\mathcal{S} := \mathcal{S}_F$ from X to $\mathcal{C}(0,T;X)$ denote the solution map associating with every initial state $x_0 \in X$ the (possibly empty) set $\mathcal{S}(x_0)$ of solutions to the differential inclusion (15) starting at x_0 at the initial time $t = 0$.

One can conceive two dual ways for defining composition products of set-valued maps G from a Banach space X to a Banach space Y and a set-valued map H from Y to a Banach space Z (which naturally coincide when H and G are single-valued):

Definition 2.1 *Let X, Y, Z be Banach spaces and $G : X \rightsquigarrow Y$, $H : Y \rightsquigarrow Z$ be set-valued maps.*

1. *the usual composition product (called simply the* **product***) $H \circ G : X \rightsquigarrow Z$ of H and G at x is defined by*

$$(H \circ G)(x) \;:=\; \bigcup_{y \in G(x)} H(y)$$

2. *the* **square product** *$H \square G : X \rightsquigarrow Z$ of H and G at x is defined by*

$$(H \square G)(x) \;:=\; \bigcap_{y \in G(x)} H(y)$$

Remark

1. The observability problems that we address involve the inversion of these Input-Output maps.

There are two ways to adapt to the set-valued case the formula which states that the inverse of a product is the product of the inverses (in reverse order), since we know that there are two ways of defining the inverse image by a set-valued map S of a subset M:

$$\begin{cases} a) & S^{-}(M) := \{ x \mid S(x) \cap M \neq \emptyset \} \\ b) & S^{+}(M) := \{ x \mid S(x) \subset M \} \end{cases}$$

We then observe the following formulas of the inverse of composition products:

$$\begin{cases} i) & (H \circ S)^{-1}(y) = S^{-}(H^{-1}(y)) \\ ii) & (H \,\square\, S)^{-1}(y) = S^{+}(H^{-1}(y)) \end{cases}$$

This may provide a further justification of the introduction of those two "dual" composition products.

2. Recall also that a set-valued map S is upper semicontinuous if and only if the inverse images S^{-} of open subsets are open and that it is lower semicontinuous if and only if the inverse images S^{+} of open subsets are open.

3. Observe finally that square products are implicitely involved in the factorization of maps. Let X be a subset, $\mathcal{R}$ be an equivalence relation on X and ϕ denote the canonical surjection from X onto the factor space $X/\mathcal{R}$. If f is a single-valued map from X to Y, its factorization $\tilde{f} : X/\mathcal{R} \mapsto Y$ is defined by

$$\tilde{f}(\xi) := (f \,\square\, \phi^{-1})(\xi)$$

It is non trivial if and only if f is consistent with the equivalence relation $\mathcal{R}$, i.e., if and only if $f(x) = f(y)$ whenever $\phi(x) = \phi(y)$.

When $F : X \rightsquigarrow Y$ is a set-valued map, we can define its factorization $\tilde{F} : X/\mathcal{R} \rightsquigarrow Y$ by

$$\tilde{F}(\xi) := (F \,\square\, \phi^{-1})(\xi) \quad \square$$

Then we can associate with this system described through state-space representation two Input-Output maps:

Definition 2.2 *Let us consider a system (F, H) defined by the set-valued map F describing the dynamics of the differential inclusion and the observation map H.*

Let $\mathcal{S} := \mathcal{S}_F$ denote the solution map of the differential inclusion. We shall say that

1. *the product $I_- := H \circ \mathcal{S}$, from X to $\mathcal{C}(0,T;Y)$ defined by*

$$\forall\, x_0 \in X,\; I_-(x_0) \;:=\; \bigcup_{x(\cdot)\in\mathcal{S}(x_0)} H(x(\cdot))$$

   is the **Sharp** *Input-Output map.*

2. *the "square product" $I_+ := H \,\square\, \mathcal{S}$, from X to $\mathcal{C}(0,T;Y)$ defined by*

$$\forall\, x_0 \in X,\; I_+(x_0) \;:=\; \bigcap_{x(\cdot)\in\mathcal{S}(x_0)} H(x(\cdot))$$

   is the **Hazy** *Input-Output map.*

Remark

Observe that when the observation map is single-valued, the use of a non trivial hazy Input-Output map requires that all solutions $x(\cdot) \in \mathcal{S}(x_0)$ yield the same observation $y(\cdot) = H(x(\cdot))$. Hence we have to study when this possibility occurs, by projecting the differential inclusion (15) onto a differential equation which "tracks" all the solutions to the differential inclusion. This is the purpose of the next section. □

3 Projection of a System onto the Observation Space

Our first task is to provide conditions implying that the hazy Input-Output map $I_+ := H \,\square\, \mathcal{S}$ is not trivial, above all when the observation map is single-valued.

We shall tackle this issue by "projecting" the differential inclusion given in the state space X onto a differential inclusion in the observation space Y in such a way that solutions to the projected differential inclusion are observations of solutions to the original differential inclusion.

Let us consider a differential inclusion

$$(16) \qquad x'(t) \;\in\; F(t, x(t)),\; x(0) \;=\; x_0$$

where $F : [0,T] \times X \rightsquigarrow X$ is a nontrivial set-valued map and an observation map $H : X \rightsquigarrow Y$ from X to another finite dimensional vector-space Y.

We project the differential inclusion (16) to a differential inclusion (or a differential equation) on the observation space Y described by a set-valued map G (or a single-valued map g)

$$(17) \qquad y'(t) \in G(t, y(t)) \ (\text{ or } y'(t) = g(t, y(t)) \ ), \ y(0) = y_0$$

which allows to track partially or completely solutions $x(\cdot)$ to the differential inclusion (16) in the following sense:

$$(18) \begin{cases} a/ & \forall (x_0, y_0) \in \text{Graph}(H) \quad \text{there exist solutions } x(\cdot) \text{ and } y(\cdot) \\ & \text{to (16) and (17) such that } \forall t \in [0,T], \ y(t) \in H(x(t)) \\ b/ & \forall (x_0, y_0) \in \text{Graph}(H) \quad \text{all solutions } x(\cdot) \text{ and } y(\cdot) \\ & \text{to (16) and (17) satisfy } \forall t \in [0,T], \ y(t) \in H(x(t)) \end{cases}$$

The second property means that the differential inclusion (17) is so to speak "blind" to the solutions to the differential inclusion (16). When it is satisfied, we see that for all $x_0 \in H^{-1}(y_0)$, all the solutions to the differential inclusion (16) do satisfy

$$\forall t \in [0,T], \ y(t) \in H(x(t))$$

In the next Proposition we denote by $DH(x,y)$ the contingent derivative of H at (x,y) (see Appendix for the definition of DH)

Proposition 3.1 *Let us consider a closed set-valued map H from X to Y.*

1. *Let us assume that F and G are nontrivial upper semicontinuous set-valued maps with nonempty compact convex images and with linear growth. We posit the assumption*

   $$(19) \ \forall (x,y) \in \text{Graph}(H), \ G(t,y) \cap (DH(x,y) \circ F)(t,x) \neq \emptyset$$

   Then property (18) a/ holds true.

2. *Let us assume that $F \times G$ is lipschitzean on a neighborhood of the graph of H and has a linear growth. We posit the assumption*

   $$(20) \quad \forall (x,y) \in \text{Graph}(H), \ G(t,y) \subset (DH(x,y) \square F)(t,x)$$

   Then property (18) b/ is satisfied.

Proof

It follows obviously from the viability and invariance theorems of the graph of H for the set-valued map $F \times G$.

1. When $G(t,y)$ intersects $(DH(x,y)\circ F)(t,x) = \bigcup_{v\in F(t,x)} DH(x,y)(v)$, we deduce that Graph$(H)$ is a viability domain of $F\times G$. Hence we apply the Viability Theorem (See [14], [1, Theorem 4.2.1, p.180]).

2. When G is lipschitzean and satisfies (20), we deduce that Graph(H) is invariant by $F\times G$. Hence we apply the Invariance Theorem (See [8], [1, Theorem 4.6.2]). □

In particular, we have obtained a sufficient condition for the hazy Input-Output set-valued map I_+ to be non trivial.

First, it will be convenient to introduce the following definition.

Definition 3.1 *Let us consider $F:[0,T]\times X \rightsquigarrow X$ and $H:[0,T]\times X \rightsquigarrow Y$. We shall say that a set-valued map $G:[0,T]\times Y \rightsquigarrow Y$ is a* **lipschitzean square projection** *of a set-valued map $F:[0,T]\times X \rightsquigarrow X$ by H if and only if*

$$\begin{cases} i) & F\times G \text{ is lipschitzean around } [0,T]\times \mathrm{Graph}(H) \\ ii) & \forall (x,y)\in \mathrm{Graph}(H),\ G(t,y) \subset (DH(x,y)\,\square\, F)(t,x) \end{cases}$$

Therefore, for being able to use nontrivial hazy Input-Output maps, we shal use the following consequence of Proposition 3.1

Proposition 3.2 *Let us assume that $F:[0,T]\times X \rightsquigarrow X$ and $H:X\rightsquigarrow Y$ are given. If there exists a lipschitzean square projection of F by H, then the hazy Input-Output map $I_+ := H\,\square\,S$ has non empty values for any initial value $y_0\in H(x_0)$.*

Remark

When the observation map H is single-valued and differentiable, then conditions (19) and (20) become respectively

$$\begin{cases} i) & \forall\, y\in H^{-1}(x),\ G(t,y) \cap \left(\bigcup_{v\in F(t,x)} H'(x)(v)\right) \neq \emptyset \\ & \text{or } G(t,y) \cap (H'(x)\circ F)(t,x) \neq \emptyset \\ ii) & \forall\, y\in H^{-1}(x),\ G(t,y) \subset \bigcap_{v\in F(t,x)} H'(x)(v) \\ & =: (H'(x)\,\square\, F)(t,x) \end{cases}$$

When $G=g$ is a single-valued map, we obtain naturally the following consequence.

Corollary 3.1 *Let us consider a closed set-valued map H from X to Y.*

1. *Let us assume that F is a nontrivial upper semicontinuous set-valued map with nonempty compact convex images and with linear growth and that there exists is a continuous selection g with linear growth of the product*

$$\forall\ (x,y) \in \mathrm{Graph}(H),\ \ g(t,y)\ \in (DH(x,y) \circ\ \ F)\,(t,x)$$

   Then property (18) a/ holds true.

2. *Let us assume that $F \times g$ is lipschitzean on a neighborhood of the graph of H with linear growth. If g satisfies*

$$\forall\ (x,y) \in\ \mathrm{Graph}(H),\ \ g(t,y)\ \in (DH(x,y) \square\ F)\,(t,x)$$

   then property (18) b/ is satisfied.

Remark

Naturally, these formulas have their analogues when the observation maps are time-dependant.

Conditions (19) and (20) become respectively

$$\begin{cases} i) & \forall\ (t,x,y) \in \mathrm{Graph}(H),\ \ G(t,y)\ \cap \left(\bigcup_{v\in F(t,x)} DH(t,x,y)(1,v)\right)\ \neq\ \emptyset \\ ii) & \forall\ (t,x,y) \in\ \mathrm{Graph}(H),\ G(t,y)\ \subset \bigcap_{v\in F(t,x)} DH(t,x,y)(1,v) \end{cases}$$

When the observation map H is single-valued and differentiable, then these conditions can be written in the form

$$\begin{cases} i) & \forall\ (t,x) \in\ \mathrm{Dom}(H), \\ & G(t,y)\ \ \cap \left(\frac{\partial}{\partial t}H(t,x) + \bigcup_{v\in F(t,x)} H'_x(t,x)v\right)\ \neq\ \emptyset \\ & \text{or } G(t,y)\ \ \cap \left(\frac{\partial}{\partial t}H(t,x) + (H'(t,x)\circ F)(t,x)\right)\ \neq\ \emptyset \\ ii) & \forall\ (t,x) \in \mathrm{Dom}(H), \\ & G(t,y)\ \ \subset \frac{\partial}{\partial t}H(t,x) + \bigcap_{v\in F(t,x)} H'_x(t,x)v \\ & =: \frac{\partial}{\partial t}H(t,x) + (H'(t,x)\square F)(t,x) \end{cases}$$

Remark

We observe that when the set-valued maps F and G are time-independant, Proposition 3.1 can be reformulated in terms of commutativity of schemes for square products.

Let us denote by Φ the solution map associating to any y_0 a solution to the differential inclusion (equation) (17) starting at y_0 (when G is single-valued such solution is unique).

Then we can deduce that property (18) b/ is equivalent to

$$\forall\, y_0 \in \mathrm{Im}(H), \quad \Phi(y_0) \subset ((H \square \mathcal{S}) \square H^{-1})(y_0)$$

Condition (20) becomes: for all $y \in \mathrm{Im}(H)$,

$$G(y) \subset \bigcap_{x \in H^{-1}(y)} \bigcap_{v \in F(x)} DH(x,y)(v) := (DH(x,y) \square F) \square H^{-1}(y)$$

In other words, the second part of Proposition 3.1 implies that if the scheme

$$\begin{array}{ccccc} & X & \stackrel{F}{\leadsto} & X & \\ H^{-1} & \uparrow & & \downarrow & DH(x,y) \\ & Y & \stackrel{G}{\leadsto} & Y & \end{array}$$

is "commutative for the square products", then the derived scheme

$$\begin{array}{cccc} & X & \stackrel{\mathcal{S}}{\leadsto} & \mathcal{C}(0,T;X) \\ H & \downarrow\uparrow\ H^{-1} & & \downarrow H \\ & Y & \stackrel{\Phi}{\leadsto} & \mathcal{C}(0,T;Y) \end{array}$$

is also commutative for the square products. □

4 Hazy and Sharp Observability

The observability concepts deal with the possibility of recovering the input — here, the initial state —, from the observation of the evolution of the state. In other words, they are related to the injectivity of the sharp and hazy Input-Output set-valued maps, or, more generally, to the univocity (or single-valuedness) of the inverses of those Input-Output maps.

So, we start with precise definitions.

Definition 4.1 *Let $\mathcal{F} : X \leadsto Y$ be a set-valued map. We shall say that it enjoys* **local inverse univocity** *around an element (x^*, y^*) of its graph if and only if there exists a neighborhood $N(x^*)$ such that*

$$\{\, x \mid \text{such that } y^* \in \mathcal{F}(x) \,\} \cap N(x^*) = \{x^*\}$$

If the neighborhood $N(x^*)$ coincides with the domain of $\mathcal{F}$, $\mathcal{F}$ is said to have **(global) inverse univocity**.

We shall say that it is **locally injective** *around x^* if and only if there exists a neighborhood $N(x^*)$ such that, for all $x_1 \neq x_2 \in N(x^*)$, we have $\mathcal{F}(x_1) \cap \mathcal{F}(x_2) = \emptyset$. It is said to be* **(globally) injective** *if we can take for neighborhood $N(x^*)$ the whole domain of $\mathcal{F}$.*

With these definitions at hand, we are able to adapt some of the observability concepts to the set-valued case.

Definition 4.2 *Assume that the sharp and hazy Input-Output maps are defined on nonempty open subsets. Let $y^* \in H(\mathcal{S}(x_0))$ be an observation associated with an initial state x_0.*

We shall say that the system is **sharply observable at** *(respectively* **locally sharply observable at***) x_0 if and only if the sharp Input-Output map I_- enjoys the global inverse univocity (respectively* **local***).*

Hazily observable *and locally hazily observable systems are defined in the same way, when the sharp Input-Output map is replaced by the hazy Input-Output map I_+.*

The system is said to be **hazily (locally) observable around** *if the hazy Input-Output map I_+ is (locally) injective.*

Remark

Several obvious remarks are in order. We observe that the system is sharply locally observable **at** x_0 if and only if there exists a neighborhood $N(x_0)$ of x_0 such that

$$\text{if } x(\cdot) \in \mathcal{S}(N(x_0)) \text{ is such that } y^*(\cdot) \in H(x(\cdot)), \text{ then } x(0) = x_0$$

i.e., sharp observability means that an observation $y^*(\cdot)$ which characterizes the input x_0.

The system is hazily locally observable **at** $(x_0$ if and only if there exists a neighborhood $N(x_0)$ of x_0 such that, for all $x_1 \in N(x_0^*)$,

$$\text{if } \forall\, x(\cdot) \in \mathcal{S}(x_1),\ y^*(\cdot) \in H(x(\cdot)), \text{ then } x_1 = x_0$$

It is also clear that sharp local (respectively global) observability implies hazy local (respectively global) observability.

We mention that if we consider two systems $\mathcal{F}_1$ and $\mathcal{F}_2$ such that

$$\forall\, x \in X, \quad \mathcal{F}_1(x) \;\subset\; \mathcal{F}_2(x)$$

then

1. If $\mathcal{F}_2$ is sharply locally (respectively globally) observable, so is $\mathcal{F}_1$
2. If $\mathcal{F}_1$ is hazily locally (respectively globally) observable, so is $\mathcal{F}_2$ □

We shall derive local observability and injectivity of a set-valued map $\mathcal{F} : X \rightsquigarrow Y$ from a general principle based on the differential calculus of set-valued maps.

For that purpose, we introduce its contingent and paratingent derivatives $D\mathcal{F}(x^\star, y^\star)$ and $P\mathcal{F}(x^\star, y^\star)$, which are closed processes from X to Y (see Appendix for precise definitions).

Since $0 \in D\mathcal{F}(x^\star, y^\star)(0)$, we observe that to say that the "linearized system" $D\mathcal{F}(x^\star, y^\star)$ enjoys the inverse univocity amounts to saying that the inverse image $D\mathcal{F}(x^\star, y^\star)^{-1}(0)$ contains only one element, i.e., that its kernel $\mathrm{Ker} D\mathcal{F}(x^\star, y^\star)$ is equal to 0, where the kernel is naturally defined by

$$\mathrm{Ker} D\mathcal{F}(x^\star, y^\star) \;:=\; D\mathcal{F}(x^\star, y^\star)^{-1}(0)$$

Theorem 4.1 *Let $\mathcal{F}$ be a set-valued map from a finite dimensional vector-space X to a Banach space Y and $(x^\star, y^\star)$ belong to its graph.*

1. *If the kernel of the contingent derivative $D\mathcal{F}(x^\star, y^\star)$ of $\mathcal{F}$ at $(x^\star, y^\star)$ is equal to $\{ 0 \}$, then there exists a neighborhood $N(x^\star)$ such that*

   (21) $$\{\, x \text{ such that } y^\star \;\in\; \mathcal{F}(x) \,\} \;\cap\; N(x^\star) \;=\; \{x^\star\}$$

2. *Let us assume that there exits $\gamma > 0$ such that $\mathcal{F}(x^\star + \gamma B)$ is relatively compact and that $\mathcal{F}$ has a closed graph.*
   If for all $y \in \mathcal{F}(x^\star)$ the kernels of the paratingent derivatives $P\mathcal{F}(x^\star, y)$ of $\mathcal{F}$ at $(x^\star, y)$ are equal to $\{0\}$, then $\mathcal{F}$ is locally injective around $x^\star$.

Proof

1. Assume that the conclusion (21) is false . Then there exists a sequence of elements $x_n \neq x^\star$ converging to $x^\star$ satisfying

$$\forall\, n \geq 0, \quad y^\star \;\in\; \mathcal{F}(x_n)$$

Let us set $h_n := \|x_n - x^\star\|$, which converges to 0, and

$$u_n := (x_n - x^\star)/h_n$$

The elements u_n do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) u_n does converge to some u different from 0. Since the above equation can be written

$$\forall\, n \geq 0, \quad y^\star + h_n 0 \;\in\; \mathcal{F}(x^\star + h_n u_n)$$

we deduce that

$$0 \;\in\; D\mathcal{F}(x^\star, y^\star)(u)$$

Hence we have proved the existence of a non zero element of the kernel of $D\mathcal{F}(x^\star, y^\star)$, which is a contradiction.

2. Assume that $\mathcal{F}$ is not locally injective. Then there exists a sequence of elements $x_n^1, x_n^2 \in N(x^\star)$, $x_n^1 \neq x_n^2$, converging to $x^\star$ and y_n satisfying

$$\forall\, n \geq 0, \quad y_n \;\in\; \mathcal{F}(x_n^1) \cap \mathcal{F}(x_n^2)$$

Let us set $h_n := \|x_n^1 - x_n^2\|$, which converges to 0, and

$$u_n := (x_n^1 - x_n^2)/h_n$$

The elements u_n do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) u_n does converge to some u different from 0.

Then for all large n

$$y_n \in \mathcal{F}(x_n^1) \cap \mathcal{F}(x_n^2) \;:=\; \mathcal{F}(x_n^2 + h_n u_n) \cap \mathcal{F}(x_n^2) \;\subset\; \mathcal{F}(x^\star + \gamma B)$$

we deduce that a subsequence (again denoted) y_n converges to some $y \in \mathcal{F}(x^\star)$ (because Graph($\mathcal{F}$) is closed).

Since the above equation implies that

$$\forall\, n \geq 0, \quad y_n + h_n 0 \;\in\; \mathcal{F}(x_n^2 + h_n u_n)$$

and we deduce that

$$0 \;\in\; P\mathcal{F}(x^\star, y)(u)$$

Hence we have proved the existence of a non zero element of the kernel of $P\mathcal{F}(x^\star, y)$, which is a contradiction. □

When $\mathcal{F}$ is convex (i.e., its graph is convex), we have a simple criterion for global observability:

Proposition 4.1 *Let $\mathcal{F}$ be a convex set-valued map from a Banach space X to a Banach space Y and $(x^\star, y^\star)$ belong to its graph. If the kernel of its algebraic derivative*[10]

$D_a F(x^\star, y^\star)$ is equal to 0, then

$$x \neq x^\star \implies y^\star \notin \mathcal{F}(x)$$

Proof

If not, there exists $x \neq x^\star$ such that $y^\star \in \mathcal{F}(x)$. We set $u := x - x^\star$. Equality

$$y^\star + 0 = y^\star \in \mathcal{F}(x) = \mathcal{F}(x^\star + u)$$

implies that u, which is different from 0, does belong to the kernel of $D_a \mathcal{F}(x^\star, y^\star)$. □

Therefore, by using this theorem for proving sufficient conditions for sharp and/or hazy observability, we need

1. to have chain rule formulas for composition and square products of set-valued maps,
2. characterize the derivatives of the solution map in terms of solutions to the associated variational equations.

The next proposition provides chaine rule formulas for square products which are needed for estimating the contingent and paratingent derivatives of the hazy Input-Output map I_+ in terms of the adjacent and circatangent derivatives of the map G at $(x^\star, y^\star)$ (see Appendix for the precise definitions).

Proposition 4.2 *Let us consider a set-valued map G from a Banach space X to a Banach space Y and a single-valued map H from Y to a Banach space Z. Assume that G is lipschitzean around $x^\star$. If H is differentiable around some $y^\star \in G(x^\star)$, then*

[10] It is defined by

$$v \in D_a \mathcal{F}(x,y)(u) \iff \exists\, h > 0 \text{ such that } y + hv \in \mathcal{F}(x + hu)$$

1. *the contingent derivative of $H \square G$ is contained in the square product of the derivative of H and the adjacent derivative of G: for all $u \in$ Dom$(D^\flat G(x^\star, y^\star))$ we have*

$$D(H \square G)(x^\star, H(y^\star))(u) \quad \subset \quad H'(y^\star) \square D^\flat G(x^\star, y^\star)(u)$$

2. *if H is continuously differentiable around $y^\star$ then the paratingent derivative of $H \square G$ is contained in the square product of the derivative of H and the circatangent derivative of G: $\forall\, u \in$ Dom $(CG(x^\star, y^\star))$ we have*

$$P(H \square G)(x^\star, H(y^\star))(u) \quad \subset \quad H'(y^\star) \square CG(x^\star, y^\star)(u)$$

Proof

1. Let $u \in$ Dom $D^\flat G(x^\star, y^\star)$ and w belong to $D(H \square G)(x^\star, H(y^\star))(u)$. Hence there exist a sequence $h_n > 0$ converging to 0 and sequences of elements u_n and w_n converging to u and w respectively such that

$$\forall\, n \geq 0, \;\; H(y^\star) + h_n w_n \quad \in \quad \bigcap_{y \in G(x^\star + h_n u_n)} H(y)$$

Take now any v in $D^\flat(G)(x^\star, y^\star)(u)$. Since G is lipschitzean around $x^\star$, there exists a sequence of elements v_n converging to v such that

$$\forall\, n \geq 0, \;\; y^\star + h_n v_n \quad \in \quad G(x^\star + h_n u_n)$$

Therefore,

$$\forall\, n \geq 0, \;\; H(y^\star) + h_n w_n \quad = \quad H(y^\star + h_n v_n)$$

Since H is differentiable around $y^\star$, we infer that

$$H'(y^\star) v \quad = \quad w$$

Since this is true for every element v of $D^\flat G(x^\star, y^\star)(u)$, we deduce that

$$w \in \bigcap_{v \in D^\flat G(x^\star, y^\star)(u)} H'(y^\star) v = H'(y^\star) \square D^\flat G(x^\star, y^\star)(u)$$

2. Let $u \in \text{Dom } CG(x^\star, y^\star)$ and w belong to $P(H \square G)(x^\star, H(y^\star))(u)$. Hence there exist a sequence $h_n > 0$ converging to 0 and sequences of elements $(x_n, z_n) \in \text{Graph}(H \square G)$, u_n and w_n converging to $(x^\star, z^\star)$, u and w respectively such that

$$\forall n \geq 0, \quad z_n + h_n w_n \quad \in \bigcap_{y \in G(x_n + h_n u_n)} H(y)$$

The set-valued map G being lipschitzean, there exists a sequence of elements $y_n \in G(x_n)$ converging to $y^\star$. By definition of the square product, we know that $z_n = H(y_n)$.

Take now any v in $CG(x^\star, y^\star)(u)$. Since G is lipschitzean around $x^\star$, there exists a sequence of elements v_n converging to v such that

$$\forall n \geq 0, \quad y_n + h_n v_n \quad \in \quad G(x_n + h_n u_n)$$

Therefore,

$$\forall n \geq 0, \quad H(y_n) + h_n w_n \quad = \quad H(y_n + h_n v_n)$$

Since H is continuously differentiable around $y^\star$, we infer that

$$H'(y^\star)v \quad = \quad w$$

Since this is true for every element v of $CG(x^\star, y^\star)(u)$, we deduce that

$$w \in \bigcap_{v \in CG(x^\star, y^\star)(u)} H'(y^\star)v = H'(y^\star) \square CG(x^\star, y^\star)(u) \qquad \square$$

For the usual product, it is easy to check that:

$$H'(y) \circ DG(x, y)(u) \quad \subset \quad D(H \circ G)(x, H(y))(u)$$

Naturally, equality holds true for algebraic derivatives: If $H \in \mathcal{L}(Y, Z)$ is a linear operator, we check that

(22) $$H \circ D_a G(x, y)(u) \quad = \quad D_a(H \circ G)(x, H(y))(u)$$

We do not know for the time other elegant criteria implying the chain rule (39) for the usual composition product of set-valued maps in infinite dimensional spaces[11].

Estimates of the various derivatives of the solution map $\mathcal{S}$ in terms of the solution maps of the variational inclusions are provided in the next section.

5 Variational Inclusions

We now provide estimates of the contingent, adjacent and circatangent derivatives of the solution map $\mathcal{S}$ associated to the differential inclusion

$$(23) \qquad x'(t) \in F(t, x(t))$$

We shall express these estimates in terms of the solution maps of adequate linearizations of differential inclusion (23) of the form

$$w'(t) \in F'(t, x(t), x'(t))(w(t))$$

where for almost all t, $F'(t, x, y)(u)$ denotes one of the (contingent, adjacent or circatangent) derivatives of the set-valued map $F(t, \cdot, \cdot)$ at a point (x, y) of its graph (in this section the set-valued map F is regarded as a family of set-valued maps $x \rightsquigarrow F(t, x)$) and *the derivatives are taken with respect to the state variable only).*

These linearized differential inclusions can be called the **variational equations**, since they extend (in various ways) the classical variational equations of ordinary differential equations.

[11]Let us mention however the following result involving the **co-subdifferential** $DG(x_0, y_0)^{\circ\star}$, which is the closed convex process from $Y^\star$ to $X^\star$ defined by

$$\begin{cases} p \in DG(x, y)^{0\star}(q) & \text{if and only if} \\ \forall\, (x', y') \in \text{Graph}(G), & < p, x' - x > \leq < q, y' - y > \end{cases}$$

Let us assume that H is a continuous linear operator $H \in \mathcal{L}(Y, Z)$ from Y to Z. Equality

$$D(H \circ G)(x_0, By_0)(u) \;=\; \overline{H \circ DG(x_0, y_0)}(u)$$

holds true if X and Y are reflexive Banach spaces and the co-subdifferential of G at (x_0, y_0) satisfies

$$\text{Im}(H^\star) + \text{Dom}(DG(x_0, y_0)^{\circ\star}) \;=\; Y^\star$$

Furthermore, this condition implies that the kernels of $D(H \circ G)(x_0, By_0)$ and $\overline{H \circ DG(x_0, y_0)}$ are equal to $\{0\}$ (see [7]).

Let $\overline{x}$ be a solution of the differential inclusion (23). We assume that F satisfies the following assumptions:

$$(24)\begin{cases} i) & \forall x \in X \text{ the set-valued map } F(\cdot, x) \text{ is measurable} \\ ii) & \forall\, t \in [0,T],\ \forall\, x \in X.\ \ F(t,x) \text{ is a closed set} \\ iii) & \exists \beta > 0,\ k(\cdot) \in L^1(0,T) \text{ such that for almost all } t \in [0,T] \\ & \text{the map } F(t,\cdot) \text{ is } k(t) - \text{Lipschitz on } \overline{x}(t) + \beta B \end{cases}$$

Consider the **adjacent variational inclusion**, which is the "linearized" along the trajectory $\overline{x}$ inclusion

$$(25)\qquad \begin{cases} w'(t) & \in & D^\flat F(t,\overline{x}(t),\overline{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) & = & u \end{cases}$$

where $u \in X$. In Theorems 5.1, 5.2 below we consider the solution map $\mathcal{S}$ as the set-valued map from $\mathbf{R}^n$ to the Sobolev space $W^{1,1}(0,T;\mathbf{R}^n)$.

Theorem 5.1 (Adjacent variational inclusion) *If the assumptions (24) hold true then for all $u \in X$, every solution $w \in W^{1,1}(0,T;X)$ to the linearized inclusion* (25) *satisfies $w \in D^\flat S(\overline{x}(0),\overline{x})(u)$*

In other words,

$$\{w(\cdot) \mid w'(t) \in D^\flat F(t,\overline{x}(t),\overline{x}'(t))(w(t)),\ w(0) = u\} \subset D^\flat \mathcal{S}(\overline{x}(0),\overline{x})(u)$$

Proof

Filippov's theorem (see for example [1, Theorem 2.4.1, p.120]) implies that the map $u \to \mathcal{S}(u)$ is lipschitzean on a neighborhood of $\overline{x}(0)$. Let $h_n > 0$, $n = 1,2,...$ be a sequence converging to 0. Then, by the very definition of the adjacent derivative, for almost all $t \in [0,T]$,

$$(26)\qquad \lim_{n\to\infty} d\left(w'(t), \frac{F(t,\overline{x}(t) + h_n w(t)) - \overline{x}'(t)}{h_n}\right) = 0$$

Moreover, since $\overline{x}'(t) \in F(t,\overline{x}(t))$ a.e. in $[0,T]$, by (24) , for all sufficiently large n and almost all $t \in [0,T]$

$$d\left(\overline{x}'(t) + h_n w'(t), F(t,\overline{x}(t) + h_n w(t))\right) \le h_n \left(\|w'(t)\| + k(t)\,\|w(t)\|\right)$$

This, (26) and the Lebesgue dominated convergence theorem yield

$$(27)\qquad \int_0^T d\left(\overline{x}'(t) + h_n w'(t), F(t,\overline{x}(t) + h_n w(t))\right) dt = o(h_n)$$

where $\lim_{n\to\infty} o(h_n)/h_n = 0$. By the Filippov Theorem (see for example [1, Theorem 2.4.1, p.120]) and by (27) there exist $M \geq 0$ and solutions $y_n \in S(\overline{x}(0) + h_n u)$ satisfying

$$\|y'_n - \overline{x}' - h_n w'\|_{L^1(0,T;X)} \leq M o(h_n)$$

Since $(y_n(0) - \overline{x}(0))/h_n = u = w(0)$ this implies that

$$\lim_{n\to\infty} \frac{y_n - \overline{x}}{h_n} = w \text{ in } C(0,T;x); \quad \lim_{n\to\infty} \frac{y'_n - \overline{x}'}{h_n} = w' \text{ in } L^1(0,T;X)$$

Hence

$$\lim_{n\to\infty} d\left(w, \frac{S(\overline{x}(0) + h_n u) - \overline{x}}{h_n}\right) = 0$$

Since u and w are arbitrary the proof is complete. □

Consider next the **circatangent variational inclusion**, which is the linearization involving circatangent derivatives:

$$(28) \qquad \begin{cases} w'(t) \in CF(t, \overline{x}(t), \overline{x}'(t))(w(t)) & \text{a.e. in } [0,T] \\ w(0) = u \end{cases}$$

where $u \in X$.

Theorem 5.2 (Circatangent variational inclusion) *Assume that conditions (24) hold true. Then for all $u \in X$, every solution $w \in W^{1,1}(0,T;X)$ to the linearized inclusion* (28) *satisfies $w \in CS(\overline{x}(0), \overline{x})(u)$.*

In other words,

$$\{w(\cdot) \mid w'(t) \in CF(t, \overline{x}(t), \overline{x}'(t))(w(t)),\ w(0) = u\} \subset CS(\overline{x}(0), \overline{x})(u)$$

Proof

By Filippov's theorem the map $u \to S(u)$ is lipschitzean on a neighborhood of $\overline{x}(0)$. Consider a sequence x_n of trajectories of (23) converging to $\overline{x}$ in $W^{1,1}(0,T;X)$ and let $h_n \to 0+$. Then there exists a subsequence $x_j = x_{n_j}$ such that

$$(29) \qquad \lim_{j\to\infty} x'_j(t) = x'_0(t) \text{ a.e. in } [0,T]$$

Set $\lambda_j = h_{n_j}$. Then, by definition of circatangent derivative and by (29), for almost all $t \in [0,T]$

$$(30) \qquad \lim_{j\to\infty} d\left(w'(t), \frac{F(t, x_j(t) + \lambda_j w(t)) - x'_j(t)}{\lambda_j}\right) = 0$$

Moreover, using the fact that $x_j'(t) \in F(t, x_j(t))$ a.e. in $[0,T]$, we obtain that for almost all $t \in [0,T]$

$$d\left(x_j'(t) + \lambda_j w'(t), F(t, x_j(t) + \lambda_j w(t))\right) \leq \lambda_j \left(\|w'(t)\| + k(t)\,\|w(t)\|\right)$$

This, (30) and the Lebesgue dominated convergence theorem yield

$$(31) \qquad \int_0^T d\left(x_j'(t) + \lambda_j w'(t), F(t, x_j(t) + \lambda_j w(t))\right) dt = o(\lambda_j)$$

where $\lim_{j\to\infty} o(\lambda_j)/\lambda_j = 0$. By the Filippov Theorem and (31), there exist $M \geq 0$ and solutions $y_j \in S(x_j(0) + \lambda_j u)$ satisfying

$$\left\|y_j' - x_j' - \lambda_j w'\right\| \leq Mo(h_j)$$

Since $(y_j(0) - x_j(0))/\lambda_j = u = w(0)$, this implies that

$$\lim_{j\to\infty} \frac{y_j - x_j}{h_{n_j}} = w \text{ in } C(0,T;X); \quad \lim_{j\to\infty} \frac{y_j' - x_j'}{h_{n_j}} = w' \text{ in } L^1(0,T;X)$$

Hence

$$(32) \qquad \lim_{j\to\infty} d\left(w, \frac{S(x_j(0) + h_{n_j}u) - x_j}{h_{n_j}}\right) = 0$$

Therefore we have proved that for every sequence of solutions x_n to (23) converging to $\bar{x}$ and every sequence $h_n \to 0+$, there exists a subsequence $x_j = x_{n_j}$ which satisfies (32). This yields that for every sequence of solutions x_n converging to $\bar{x}$ and $h_n \to 0+$

$$\lim_{n\to\infty} d\left(w, \frac{S(x_n(0) + h_n u) - x_n}{h_n}\right) = 0$$

Since u and w are arbitrary the proof is complete. □

We consider now the **contingent variational inclusion**

$$(33) \qquad \begin{cases} w'(t) & \in & \overline{co}DF(t, \bar{x}(t), \bar{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) & = & u \end{cases}$$

Theorem 5.3 (Contingent variational inclusion) *Let us consider the solution map S as a set-valued map from $\mathbf{R}^n$ to $W^{1,\infty}(0,T;\mathbf{R}^n)$ supplied with the weak-* topology and let $\bar{x}(\cdot)$ be a solution of the differential inclusion (23) starting at x_0. Then the contingent derivative $DS(x_0, \bar{x}(\cdot))$ of the solution* *map is contained in the solution map of the contingent variational inclusion (33), in the sense that*

$$(34)\quad \left| \begin{array}{l} DS(x_0,\bar{x}(\cdot))(u) \subset \\ \{ w(\cdot) \mid w'(t) \in \overline{co}DF(t,\bar{x}(t),\bar{x}'(t))(w(t)),\ \ w(0)=u \} \end{array} \right.$$

Proof

Fix a direction $u \in \mathbf{R}^n$ and let $w(\cdot)$ belong to $DS(x_0,\bar{x}(\cdot))(u)$. By definition of the contingent derivative, there exist sequences of elements $h_n \to 0+$, $u_n \to u$ and $w_n(\cdot) \to w(\cdot)$ in the weak-$\star$ topology of $W^{1,\infty}(0,T;\mathbf{R}^n)$ and $c > 0$ satisfying

$$(35)\quad \left\{ \begin{array}{lllll} i) & \|w_n'(t)\| & \leq & c & \text{a.e. in } [0,T] \\ ii) & \bar{x}'(t)+h_n w_n'(t) & \in & F(t,\bar{x}(t)+h_n w_n(t)) & \text{a.e. in } [0,T] \\ iii) & w_n(0) & = & u_n & \end{array} \right.$$

Hence

$$(36)\quad \left| \begin{array}{lll} i) & w_n(\cdot) & \text{converges pointwise to } w(\cdot) \\ ii) & w_n'(\cdot) & \text{converges weakly in } L^1(0,T;\mathbf{R}^n) \text{ to } w'(\cdot) \end{array} \right.$$

By Mazur's Theorem and (36) ii), a sequence of convex combinations

$$v_m(t) \ := \ \sum_{p=m}^{\infty} a_m^p w_p'(t)$$

converges strongly to $w'(\cdot)$ in $L^1(0.T;X)$. Therefore a subsequence (again denoted) $v_m(\cdot)$ converges to $w'(\cdot)$ almost everywhere. By (35) i), ii) for all p and almost all $t \in [0,T]$

$$w_p'(t) \ \in \ \left(\frac{1}{h_p} F(t,\bar{x}(t)+h_p w_p(t)) - \bar{x}'(t) \right) \cap cB$$

Let $t \in [0,T]$ be a point where $v_m(t)$ converges to $w'(t)$ and $\bar{x}'(t) \in F(t,\bar{x}(t))$. Fix an integer $n \geq 1$ and $\epsilon > 0$. By (36) i), there exists m such that $h_p \leq 1/n$ and $\|w_p(t) - w(t)\| \leq 1/n$ for all $p \geq m$.

Then, by setting

$$\Phi(y,h) := \left(\frac{1}{h} F(t,\bar{x}(t)+hy) - \bar{x}'(t) \right) \cap cB$$

we obtain that

$$v_m(t) \in K_n := co\left(\bigcup_{h\in]0,\frac{1}{n}],y\in w(t)+\frac{1}{n}B}\Phi(y,h)\right)$$

and therefore, by letting m go to ∞, that

$$w'(t) \in \overline{co}\left(\bigcup_{h\in]0,\alpha],y\in w(t)+\frac{1}{n}B}\Phi(y,h)\right)$$

Since this is true for any n, we deduce that $w'(t)$ belongs to the convex upper limit[12]:

$$w'(t) \in \bigcap_{n\geq 1}\overline{co}\left(\bigcup_{h\in]0,\frac{1}{n}],y\in w(t)+\frac{1}{n}B}\Phi(y,h)\right)$$

Since the subsets $\Phi(y,h)$ are contained in the ball of radius c, we infer that $w'(t)$ belongs to the closed convex hull of the Kuratowski upper limit[13]:

$$w'(t) \in \overline{co}\bigcap_{\epsilon>0,n\geq 1}\left(\bigcup_{h\in]0,\frac{1}{n}],y\in w(t)+\frac{1}{n}B}\Phi(y,h)+\epsilon B\right)$$

[12] Let K_n be a sequence of subsets of a Banach space X. We say that the set

$$\text{co-limsup}_{n\to\infty}K_n := \bigcap_{N>0}\overline{co}\bigcup_{n>N}K_n$$

is the **convex upper limit** of the sequence K_n. Recall that the **Kuratowski upper limit** of the K_n's is defined by

$$\limsup_{n\to\infty}K_n := \bigcap_{\epsilon>0}\bigcap_{N>0}\bigcup_{n\geq N}(K_n+\epsilon B)$$

It is clear that the convex upper limit is closed and convex. Moreover since $\overline{co}\bigcup_{n\geq N}(K_n+\epsilon B) = \overline{co}\bigcup_{n\geq N}K_n+\epsilon B$ we obtain

$$\text{co-limsup}_{n\to\infty}K_n := \bigcap_{\epsilon>0}\bigcap_{N>0}\overline{co}\bigcup_{n>N}(K_n+\epsilon B)$$

Hence the convex upper limit contains the closed convex hull of the Kuratowski upper limit.

[13] The convex hull of an upper limit and the convex upper limit are related by the following

Lemma 5.1 *Let us consider a sequence of subsets K_n contained in a bounded subset of a*

We observe now that

$$\bigcap_{\epsilon>0,n\geq 1}\left(\bigcup_{h\in[0,\frac{1}{n}],y\in w(t)+\frac{1}{n}B}\Phi(y,h)+\epsilon B\right) \subset DF(t,\overline{x}(t),\overline{x}'(t))(w(t))$$

to conclude that $w(\cdot)$ is a solution to the differential inclusion

$$\begin{cases} w'(t) & \in & \overline{co}DF(t,\overline{x}(t),\overline{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) & = & u \end{cases}$$

Since $w\in DS(x_0,\overline{x}(\cdot))(u)$ is arbitrary we proved (34). □

6 Local Observability Theorems

We piece together in this section the general principle on local inverse univocity and local injectivity (Theorem 4.1), the chain rule formulas (Proposition 4.2) and the estimates of the derivatives of the solution map in terms of solution maps of the variational equations (Theorems 5.1, 5.2 and 5.3) to prove the statements on local hazy and sharp observability we have announced.

Througout the whole section we assume that H is differentiable and F has a linear growth. We impose also some regularity assumptions on the

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finite dimensional vector-space X. Then

$$\text{co-limsup}_{n\to\infty}K_n = \overline{co}(\limsup_{n\to\infty}K_n)$$

Proof

Since an element x of co-lim $\sup_{n\to\infty}K_n$ is the limit of a subsequence of convex combinations v_N of elements of $\bigcup_{n>N}K_n$ and since the dimension of X is an integer p, Carathéodory's Theorem allows to write that

$$v_N := \sum_{j=0}^{p}a_j^N x_{N_j}; \quad \text{where } \sum_{j=0}^{p}a_j^N = 1; \; a_j^N\geq 0$$

where $N_j\geq N$ and where x_{N_j} belongs to K_{N_j} The vector a^N of $p+1$ components a_j^N contains a converging subsequence (again denoted) a^N which converges to some non negative vector a of $p+1$ components a_j such that $\sum_{j=0}^{p}a_j = 1$.

The subsets K_n being contained in a given compact subset, we can extract successively subsequences (again denoted) x_{N_j} converging to elements x_j, which belong to the Kuratowski upper limit of the subsets K_n. Hence x is equal to the convex combination $\sum_{j=0}^{p}a_jx_j$ and the lemma is proved. □

derivatives of F. In the next theorem it is assumed that F is derivable in the sense that its contingent and adjacent derivatives do coincide (see Appendix for the definition of derivability).

Theorem 6.1 *Let us assume that F is derivable, satisfies assumptions (12), that it has a lipschitzean square projection G by H. Let $\overline{x}(\cdot) \in S(x_0)$. If the contingent variational inclusion*

$$(37) \qquad \text{for almost all } t \in [0,T], \quad w'(t) \in DF(t, \overline{x}(t), \overline{x}'(t))(w(t))$$

is globally hazily observable through $H'(\overline{x}(\cdot))$ at 0, then the system (23) is locally hazily observable through H **at** *x_0.*

Proof

We apply the general principle (Theorem 4.1) to the hazy Input-Output map $I_+ := H \square S$, which is defined since we assumed that there exists a square projection G (see Definition 3.1 and Proposition 3.2). We have to prove that the kernel of the contingent derivative $DI_+(x_0, y_0)$ of I_+ (where $y_0 := H(\overline{x}(\cdot))$) is equal to 0. By Filippov's Theorem, the solution map S is lipschitzean around x_0. Then we can apply Proposition 4.2 which states that for all $u \in \text{Dom}(D^\flat S(x_0, \overline{x}(\cdot)))$

$$DI_+(x_0, y_0)(u) \subset \left(H'(\overline{x}(\cdot)) \square D^\flat S(x_0, \overline{x}(\cdot))\right)(u)$$

By Theorem 5.1, we know that for any $u \in X$, the set $\Phi(u)$ of solutions to the adjacent variational inclusion (25) starting at u is contained in the adjacent derivative of S:

$$(38)\left\{\begin{array}{l} \Phi(u) := \{w(\cdot) \mid w'(t) \in D^\flat F(t, \overline{x}(t), \overline{x}'(t))(w(t)) \ \& \ w(0) = u\} \\ = \{w(\cdot) \mid w'(t) \in DF(t, \overline{x}(t), \overline{x}'(t))(w(t)) \ \& \ w(0) = u\} \\ \subset D^\flat S(x_0, \overline{x})(u) \end{array}\right.$$

We also know that for all $(x,y) \in \text{Graph}(F(t,\cdot))$, the contingent derivative $DF(x,y)$ is $k(t)$−Lipschitz (see Appendix). Hence, by the Filippov theorem ([1, Theorem 2.4.1, p.120]) for every $u \in \mathbf{R}^n$ the contingent variational inclusion (37) has a solution starting at u. Therefore, by (38), $\text{Dom}(D^\flat S(x_0, \overline{x}(\cdot)))$ is equal to the whole space. This yiedls

$$\forall\, u \in \mathbf{R}^n, \quad DI_+(x_0, y_0)(u) \subset (H'(\overline{x}) \square \Phi)(u)$$

so that the kernel of $DI_+(x_0, y_0)$ is contained in the kernel of $H'(\overline{x}) \square \Phi$. But to say that the kernel of $H'(\overline{x}) \square \Phi$ is equal to 0 amounts to saying that the linearized system (37) is hazily globally observable at 0 through $H'(\bar{x}(\cdot))$. Hence the kernel of $DI_+(x_0, y_0)$ is equal to 0. and thus. the inverse image of hazy Input-Output map contains locally a unique element. □

Remark

The above result remains true if instead of derivability of F we assume that $\text{Dom}(D^p \mathcal{S}(x_0, \bar{x}(\cdot))) = \mathbf{R}^n$. □

In the next theorem we assume that F is sleek, so that its contingent and circatangent derivatives do coincide (see Appendix).

Theorem 6.2 *Let us assume that F is sleek, has convex images, satisfies assumptions (12), and that it has a lipschitzean square projection G by H. If for all $\bar{x}(\cdot) \in \mathcal{S}(x_0)$ the contingent variational inclusion (37) is globally hazily observable through $H'(\bar{x}(\cdot))$* **at** 0 *, then the system (23) is hazily observable through H* **around** x_0.

Proof

We apply the second part of the general principle on local injectivity (Theorem 4.1) to the hazy Input-Output map $I_+ := H \,\square\, \mathcal{S}$, which is defined since we assumed that there exists a square projection G. We have to prove that the kernels of the paratingent derivatives $PI_+(x_0, y)$ of I_+ are equal to 0 (where $y(\cdot) := H(\bar{x}(\cdot))$ and $\bar{x}(\cdot) \in \mathcal{S}(x_0)$) . In the way similar to [1, Theorem 2.2.1, p.104], we prove that for all $\gamma > 0$ the set $\mathcal{S}(x_0 + \gamma B)$ is compact in $C(0, T; \mathbf{R}^n)$. Hence $I_+(x_0 + \gamma B)$ is relatively compact in $C(0, T; \mathbf{R}^n)$. By Filippov's Theorem, the solution map $\mathcal{S}$ is lipschitzean around x_0. This and compactness of $\mathcal{S}(x_0 + \gamma B)$ imply that $\text{Graph}(I_+)$ is a closed set. Then we can apply the second part of Proposition 4.2 which states that for all $u \in \text{Dom}(C\mathcal{S}(x_0, \bar{x}(\cdot)))$

$$PI_+(x_0, y)(u) \subset (H'(\bar{x}(\cdot)) \,\square\, C\mathcal{S}(x_0, \bar{x}(\cdot)))(u)$$

By Theorem 5.2, we know that for all u, the set $\Phi(u)$ of solutions to the circatangent variational inclusion (24) starting at u is contained in the circatangent derivative of $\mathcal{S}$:

$$\begin{cases} \Phi(u) := \{w(\cdot) \mid w'(t) \in CF(\bar{x}(t), \bar{x}'(t))(w(t)) \ \& \ w(0) = u\} \\ = \{w(\cdot) \mid w'(t) \in DF(\bar{x}(t), \bar{x}'(t))(w(t)) \ \& \ w(0) = u\} \\ \subset C\mathcal{S}(x_0, \bar{x})(u) \end{cases}$$

But from the proof of Theorem 6.1 we know that $\text{Dom}(\Phi) = \mathbf{R}^n$. Therefore,

$$PI_+(x_0, y)(u) \subset (H'(\overline{x}) \square \Phi)(u)$$

so that the kernel of $PI_+(x_0, y)$ is contained in the kernel of $H'(\overline{x}) \square \Phi$. But to say that the kernel of $H'(\overline{x}) \square \Phi_0$ is equal to 0 amounts to saying that the linearized system (37) is hazily globally observable through $H'(\overline{x})$. Hence the kernel of $PI_+(x_0, y)$ is equal to 0, and thus, the hazy Input-Output map is locally injective. □

We consider now the sharp Input-Output map.

Theorem 6.3 *Let us assume that the graphs of the set-valued maps $F(t, \cdot) : X \rightsquigarrow X$ are closed and convex. Let H be a linear operator from X to another finite dimensional vector-space Y. Let $\overline{x}(\cdot)$ be a solution to the differential inclusion (23). If the contingent variational inclusion (37) is globally sharply observable through H around 0, then the system (23) is globally sharply observable through H around x_0.*

Proof

We apply Proposition 4.1 to the sharp Input-Output map $I_- := H \circ S$. We have to prove that the kernel of the algebraic derivative $D_a I_-(x_0, y_0)$ of I_- (where $y_0 := H(\overline{x})$) is equal to 0. Consider S as a map from $\mathbf{R}^n$ to the Sobolev space $W^{1,1}(0, T; \mathbf{R}^n)$.

Since the graph of the solution map S is convex (for the graphs of the set-valued map F is assumed to be convex), and since the map H is linear, we know that the chain rule (22) holds true:

$$(39) \qquad DI_-(x_0, y_0)(u) \;=\; (H \circ D_a S(x_0, \overline{x}(\cdot)))(u)$$

It remains to check that the algebraic derivative $D_a S(x_0, \overline{x}))(u)$ of S is contained in the subset $\Psi_a(u)$ of solutions to the algebraic variational inclusion starting at u:

$$\begin{cases} D_a S(x_0, \overline{x}(\cdot))(u) \;\subset \Psi_a(u) \;:= \\ \{w(\cdot) \mid w'(t) \;\in\; D_a F(\overline{x}(t), \overline{x}'(t))(w(t)) \;\;\&\;\; w(0) \;=\; u\} \end{cases}$$

Since the algegraic derivative of a convex set-valued map is contained in the contingent derivative, then the set $\Psi_a(u)$ is contained in the subset $\Psi(u)$ of solutions to the contingent variational inclusion (34) starting at u. Hence the kernel of $DI_-(x_0, y_0)$ is contained in the kernel of $H \circ \Psi$. But to say that the kernel of $H \circ \Psi$ is equal to 0 amounts to saying that the contingent variational inclusion system (37) is sharply globally observable through H. Therefore the kernel of $D_a I_-(x_0, y_0)$ is equal to 0, and thus, the inverse image of sharp Input-Output map contains a unique element. This concludes the proof. □

Remark

We do not know for the time other elegant criteria implying the chain rule (39) for the usual composition product of set-valued maps in infinite dimensional spaces[14].

If we assume that the chain rule holds true, we can state the following proposition, a consequence of the general principle (Theorem 4.1) and of Theorem 5.3 on the estimate of the contingent derivative of the solution map.

Proposition 6.1 *Let us assume that the solution map of the differential inclusion (23) and the differentiable observation map H do satisfy the chain rule*

$$DI_-(x_0, y_0)(u) \quad = \quad (H'(\overline{x}) \circ S(x_0, \overline{x}(\cdot)))(u)$$

If the contingent variational inclusion

$$\text{for almost all } t \in [0, T], \quad w'(t) \in \overline{co} DF(t, \overline{x}(t), \overline{x}'(t))(w(t))$$

is globally sharply observable through $H'(\overline{x}(\cdot))$ around 0, then the system (23) is locally sharply observable through H around x_0. □

[14]Let us mention however the following result involving the **co-subdifferential** $DG(x_0, y_0)^{\circ *}$, which is the closed convex process from Y^* to X^* defined by

$$\begin{cases} p \in DG(x, y)^{\circ *}(q) & \text{if and only if} \\ \forall (x', y') \in \mathrm{Graph}(G), & < p, x' - x > \leq < q, y' - y > \end{cases}$$

Let us assume that H is a continuous linear operator $H \in \mathcal{L}(Y, Z)$ from Y to Z. Equality

$$D(H \circ G)(x_0, By_0)(u) \quad = \quad \overline{H \circ DG(x_0, y_0)}(u)$$

holds true if X and Y are reflexive Banach spaces and the co-subdifferential of G at (x_0, y_0) satisfies

$$\mathrm{Im}(H^*) + \mathrm{Dom}(DG(x_0, y_0)^{\circ *}) \quad = \quad Y^*$$

Furthermore, this condition implies that the kernels of $D(H \circ G)(x_0, By_0)$ and $\overline{H \circ DG(x_0, y_0)}$ are equal to $\{0\}$ (see []).

However, we can bypass the chain rule formula and attempt to obtain directly other criteria of local sharp observability in the nonconvex case.

Theorem 6.4 *Assume that F has closed convex images, is continuous, Lipschitz in the second variable with a constant independent of t and that the growth of F is linear with respect to the state. Let H be a twice continuously differentiable function from X to another finite dimensional vector-space Y. Consider an observation $y^\star \in I_-(x_0)$ and assume that for every solution $\overline{x}(\cdot)$ to the differential inclusion (23) satisfying $y^\star(\cdot) = H(\overline{x}(\cdot))$ and for all $t \in [0,T]$ we have*

$$(40) \qquad (\mathrm{Ker}\, H'(\overline{x}(t))) \subset (F(t,\overline{x}(t)) - F(t,\overline{x}(t)))^\perp$$

If for all $\overline{x}$ as above the contingent variational inclusion

$$(41) \qquad \text{for almost all } t \in [0,T], \; w'(t) \in \overline{\mathrm{co}}\, DF(t,\overline{x}(t),\overline{x}'(t))(w(t))$$

is globally sharply observable through $H'(\overline{x}(t))$ around 0, then the system (23) is locally sharply observable through H at $(x_0, y^\star)$.

Proof

Assume for a moment that the inclusion (23) is not locally sharply observable through H at $(x_0, y^\star)$. Then there exists a sequence $x_0^n \neq x_0$, $x_0^n \rightarrow x_0$ such that $y^\star \in I_-(x_0^n)$, i.e., for some $x_n \in S(x_0^n)$

$$(42) \qquad y^\star = H(x_n(\cdot))$$

Taking a subsequence if needed and keeping the same notations, we may assume that $x_n \rightarrow \overline{x}$ weakly in $W^{1,\infty}(0,T;\mathbf{R}^n)$. Then (42) yields

$$(43) \qquad y^\star = H(\overline{x}(\cdot)) \; ; \quad H'(\overline{x}(t))\overline{x}'(t) = H'(x_n(t))x_n'(t)$$

We shall prove that the convergence is actually strong and even more, that there exists a constant $c > 0$ such that

$$(44) \qquad \|x_n'(t) - \overline{x}'(t)\| \leq c\, \|x_n(t) - \overline{x}(t)\| \quad \text{a.e. in } [0,T]$$

Indeed otherwise there exist sequences t_k and n_k such that

$$\begin{cases} x_{n_k}'(t_k) \in F(t_k, x_{n_k}(t_k)) \; ; \quad \overline{x}'(t_k) \in F(t_k, \overline{x}(t_k)) \\ \|x_{n_k}'(t_k) - \overline{x}'(t_k)\| \geq k\, \|x_{n_k}(t_k) - \overline{x}(t_k)\| \end{cases}$$

Taking a subsequence and keeping the same notations, by continuity of F, we may assume that for some $t \in [0,T]$, $p \in F(t, \overline{x}(t))$

$$(45) \qquad t_k \to t\ ; \quad \overline{x}'(t_k) \to p$$

Let ρ denote the Lipschitz constant of F with respect to x and let $y(t_k) \in F(t_k, \overline{x}'_{n_k})$ be such that

$$(46) \qquad \left\| y(t_k) - \overline{x}'_{n_k}(t_k) \right\| \ \le\ \rho \ \|x_{n_k}(t_k) - \overline{x}(t_k)\|$$

Since H' is locally Lipschitz and x'_{n_k} are equibounded, from the last inequality and (43) we deduce that for some constants M, $M_1 > 0$

$$(47) \quad \begin{cases} \|H'(\overline{x}(t_k))(y(t_k) - \overline{x}'(t_k))\| \\ \le \left\| H'(\overline{x}(t_k))(x'_{n_k}(t_k) - \overline{x}'(t_k)) \right\| + \rho \|H'(\overline{x}(t_k))\| \, \|x_{n_k}(t_k) - \overline{x}(t_k)\| \\ \le \left\| H'(x_{n_k}(t_k))x'_{n_k}(t_k) - H'(\overline{x}(t))\overline{x}'(t_k) \right\| + M \|x_{n_k}(t_k) - \overline{x}(t_k)\| \, \left\| x'_{n_k}(t_k) \right\| \\ + \rho \ \|H'(\overline{x}(t_k))\| \, \|x_{n_k}(t_k) - \overline{x}(t_k)\| \le M_1 \ \|x_{n_k}(t_k) - \overline{x}(t_k)\| \end{cases}$$

From (46) and the choice of t_k, we obtain

$$(48) \qquad \frac{\|y(t_k) - \overline{x}'(t_k)\|}{\|x_{n_k}(t_k) - \overline{x}(t_k)\|} \to \infty \text{ when } k \to \infty$$

It is also not restrictive to assume that for some u of $\|u\| = 1$

$$(49) \qquad u_k := \frac{y(t_k) - \overline{x}'(t_k)}{\|y(t_k) - \overline{x}'(t_k)\|} \to u$$

Then (47), (48) yield

$$u \in \operatorname{Ker} H'(\overline{x}(t))$$

On the other hand u_k is contained in the space spanned by $F(t_k, \overline{x}(t_k)) - F(t_k, \overline{x}(t_k))$ and, by continuity of F, u is contained in the space spanned by $F(t, \overline{x}(t)) - F(t, \overline{x}(t))$. Since $u \ne 0$ this contradicts (40) and therefore (44) follows.

From the Gronwall inequality and (44) we deduce that for some $M_2 > 0$

$$\|x_n(t) - \overline{x}(t)\| \ \le\ M_2 \, \|x_n(0) - \overline{x}(0)\|$$

Setting $h_n = \|x_0^n - x_0\|$ we obtain

$$\|x_n - \overline{x}\|_{W^{1,\infty}(0,T)} \ \le\ M_2 \, h_n$$

Taking a subsequence and keeping the same notations we may assume that

$$\frac{x_n - \overline{x}}{h_n} \rightharpoonup w \text{ weakly in } W^{1,\infty}(0,T)$$

By Theorem 5.3, w is a solution of the contingent variational inclusion (33). Hence w is a solution of (41). Moreover $w(0) = u \neq 0$. Since $H(x_n(\cdot)) = H(\overline{x}(\cdot))$ taking the derivatives we obtain that for every $t \in [0,T]$, $H'(\overline{x}(t))w(t) = 0$. This contradicts the assumption (40) of theorem and completes the proof. □

Example: Observability around an Equilibrium

Let us consider the case of a time-independent system (F, H): this means that the set-valued map $F : X \rightsquigarrow X$ and the observation map $H : X \rightsquigarrow Y$ do not depend upon the time.

We shall observe this system around an equilibrium $\bar{x}$ of F, i.e., a solution to the equation

$$(50) \qquad 0 \in F(\bar{x})$$

For simplicity, we shall assume that the set-valued map F is **sleek** at the equilibrium. Hence all the derivatives of F at $(\bar{x}, 0)$ do coincide with the contingent derivative $DF(\bar{x}, 0)$, which is a closed convex process from X to itself.

The theorems on local observability reduce the local observability around the equilibrium $\bar{x}$ to the study of the observability properties of the variational inclusion

$$(51) \qquad w'(t) \in DF(\bar{x}, 0)(w(t))$$

through the observation map $H'(\bar{x})$ around the solution 0 of this variational inclusion.

We mention below a characterization of sharp observability of the variational inclusion in terms of "viability domains" of the restriction of the derivative $DF(\bar{x}, 0)$ to the kernel of $H'(\bar{x})$.

Recall that a subset $P \subset \ker H'(\bar{x})$ is a " viability domain" if

$$\forall\, w \in P, \;\; DF(\bar{x}, 0)(w) \cap T_P(w) \neq \emptyset$$

where $T_P(w)$ denotes the "contingent cone to P ar $w \in P$".

Proposition 6.2 *Let us assume that F is sleek at its equilibrium $\bar{x}$ and that H is differentiable at $\bar{x}$. Then the variational inclusion (51) is sharply* *observable at* 0 *if and only if the* **largest closed viability domain** *of the restriction to* $\ker H'(\bar{x})$ *of the contingent derivative* $DF(\bar{x},0)$ *is equal to zero.*

Proof

Let us denote by E the restiction of the contingent derivative $DF(\bar{x},0)$ to the kernel of $H'(\bar{x})$ defined by:

$$(52) \qquad E(u) \;:=\; \begin{cases} DF(\bar{x},0)(u) & \text{if } \; u \in \ker H'(\bar{x}) \\ \emptyset & \text{if } \; u \notin \ker H'(\bar{x}) \end{cases}$$

We consider the associated differential inclusion

$$(53) \qquad w'(t) \;\in\; E(w(t))$$

We know that the **largest closed viability domain** of the closed convex process E is the domain of the solution map of the associated differential inclusion (53). (See [6]).

But if we denote by $\mathcal{R}$ the solution map of the variational inclusion (51) and by $\mathcal{B}$ the set of functions $x(\cdot)$ such that

$$\forall\, t \in [0,T], \;\; x(t) \in \ker H'(\bar{x})$$

we observe that the solution map of the differential inclusion (53) is the set-valued map $u \rightsquigarrow \mathcal{R}(u) \cap \mathcal{B}$. Hence its domain is the set $\mathcal{R}^{-}(\mathcal{B})$. Since

$$\mathcal{R}^{-}(\mathcal{B}) \;=\; \ker(H'(\bar{x}) \circ \mathcal{R})$$

we infer that **the largest viability domain of E is the kernel of the sharp Input-Output map $H'(\bar{x}) \circ \mathcal{R}$.**

Consequently, the variational inclusion (51) being sharply observable if and only if the kernel of $H'(\bar{x}) \circ \mathcal{R}$ is equal to zero, our Proposition ensues. □

Remark

In the same way, the variational inclusion (51) is hazily observable if and only if the kernel of $H'(\bar{x}) \,\square\, \mathcal{R}$ is equal to zero.

There are also some relations between the kernel of the hazy Input-Output map $H'(\bar{x}) \,\square\, \mathcal{R}$ and invariance domains of the restriction of the derivative to the kernel of $H'(\bar{x})$. First, we remark that

$$\mathcal{R}^{+}(\mathcal{B}) \;=\; \ker(H'(\bar{x}) \,\square\, \mathcal{R})$$

i.e., that the kernel of $H'(\bar{x}) \square \mathcal{R}$ is the largest set enjoying the "invariance property": for any $u \in \ker H'(\bar{x})$, all solutions to the differential inclusion (53) remain in this kernel.

When E is lipschitzean in a neighborhood of $\ker H'(\bar{x})$, any closed subset $P \subset \ker H'(\bar{x})$ which is "invariant" in the sense that

$$\forall\, w \in P,\;\; DF(\bar{x},0)(w) \subset T_P(w)$$

enjoy the invariance property. The converse is true only if we assume that the domain of $DF(\bar{x},0)$ is the whole space.

Then, if such is the case, *the variational inclusion is hazily observable if and only if the largest closed invariance domain of the restriction to* $\ker H'(\bar{x})$ *of the derivative* $DF(\bar{x},0)$ *is equal to zero.* □

Remark

We have proved in [4] that under some further conditions, the sharp observability of the variational inclusion at 0 is equivalent to the controllability of the adjoint system

$$(54)\qquad -p'(t) \;\in\; DF(\bar{x},0)^\star(p(t)) + H'(\bar{x})^\star u(t),\;\; u(t) \;\in\; Y^\star$$

Proposition 6.3 *We posit the assumptions of Proposition 6.2, we assume that* $DF(\bar{x},0)(0) = 0$ *and we suppose that*

$$(55)\qquad \ker H'(\bar{x}) + \mathrm{Dom}(DF(\bar{x},0)) = X$$

Then the concepts of sharp and hazy observability of the variational inclusion coincide and are equivalent to the controllability of the adjoint system

(About eleven characterizations of this property are supplied in [4]).

Proof

Assumption (55) implies that the transpose $E^\star$ of the restriction E of the closed convex process $DF(\bar{x},0)$ to $\ker H'(\bar{x})$ is given by the formula

$$(56)\qquad (DF(\bar{x},0)|_{\ker H'(\bar{x})})^\star \;=\; DF(\bar{x},0)^\star + \mathrm{Im}(H'(\bar{x})^\star)$$

(see [7, Corollary 3.3.17, p.142])

We also know (see [4, Proposition 1.12, p.1198]) that if the domain of the transpose $E^\star$ of E is the whole space, then a vector subspace P is an invariance domain of E if and only if its orthogonal $P^\perp$ is a viability domain[15] of E^*. Since the domain of E^* is equal to the domain of $DF(\bar{x},0)^*$ (thanks to formula (56), this condition is equivalent to $DF(\bar{x},0) = 0$).

Hence the variational inclusion (51) being sharply observable at 0 if and only if the largest closed viability domain of E is equal to 0 (by the above Proposition 6.2), we deduce that this happens if and only if the smallest invariance domain of E^* is equal to 0, i.e., if and only if the adjoint system (54) is controllable.

The assumption that $DF(\bar{x},0)(0) = 0$ implies that the restriction[16] $DF(\bar{x},0)|_{\ker H'(\bar{x})}$ of $DF(\bar{x},0)$ to the kernel of $H'(\bar{x})$ is single-valued, (and thus, a linear operator), so that both concepts of sharp and hazy observability do coincide.

Therefore, our statement follows from [4, Theorem 5.5., p1207]. □

7 Appendix: Derivatives of Set-Valued Maps

Definition 7.1 *Let (x,y) belong to the graph of a set-valued map $F : X \rightsquigarrow Y$ from a normed space X to another Y. Then the* **contingent derivative** *$DF(x,y)$ of F at (x,y) is the set-valued map from X to Y defined by*

$$v \in DF(x,y)(u) \iff \liminf_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0$$

and the **paratingent**[17] **derivative** *$PF(x,y)$ of F at (x,y) is the set-valued map from X to Y defined by*

$$v \in PF(x,y)(u) \iff \liminf_{h\to 0+, (x',y')\to_F (x,y), u'\to u} d\left(v, \frac{F(x'+hu')-y'}{h}\right) = 0$$

where $\to_F$ denotes the convergence in Graph(F)

When F is lipschitzean around $x \in \mathrm{Int}(\mathrm{Dom}(F))$, the above formulas become

$$\begin{cases} i) & v \in DF(x,y)(u) \iff \liminf_{h\to 0+} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0 \\ ii) & v \in PF(x,y)(u) \iff \liminf_{h\to 0+, (x',y')\to_F (x,y)} d\left(v, \frac{F(x'+hu)-y'}{h}\right) = 0 \end{cases}$$

[15] This is also true when the domain of E is the whole space. But this does not apply to our case, since the domain of E is the kernel of $H'(\bar{x})$.

[16] this does not require that $DF(\bar{x},0)(0) = 0$ is single-valued on its domain when the latter is not a vector subspace.

[17] see [?] for the study of paratingent cones and the applications of Choquet's Theorem.

Moreover if k denotes the Lipschitz constant of F at x, then for every $y \in F(x)$ the derivative $DF(x,y)$ has nonempty images and is k-lipschitzean (see [13]).

Despite the fact that both adjacent and circatangent derivatives can be defined for any set-valued map F, the formulas are simpler when we deal with lipschitzean set-valued maps. Since we use them only in this context in this paper, we provide their definitions in this limited case.

Definition 7.2 *Let (x,y) belongs to the graph of a set-valued map $F : X \rightsquigarrow Y$ from a normed space X to another Y. Assume that F is lipschitzean around an element $x \in$ Int(Dom (F)), then the* **adjacent derivative** *$D^\flat F(x,y)$ and the* **circatangent derivative** *$CF(x,y)$ are the set-valued maps from X to Y respectively defined by*

$$v \in D^\flat F(x,y)(u) \iff \lim_{h \to 0+} \left(v, \frac{F(x+hu)-y}{h} \right) = 0$$

and

$$v \in CF(x,y)(u) \iff \lim_{h \to 0+,(x',y') \to_F (x,y)} d\left(v, \frac{F(x'+hu)-y'}{h} \right) = 0$$

Several remarks are in order. First, all these derivatives are positively homogeneous and their graphs are closed.

We observe the obvious inclusions

$$CF(x,y)(u) \subset D^\flat F(x,y)(u) \subset DF(x,y)(u) \subset PF(x,y)(u)$$

and that the definitions of contingent and adjacent derivatives on one hand, the paratingent and circatangent derivatives, on the other one, are symmetric. When $F := f$ is single-valued, we set

$$Df(x) := Df(x,f(x)), \quad D^\flat f(x) := D^\flat f(x,f(x)), \quad Cf(x) := Cf(x,f(x))$$

We see easily that

$$\begin{cases} Df(x)(u) & = f'(x)u \quad \text{if } f \text{ is Gâteaux differentiable} \\ D^\flat f(x)(u) & = f'(x)u \quad \text{if } f \text{ is Fréchet differentiable} \\ Cf(x)(u) & = f'(x)u \quad \text{if } f \text{ is continuously differentiable} \end{cases}$$

The choice of these strange limits is dictated by the fact that the graph of each of these derivatives is the corresponding tangent cone to the graph of F at (x,y). (The graphs of the circatangent derivatives are the Clarke tangent cones to the graphs, which are always convex.)

This allows also to define and use derivatives of restrictions $F := f|_K$ of single-valued maps f to subsets $K \subset X$, which are defined by

$$f|_K(x) \quad := \quad \begin{cases} f(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

If f is continuously differentiable around a point $x \in K$, then **the derivative of the restriction is the restriction of the derivative to the corresponding tangent cone**.

The most familiar instance of set-valued maps is the inverse of a non injective single-valued map. The **derivative of the inverse of a set-valued map F is the inverse of the derivative**:

$$\begin{cases} P(F)^{-1}(y,x) & = & PF(x,y)^{-1} \\ D(F)^{-1}(y,x) & = & DF(x,y)^{-1} \\ D^{\flat}(F)^{-1}(y,x) & = & D^{\flat}F(x,y)^{-1} \\ C(F)^{-1}(y,x) & = & CF(x,y)^{-1} \end{cases}$$

and enjoy a now well investigated calculus.

The circatangent derivatives are closed convex processes, because their graph are closed convex cones, i.e., they are set-valued anlogues of the continuous linear operators. We refer to [21], [2, Chapter 7] for various properties of closed convex processes.

We say that a set-valued map F is **derivable** at $(x,y) \in \text{Graph}(F)$ if $DF(x,y) = D^{\flat}F(x,y)$ and that it is **derivable** if it is derivable at every point of its graph.

We say that a set-valued map F is **sleek** at $(x,y) \in \text{Graph}(F)$ if

$$\text{Graph}(F) \ni (x',y') \rightsquigarrow \text{Graph}(DF)(x',y') \quad \text{is lower semicontinuous at} \quad (x,y)$$

and it is **sleek** if it is sleek at every point of its graph. In this case, we can prove that **the contingent, adjacent and circatangent derivatives coincide**.

References

[1] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIAL INCLUSIONS . Springer- Verlag (Grundlehren der Math. Wissenschaften, Vol.264, pp.1-342)

[2] AUBIN J.-P. & EKELAND I. (1984) APPLIED NONLINEAR ANALYSIS . Wiley-Interscience

[3] AUBIN J.-P. & FRANKOWSKA H. (1987) *On the inverse fonction Theorem* . J. Math. Pures Appliquées. 66, 71-89

[4] AUBIN J.-P. , FRANKOWSKA H. & OLECH C. (1986) *Controllability of convex processes* . SIAM J. of Control and Optimization, 24, 1192-1211

[5] AUBIN J.-P. , FRANKOWSKA H. & OLECH C. (1986) *Contrôlabilité des processus convexes* . Comptes-rendus de l'Académie des Sciences, Paris, 301, 153-156

[6] AUBIN J.-P. (1987) *Smooth and heavy solutions to control problems.* Proceedings of the conference on Functional Analysis at Santa Barbara, June 24-26, 1985, (IIASA WP-85-44)

[7] AUBIN J.-P. (to appear) VIABILITY THEORY.

[8] CLARKE F. H. (1975) *Generalized gradients and applications.* Trans. of AMS, 205, 247-262

[9] DOLECKI S. & RUSSEL D. L. (1977) *A general theory of observation and control.* SIAM J. Control and Optimization 15, 185-220

[10] FRANKOWSKA H. (1986) *Contingent cones to reachable sets of control systems* . Preprint CRM-1381, Université de Montréal

[11] FRANKOWSKA H. (1986) *Local controllability of control systems with feedback* . Preprint CRM-1364, Université de Montréal

[12] FRANKOWSKA H. (1986) *Local controllability and infinitesimal generators of semigroups of set-valued maps* . SIAM J. of Control and Optimization

[13] FRANKOWSKA H. (1987) *The maximum principle for the differential inclusions with end point constraints* . SIAM J. of Control and Optimization

[14] HADDAD G. (1981) *Monotone trajectories of differential inclusions with memory* . Israel J. Maths, 39, 38-100

[15] HERMANN R. & KRENER A.J. (1977) *Nonlinear controllability and observability.* IEEE Trans. Autom. Control AC-22, 728-740

[16] ISIDORI A. (1985) NONLINEAR CONTROL SYSTEMS: AN INTRODUCTION . Springer-Verlag Lecture Notes in Control and Information Sciences, Vol.72

[17] KRENER A. J. & ISIDORI A. (1983) *Linearization by output injection and nonlinear observers.* Syst. & Control letters, 3, 47-52

[18] KRENER A. J. & RESPONDEK W. (1985) *Non linear observers with linearizable error dynamics.* SIAM J. Control & Opt.

[19] KURZHANSKII A. B. (1977) CONTROL AND OBSERVATION UNDER CONDITIONS OF UNCERTAINTY. Nauka

[20] LEITMANN G., (1981) THE CALCULUS OF VARIATIONS AND OPTIMAL CONTROL. Plenum Press

[21] ROCKAFELLAR R. T. (1967) MONOTONE PROCESSES OF CONVEX AND CONCAVE TYPE. Mem. of AMS # 77

[22] ROCKAFELLAR R. T. (1979) LA THÉORIE DES SOUS-GRADIENTS . Presses de l'Université de Montréal

[23] SHI SHUZHONG (1987) *Choquet Theorem and Nonsmooth Analysis.* Cahiers de Mathématiques de la Décision

[24] SHI SHUZHONG (1987) *Théorème de Choquet et Analyse non régulière.* Comptes-rendus de l'Académie des Sciences, PARIS, 305, 41-44