THE ⊛ – KERNEL FOR A QUASIDIFFERENTIABLE FUNCTION

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FOREWORD

Generally speaking, the quasidifferentials for a quasidifferentiable function in the sense of Demyanov and Rubinov are not unique. Therefore, it is difficult to study the continuity of quasidifferentials. Does there exists a kind of kernel for the quasidifferentials of a quasidifferentiable function at a point? If so, what kind of structure does it possess? The main purpose in this paper is to explore ways and means of finding the kernel quasidifferentials in the sense of Demyanov and Rubinov for a certain class of quasidifferentiable functions. The results given here indicate that there exists a kind of kernel - the so-called *star-kernel* for quasidifferentials, which is defined through a *star-equivalent bounded subfamily* of a quasidifferentiable function at a given point. A directional subderivative and superderivative of a quasidifferentiable function are proposed here that are unique. The continuity of the kernel is also studied briefly.

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ABSTRACT

This paper attempts to explore ways and means of finding the kernels of quasidifferentials. The results here show that there exists a kind of kernel called ⊛ - kernel for the quasidifferentials, with a ⊛ - equivalent bounded subfamily, of a quasidifferentiable function at a point. The directional subderivative and superderivative of a quasidifferentiable function are proposed. The continuity of the kernel also is mentioned in this paper.

Key words: Quasidifferentiable function, quasidifferential calculas, convex analysis, generalized gradient, upper and lower semicontinuous.

CONTENTS

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1. INTRODUCTION

It is well known that for any quasidifferentiable function in the sense of [2] its directional derivative can be expressed as the form of sum of a pair of sublinear operator and superlinear operator, or the one of difference of two sublinear operators, [2], [3], [5]. This sort of structure of derivatives of quasidifferentiable functions brings on that a quasidifferential of a quasidifferentiable function, called bidifferential also in [4], is not unique, but the quasidifferential equivalent class of a quasidifferentiable function is unique. Therefore, it is difficult to study the continuity of a quasidifferential mapping and other problems concerned. Recently a new result has been obtained in [5].

We observe a convex function f defined in $\mathbf{R}^n$. Obviously, the directional derivative of f at x in a direction $d \in \mathbf{R}^n$ can be expressed as

$$f'(x:d) = \max_{v \in \partial f(x)} <v,\, d> \quad . \tag{1.1}$$

If the convex function is regarded as a quasidifferentiable function, then the expression (1.1) may be converted into

$$f'(x:d) = \max_{v \in \underline{\partial} f(x)} <v,\, d> + \min_{w \in \bar{\partial} f(x)} <w,\, d> \quad . \tag{1.2}$$

From (1.1) and (1.2) we have

$$\underline{\partial} f(x) = \partial f(x) - \bar{\partial} f(x) \quad . \tag{1.3}$$

The expression (1.2) is more complicated than (1.1). This shows that it is possible for a simple problem to become a quite complicated one when it is treated by quasidifferentials, even if in the case where a continuously differentiable function f_{c^1} is treated as a quasidifferentiable one, i.e.,

$$\underline{\partial} f_{c^1}(x) = \nabla f_{c^1}(x) - \bar{\partial} f_{c^1}(x) \quad . \tag{1.4}$$

But it seems that (1.3) and (1.4) may be used to explore a kind of intrinsic character of the quasidifferentials of a q.d. function at a point x, although they are more complicated than (1.1). It is easy to be seen that if the following forms are investigated

$$\begin{aligned} &\underline{\partial}\, f(x) + \bar{\partial}\, f(x) = \partial\, f(x) + (\bar{\partial}\, f(x) - \bar{\partial}\, f(x)) \ , \\ &\bar{\partial}\, f(x) - \bar{\partial}\, f(x) \ , \end{aligned} \tag{1.5}$$

corresponding to (1.3), and

$$\begin{aligned} &\underline{\partial}\, f_{C^1}(x) + \bar{\partial}\, f_{C^1}(x) = \nabla\, f_{C^1}(x) + (\bar{\partial}\, f_{C^1}(x) - \bar{\partial}\, f_{C^1}(x)) \ , \\ &\bar{\partial}\, f_{C^1}(x) - \bar{\partial}\, f_{C^1}(x) \ , \end{aligned} \tag{1.6}$$

corresponding to (1.4), then it would be found that

$$\begin{aligned} &\partial f(x) = \cap\ (\underline{\partial} f(x) + \bar{\partial} f(x)) \ , \\ &\{0\} = \cap\ (\bar{\partial} f(x) - \bar{\partial} f(x)) \ , \end{aligned} \tag{1.7}$$

where the intersections are taken with respect to the quasidifferentials of f at a point x. Similarly,

$$\begin{aligned} &\{\nabla\, f_{C^1}(x)\} = \cap\ (\underline{\partial} f_{C^1}(x) + \bar{\partial} f_{C^1}(x)) \ , \\ &\{0\} = \cap\ (\bar{\partial} f_{C^1}(x) - \bar{\partial} f_{C^1}(x)) \ . \end{aligned} \tag{1.8}$$

It is reasonable from (1.7) and (1.8) that $[\partial f(x), 0]$ is regarded as a kind of kernel of the quasidifferentials of a q.d. function f at x in the case where f is convex, and $[\nabla\, f(x), 0]$ is regarded as a kind of kernel of the quasidifferentials for a continuously differentiable function f at x.

We are very interested in the question, that is, if there is a kind of kernel for the quasidifferentials of a generally q.d. function f at x. If so, what kind of structure does it possess? The main purpose in this paper is to find a kind of kernel for a certain class of quasidifferentiable functions whose quasidifferentials have ⊛ – equivalent bounded subfamilies. Some of their properties are also represented in this paper. The space we will use in this paper is the n-dimensional Euclidean space $\mathbf{R}^n$.

2. DIRECTIONAL SUBDERIVATIVE AND SUPERDERIVATIVE

Let f be a quasidifferentiable function defined on an open set $S \subset \mathbf{R}^n$ and $x \in S$. We denote by $\mathcal{D}f(x)$ the class of all equivalent quasidifferentials of f at x, by $\underline{\mathcal{D}}f(x)$ the family of all subdifferentials of f at x, by $\bar{\mathcal{D}}f(x)$ the family of all superdifferentials of f at x, i.e.,

$$\mathcal{D}f(x) := \{[\underline{\partial} f(x),\, \bar{\partial} f(x)] \mid f'(x : d) = p(d) + q(d) =$$

$$= \max_{v \in \underline{\partial} f(x)} <v,\, d> + \min_{w \in \bar{\partial} f(x)} <w,\, d>, \forall\, d \in \mathbf{R}^n\} \ ,$$

$$\underline{\mathcal{D}}f(x) := \{\underline{\partial} f(x) \mid \exists \text{ a convex compact set } \bar{\partial} f(x) : [\underline{\partial} f(x),\, \bar{\partial} f(x)] \in \mathcal{D}f(x)\} \ ,$$

$$\bar{\mathcal{D}}f(x) := \{\bar{\partial} f(x) \mid \exists \text{ a convex compact set } \underline{\partial} f(x) : [\underline{\partial} f(x),\, \bar{\partial} f(x)] \in \mathcal{D}f(x)\} \ ,$$

where $p(d)$ is a sublinear operator and $q(d)$ is a superlinear operator. According to the definition of quasidifferentiable functions, if f is a quasidifferentiable at x, then its directional derivative at this point in a direction $d \in \mathbf{R}^n$ can be represented as

$$f'(x;\, d) = \max_{v \in \underline{\partial} f(x)} <v,\, d> + \min_{w \in \bar{\partial} f(x)} <w,\, d>$$

or equivalently,

$$f'(x;\, d) = p_1(d) - p_2(d) =$$

$$= \max_{v \in \underline{\partial} f(x)} <v,\, d> - \max_{w \in -\bar{\partial} f(x)} <w,\, d> \ , \tag{2.1}$$

where both of $p_1(d)$ and $p_2(d)$ are sublinear operators. Of the two expressions the latter, the expression (2.1), is convenient sometimes to be used. For instance, necessary conditions given in [6] can be obtained easily in terms of the form (2.1) and [7, Sec. 13], similar to ones in [3, §16].

Let $[\underline{\partial} f(x),\, \bar{\partial} f(x)] \in \mathcal{D}f(x)$. Since

$$(\underline{\partial} f(x) + \bar{\partial} f(x)) - \bar{\partial} f(x) = \underline{\partial} f(x) - (\bar{\partial} f(x) - \bar{\partial} f(x)) \ ,$$

It follows from properties of quasidifferentiable functions that

$$[\underline{\partial} f(x) + \bar{\partial} f(x),\, \bar{\partial} f(x) - \bar{\partial} f(x)] \in \mathcal{D}f(x) \ ,$$

e.g. [2]. Thus the expression (2.1) can be replaced by

$$f'(x;\, d) = \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d> - \max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> \tag{2.2}$$

It is clear that $0 \in \bar{\partial} f(x) - \bar{\partial} f(x)$. Hence, for an $\bar{\partial} f(x) \in \bar{D} f(x)$ the second term on the right hand side of (2.2),

$$\max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> \ ,$$

is always nonnegative. One has

$$f'(x;\, d) \le \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d>, \forall [\underline{\partial} f(x),\, \bar{\partial} f(x)] \in Df(x) \ .$$

Taking the infirmum to the inequality above over $Df(x)$, we obtain

$$f'(x;\, d) \le \inf_{Df(x)} \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d> \ .$$

Define

$$\underline{f}'(x;\, d) := \inf_{Df(x)} \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d> \ .$$

The function $\underline{f}'(x;\, d)$ of $d \in \mathbf{R}^n$ is called the directional subderivative of f at x. On the other hand, since

$$\begin{aligned} \max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> &= \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d> - f'(x;\, d) \\ &\ge \underline{f}'(x;\, d) - f'(x;\, d) \ , \end{aligned}$$

the set

$$\left\{ \max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> \mid \bar{\partial} f(x) \in \bar{D} f(x) \right\}$$

has a finite infirmum for every $d \in \mathbf{R}^n$. By $\bar{f}'(x;\, d)$ we denote it i.e.,

$$\bar{f}'(x;\, d) := \inf_{Df(x)} \max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> \ .$$

It is called the directional superderivative of f at x. Now the directional derivative of f at x in a direction $d \in \mathbf{R}^n$ can be rewritten as

$$\begin{aligned} f'(x;\, d) &= \inf_{Df(x)} \max_{v \in \underline{\partial} f(x) + \bar{\partial} f(x)} <v,\, d> - \inf_{Df(x)} \max_{w \in \bar{\partial} f(x) - \bar{\partial} f(x)} <w,\, d> = \\ &= \underline{f}'(x;\, d) - \bar{f}'(x;\, d) \ . \end{aligned}$$

For the convenience of simplicity, without confusion subderivative and superderivative will be often used instead of directional subderivative and directional superderivative, respectively, later on.

It has been clarified that for every $d \in \mathbf{R}^n$, $\underline{f}'(x;\ d)$ and $\overline{f}'(x;\ d)$ are finite and the superderivative is nonnegative. Furthermore, $\overline{f}'(x;\ \cdot)$ is bounded on $bd\, B_1(0)$, where $B_1(0)$ is the unit ball in $\mathbf{R}^n$ with origin as the center. In fact, since

$$0 \leq \overline{f}'(x;\ b) \leq \max_{w \in \overline{\partial} f(x) - \overline{\partial} f(x)} <w,\ b> \leq$$

$$\leq \max \{\| w \| \mid w \in \overline{\partial} f(x) - \overline{\partial} f(x)\},\ \forall b \in bd\, B_1(0)$$

one has that $\overline{f}'(x;\ \cdot)$ is bounded on $bd\, B_1(0)$. It may be proved from [5, Prop. 1.1] that $f'(x;\ \cdot)$ is bounded on $bd\, B_1(0)$ and Lipschitzian. Therefore, the subderivative $\underline{f}'(x;\ \cdot)$ is bounded on $bd\, B_1(0)$ too.

DEFINITION 2.1 [3, §9] *Let $\wedge$ be an arbitrary set. A family $\{p_\lambda \mid \lambda \in \wedge\}$, where p_λ is a u.c.a. of a function f at x, is called an exhaustive family of u.c.a.s of f at x if*

$$\inf_{\lambda \in \wedge} p_\lambda(d) = f_x{}'(d),\ \forall\, d \in \mathbf{R}^n \ ,$$

where $f_x{}'(d)$ is the same as $f'(x;\ d)$.

DEFINITION 2.2 *Let f be a quasidifferentiable function at x. A family $\hat{\underline{P}}$ such that*

$$\hat{\underline{P}} \subset \underline{P} := \left\{ \max_{v \in \underline{\partial} f(x) + \overline{\partial} f(x)} <v,\ \cdot> \mid [\underline{\partial} f(x),\ \overline{\partial} f(x)] \in \mathcal{D}f(x) \right\} ,$$

$$\underline{f}'(x;\ \cdot) = \inf_{p \in \hat{\underline{P}}} p(\cdot)$$

is referred to as a subexhaustive family of u.c.a.s of f at x, and a family $\hat{\overline{P}}$ such that

$$\hat{\overline{P}} \subset \overline{P} := \left\{ \max_{w \in \overline{\partial} f(x) - \overline{\partial} f(x)} <w,\ \cdot> \mid \overline{\partial} f(x) \in \mathcal{D}f(x) \right\} ,$$

$$\overline{f}'(x;\ \cdot) = \inf_{p \in \hat{\overline{P}}} p(\cdot)$$

is referred to as a superexhaustive family of u.c.a.s of f at x.

PROPOSITION 2.3 *For any quasidifferentiable function f at x there exist a subexhaustive family of u.c.a.s $\underline{P}$ of f at x and a superexhaustive family of u.c.a.s $\overline{P}$ of f at x, such that*

$$f'(x;\ \cdot) = \inf_{p_1 \in \underline{P}} p_1(\cdot) - \inf_{p_2 \in \overline{P}} p_2(\cdot) \ .$$

The Theorem 9.1 in [3] pointed out that for a directionally differentiable function f at a point x if the derivative $f'(x;\cdot)$ is continuous then the function f has exhaustive families of u.c.a.s at x. Contrary, if there exists a bounded exhaustive family of u.c.a.s for a directionally differentiable function f at x then $f'(x;\cdot)$ is continuous. It can be proved. But we will give another proposition below for our purpose.

PROPOSITION 2.4 *Suppose a function f is quasidifferentiable at x. If there exists a bounded subexhaustive family of u.c.a.s of f at x included in $\underline{P}$; then the sub derivative function $\underline{f}'(x;\cdot)$ is Lipschitzian, and if there exists a bounded superexhaustive family of u.c.a.s of f at x included in $\bar{P}$, then the superderivative function $\bar{f}'(x;\cdot)$ is Lipschitzian.*

PROOF Given a $d \in \mathbf{R}^n$. Let $\hat{\underline{P}} \subset \underline{P}$ be a bounded subexhaustive family mentioned in this proposition. We will prove that $\underline{f}'(x;d)$ is Lipschitzian in directions. We choose sequences $\{\hat{p}_i\}_1^\infty \subset \hat{\underline{P}}$ and $\{\epsilon_i > 0\}_1^\infty$ such that

$$\begin{aligned}\lim_{i\to\infty} \hat{p}_i(d) &= \inf_{p\in\hat{\underline{P}}} p(d) = \\ &= \inf_{p\in\underline{P}} p(d) = \\ &= \underline{f}'(x;d) \ ,\end{aligned}$$

$$\epsilon_i \downarrow 0, \ \text{as } i \to \infty \ ,$$

and for any i

$$\inf_{p\in\hat{\underline{P}}} p(d) > \hat{p}_i(d) - \epsilon_i \ . \tag{2.3}$$

Consider the difference $\underline{f}'(x;d+q) - \underline{f}'(x;d)$, where $q \in \mathbf{R}^n$. Since

$$\underline{f}'(x;d+q) \le \hat{p}_i(d+q), \forall i \ ,$$

one has from (2.3) that

$$\begin{aligned}\underline{f}'(x;d+q) - \underline{f}'(x;d) &= \inf_{p\in\hat{\underline{P}}} p(d+q) - \inf_{p\in\hat{\underline{P}}} p(d) \le \\ &\le \hat{p}_i(d+q) - \hat{p}_i(d) + \epsilon_i \le \\ &\le \hat{p}_i(q) + \epsilon_i \\ &\qquad \forall i \ .\end{aligned} \tag{2.4}$$

On the other hand, we make an investigation of the difference $\underline{f}'(x;\, d) - \underline{f}'(x;\, d+q)$ since for any sublinear operator p

$$p(d) \leq p(d+q) + p(-q) \quad ,$$

the following inequality holds

$$\begin{aligned}
\underline{f}'(x;\, d) - \underline{f}'(x;\, d+q) &= \inf_{p \in \underline{\hat{P}}} p(d) - \inf_{p \in \underline{\hat{P}}} p(d+q) \leq \\
&\leq \inf_{p \in \underline{\hat{P}}} p(d) - \inf_{p \in \underline{\hat{P}}} [p(d) - p(-q)] \leq \\
&\leq \inf_{p \in \underline{\hat{P}}} p(d) - \inf_{p \in \underline{\hat{P}}} [p(d) - \sup_{p \in \underline{\hat{P}}} \| p \| \| q \|] \leq \\
&\leq \sup_{p \in \underline{\hat{P}}} \| p \| \| q \| \quad , \\
&\forall\, q \in \mathbf{R}^n \ . \qquad (2.5a)
\end{aligned}$$

Combining (2.4) and (2.5), we get

$$|\, \underline{f}'(x;\, d+q) - \underline{f}'(x;\, d)\,| \leq \max\left\{ \hat{p}_i(q) + \epsilon_i ,\ \sup_{p \in \underline{\hat{P}}} \| p \| \| q \| \right\} \ .$$

Let M be a bound of $\underline{\hat{P}}$. Thus

$$\begin{aligned}
|\, \underline{f}'(x;\, d+q) - \underline{f}'(x;\, d)\,| &\leq \sup_{p \in \underline{\hat{P}}} \| p \| \| q \| + \epsilon_i \leq \\
&\leq M \| q \| + \epsilon_i \quad , \\
&\forall\, i \ . \qquad (2.5b)
\end{aligned}$$

The inequality above holds for any i, so for any $q \in \mathbf{R}^n$ one has

$$|\, \underline{f}'(x;\, d+q) - \underline{f}'(x;\, d)\,| \leq M \| q \| \ .$$

Hence, $\underline{f}'(d;\, \cdot)$ is Lipschitzian. As for $\overline{f}'(x;\, \cdot)$ the proof of the second assertion is the same as the one of $\underline{f}'(x;\, \cdot)$. $\square$

COROLLARY 2.5 *If there exists a bounded subexhaustive family of u.c.a.s of f at x included in $\underline{P}$ (or if there exists a bounded superexhaustive family of u.c.a.s of f at x included in $\overline{P}$), then $\overline{f}'(x;\, \cdot)$ (or $\underline{f}'(x;\, \cdot)$) is Lipschitzian.*

Suppose f is defined on $\mathbf{R}^n$ and is quasidifferentiable, then a necessary condition for a solution $x^* \in \mathbf{R}^n$ of the extremum problem

$$\min_{x \in \mathbf{R}^n} f(x) \tag{2.6}$$

is well known that for any $d \in \mathbf{R}^n$ it must be satisfied

$$f'(x^*;\, d) \geq 0 \ . \tag{2.7}$$

Assume, furthermore, that one of the subderivative and the superderivative of f at x is continuous. It follows from [5, Prop. 1.1] that the other is also continuous. They are nonnegative because of

$$\underline{f}'(x^*;\, \cdot) \geq f'(x^*;\, \cdot) \ .$$

Thus the two derivatives $\underline{f}'(x^*;\, \cdot)$ and $\overline{f}'(x^*;\, \cdot)$ are nonnegative continuous and positively homogeneous. It follows from a theorem, due to [3] or [8], that there exist two star - shaped sets $\underline{\Omega}$ and $\overline{\Omega}$ such that $\underline{f}'(x^*,\, \cdot)$ and $\overline{f}'(x^*;\, \cdot)$ are the gauge functions of $\underline{\Omega}$ and $\overline{\Omega}$, respectively, i.e.,

$$\underline{f}'(x^*;\, \cdot) = |\cdot|_{\underline{\Omega}} \ ,$$

$$\overline{f}'(x^*;\, \cdot) = |\cdot|_{\overline{\Omega}} \ ,$$

where $\underline{\Omega} = \{u \,|\, \underline{f}'(x^*;\, u) \leq 1\}$ and $\overline{\Omega} = \{u \,|\, \overline{f}'(x^*;\, u) \leq 1\}$ and

$$|\, d\,|_{\underline{\Omega}} = inf\{\lambda > 0 \,|\, \underline{f}'(x^*;\, d) \leq \lambda\}$$

or

$$|\, d\,|_{\underline{\Omega}} = inf\{\lambda \,|\, \lambda > 0,\, d \in \lambda \underline{\Omega}\} \ ,$$

i.e., the Minkowskian gauge function. The necessary condition (2.7) can be converted into the following condition

$$|\, d\,|_{\underline{\Omega}} \geq |\, d\,|_{\overline{\Omega}},\, \forall\, d \in \mathbf{R}^n \ .$$

According to the properties of gauges one has

$$\underline{\Omega} \subset \overline{\Omega} \ .$$

Finally, it is easy to know that if the one of $\underline{f}'(x^*;\, \cdot)$ and $\overline{f}'(x^*;\, \cdot)$ is continuous, then for unconstrained optimization the two necessary conditions

$$-\, \bar{\partial} f(x^*) \subset \underline{\partial} f(x^*)$$

and

$$\underline{\Omega} \subset \bar{\Omega}$$

are equivalent. Furthermore, if one of sets $\underline{\Omega}$ and $\bar{\Omega}$ is convex, then one has

$$\underline{\Omega} \subset \bar{\Omega} \subset \underline{\partial} f(x^*) + \bar{\partial} f(x^*)$$

$$\forall [\underline{\partial} f(x^*), \bar{\partial} f(x^*)] \in Df(x^*) \quad ,$$

and

$$\underline{\Omega} \subset \bar{\Omega} \subset \bar{\partial} f(x^*) - \bar{\partial} f(x^*)$$

$$\forall \bar{\partial} f(x^*) \in \bar{D}f(x^*) \quad .$$

In other words,

$$\inf \left\{ \left[\bigcap_{\bar{\partial} f(x^*) \in \bar{D}f(x^*)} (\bar{\partial} f(x^*) - \bar{\partial} f(x^*)) \right] \cap \left[\bigcap_{[\underline{\partial} f(x^*), \bar{\partial} f(x^*)] \in Df(x^*)} (\underline{\partial} f(x^*) + \bar{\partial} f(x^*)) \right] \right\} \neq \emptyset \quad .$$

3. A ⊛ -KERNEL FOR $Df(x)$ WITH A ⊛ -EQUIVALENT BOUNDED SUBFAMILY

Let $\hat{D}f(x)$ be a subfamily of $Df(x)$. $\hat{D}f(x)$ is said to be a ⊛ -equivalent bounded subfamily if the following conditions are satisfied:

(C1) there exists a positive number M such that

$$\underline{\partial} f(x) \cup \bar{\partial} f(x) \subset B_M(0) \quad ,$$

$$\forall [\underline{\partial} f(x), \bar{\partial} f(x)] \in \hat{D}f(x) \quad .$$

where $B_M(0)$ denotes the Euclidean ball in $\mathbf{R}^n$ with the center at origin;

(C2) the subfamily $\{\underline{\partial} f(x) + \bar{\partial} f(x) \mid [\underline{\partial} f(x), \bar{\partial} f(x)] \in \hat{D}f(x)\}$ and the subfamily $\{\bar{\partial} f(x) - \bar{\partial} f(x) \mid \bar{\partial} f(x) \in \hat{\bar{D}}f(x)$, where $\hat{\bar{D}}f(x) = \{\bar{\partial} f(x) \mid \exists\, \underline{\partial} f(x) : [\underline{\partial} f(x), \bar{\partial} f(x)] \in \hat{D}f(x)\}$, form a subexhaustive family and a superexhaustive family of u.c.a.s of f at x, respectively, i.e.,

$$\underline{f}'(x; \cdot) = \inf_{Df(x)} \max_{u \in \underline{\partial} f(x) + \bar{\partial} f(x)} \langle u, \cdot \rangle =$$

$$= \inf_{\hat{D}f(x)} \max_{u \in \underline{\partial} f(x) + \bar{\partial} f(x)} \langle u, \cdot \rangle$$

and

$$\bar{f}'(x;\,\cdot) = \inf_{Df(x)} \max_{u \in \bar{\partial} f(x) - \bar{\partial} f(x)} <u,\,\cdot> =$$

$$= \inf_{\hat{D}f(x)} \max_{u \in \bar{\partial} f(x) - \bar{\partial} f(x)} <u,\,\cdot> \ .$$

For the convenience of discussion without loss of generality assume that the subfamily

$$D_M f(x) := \{[\underline{\partial} f(x),\, \bar{\partial} f(x)] \in Df(x) \,|\, \forall\, u \in \underline{\partial} f(x) \cup \bar{\partial} f(x) : \|\, u \,\| \le M\}$$

is a ⊛ –equivalent bounded subfamily of $Df(x)$, i.e., let

$$\hat{D}f(x) = D_M f(x) \ .$$

Some notations and definitions will be introduced below in order to find a ⊛ –kernel for $Df(x)$. To begin with, define two sets of sequences for any $(u,\, x) \in \mathbf{R}^n \times \mathbf{R}^n$ as follows,

$$\underline{U}(u,\,x) := \left\{ \{u_i\}_1^\infty \,\middle|\, \begin{array}{c} \exists \{[\underline{\partial}_i f(x),\, \bar{\partial}_i f(x)]\}_1^\infty \subset D_M f(x) \,\exists \{d_i\} \subset \mathbf{R}^n : \\ d_i \to d \in \mathbf{R}^n, \text{ as } i \to \infty \ , \\ u_i \to u \in \mathbf{R}^n, \text{ as } i \to \infty \ , \\ u_i \in \operatorname*{Arg\,max}_{u \in \underline{\partial}_i f(x) + \bar{\partial}_i f(x)} <u,\, d_i> \ , \\ <u,\, d> = \lim_{i \to \infty} <u_i,\, d_i> = \underline{f}'(x;\, d) \end{array} \right\} \tag{3.1}$$

and

$$\bar{U}(u,\,x) := \left\{ \{u_i\}_1^\infty \,\middle|\, \begin{array}{c} \exists \{[\underline{\partial}_i f(x),\, \bar{\partial}_i f(x)]\}_1^\infty \subset D_M f(x) \,\exists \{d_i\} \subset \mathbf{R}^n : \\ d_i \to d \in \mathbf{R}^n, \text{ as } i \to \infty \ , \\ u_i \to u \in \mathbf{R}^n, \text{ as } i \to \infty \ , \\ u_i \in \operatorname*{Arg\,max}_{u \in \bar{\partial}_i f(x) - \bar{\partial}_i f(x)} <u,\, d_i> \ , \\ <u,\, d> = \lim_{i \to \infty} <u_i,\, d_i> = \bar{f}'(x;\, d) \end{array} \right\} \tag{3.2}$$

Define

$$\underline{U}_0'(x) := \{u \in \mathbf{R}^n \,|\, \underline{U}(u,\,x) \ne \emptyset\}$$

and

$$\bar{U}_0'(x) := \{u \in \mathbf{R}^n \,|\, \bar{U}(u,\,x) \ne \emptyset\} \ .$$

Let $\underline{\cup}_0(x)$ and $\bar{\cup}_0(x)$ be the smallest equivalent subsets of $\underline{\cup}_0'(x)$ and $\bar{\cup}_0'(x)$, respectively, where "equivalent" means that, for instance,

$$\forall\, b \in bd\, B(0,\, 1)\, \exists\, u \in \underline{\cup}_0(x):$$

$$<u,\, b> \,=\, \underline{f}'(x;\, b) \quad ,$$

and "smallest" means that, for instance, for any equivalent subset $\underline{\cup}_0''(x)$ of $\underline{\cup}_0'(x)$ one has

$$co\, \underline{\cup}_0(x) \subset co\, \underline{\cup}_0''(x) \quad .$$

$\underline{\cup}(x)$ and $\bar{\cup}(x)$ are defined by convex hulls, of $\underline{\cup}_0(x)$ and $\bar{\cup}_0(x)$ respectively. It is obvious that for any $d \in \mathbf{R}^n$ there exist vectors $\underline{u} \in \underline{\cup}(x)$, $\bar{u} \in \bar{\cup}(x)$ such that

$$<\underline{u},\, d> \,=\, \underline{f}'(x;\, d)$$

and

$$<\bar{u},\, d> \,=\, \bar{f}'(x;\, d)$$

because of the boundedness of $\mathcal{D}_M f(x)$. Hence, $\underline{\cup}(x)$ and $\bar{\cup}(x)$ are nonempty, bounded and convex. The following new functions are necessary to be introduced. The new functions $\underline{\varphi}(\cdot \circledast\!-\, \cdot)$ and $\bar{\varphi}(\cdot \circledast\!-\, \cdot)$ (simply, $\circledast$ –operation) are defined as follows.

$$\underline{\varphi}(\cdot \circledast \cdot): \mathbf{R}^n \times \mathbf{R}^n \to \overline{\mathbf{R}} = [-\infty,\, +\infty]$$

represented by

$$\underline{\varphi}(u \circledast d) = \inf_{\substack{\{\tilde{u}_j\}_1^\infty \in \Sigma \lambda_i \underline{U}(u_i,\, x) \\ u = \Sigma \lambda_i u_i,\ u_i \in \underline{\cup}_0(x) \\ \Sigma \lambda_i = 1,\ \lambda_i \geq 0}} \ \liminf_{\tilde{u}_j \to u} \ \max_{\substack{u \in \Sigma \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)] \\ (\tilde{u}_j \in \Sigma \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)])}} <u,\, d>$$

$$\forall\, (u,\, d) \in \underline{\cup}(x) \times \mathbf{R}^n \tag{3.3}$$

and

$$\bar{\varphi}(\cdot \circledast \cdot): \mathbf{R}^n \times \mathbf{R}^n \to \overline{\mathbf{R}} = [-\infty,\, +\infty]$$

represented by

$$\bar{\varphi}(u \circledast d) = \inf_{\substack{\{\tilde{u}_j\} \in \Sigma \lambda_i \bar{U}(u_i,\, x) \\ u = \Sigma \lambda_i u_i,\ u_i \in \bar{\cup}_0(x) \\ \Sigma \lambda_i = 1,\ \lambda_i \geq 0}} \ \liminf_{\tilde{u}_j \to u} \ \max_{\substack{u \in \Sigma \lambda_i [\bar{\partial}_{ij} f(x) - \bar{\partial}_{ij} f(x)] \\ (\tilde{u}_j \in \Sigma \lambda_i [\bar{\partial}_{ij} f(x) - \bar{\partial}_{ij} f(x)])}} <u,\, d>$$

$$\forall\, (u,\, d) \in \bar{\cup}(x) \times \mathbf{R}^n \quad . \tag{3.4}$$

Define $\underline{\varphi}(u \circledast \cdot) = -\infty$ if $u \notin \underline{\cup}(x)$, and $\bar{\varphi}(u \circledast \cdot) = -\infty$ if $u \notin \bar{\cup}(x)$. Obviously, $\underline{\varphi}(u \circledast \cdot)$ and $\bar{\varphi}(u \circledast \cdot)$ are positively homogeneous, i.e., $\underline{\varphi}(u \circledast \lambda\, \cdot) = \lambda\, \underline{\varphi}(u \circledast \cdot)$ and

$\bar{\varphi}(u \circledast \lambda \cdot) = \lambda \bar{\varphi}(u \circledast \cdot)$, where $\lambda > 0$. For the convenience of writing the form

$$<u \circledast d>$$

will be used instead of the forms $\underline{\varphi}(u \circledast d)$ and $\bar{\varphi}(u \circledast d)$ from time to time.

LEMMA 3.1 $\quad$ *$<\cdot \circledast d>$ is convex in $\underline{\cup}(x)$ and $\bar{\cup}(x)$, respectively.*

PROOF $\quad$ Let $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Suppose u^1 and u^2 are in $\underline{\cup}(x)$. Let $u = \alpha u^1 + \beta u^2$. Since

$$T = \left\{ \Sigma \lambda_i \underline{U}(u_i, x) \;\middle|\; \begin{array}{l} u = \Sigma \lambda_i u_i, \\ u_i \in \underline{\cup}_0(x) \\ \Sigma \lambda_i = 1,\ \lambda_i \geq 0 \end{array} \right\} \supset$$

$$\alpha T_1 + \beta T_2 = \left\{ \begin{array}{l} \alpha \Sigma \lambda_i^1 \underline{U}(u_i^1, x) + \\ \\ \beta \Sigma \lambda_i^2 \underline{U}(u_i^2, x) \end{array} \;\middle|\; \begin{array}{l} u^1 = \Sigma \lambda_i^1 u_i^1,\ u_i^1 \in \underline{\cup}_0(x) \\ u^2 = \Sigma \lambda_i^2 u_i^2,\ u_i^2 \in \underline{\cup}_0(x) \\ \Sigma \lambda_i^1 = 1,\ \lambda_i^1 \geq 0 \\ \Sigma \lambda_i^2 = 1,\ \lambda_i^2 \geq 0 \end{array} \right\}, \tag{3.5}$$

one has

$$\begin{aligned} \underline{\varphi}((\alpha u^1 + \beta u^2) \circledast d) &= \inf_{\{\tilde{u}_j\} \in T} [\cdots] \leq \\ &\leq \inf_{\{\tilde{u}_j\} \in \alpha T_1 + \beta T_2} [\cdots] = \\ &= \inf_{\{\tilde{u}_j\} \in \alpha T_1 + \beta T_2} \ \liminf_{\tilde{u}_j \to u} \\ &\quad \max_{u \in \alpha \sum_i \lambda_i^1 (\underline{\partial}_{ij}^1 f(x) + \bar{\partial}_{ij}^1 f(x)) + \beta \sum_i \lambda_i^2 (\underline{\partial}_{ij}^2 f(x) + \bar{\partial}_{ij}^2 f(x))} <u, d> = \\ &= \alpha \underline{\varphi}(u^1 \circledast d) + \beta \underline{\varphi}(u^2 \circledast d) \ . \end{aligned}$$

In other words,

$$<(\alpha u^1 + \beta u^2) \circledast d> \leq \alpha <u^1 \circledast d> + \beta <u^2 \circledast d> \ ,$$

$<u \circledast d>$ is convex in $\underline{\cup}(x)$ and $\bar{\cup}(x)$. $\square$

LEMMA 3.2 *For any $u \in \underline{\cup}(x)$ the relation*

$$<u,\, d> \;\leq\; <u \circledast d>, \forall\, d \in \mathbf{R}^n \tag{3.6}$$

is always true.

PROOF Since in (3.3) the inequality

$$\max_{u \in \Sigma \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)]} <u,\, d> \;\geq\; <\tilde{u}_j,\, d>, \forall\, j \in \{1,\, 2, \ldots\} \forall\, d \in \mathbf{R}^n$$

is always true and $\lim\limits_{j \to \infty} \tilde{u}_j = u$, one has

$$\underline{\varphi}(u \circledast d) \;\geq\; <u,\, d> \quad .$$

This is what we want. □

LEMMA 3.3 *For any $d \in \mathbf{R}^n$ we have*

$$\underline{f}'(x;\, d) = \min_{u \in \underline{\cup}(x)} <u \circledast d>$$

and

$$\bar{f}'(x;\, d) = \min_{u \in \bar{\cup}(x)} <u \circledast d>$$

PROOF Since for any

$$\{\tilde{u}_j\}_1^\infty \in \Sigma\, \lambda_i \underline{U}(u_i,\, x)$$

such that $u = \Sigma\, \lambda_i u_i$, $u_i \in \underline{\cup}_0(x)$, $\Sigma\, \lambda_i = 1$, $\lambda_i \geq 0$, we have

$$\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x) \in \underline{D} f(x) \quad ,$$

it follows from the properties of quasidifferentiable functions that

$$\Sigma\, \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)] \in \underline{D} f(x) \quad .$$

Therefore for any $d \in \mathbf{R}^n$

$$\inf_{\substack{\{\tilde{u}_j\}_1^\infty \in \Sigma \lambda_i \underline{U}(u_i,\, x) \\ \cdots}} \left[\liminf_{\tilde{u}_j \to u} \max_{u \in \Sigma \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)]} <u,\, d>\right] \geq$$

$$\geq \inf_{\substack{\{\tilde{u}_j\}_1^\infty \in \Sigma \lambda_i \underline{U}(u_i,\, x) \\ \cdots}} \left[\liminf_{\tilde{u}_j \to u} \underline{f}'(x;\, d)\right] =$$

$$\lambda_{ij} \to \lambda_j, \text{ as } i \to \infty, \; j = 1, 2, \ldots, n+1 \;,$$

$$\Sigma\, \lambda_j = 1 \;,$$

d_{ij} (it can be replaced by b_{ij}, where $b_{ij} \in bd\, B_1(0)$) converges to d_j, as $i \to \infty$. Taking a sequence $\{\epsilon_i > 0\} \downarrow 0$, for each u_{ij} one can choose an element u_{ij}^k such that

$$< u_{ij}^k,\, d_{ij}^k > \; \geq \underline{f}'(x;\, d_{ij}^k) \geq \; < u_{ij}^k,\, d_{ij}^k > - \; \epsilon_i,\; k \geq i \;,$$

$$u_{ij}^k \in \operatorname*{Arg\,max}_{u \in \underline{\partial}_{ij}^k f(x) + \bar{\partial}_{ij}^k f(x)} < u,\, d_{ij}^k >$$

$$d_{ij}^k \to d_j, \text{ as } i \to \infty$$

$$u_{ij}^k \to u_j, \text{ as } i \to \infty \;.$$

Thus

$$\lim_{i \to \infty} < u_{ij}^k,\, d_{ij}^k > \; = \lim_{i \to \infty} \underline{f}'(x;\, d_{ij}^k) \;.$$

It follows from Prop. 2.4 that

$$< u_j,\, d_j > \; = \underline{f}'(x;\, d_j) \;.$$

Finally u can be represented as

$$u = \Sigma\, \lambda_j u_j \;,$$

where $u_j \in \underline{\cup}_0(x)$. This shows that $\underline{\cup}_0(x)$ and $\bar{\cup}_0(x)$ are closed. Because our discussion is confined within $\mathbf{R}^n$, $\underline{\cup}_0(x)$ and $\bar{\cup}_0(x)$ are compact. □

From the lemmas given above we obtain the following theorem.

THEOREM 3.5 *For any quasidifferentiable function f define don some open set S with a $\circledast$ -equivalent bounded subfamily there exists a pair of nonempty compact convex sets, $\underline{\cup}(x)$, $\bar{\cup}(x)$, at each point $x \in S$ such that*

$$f'(x;\, d) = \min_{u \in \underline{\cup}(x)} < u \circledast d > - \min_{u \in \bar{\cup}(x)} < u \circledast d > \;, \tag{3.9}$$

$$\forall\, d \in \mathbf{R}^n$$

where $< \cdot \circledast d >$ are convex functions in $\underline{\cup}(x)$ and $\bar{\cup}(x)$, respectively.

The expression of the directional derivative (3.9) of f and x in a direction $d \in \mathbf{R}^n$ can be represented as a form of the Euclidean inner product. In fact, let

$$\underline{M}(u) := co \left\{ w \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \exists \{w_k\}_1^\infty \exists b \in bd\, B_1(0): \\ w = \lim\limits_{k \to \infty} w_k , \\ <w,\, b> = \underline{\varphi}(u \circledast b) , \\ w_j \in \operatorname*{Arg\,max}\limits_{u \in \sum\limits_i \lambda_i [\underline{\partial}_{ij} f(x) + \bar{\partial}_{ij} f(x)]} <u,\, b> \quad \text{in}(3.3) \end{array} \right\},$$

where $u \in \underline{\cup}(x)$, and let

$$\bar{M}(u) := co \left\{ w \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \exists \{w_k\}_1^\infty \exists b \in bd\, B_1(0): \\ w = \lim\limits_{k \to \infty} w_k , \\ <w,\, b> = \bar{\varphi}(u \circledast b) , \\ w_j \in \operatorname*{Arg\,max}\limits_{u \in \sum\limits_i \lambda_i [\bar{\partial}_{ij} f(x) - \bar{\partial}_{ij} f(x)]} <u,\, b> \quad \text{in}(3.4) \end{array} \right\},$$

where $u \in \bar{\cup}(x)$. It may be checked that for each $u \in \underline{\cup}(x)$ one has

$$\underline{\varphi}(u \circledast d) = <w,\, d> = \max_{w \in \underline{M}(u)} <w,\, d>, \forall\, d \in \mathbf{R}^n \tag{3.10}$$

and for each $u \in \bar{\cup}(x)$ one has

$$\bar{\varphi}(u \circledast d) = <w,\, d> = \max_{w \in \bar{M}(u)} <w,\, d>, \forall\, d \in \mathbf{R}^n \ . \tag{3.11}$$

Thus (3.9) can be converted into

$$f'(x;\, d) = \min_{u \in \underline{\cup}(x)} \max_{w \in \underline{M}(u)} <w,\, d> - \min_{u \in \bar{\cup}(x)} \max_{w \in \bar{M}(u)} <w,\, d> \ . \tag{3.12}$$

It tuns out that $f'(x;\, d)$ may be decomposed as a difference of two minimaxes.

DEFINITION 3.6 $\underline{\cup}(x)$ *and* $\bar{\cup}(x)$ *are called the* $\circledast$ *-subkernel and* $\circledast$ *-superkernel of quasidifferentials of* f *at* x*, denoted, respectively, by* $\partial_{\circledast} f(x)$ *and* $\partial^{\circledast} f(x)$*. The pair* $[\partial_{\circledast} f(x),\, \partial^{\circledast} f(x)]$ *is referred to as the* $\circledast$ *-kernel of quasidifferentials of* f *at* x*, denoted by* $(D \circledast f)(x)$ *or* $D \circledast f(x)$.

We now have that

$$\underline{f}'(x;\, d) = \min_{u \in \partial_{\circledast} f(x)} <u \circledast d> =$$

$$= \min_{u \in \partial \circledast f(x)} \max_{w \in \bar{M}(u)} <w,\, d> \ .$$

and

$$\bar{f}'(x;\, d) = \min_{u \in \partial^{\circledast} f(x)} < u \circledast d>$$

$$= \min_{u \in \partial^{\circledast} f(x)} \max_{w \in \bar{M}(u)} < w,\, d> \ .$$

EXAMPLES 3.7 *Some examples could be referred to in [9].*

PROPOSITION 3.8 *Let f_1 and f_2 be quasidifferentiable at x with $\circledast$ -equivalent bounded subfamilies, λ be a scalar. Then*

$$\partial_{\circledast}(f_1 + f_2)(x) = \partial_{\circledast} f_1(x) + \partial_{\circledast} f_2(x) \ ,$$

$$\partial^{\circledast}(f_1 + f_2)(x) = \partial^{\circledast} f_1(x) + \partial^{\circledast} f_2(x) \ ,$$

i.e.,

$$D \circledast (f_1 + f_2)(x) = D \circledast f_1(x) + D \circledast f_2(x) \ ,$$

and

$$\partial_{\circledast}(\lambda f_1)(x) = \lambda\, \partial_{\circledast} f_1(x) \ ,$$

$$\partial^{\circledast}(\lambda f)(x) = \begin{cases} \lambda\, \partial^{\circledast} f_1(x), & \lambda \geq 0 \\ |\lambda|\, \partial^{\circledast}(-\, f_1)(x), & \lambda < 0 \end{cases}$$

i.e.,

$$D \circledast (\lambda f_1)(x) = |\lambda|\, D \circledast ((\operatorname{sign} \lambda) f_1)(x) \ .$$

COROLLARY 3.9 *Let $f = \Sigma\, \lambda_i f_i$. Then one has*

$$D \circledast (\Sigma\, \lambda_i f_i)(x) = \Sigma\, |\lambda_i|\, D \circledast ((\operatorname{sign} \lambda_i) f_i)(x) \ .$$

REMARK 1 For each $\underline{u} \in \partial_{\circledast} f(x)$ and each $\bar{u} \in \partial^{\circledast} f(x)$, the functions $\underline{\varphi}(\underline{u} \circledast \cdot)$ and $\bar{\varphi}(\bar{u} \circledast \cdot)$ are Lipschitzian.

2 For each $d \in \mathbf{R}^n$ the functions $\underline{\varphi}(\cdot \circledast d)$ and $\bar{\varphi}(\cdot \circledast d)$ are proper upper semicontinuous.

3 The conditions

$$<u,\, d> = \lim_{i \to \infty} <u_i,\, d_i> = \underline{f}'(x;\, d) \ ,$$

$$<u,\, d> = \lim_{i \to \infty} <u_i,\, d_i> = \bar{f}'(x;\, d)$$

in (3.1) and (3.2), respectively, can be omitted and the sequence $\{d_i\}$ convergent to d can be replaced only by d. But in this case it is necessary that

$$\underline{U}(x) = co\, \underline{U}_0(x),\, \overline{U}(x) = co\, \overline{U}_0(x)$$

are replaced by

$$\underline{U}(x) = cl\, co\, \underline{U}_0(x),\, \overline{U}(x) = cl\, co\, \overline{U}_0(x) \ ,$$

respectively.

4. OTHER RESULTS

The function of $d \in \mathbf{R}^n$

$$\underline{f}'(x;\, \cdot) - \bar{f}'(x;\, \cdot)$$

is directional differentiable at origin and

$$(\underline{f}'(x;\, \cdot) - \bar{f}'(x;\, \cdot))'(0;\, d) =$$

$$= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left\{ [\underline{f}'(x;\, \lambda d) - \bar{f}'(x;\, \lambda d)] - [\underline{f}'(x;\, 0) - \bar{f}'(x;\, 0)] \right\}$$

$$= \underline{f}'(x;\, d) - \bar{f}'(x;\, d) \ .$$

Let x^* be a minimum point. Since $\underline{f}'(x;\, \cdot)$ and $\bar{f}'(x;\, \cdot)$ are Lipschitzian and

$$(\underline{f}'(x;\, \cdot) - \bar{f}'(x;\, \cdot))^0(0;\, d) \geq \underline{f}'(x;\, d) - \bar{f}'(x;\, d) \ ,$$

where $(\cdots)^0(0;\, d)$ means the generalized directional derivative at origin in a direction $d \in \mathbf{R}^n$ in the Clarke's sense [1], one has

$$\delta^*(\cdot \mid \partial_c[\underline{f}'(x^*;\, \cdot) - \bar{f}'(x^*;\, \cdot)](0)) \geq 0 \ .$$

i.e.,

$$0 \in \partial_c[\underline{f}'(x^*;\, \cdot) - \bar{f}'(x^*;\, \cdot)](0)$$

where $\partial_c f$ is the Clarke's generalized gradient of f. Thus

$$\partial_c \underline{f}'(x^*; \cdot)(0) \cap \partial_c \overline{f}'(x^*; \cdot)(0) \neq \emptyset \ . \tag{4.1}$$

For any $(u,\, d) \in \partial_{\circledast} f(x) \times \mathbf{R}^n$ we have

$$(\underline{\varphi}(u \circledast \cdot))'(0;\, d) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} [\underline{\varphi}(u \circledast \lambda d) - \underline{\varphi}(u \circledast 0)] =$$

$$= \underline{\varphi}(u \circledast d) \ .$$

Since for any $(u,\, d) \in \partial \circledast f(x) \times \mathbf{R}^n$

$$\underline{\varphi}(u \circledast d) \leq (\underline{\varphi}(u \circledast \cdot))^0(0;\, d) =$$

$$= \limsup_{\substack{\lambda \downarrow 0 \\ d' \to 0}} \frac{1}{\lambda} [\underline{\varphi}(u \circledast (d' + \lambda d)) - \underline{\varphi}(u \circledast d')] =$$

$$= \limsup_{\substack{\lambda \downarrow 0 \\ d' \to 0}} \frac{1}{\lambda} [\max_{w \in \underline{M}(u)} <w,\, d' + \lambda d> - \max_{w \in \underline{M}(u)} <w,\, d'>] \leq$$

$$\leq \limsup_{\substack{\lambda \downarrow 0 \\ d' \to 0}} \frac{1}{\lambda} \max_{w \in \underline{M}(u)} <w,\, \lambda d> \leq$$

$$\leq \max_{w \in \underline{M}(u)} <w,\, d> =$$

$$= \underline{\varphi}(u \circledast d) \ ,$$

one has

$$\partial_c \underline{\varphi}(u \circledast \cdot)(0) = \underline{M}(u)$$

We now have the following theorem.

THEOREM 4.1 *Let x^* be a local minimum point for an unconstrained problem. Then (4.1) holds, and furthermore for any $\underline{u}$ and any $\bar{u}$ such that*

$$\underline{\varphi}(\underline{u} \circledast d) = \min_{u \in \partial_{\circledast} f(x)} \underline{\varphi}(u \circledast d) =$$

$$= \min_{u \in \partial_{\circledast} f(x)} \max_{w \in \partial_c \underline{\varphi}(u \circledast \cdot)(0)} <w,\, d>$$

and

$$\bar{\varphi}(\bar{u} \circledast d) = \min_{u \in \partial^{\circledast} f(x)} \bar{\varphi}(u \circledast d) =$$

$$= \min_{u \in \partial^{\circledast} f(x)} \max_{w \in \partial_c \bar{\varphi}(u \circledast \cdot)(0)} <w, d> \ ,$$

the inclusion relation

$$\partial_c \underline{\varphi}(\underline{u} \circledast \cdot)(0) \cap \partial_c \bar{\varphi}(\bar{u} \circledast \cdot)(0) \neq \emptyset \ ,$$

holds, i.e.,

$$\underline{M}(\underline{u}) \cap \bar{M}(\bar{u}) \neq \emptyset$$

holds. □

The following lemma is easy to be deduced in terms of Lem. 3.3, (3.10), (3.11) and the definitions of $\circledast$ –operations.

LEMMA 4.2 *Suppose $u \in \mathbf{R}^n$. This lemma consists of:*

1 $u \in \partial_{\circledast} f(x)$ *if and only if*

$$\underline{M}(u) \neq \emptyset, \text{ i.e., } \partial_c \underline{\varphi}(u \circledast \cdot)(0) \neq \emptyset \ ,$$

and $u \in \partial^{\circledast} f(x)$ *if and only if*

$$\bar{M}(u) \neq \emptyset, \text{ i.e., } \partial_c \bar{\varphi}(u \circledast \cdot)(0) \neq \emptyset \ ;$$

2 $u \in \partial_{\circledast} f(x)$ *if and only if*

$$u \in \underline{M}(u), \text{ i.e., } u \in \partial_c \underline{\varphi}(u \circledast \cdot)(0) \ ,$$

and $u \in \partial^{\circledast} f(x)$ *if and only if*

$$u \in \bar{M}(u), \text{ i.e., } u \in \partial_c \bar{\varphi}(u \circledast \cdot)(0) \ ;$$

3 $w \in \underline{M}(u)$ *(or $\partial_c \underline{\varphi}(u \circledast \cdot)(0)$) if and only if for any $d \in \mathbf{R}^n$ the inequality*

$$\underline{\varphi}(u \circledast d) \geq <w, d>$$

holds, and $w \in \bar{M}(u)$ (or $\partial_c \bar{\varphi}(u \circledast \cdot)(0)$) if and only if the inequality

$$\bar{\varphi}(u \circledast d) \geq <w, d>, \forall\, d \in \mathbf{R}^n$$

holds;

4 $u \in \partial_{\circledast} f(x)$ *if and only if*

$$\underline{f}'(x;\, d) \le \max_{w \in \underline{M}(u)} <w,\, d> =$$

$$= \max_{w \in \partial_c \underline{\varphi}(u \circledast \cdot)(0)} <w,\, d>,\, \forall\, d \in \mathbf{R}^n \tag{4.2}$$

and $u \in \partial^{\circledast} f(x)$ *if and only if*

$$\bar{f}'(x;\, d) \le \max_{w \in \bar{M}(u)} <w,\, d> =$$

$$= \max_{w \in \partial_c \bar{\varphi}(u \circledast \cdot)(0)} <w,\, d>,\, \forall\, d \in \mathbf{R}^n \ . \tag{4.3}$$

□

By $\underline{\varphi}(x,\, u \circledast d)$ we replace $\underline{\varphi}(u \circledast d)$ when x varies. $\underline{\varphi}(u \circledast d)$ is used in the case where our discussions concerned is restricted at a point. As for $\underline{M}(x,\, u)$ the use is the same as that of $\underline{\varphi}(x,\, u \circledast d)$.

PROPOSITION 4.3 *Suppose* $\underline{\varphi}(x,\, u \circledast d)$ *is upper semicontinuous in* $(x,\, u) \in S \times \partial_{\circledast} f(x)$ *for each* $d \in \mathbf{R}^n$. *If* $\mathcal{D}_M f(x)$ *is bounded uniformly in a neighborhood of* x, $N_x(\delta)$, *then* $\partial_{\circledast} f(\cdot)$ *is upper semicontinuous in* $N_x(\delta)$.

PROPOSITION 4.4 *Suppose* $\underline{f}'(x\colon d)$ *is lower semicontinuous in* $x \in S$ *for each* $d \in \mathbf{R}^n$. *If* $\mathcal{D}_M f(x)$ *is bounded uniformly in a neighborhood of* x, $N_x(\delta)$, *and* $\underline{M}(x,\, u)$ *is upper semicontinuous in* $(x,\, u) \in S \times \partial_{\circledast} f(x)$ *for each* $d \in \mathbf{R}^n$, *then* $\partial_{\circledast} f(\cdot)$ *is upper semicontinuos in* $N_x(\delta)$.

For $\partial^{\circledast} f(\cdot)$ we have similar assertions. Given an interval $[x,\, y] \subset \mathbf{R}^n$, where $x \neq y$, a Mean Value Theorem can be obtained: there exists $\xi \in (0,\, 1)$ such that

$$f(y) - f(x) = <\underline{u} - \bar{u},\, y - x> \ ,$$

where $\underline{u} \in \partial_{\circledast} f(x + \xi(y - x))$ and $\bar{u} \in \partial^{\circledast} f(x + \xi(y - x))$, or

$$f(y) - f(x) \in <\partial_{\circledast} f(x + \xi(y - y)) - \partial^{\circledast} f(x + \xi(y - x)),\, y - x> \ .$$

Let $\underline{\Psi}(u,\, d)$ and $\bar{\Psi}(u,\, d)$ be the functions

$$\underline{\Psi}(u,\, d) = \begin{cases} \underline{\varphi}(u \circledast d), & u \in \partial_{\circledast} f(x) \\ +\infty \ , & \text{otherwise} \end{cases}$$

holds;

4 $u \in \partial_{\circledast} f(x)$ *if and only if*

$$\underline{f}'(x;\, d) \le \max_{w \in \underline{M}(u)} <w,\, d> =$$

$$= \max_{w \in \partial_c \underline{\varphi}(u \circledast \cdot)(0)} <w,\, d>,\ \forall\, d \in \mathbf{R}^n \tag{4.2}$$

and $u \in \partial^{\circledast} f(x)$ *if and only if*

$$\bar{f}'(x;\, d) \le \max_{w \in \bar{M}(u)} <w,\, d> =$$

$$= \max_{w \in \partial_c \bar{\varphi}(u \circledast \cdot)(0)} <w,\, d>,\ \forall\, d \in \mathbf{R}^n\ . \tag{4.3}$$

□

By $\underline{\varphi}(x,\, u \circledast d)$ we replace $\underline{\varphi}(u \circledast d)$ when x varies. $\underline{\varphi}(u \circledast d)$ is used in the case where our discussions concerned is restricted at a point. As for $\underline{M}(x,\, u)$ the use is the same as that of $\underline{\varphi}(x,\, u \circledast d)$.

PROPOSITION 4.3 *Suppose* $\underline{\varphi}(x,\, u \circledast d)$ *is upper semicontinuous in* $(x,\, u) \in S \times \partial_{\circledast} f(x)$ *for each* $d \in \mathbf{R}^n$. *If* $D_M f(x)$ *is bounded uniformly in a neighborhood of* x, $N_x(\delta)$, *then* $\partial_{\circledast} f(\cdot)$ *is upper semicontinuous in* $N_x(\delta)$.

PROPOSITION 4.4 *Suppose* $\underline{f}'(x\colon d)$ *is lower semicontinuous in* $x \in S$ *for each* $d \in \mathbf{R}^n$. *If* $D_M f(x)$ *is bounded uniformly in a neighborhood of* x, $N_x(\delta)$, *and* $\underline{M}(x,\, u)$ *is upper semicontinuous in* $(x,\, u) \in S \times \partial_{\circledast} f(x)$ *for each* $d \in \mathbf{R}^n$, *then* $\partial_{\circledast} f(\cdot)$ *is upper semicontinuos in* $N_x(\delta)$.

For $\partial^{\circledast} f(\cdot)$ we have similar assertions. Given an interval $[x,\, y] \subset \mathbf{R}^n$, where $x \ne y$, a Mean Value Theorem can be obtained: there exists $\xi \in (0,\, 1)$ such that

$$f(y) - f(x) = <\underline{u} - \bar{u},\, y - x>\ ,$$

where $\underline{u} \in \partial_{\circledast} f(x + \xi(y - x))$ and $\bar{u} \in \partial^{\circledast} f(x + \xi(y - x))$, or

$$f(y) - f(x) \in <\partial_{\circledast} f(x + \xi(y - y)) - \partial^{\circledast} f(x + \xi(y - x)),\, y - x>\ .$$

Let $\underline{\Psi}(u,\, d)$ and $\bar{\Psi}(u,\, d)$ be the functions

$$\underline{\Psi}(u,\, d) = \begin{cases} \underline{\varphi}(u \circledast d), & u \in \partial_{\circledast} f(x) \\ +\infty\ , & \text{otherwise} \end{cases}$$

$$\bar{\Psi}(u, d) = \begin{cases} \bar{\varphi}(u \circledast d), & u \in \partial^{\circledast} f(x) \\ +\infty , & \text{otherwise} , \end{cases}$$

respectively.

REMARK Let $s = \mathbf{R}^n$ and $x \in \mathbf{R}^n$. The functions $\underline{\Psi}(u, d)$ and $\bar{\Psi}(u, d)$ are closed proper convex functions for each $d \in \mathbf{R}^n$. Their conjugate functions are

$$\underline{\Psi}^*(u^*, d) = \sup_u \{<u, u^*> - \underline{\Psi}(u, d)\}$$

and

$$\bar{\Psi}^*(u, d) = \sup_u \{<u, u^*> - \bar{\Psi}(u, d)\} ,$$

respectively. It is enough to discuss $\underline{\Psi}(\cdot, d)$ and $\underline{\Psi}^*(\cdot^*, d)$. The minimum set of $\underline{\Psi}(\cdot, d)$ is a non-empty bounded set. According to the Th. 27.1 in [7], one has

$$0 \in \operatorname{int}(\operatorname{dom} \underline{\Psi}^*(\cdot, d)) .$$

In addition, all of the cluster points of a sequence $\{u_i\}$ such that

$$\underline{\Psi}(u_i, d) \xrightarrow[i \to \infty]{} \underline{f}'(x; d)$$

are in the minimum set of $\underline{\varphi}(\cdot \circledast d)$, [7, Corol. 27. 2.1]. Since for $u^* = d$ we have

$$\begin{aligned} \sup_u \{<u, u^*> - \underline{\Psi}(u, d)\} &= \max_{u \in \mathbf{R}^n} \{<u, u^*> - \underline{\Psi}(u, d)\} = \\ &= <\underline{u}, d> - \underline{\varphi}(u \circledast d) = \\ &= 0 , \end{aligned}$$

where $\underline{u}$ is such that $\underline{u} \in \partial_{\circledast} f(x)$ and

$$\begin{aligned} <\underline{u}, d> = \underline{\varphi}(\underline{u} \circledast d) &= \min_{u \in \partial_{\circledast} f(x)} \underline{\varphi}(u \circledast d) = \\ &= \min_{u \in \mathbf{R}^n} \underline{\Psi}(u, d) = \underline{f}'(x; d) , \end{aligned} \tag{4.4}$$

one has

$$u^* = d \in \partial \underline{\Psi}(\cdot, d)(\underline{u}) ,$$

[7, Th. 23.5]. Since $\Psi(\cdot, d)$ is closed, the relation

$$\underline{u} \in \partial \underline{\Psi}^*(\cdot^*, d)(d)$$

holds, Moreover we have

$$\underline{u} \in \partial \underline{\Psi}^*(\cdot^*, d)(0) \ .$$

It is clear that

$$\partial \underline{\Psi}^*(\cdot^*, d)(0) \subset \partial \underline{\Psi}^*(\cdot^*, d)(d)$$

because of the minimum set of $\underline{\Psi}(\cdot, d)$ being $\partial \underline{\Psi}^*(\cdot^*, d)(0)$, and

$$0 \in \partial \underline{\Psi}(\cdot, d)(\underline{u}) \ .$$

Therefore

$$[0, d] \in \partial \underline{\Psi}(\cdot, d)(\underline{u}) \ .$$

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