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VIABILITY TUBES AND THE TARGET PROBLEM

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FOREWORD

Viability tubes and invariant tubes of a differential inclusion are defined and then used to build "bridges[1]" between an initial set K and a "target" C that at least one trajectory (respectively all trajectories) follows for leaving K and reaching C in finite or infinite horizon. (This is the target or $K - C$ problem). We study some asymptotic properties of these tubes (it is shown in particular that targets are necessarily viability domains) and viability tubes are characterized by showing that the indicator functions of their graphs are solutions to the "contingent Hamilton-Jacobi equation". Some examples of viability tubes are provided .

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Contents

[1]This terminology is been used by the Russian School of Sverdlovsk. In another neighboring context, tubes are called funnels, again by Russian mathematicians

1 Introduction

Let X be a finite dimensional vector space and $F : [0, \infty[\times X \leadsto X$ a set-valued map which associates with any state $x \in X$ and any time t the subset $F(t, x)$ of velocities of the system. The evolution of the system is governed by the differential inclusion[2]

$$(1) \qquad x'(t) \;\in\; F(t, x(t)), \;\; x(0) \;=\; x_0$$

We consider now "tubes", i.e., set-valued maps $t \leadsto P(t)$ from $[0, \infty[$ to X.

We say that a solution $t \mapsto x(t) \in X$ is "viable" (in the tube P) if

$$\forall\, t \geq 0, \;\; x(t) \;\in\; P(t)$$

A tube P enjoys the **viability property** if and only if, for all $t_0 \geq 0$ and $x_0 \in P(t_0)$, there exists at least a solution $x(\cdot)$ to the differential inclusion (1) which is viable.

Remark A simple-valued tube $t \leadsto \{x(t)\}$ enjoys the viability property if and only if $x(\cdot)$ is a solution to the differential inclusion (1). So it is legitimate to regard a tube having the viability property as a "multivalued solution" to the differential inclusion (1). □

The knowledge of a tube enjoying the viability property allows to infer some informations upon the asymptotic behavior of some solutions to the differential inclusion (1), as we do with Lyapunov functions. They also share the same disadvantages: the dynamics F being given, how do we construct the tubes of F ?

We shall begin by characterizing such tubes as "viability tubes". For that purpose, we need an adequate concept of derivative of set-valued map, the "contingent derivative[3] ".

[2] Examples of differential inclusions are provided by control problems, specially by control problems with a priori feedbacks, which can no longer be parametrized in a smooth way, or by differential games, or by systems evolving under uncertainty. See Aubin-Cellina [1984] for further motivations.

[3] It is defined as follows: If $x \in P(t)$, v belongs to $DP(t, x)(1)$ if and only if $\liminf_{h \to 0+} d\left(v, \frac{P(t+h)-x}{h}\right) = 0$

Viability tubes are those tubes satisfying

$$(2) \qquad \forall\, t \geq 0,\ \forall\, x \in P(t),\quad F(t,x) \cap DP(t,x)(1)\ \neq; \emptyset$$

We can regard (2) as a "differential equation for tubes", which provides another approach than the "funnel equations" to study the evolution of tubes.

A first application of these tubes can be made in control and differential games, for "guiding" at least one solution from an initial set K to a target C. In the finite horizon case, we look for tubes P satisfying the boundary conditions $P(0) = K$ and $P(T) = C$. This the reason why Russian mathematicians called them **bridges**. In infinite horizon, we need to study the asymptotic properties of $P(t)$ when $t \to \infty$.

We prove in the third section that the "Kuratowski upper limit" when $t \to \infty$ of a viability tube $P(t)$ is a viability domain: hence **targets** of a differential inclusion are necessarily viability domains. We construct in the fourth section the largest viability tube "converging" to a given target.

We also provide in the fifth section a surjectivity criterion which is useful for solving such problems.

We can also characterize viability tubes $P(t)$ by the indicator functions V_P of their graphs, defined by:

$$V_P(t,x) \ := \ \begin{cases} 0 & \text{if} \\ +\infty & \text{if } x \notin P(t) \end{cases}$$

We thus observe that P is a viability tube if and only if V_P is a solution to the "contingent Hamilton-Jacobi equation[4]"

$$(3) \qquad \inf_{v \in F(t,x)} D_{\uparrow}V(t,x)(1,v) \ = \ 0$$

This issue relating this new approach to classical concepts is the topic of the sixth section.

[4]where

$$D_{\uparrow}V(t,x)(1,v) \ := \ \liminf_{h \to 0+,\, v' \to v} \frac{V(t+h, x+hv') - V(t,x)}{h}$$

is the "contingent epiderivative,, of V at (t,x) in the direction $(1,v)$

We then investigate in the seventh section tubes enjoying a dual property, the **invariance property:** for all $t_0 \geq 0$ and $x_0 \in P(t_0)$, **all** solutions to the differential inclusion are viable. We justify in section 7 the claim that viability tubes and invariant tubes are in some convenient sense "dual". When $F(t,x) := A(t)x$ is a "set-valued linear operator" (called a **closed convex process**), we can define its "transpose". Therefore, we associate with the "linear differential inclusion"

$$x'(t) \in A(t)x(t)$$

its "adjoint" differential inclusion

$$-p'(t) \in A(t)^{\star}p(t)$$

We show that if a tube $t \rightsquigarrow R(t)$, the values of which are closed convex cones, enjoys the invariance property (for the original system), its polar tube $t \rightsquigarrow R(t)^{+}$, where $R(t)^{+}$ is the positive polar cone to $R(t)$, is a viability tube of the adjoint differential inclusion.

We end this exposition of viability tubes with one family of examples. In section 9, we investigate "finite horizon" tubes of the form $P(t) := \phi(t,G,D)$ where $\phi(0,C,D) = C$ and $\phi(T,C,D) = D$, which "carry" a subset C to a subset D.

2 Viability Tubes

Let X be a finite dimensional vector space. We consider a set-valued map $F : [0,T] \times X \rightsquigarrow X$ which associates with every (t,x) the subset $F(t,x)$ of velocities of the system at time t when its state is $x \in X$. We shall study the **differential inclusion**

$$(4) \qquad \begin{cases} i) & \text{for almost all } \; t \in [0,T], \; x'(t) \; \in \; F(t,x) \\ ii) & x(t_0) \; = \; x_0 \end{cases}$$

It will be convenient to regard a set-valued map P from $[0,T]$ to X as a "tube".

Definition 2.1 *We say that a tube P enjoys the* **viability property** *if and only if for all $t_0 \in [0,T]$, $x_0 \in P(t_0)$, there exists a solution $x(\cdot)$ to (4) which is "viable" in the sense that*

$$(5) \qquad \begin{cases} i) & \forall\, t \in [t_0, T[, \; x(t) \; \in \; P(t) \\ ii) & \text{if } \; T < +\infty, \; \forall\, t \geq T, \; x(t) \; \in P(T) \end{cases}$$

Recall that a subset K has the "viability property" if and only if the "constant tube" $t \rightsquigarrow P(t) := K$ does enjoy it. For time independent systems, we know how to characterize closed subsets K which enjoy the viability property (see Haddad [1981], Aubin-Cellina [1984]). For that purpose, we introduce the "contingent cone" $T_K(x)$ to K at x, the closed cone of vectors $v \in X$ such that

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} = 0$$

A subset K is said to be a **viability domain** of a set-valued map $F : X \rightsquigarrow X$ if and only if

$$\forall\, x \in K, \;\; F(x) \cap T_K(x) \neq \emptyset$$

When F is upper semicontinuous with compact convex images, such that $\|F(x)\| \leq (\|x\| + 1)$, Haddad's viability Theorem states that a closed subset K enjoys the viability property if and only if it is a viability domain.

Our first task is to characterize tubes enjoying the viability property thanks to its "contingent derivative" (see Aubin [1981], Aubin-Ekeland [1984]). We recall that

$$v \in DP(t,x)(\tau) \iff \liminf_{h\to 0+,\tau'\to\tau} d\left(v, \frac{P(t+\tau' h)-x}{h}\right) = 0 \tag{6}$$

We observe that it is enough to know this contingent derivative in the only directions $1, 0$ and -1. In particular, we note that

$$\begin{cases} i) & DP(t,x)(1) = \{v \in X \mid \liminf_{h\to 0+,\tau'\to 1} d\left(v, \frac{P(t+\tau' h)-x}{h}\right) = 0\} \\ ii) & T_{P(t)}(x) \subset DP(t,x)(0) \end{cases}$$

(Equality in (2)i) holds when P is Lipschitzean in a neighborhood of x).

We observe that the graph of $DP(t,x)$ is the contingent cone to the graph of P at (t,x).

Definition 2.2 *A tube* $P : [0,T] \leadsto X$ *is called a* **viability tube** *of a set-valued map* $F : [0,T] \times X \leadsto X$ *if its graph is contained in the domain of* F *and if*

$$\begin{cases} i) & \forall\, t \in [0,T[,\ \ \forall\, x \in P(t),\ \ F(t,x) \cap DP(t,x)(1) \neq \emptyset \\ ii) & \text{if } \ T < \infty,\ \ \forall\, x \in P(T),\ \ F(T,x) \cap DP(T,x)(0) \neq \emptyset \end{cases} \tag{7}$$

A tube is said to be **closed** *if and only if its graph is closed.*

Haddad's viability Theorem for autonomous systems and other results imply easily the following:

Theorem 2.1 *Assume that the set-valued map $F : [0,\infty[\times X \rightsquigarrow X$ satisfies:*

$$(8) \qquad \begin{cases} i) & F \text{ upper semi-continuous with closed convex values} \\ ii) & \|F(t,x)\| \;\leq\; a(\|x\|+1) \end{cases}$$

1. *A necessary and sufficient condition for a closed tube to enjoy the viability property is that it is a viability tube.*
2. *There exists a largest closed viability tube contained in the domain of F.*
3. *If P_n is a sequence of closed viability tubes, then the tube P defined by the Kuratowski upper limit*

$$\mathrm{Graph}(P) \;:=\; \limsup_{n\to\infty} \mathrm{Graph}(P_n)$$

is also a (closed) viability tube.

Proof We introduce the set-valued map G from $\mathrm{Graph}(P)$ to $\mathbf{R}_+ \times \mathbf{R}^n$ defined by

$$G(s,x) \;:=\; \begin{cases} \{1\} \times F(s,x) & \text{if } s \in [0,T[\\ [0,1] \times F(T,x) & \text{if } s = T \\ \{0\} \times F(T,x) & \text{if } s > T \end{cases}$$

We observe that $(s(\cdot), x(\cdot))$ is a solution to the differential inclusion

$$(s'(t), x'(t)) \;\in\; G(s(t), x(t))$$

starting at $(s(t_0), x(t_0)) = (t_0, x_0)$ if and only if $x(\cdot)$ is a solution to the differential inclusion (4). We also note that the tube P has the viability property if and only if its graph enjoys the viability property for G and that P is a viability tube if and only if its graph is a viability domain of G. It thus remains to translate the time independent results.

3 Asymptotic properties of viability tubes

We shall now study the behavior of viability tubes when $t \to \infty$.

Theorem 3.1 *Consider a set-valued map F from X to X, which is assumed to be upper semicontinuous, convex compact valued and satisfies*

$$\forall\, x \in \mathrm{Dom}(F),\ \ \|F(x)\| \ \leq \ a(\|x\|+1)$$

Then the Kuratowski upper limit

$$C := \limsup_{t\to\infty} P(t)$$

is a viability domain of F.

Proof We shall prove that C enjoys the viability property. Let ξ belong to C. Then $\xi = \lim \xi_n$ where $\xi_n \in P(t_n)$. We consider the solutions $x_n(\cdot)$ to the differential inclusion

$$x_n'(t) \ \in F(x_n(t)),\ \ x_n(t_n) \ = \ \xi_n$$

which are viable in the sense that $\forall\ t \geq t_n,\ \ x_n(t) \in P(t)$. The function $y_n(\cdot)$ defined by $y_n(t) := x_n(t+t_n)$ are solutions to

$$y_n'(t) \ \in \ F(y_n(t)),\ \ y_n(0) \ = \ \xi_n$$

The assumptions of Theorem3.1 imply that these solutions remain in a compact subset of $\mathcal{C}(0,\infty;X)$. Therefore, a subsequence (again denoted) converges to y, which is a solution to

$$y'(t) \ \in \ F(y(t)),\ \ y(0) \ = \ \xi$$

Furthermore, this solution is viable in C since for all $t \geq 0$, $y(t)$ is the limit of a subsequence of $y_n(t) = x_n(t+t_n) \in P(t+t_n)$, and thus belongs to C. □

4 The target problem

We shall study the "target problem"

> A closed viability domain C of F being given regarded as a **target**, find the largest closed viability tube P_C ending at C in the sense that $P_C(T) = C$ if $T < +\infty$ or $\limsup_{t\to\infty} P_C(t) = C$ if $T = +\infty$.

Knowing such a tube P_C, we thus deduce that starting at time 0 from $K := P_C(0)$, a solution to the differential inclusion $x' \in F(x)$ must bring this initial state to the target.

Proposition 4.1 *The assumptions are those of Theorem3.1. We can associate with any closed viability domain C of F a largest viability tube P_C ending at C. This tube is closed if we assume, for instance, that for any compact subset K, the set S of solutions to*

$$x'(t) \;\in\; F(x(t)), \;\; x(0) \;\in\; K$$

is compact in the Banach space $\mathcal{B}(0,\infty;X)$ of bounded functions.

Proof The solution is obvious when $T < +\infty$: We take

$$P_C(t) \;\;:=\;\; \{x(t) \;\; | x'(t) \in F(x(t)), \;\; x(T) \in C\}$$

It has the viability property: if (t, ξ) belongs to the graph of P_C, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ such that $x(t) = \xi$ and $x(T) \in C$ satisfying $x(s)$ belongs to $P_C(s)$ for all $s \geq t$ by the very definition of P_C. Hence it is viability tube ending at C.

It is the largest one: if P is any viability tube, then, for all $(t, \xi) \in \text{Graph}(P)$, there exists, thanks to the viability Theorem, a solution $x(\cdot)$ to $x' \in F(x)$ such that $x(s) \in P(s)$ for all $s \geq t$. Since $x(T) \in P(T) \subset C$, so that ξ belongs to $P_C(t)$.

The graph of P_C is closed : if $\xi_n \in P_C(t_n)$ and if (t_n, ξ_n) converges to (t, ξ), we see that (t, ξ) belongs to the graph of P_C. For there exists a sequence of solutions $x_n(\cdot)$ to $x_n' \in F(x_n)$ satisfying $x_n(t_n) = \xi_n$ and $x_n(T) \in C$. Since these solutions remain in a compact subset of $\mathcal{C}(0, T; X)$, a subsequence (again denoted) $x_n(\cdot)$ converges uniformly to a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ which satisfies $x(t) = \xi$ and $x(t) = \lim_{n\to\infty} x_n(t) \in C$.

We also observe that

$$(9) \qquad P_C(t) \; = \; \{y(T-t) \mid y'(t) \; \in \; -F(y(t)), \;\; \& y(0) \in C\}$$

Those two subsets do coincide because $x(\cdot)$ is a solution to $x' \in F(x)$ if and only if the function $y(\cdot)$ defined by $y(t) := x(T-t)$ is a solution to $y' \in -F(y)$ such that $y(0) = x(T)$.

Consider now the case when $T = \infty$ and denote by L the set-valued map associating with any continuous function $x(\cdot) \in \mathcal{C}(0, \infty; X)$ its limit set

$$L(x) \;\; := \;\; \limsup_{t \to \infty} \{x(t)\} \;\; = \;\; \bigcap_{T > 0} \overline{(x([T, \infty[)}$$

The same arguments as those in the finite horizon case imply that the tube P_C defined by

$$(10) \qquad P_C(t) \;\; := \;\; \{x(t) \mid x'(t) \; \in \; F(x(t)), \;\; \& \;\; L(x) \;\; \subset \;\; C\}$$

is the largest viability tube "converging" to C.

We have to show that it is closed. As in the finite horizon case, we consider a sequence $(t_n, x_n) \in \text{Graph}(P_C)$ which converges to (t, x) and solutions $x_n(\cdot)$ to

$$x_n'(t) \;\; \in \;\; F(x_n(t)), \;\; x_n(t_n) = \xi_n \;\; \& \;\; L(x_n) \;\; \subset \;\; C$$

Since the ξ_n 's belong to a compact K, the last assumption we made implies that the solutions $x_n(\cdot)$ lie in a compact subset of $\mathcal{B}(0, \infty; X)$.

A subsequence (again denoted) $x_n(\cdot)$ converges uniformly on $[0, \infty[$ to a solution $x(\cdot)$ to $x' \in F(x), \;\; x(t) = \xi$.

We deduce that its limit set $L(x)$ is contained in C from the fact that the set-valued map L is lower semicontinuous: for if y belongs to $L(x)$ and if a sequence $x_n(\cdot)$ converges uniformly to $x(\cdot)$, then there exists $y_n \in L(x_n) \subset C$ which converges to y, and which thus belongs to C, which is assumed to be closed.

The lower semicontinuity of L follows from:

Lemma 4.1 *Let be the Banach space of bounded continuous functions. The set-valued map L is lower semicontinuous from $\mathcal{B}(0,\infty;X)$ to X.*

Proof Let $\xi \in L(x)$ and $x_n(\cdot) \in \mathcal{B}(0,\infty;X)$ converge uniformly to $x(\cdot)$ on $[0,\infty[$. There exists $t_k \to \infty$ such that $x(t_k)$ converges to ξ. Further, for all $\epsilon > 0$, there exists N such that

$$\forall\, n \geq N,\ \ \|x_n(t_k) - x(t)\| \ \leq \ \epsilon$$

Hence $\|x_n(t_k) - \xi\| \leq \epsilon$ for all t_k large enough. Since the dimension of X is finite, the subsequence $x_n(t_k)$ converges to an element ξ_n which belongs to $L(x_n)$ and thus, $\|\xi_n - \xi\| \leq 2\epsilon$ for all $n \geq N$. Hence L is lower semicontinuous. □

5 A surjectivity criterion for set-valued maps

We propose now a criterion which allows to decide whether a compact convex subset C lies in the target of a differential inclusion. It belongs to the class of surjectivity theorems for "outward maps" (see Aubin-Ekeland, [1984]). The idea is the following. We consider a set-valued map R (the reachable map in our framework) from a subset K of a Hilbert space X to another Hilbert space Y. We want to solve the following problem (The **K-C problem**):

For every y in C, find x in K such that y belongs to $R(x)$

(i.e. we can reach any element of the target C from K). Assume that we know how to solve this problem for a "nicer" set-valued map Q from K to Y (say, a map with compact convex graph).

For every y in C, find x in K such that i belongs to $Q(x)$

The next theorem states how a relation linking R and Q (R is "outward with respect to" Q) allows to deduce the surjectivity of R from the surjectivity of Q.

Theorem 5.1 *We assume that the graph of Q is convex and compact and that R is upper semicontinuous with convex values. We set*

$$K \ := \ \mathrm{Dom}(Q), \ C \ := \ \mathrm{Im}(Q)$$

If R is **outward with respect to** Q *in the sense that*

$$\forall\, x \in K,\ \forall\, y \in Q(x),\ \ y \ \in \ R(x) + T_C(y), \tag{11}$$

then R is surjective from K to C (in the sense that $C \subset R(K)$).

Proof It is a simple consequence of Theorem 6-4.12 p.343 of Aubin-Ekeland [1984]. We replace X by $X \times Y$, K by Graph(Q) (which is convex compact), A by the projection pi_Y from $X \times Y$ to Y and R by the set-valued map G from $X \times Y$ to Y defined by:

$$G(x, y) := R(x) - y_0 \quad \text{where } y_0 \text{ is given in } C$$

The outwardness condition implies that the tangential condition :

$$O \in -y + R(x) + T_C(y)$$

if satisfied. Since $y_0 - y$ belongs to $T_C(y)$ (because $y_0 \in C$), then

$$O \in -y_0 + R(x) + T_C(y) = G(x,y) + T_C(y)$$

We observe that

$$\begin{cases} T_C(y) = T_{\mathrm{Im}(Q)}(y) = T_{\pi_Y(\mathrm{Graph}(Q))}(pi_Y(x,y)) \\ = \pi_Y(T_{\mathrm{Graph}(Q)}(x,y)) \end{cases}$$

so that

$$0 \in G(x,y) + \overline{\pi_Y}(T_{\mathrm{Graph}(Q)}(x,y))$$

Theorem 6.4.12 of Aubin-Ekeland [1984] implies the existence of $(\bar{x},\bar{y})$ in the graph of Q, a solution to the inclusion $0 \in G(\bar{x},\bar{y})$, i.e., to the inclusion $y_0 \in R(\bar{x})$.

Remark The dual version of the "outwardness condition" is the following:

$$(12) \qquad \forall\, q \in N_C(y),\ \forall\, x \in A^{-1}(y),\ < q, y > \ \leq\ \sigma(R(x), q)$$

where $N_C(y)$ denotes the normal cone to the convex set C at y and

$$\sigma(R(x), q) := \sup_{y \in R(x)} < q, y >$$

is the support function of $R(x)$.

Remark By using the concept of σ-selectionable maps introduced by Haddad-Lasry [1983] (see also Aubin-Cellina [1984], p. 235), we can extend this theorem to the case when R is σ-selectionable instead of being convex-valued. We obtain:

Theorem 5.2 *We assume that the graph of Q is convex and compact and that R is σ-selectionable. If R is "strongly outward with respect to Q" in the sense that*

$$\forall\, x \in K,\ \forall\, y \in Q(x),\ R(x) \subset y - T_C(y)$$

than R is surjective from K to C.

Remark Other sufficient conditions can be proposed to guarantee the surjectivity of R. For instance, "inwardness" condition

$$C \subset \bigcap_{x \in K} (R(x) + T_C(Q(x)))$$

implies the surjectivity condition when R is upper semicontinuous with convex valued and "strong inwardness" condition

$$C - R(x) \subset \bigcap_{y \in Q(x)} T_C(y)$$

implies the surjectivity condition when R is only σ-selectionable.

To prove these statements, we use the same methods applied to the set-valued map

$$H(x, y) := R(x) - y_0$$

6 Contingent Hamilton-Jacobi Equations

We may regard condition (7)i) involved in the definition of viability tubes as a "set-valued differential inclusion", the solutions to which are "viability tubes" and condition (7)ii) as a "final" condition. Actually, conditions (7) defining "viability tubes" is a multivalued version of the Hamilton-Jacobi equation in the following sense.

We characterize a tube P by the indicator function V_P of its graph defined by

$$(13) \qquad V_P(t,x) \ := \ \begin{cases} 0 & \text{if } x \in P(t) \\ +\infty & \text{if } x \notin P(t) \end{cases}$$

The contingent epiderivative $D_\uparrow V(t,x)$ of a function V from $\mathbf{R} \times X$ to $\mathbf{R} \cup \{+\infty\}$ at (t,x) in the direction (α, v) is defined by

$$(14) \quad D_\uparrow V(t,x)(\alpha,v) := \liminf_{h \to 0+, w \to v, \beta \to \alpha} \frac{V(t+\beta h, x+hw) - V(t,x)}{h}$$

The epigraph of $D_\uparrow V(t,x)$ is the contingent cone to the epigraph of V at $(t, x, V(t,x))$.

Hence, conditions (7) can be translated in the following way:

Proposition 6.1 *A tube P is a viability tube if and only if the indicator function V_P of its graph is a solution to the* **contingent Hamilton-Jacobi** *equation:*

$$(15) \qquad \inf_{v \in F(t,x)} D_\uparrow V(t,x)(1,v) \ = \ 0$$

satisfying the final condition (when $T < \infty$):

$$(16) \qquad \inf_{v \in F(T,x)} D_\uparrow V(T,x)(0,v) \ = \ 0$$

Remark When the function V is differentiable, equation (15) can be written in the form

$$\frac{\partial V}{\partial t} + \inf_{v \in F(t,x)} \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,x) v_i \ = \ 0$$

We recognize the classical Hamilton-Jacobi equation (see Aubin-Cellina [1984], Chapter 6). A thorough study of contingent Hamilton-Jacobi equations (for lipschitzean maps $F(t,x)$) is carried out in Frankowska [1986]), where relations with viscosity solutions introduced by Crandall & Lions P.L. [1983] (see also Lions P.L. [1982]) and generalized Hamilton-Jacobi equations (Clarke & Vinter [1983], Rockafellar [to appear]) are worked out.

7 Invariant tubes

We distinguish between viability tubes and invariant tubes in the same way as viability domains and invariant domains.

Definition 7.1 *We say that a tube P enjoys the invariance property if and only if for all t_0 and $x_0 \in P(t_0)$,* **all the solutions** *to the differential inclusion (3.1) are viable in the tube P.*

We say that P is an "invariant tube" if

$$(17) \quad \begin{cases} i) & \forall\, t \in [0,T[,\ \forall\, x \in P(t),\ F(t,x) \subset DP(t,x)(1) \\ ii) & \text{if } T < +\infty,\ \forall\, x \in P(T),\ F(t,x) \subset DP(t,x)(0) \end{cases}$$

We obtain the following theorem.

Theorem 7.1 *Assume that $F : [0,T[\times\Omega \to X$ is lipschitzean with respect to x in the sense that*

$$\exists\, k(\cdot) \in \mathrm{L}^1(0,T) \ \mid\ F(t,x) \subset F(t,y) + k(t)\|x-y\|B$$

(B is a unit ball). Let $t \rightsquigarrow P(t) \subset \Omega$ be a closed tube: If P is invariant, then it enjoys the invariance property.

Proof The theorem follows from the following lemma, an extension to a result from Aubin-Clarke [1977].

Lemma 7.1 *Let P be a closed tube and $\pi_{P(t)}(y)$ denote the set of best approximations of y by elements of $P(t)$.*

$$\left\{ \liminf_{h\to 0+} \frac{d(y+hv, P(t+h)) - d(y,P(t))}{h} \ \leq\ \inf_{x\in\pi_{P(t)}(y)} d(v, DP(t,x)(1)) \right.$$

Indeed, with any solution to the differential inclusion $x'(t) \in F(t,x(t))$, we can associate the function $g(t) := d(x(t),P(t))$. Let us choose $y(t) \in \pi_{P(t)}(x(t))$. Inequalities

$$\begin{cases} \frac{g(t+h)-g(t)}{h} = \frac{d(x(t)+hx'(t)+ho(h),P(t+h)) - d(x(t),P(t))}{h} \\ \leq \|o(h)\| + \frac{d(x(t)+hx'(t),P(t+h)) - d(x(t),P(t))}{h} \leq d(x'(t), DP(t,y(t)(1)) \\ \leq d(x'(t), F(t,y(t))) \leq \sup_{v\in F(t,x(t))} d(v, F(t,y(t))) \\ \leq k(t)\|y(t)-x(t)\| = k(t)d(x(t),P(t)) = k(t)g(t) \end{cases}$$

imply that g(t) is a solution to the differential inequality

$$D_{\uparrow}g(t)(1) \ \leq\ k(t)g(t) \ \&\ g(t_0) = d(x_0,P(t_0)) = 0$$

Hence $d(x(t),P(t)) = g(t) = 0$ for all $t \in [t_0,T[$. □

Proof of Lemma 7.1 Let $y \in P(t)$ and $u \in DP(t,y)(1)$ be given. We consider sequences $h_n \to 0+$ and $u_n \to u$ such that

$$\liminf_{n\to\infty} \frac{d(y+h_n u_n, P(t+h_n))}{h_n} = 0$$

Hence, for all $v \in X$,

$$d(y+h_n v, P(t+h_n))/h_n \leq \|v-u_n\| + d(y+h_n u_n, P(t+h_n))/h_n$$

which implies the desired inequality by letting $h_n > 0$ go to 0.

Let us choose now $y \notin P(t)$ and $x \in P(t)$ such that $\|x-y\| = d(y,P(t))$. We observe that

$$\begin{cases} (d(y+hv,P(t+h)) - d(y,P(t)))/h \\ \leq \ (\|y-x\| + d(x+hv,P(t+h)) - d(y,P(t)))/h \\ = \ d(x+hv,P(t+h)/h \end{cases}$$

Since x belongs to $P(t)$, the desired inequality for x implies the one for y since

$$\begin{cases} \liminf_{h\to 0+}(d(y+hv,P(t+h)) - d(y,P(t)))/h \\ \leq \ \liminf_{h\to 0+} d(x+hv,P(t+h))/h \\ \leq \ d(v,DP(t,x)(1)) \quad \square \end{cases}$$

Remark This lemma implies that if

$$\forall\, t,\ \forall\, x \in P(t),\ F(t,x) \subset DP(t,x)(1)$$

and if

$$\forall\, t,\ x \leadsto F(t,x) \text{ is lower semicontinuous,}$$

then

$$\forall\, t,\ \forall\, x \in P(t),\ F(t,x) \subset CP(t,x)(1)$$

where

$$v \in CP(t,x)(1) \iff \lim_{h\to 0+, y\to_{P(t)} x} \frac{d(y+hv,P(t+h))}{h} = 0$$

This convergence is uniform with respect to $v \in F(t,x)$ if this subset is compact. In particular, if $x \leadsto DP(t,x)(1)$ is lower semicontinuous. then

$$DP(t,x)(1) = CP(t,x)(1) \quad \Box$$

Remark If we assume that the condition

$$\forall\ (t,y) \in \mathrm{Dom}(F),\ \ \exists\, x \in \pi_{P(t)}(y) \ \text{ such that } \ F(t,y) \ \subset \ DP(t,x)(1)$$

holds true, then the tube P is invariant by F: this knowledge of the behavior of F outside the graph of the tube P allows to dispose of the lipschitzean assumption. □

We can characterize the indicator functions of the graphs of invariant tubes in the following way:

Proposition 7.1 *A tube P is invariant by F if and only if the indicator function V_P of its graph is a solution to the equation*

$$\text{(18)} \qquad \sup_{v \in F(t,x)} D_{\uparrow}V(t,x)(1,v) \ = \ 0$$

satisfying the final condition

$$\text{(19)} \qquad \text{If } \ T < +\infty, \ \ \sup_{v \in F(t,x)} D_{\uparrow}V(T,x)(0,v) \ = \ 0$$

8 Duality relations between invariant and viability tubes

Let us consider the case when $F(t,x) := A(t)x$ is a time dependent closed convex process $A(t)$ whose domain is the whole space X. In this case, we look for tubes R the images of which are closed convex cones.

We associate with such a tube R its "polar tube" R^+ mapping any t to the (positive) polar cone

$$R(t)^+ \ := \ \{q \in X^\star \mid \forall\, y \in R(t), \ < q, y > \geq \ 0\}$$

We also associate with $A(t)$ its "transpose" $A(t)^\star$ defined by

$$\begin{cases} p \in A(t)^\star q \\ \iff \ \forall\,(x,y) \in \mathrm{Graph}(A(t)), \ < p, x > \leq < q, y > \\ \iff \ (-p, q) \ \in \ \mathrm{Graph}(A(t))^+ \end{cases}$$

We consider the "linear" differential inclusion

(20) $$x'(t) \ \in \ A(t)x(t)$$

and its "adjoint differential inclusion"

(21) $$-p'(t) \ \in \ A(t)^\star p(t)$$

We shall prove that the invariance of the tube R implies that its positive polar tube R^+ is a viability tube of the adjoint inclusion.

Theorem 8.1 *Let us assume that the domains of the closed convex processes are all equal to X and that*

$$\begin{cases} i) & \textit{the lipschitz constants of } A(t) \textit{ is bounded by } k(\cdot) \ \in \ \mathrm{L}^2([0,T]) \\ ii) & \forall\, x \in X, \ \ (t,q) \mapsto \sigma(A(t)x, q) \ \textit{ is lower semicontinuous} \end{cases}$$

Let R be a tube with closed convex cone values. If R enjoys the viability property for $A(t)$, then the tube R^+ is a viability tube of the adjoint differential inclusion and thus, enjoys the viability property in the sense that $\forall\, t \in [0,T]$, $\forall\, q \in R(t)^+$, there exists a solution $q(\cdot)$ to the adjoint inclusion such that $q(t) = q$ and

$$\forall\, \tau \in [0,t], \ \ q(\tau) \in R(\tau)^+$$

Proof We have to prove that

$$\forall\, t \in [0,T], \ \forall\, q_t \in R(t)^+, \ \ A(t)^\star q_t \cap DR^+(t, q_t)(-1) \neq \emptyset$$

Since the transpose $A(t)^\star q$ is upper semicontinuous with compact convex images, Theorem 2.1 will imply that R^+ enjoys the viability property.

Let $S \subset \mathrm{H}^1(0,T;X)$ be the set of solutions to the differential inclusion $x'(t) \in A(t)x(t)$.

We denote by γ_τ the linear operator from $\mathrm{H}^1(0,T;X)$ to X associating with every x its value $\gamma_\tau x := x(\tau)$ at $\tau \in [0,T]$. To say that R enjoys the invariance property means that for all $0 \leq s \leq t \leq T$,

$$\gamma_t(S \cap \gamma_s^{-1}R(s)) \quad \subset \quad R(t)$$

By polarity, we deduce that

$$R(t)^+ \quad \subset \quad (\gamma_t(S \cap \gamma_s^{-1}(R(s)))^+ \quad = \quad \gamma_t^{\star^{-1}}(S \cap \gamma_s^{-1}(R(s)))^+$$

We deduce from Frankowska [1986a] that

$$(S \cap \gamma_s^{-1}R(s))^+ \quad = \quad S^+ + \gamma_s^\star R(s)^+$$

Hence, for all $q_t \in R(t)^+$ and for all $s \leq t$, there exists $q_s \in R(s)^+$ such that $\gamma_t q_t - \gamma_s q_s$ belongs to S^+. Always by Frankowska [1986a], there exists a solution $p_s(\cdot)$ to the adjoint inclusion on the interval $[s,t]$

$$(22) \qquad -p_s'(\tau) \quad \in \quad A(\tau)^\star p_s(\tau) \quad \& \quad p_s(t) = q_t$$

which satisfies

$$p_s(s) \quad \in \quad R(s)^+$$

By replacing t by s and s by 0, we can extend the solution $p_s(\cdot)$ on the whole interval $[0,t]$. We now let s converge to t. Since $\mathrm{Dom}(A(t)) = X$, we know that

$$\sigma(A(t)^\star p, x) = -\sigma(A(t)x, -p)$$

Hence the lower semicontinuity of $(t,p) \mapsto \sigma(A(t)x,-p)$ implies the upper semicontinuity of $\sigma(A(t)^\star p, x)$, and thus, the upper semicontinuity of $(t,p) \mapsto A(t)^\star p$. (See Aubin-Ekeland, [1984], Theorem 3.2.10). Therefore for all $\epsilon > 0$, there exists $\eta > 0$ such that, for all $\tau \in [t-\eta, t]$ and $p \in q_t + \eta B$, we have

$$A(\tau)^\star p \quad \subset \quad A(t)^\star q_t + \epsilon B$$

The set of solutions $p_s(\cdot)$ to the adjoint inclusion being contained in a compact set of $\mathcal{C}(0,T;X)$, a subsequence (again denoted) $p_s(\cdot)$ converges uniformly to a solution $p_0(\cdot)$ to the adjoint equation.

Hence there exists $\alpha \le \eta$ such that, for all $\tau \in [t-\alpha, t]$, and for all s,

$$\|p_s(\tau) - q_t\| \quad \le \quad \eta$$

Therefore

$$\forall\, s,\ \forall\, r \in [t-\alpha, t],\ A(\tau)^\star p_s(\tau) \quad \subset \quad A(t)^\star q_t + \epsilon B$$

By integrating (22) on the interval $[t-h, t]$ with $s = t-h, h \le \alpha$, we deduce that

$$\begin{cases} v_h \;:=\; \frac{p_{t-h}(t-h) - q_t}{h} \;=\; \int_{t-h}^{t} p'_{t-h}(\tau)d\tau \\ \in\; -\frac{1}{h}\int_{t-h}^{t} A(\tau)^\star p_s(\tau)d\tau \;\subset\; -\overline{co}(A(t)^\star q_t + \epsilon B) \\ =\quad -A(t)^\star q_t + \epsilon B \end{cases}$$

This subset being compact, a subsequence v_n converges to an element $v \in A(t)^\star q_t$. Since

$$q_t + hv_n \quad = \quad p_{t-h}(t-h) \quad \in \quad R(t-h)^+$$

for all $h > 0$, we deduce that v belongs to $DR^+(t, q_t)(-1)$. □

9 Examples of viability tubes

Let us consider two closed subsets C and D of $X := \mathbf{R}^n$ and a differentiable map Φ from a neighborhood of $[0,T] \times C \times D$ to X.

We consider tubes of the form

$$(23) \qquad P(t) \;:=\; \Phi(t, C, D)$$

Proposition 9.1 *Let us assume that for all $t \leq T$, for all $x \in P(t)$, there exists $(y,z) \in C \times D$ satisfying $\Phi(t,y,z) = x$ and there exists $(u,v) \in T_{C\times D}(y,z)$ such that*

$$(24)\begin{cases} i) & \text{if } t < T, \;\; \Phi'_y(t,y,z)u + \Phi'_z(t,y,z)v \;\in\; F(t,x) - \Phi'_t(t,y,z) \\ ii) & \text{if } t = T, \;\; \Phi'_y(T,y,z) + \Phi'_z(T,y,z)v \;\in\; F(T,x) \end{cases}$$

Then the set-valued map P defined by (23) is a viability tube of F on $[0,T]$.

Proof We observe that Graph(P) is the image of $[0,T] \times C \times D$ under the map Ψ defined by

$$\Psi(t,y,z) \;=\; (t, \Phi(t,y,z))$$

By Proposition 7.6.2, p. 430 of Aubin-Ekland [1984],

$$\Psi'(t,y,z)T_{[0,T]\times C\times D}(t,y,z) \;\subset\; T_{\mathrm{Graph}(P)}(\Psi(t,y,z))$$

so that the assumptions (24) imply that P is a viability tube. □

When C and D are closed and convex, we can characterize viability tubes of the form (23) through dual conditions.

Proposition 9.2 *Let us assume that the values of F are compact and convex and that the subsets C and D are closed and convex. If for any $t \in [0,T]$, $\forall\, x \in P(t)$, there exists $(y,z) \in C \times D$ satisfying $\Phi(t,y,z) = x$ and for all*

$$p \in \Phi'_y(t,y,z)^{\star^{-1}} N_C(y) \cap \Phi'_z(t,y,z)^{\star^{-1}} N_D(z)$$

we have

$$(25)\begin{cases} i) & \forall\, t < T, \;\; < p, \Phi'_t(t,y,z) > + \sigma(F(t,\Phi(t,y,z)), -p) \;\geq\; 0 \\ ii) & \text{for } t = T, \;\; \sigma(F(T,\Phi(T,y,z)), -p) \;\geq\; 0 \end{cases}$$

then the set-valued map P defined by (23) is a viability tube of F on $[0,T]$.

Proof When C and D are convex, $T_{C\times D}(y,z) = T_C(y) \times T_D(z)$, so that conditions (24)i) and ii) can be written

$$\begin{cases} i) & (F(t,x) - \Phi'_t(t,y,z)) \cap (\Phi'_y(t,y,z)T_C(y) + \Phi'_z(t,y,z)T_D(z)) \neq \emptyset \\ ii) & F(T,x) \cap (\Phi'_y(T,y,z)T_C(y) + \Phi'_z(T,y,z)T_D(z)) \neq \emptyset \end{cases}$$

The separation theorem shows that they are equivalent to conditions (25).

Corollary 9.1 *Let us assume that C and D are closed convex subsets and that the values of F are convex and compact. Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a differentiable function satisfying either one of the following equivalent conditions: For any $t \geq 0$, $\forall\, x \in P(t)$, there exist $y \in C, z \in D$ such that $x = y + \phi(t)z$ and either*

$$(26)\begin{cases} i) & (F(t,y+\phi(t)z) - \phi'(t)z) \cap (T_C(y) + T_D(z)) \neq \emptyset \text{ if } t < T \\ ii) & (F(T,y+\phi(T)z) \cap (T_C(y) + T_D(z)) \neq \emptyset \text{ if } t = T \end{cases}$$

or, $\forall\, p \in N_C(y) \cap N_D(z)$,

$$(27)\qquad \begin{cases} i) & \phi'(t)\sigma_D(p) + \sigma(F(t,y+\phi(t)z,-p) \geq 0 \text{ if } t < T \\ ii) & \sigma(F(T,y+\phi(T)z,-p) \geq 0 \text{ if } t = T \end{cases}$$

Then the set-valued map P defined by

$$(28)\qquad P(t) := C + \phi(t)D$$

is a viability tube of F on $[0,T]$.

Let us consider the instance when $C = \{c\}$ and when 0 belongs to the interior of the closed convex subset D.

We introduce the function a_0 defined by

$$(29)\qquad \begin{cases} a_0(t,w) := \\ \sup_{z\in D} \sup_{p\in N_D(x),\, \sigma_D(p)=1} \inf_{v\in F(t,c+wz)} < p,v > \\ \sup_{z\in D} \inf_{v\in F(t,c+wz)} \sup_{p\in N_D(x),\, \sigma_D(p)=1} < p,v > \end{cases}$$

(The last equation follows from the minimax theorem.)

Let us assume that there exists a continuous function $a : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}$ satisfying $a(t,0) = 0$ for all $t \geq 0$ and

$$\forall\ (t,w) \in \mathbf{R}_+ \times \mathbf{R}_+,\ \ a(t,w) \quad \geq a_0(t,w)$$

Let ϕ be a solution to the differential equation

(30) $$\phi'(t) \ = \ a(t,\phi(t)) \ \ \& \ \ \phi(0) = \phi_0 \ \text{ given}$$

satisfying

$$a(T,\phi(T)) = 0$$

Since $\sigma_D(p) > 0$ for all $p \neq 0$, we deduce that for all $z \in D$ and all $p \in N_D(z)$,

$$\begin{cases} \phi'(t)\sigma_D(p) \ \geq \ a(t,\phi(t))\sigma_D(p) \ \geq \ a_0(t,\phi(t))\sigma_D(p) \\ \geq \ \sigma_D(p) \sup_{v \in F(t,c+\phi(t)z)} < -\frac{p}{\sigma_D(p)}, v > \\ = -\sigma(F(t,c+\phi(t)z), -p) \end{cases}$$

Hence, condition (27)i) is satisfied. We also check that

$$0 \ = \ a(T,\phi(T)) \ \ geq\, a_0(T,\phi(T)) \ \geq \ \frac{-1}{\sigma_D(p)}\sigma(F(T,c+\phi(T)z),-p)$$

Then the tube defined by $P(t) \ := \ c + \phi(t)D$ is a viability tube of F. □

For instance, if $D := B$ is the unit ball, then $\sigma_B(p) = \|p\|$ and $N_B(z) = \lambda z$ for all $z \in S := \{x \| \|x\| = 1\}$. Hence, in this case we have

$$a_0(t,w) \ := \ \sup_{\|z\|=1} \ \inf_{v \in F(t,c+wz)} \ < v, z >$$

In other words, the function a_0 defined by (29) conceals all the information needed to check whether a given subset D can generate a tube P. □

Remark When a is non-positive and satisfies $a(t,0) = 0$ for all $t \geq 0$, then there exists a non-negative non-increasing solution $\phi(\cdot)$ to the differential equation (30).

When $T = \infty$, we infer that $\int_0^\infty a(\tau,\phi(\tau))d\tau$ is finite. Let us assume that for all $w_\star \in \mathbf{R}_+$,

$$\lim_{t \to \infty, w \to w_*} a(t,w) \;=\; a_*(w_*)$$

Then the limit ϕ_* of $\phi(t)$ when $t \to \infty$ satisfies the equation

$$a_*(\phi_*) = 0$$

Otherwise, there would exist $\epsilon > 0$ and T such that $a_*(\phi_*) + \epsilon < 0$ and for all $t > T, a(t,\phi(t)) \leq a_*(\phi_*) + \epsilon$ by definition of a_*.

We deduce the contradiction

$$\phi(t) - \phi(T) \;= \int_T^t a(\tau, \phi(\tau)) d\tau \;\leq\; (t-T)(a_*(\phi_*) + \epsilon)$$

when t is large enough.

Example

Let us consider the case when F does not depend upon t. We set

$$(31) \qquad \rho_0 \;:=\; \sup_{\lambda \in \mathbf{R}} \inf_{w>0} (\lambda w - a_0(w))$$

Assume also that $\lambda_0 \in \mathbf{R}$ achieves the supremum. We can take $\psi(w) := \lambda_0 w - \rho_0$.

If $\rho_0 > 0$, the function

$$(32) \qquad \phi_T(t) \;:=\; \begin{cases} \frac{\rho_0}{\lambda_0}(1 - \exp(\lambda_0(t-T)) & \text{if } \lambda_0 \neq 0 \\ -\rho_0(t-T) & \text{if } \lambda_0 = 0 \end{cases}$$

is such that $P(t) := \{c + \phi_T(t)D\}$ is a tube of F such that $P(T) = \{c\}$.

If $\rho_0 \leq 0$ and $\lambda_0 < 0$, then the functions

$$\phi(t) : \frac{1}{\lambda_0}(\rho_0 - e^{\lambda_0 t})$$

are such that $P(t) := c + \phi(t)D$ defines a tube of F on $[0, \infty[$ such that $P(t)$ decreases to the set $P_\infty := c + \frac{\rho_0}{\lambda_0}D$. □

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