Working Paper

NON-LINEAR URN PROCESSES: ASYMPTOTIC BEHAVIOR AND APPLICATIONS

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FOREWORD

Adaptive (path dependent) processes of growth modeled by urn schemes are important for several fields of applications: biology, physics, chemistry, economics.

In this paper the authors continue their previous investigations of generalized urn schemes with balls of different colors and path-dependent increments. They consider processes with random additions of balls at each time. These processes are similar to the branching processes, however the classical theory of branching processes does not consider the case when the composition of added balls depends on the state of the process. These state dependent processes are very important for the applications and are considered in this paper.

The asymptotic behavior of the proportions of balls of each color in the total population is studied. It appears that these proportions can be expressed through the so-called urn functions which define the probabilities of adding the new ball depending on the current composition of urn population. The dynamics of the urn scheme is written in terms of a stochastic finite differential equation. The trajectories defined by this equation are attracted to the urn functions fixed points.

The techniques used to obtain these results have much in common with convergence analysis of the stochastic approximation type procedures for solving systems of nonlinear equations with discontinuous functions and stochastic quasi gradient procedures of stochastic optimization. In the general case the convergence results are formulated through nondifferentiable Lyapunov functions.

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1. INTRODUCTION

Many stochastic processes involved in the economic, chemistry, biology are of a "self-organizing" type. Such systems tend to be sensitive to early dynamical fluctuations. The cumulation of small events early on "pushes" the dynamics into the orbit one of possible patterns and thus "selects" the structure that the system eventually locks into.

Fluctuation dynamics of this type are usually modeled by non-linear differential equations with Markovian perturbations [15]. But while these continuous time formulations often work well for discrete events, continuous-time formulations usually involve approximations. Their asumptotic properties must often be specially studied and are not always easy to derive.

In this paper we continued the study of non-linear Polya processes of growth developed in previous articles [3], [4], and [2]. Within this class of stochastic processes we can investigate the emergence of structure by deriving theorems on asymptotic behavior.

We briefly discuss applications in industrial location theory, chemical kinetics, and the evolution of technological structure in the economy. The limit theorems we present generalize the strong law of large numbers to a wide class of path-dependent stochastic processes.

2. EXAMPLES

In 1923 Polya and Eggenberger [16] formulated a path-dependent process that has a particularly unusial behavior. Think of an urn of infinite capacity to which are added balls of two possible colors red and black say. Starting with one red and one black ball in the urn, add a ball each time, indefinitely, according to the rule: chose a ball in the urn at random and replace it; if it is red, add a red one; if is black add a black one. Obviously this process has increments that are path-dependent. We might then ask: does the proportion of red (or black) balls wander indefinitely between zero and one, or does a strong law operate, so that the proportion settles down to a limit causing a structure to emerge? And if there is a fixed limiting proportion, what is it? Polya proved in 1931 [18] that indeed in a scheme like this the proportion of red balls does tend to a limit X, and with probability one. But X is a random variable uniformly distributed between 0 and 1.

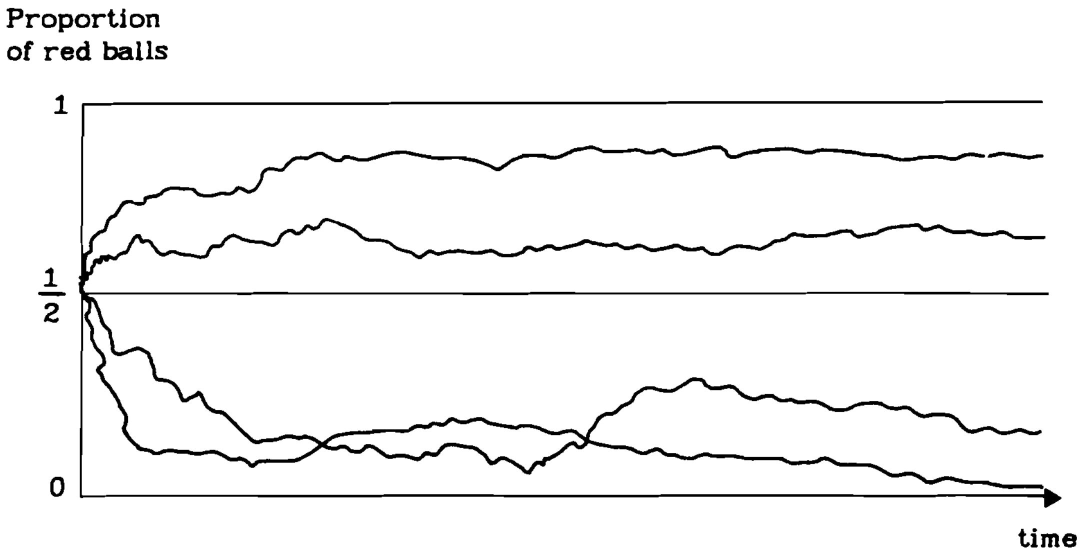


FIGURE 1

Figure 1 shows possible realizations of this basic process. We can see in this special case that the proportions do indeed settle down – a structure does emerge each time – but the structure that is "selected" is perfectly random. A particularly insightful and entertaining account of this standard Polya processes and how it might apply to the emergency and also to the misinterpretation of structure in biology and physics is given by Joel Cohen [18]. In the more general case where the urn starts off from an arbitrary number of red and black balls, proportions once again tend to a limit X, but now X has a two parameter Beta distribution [19].

Here are two examples of Polya-type path-dependense.

EXAMPLE 1 Dual Autocatalytic Chemical Reaction. A substrate molecule S is converted into an R-molecule, or into a B-molecule if it encounters a B-molecule:

$$S + R \rightarrow 2R + \text{Waste Molecule } E$$

$S + B \rightarrow 2B +$ Waste Molecule F .

Thus the probability that an R-molecule is created at any time exactly equals the concentration of R-product. A standard Polya process operates. Starting with one molecule of R and B, the process settles to a fixed concentration of R-product, but one that is anywhere between 0 and 100%.

EXAMPLE 2 Industrial Location by Spin-off. An industry builds up regionally from some set of initial firms, one per region say, but this time new firms are added by "spinning off" from parent firms one at a time. (David Cohen [20] has shown that such spin-offs have been the dominant "birth" mechanism in the US electronics industry.) Assume that each new firm stays in its parent location, and that any existing firm is as likely to spin off a new firm as any other. We again have Polya path-dependense–firms are added to regions incrementally with probabilities exactly equal to the proportions of firms in each region.

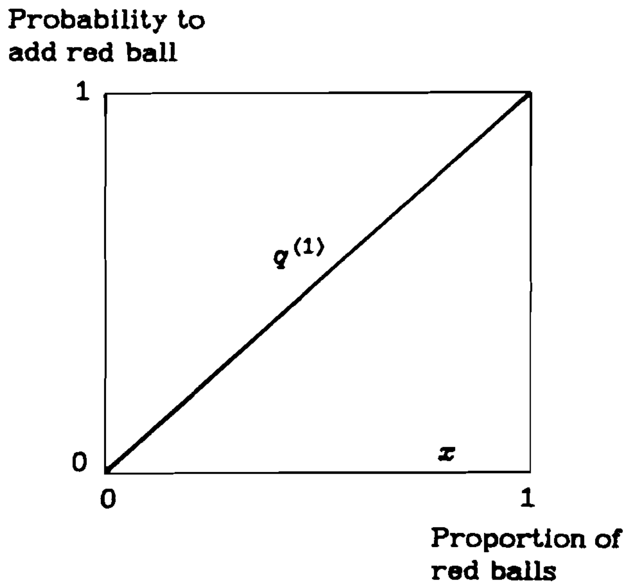


FIGURE 2

To get some intuitive feeling for this Polya urn process, notice that, as in Figure 2, the probability $q^{(1)}$ at adding a red always equals the proportion of reds. It is easy to show that this means, on average, that the process tends to stay where it is. There is no"drift". Of course, there are perturbations to the proportion red caused by the random sampling of balls; but unit additions to the urn make less and less difference to the proportions as the total number of balls grows; and therefore the effect of these perturbations dies away. The process then fluctuates less and less, and since it does not drift it settles down. Where it settles, of course, depends completely on its early random movements.

3. NON-LINEAR PATH DEPENDENCE

The standard Polya scheme requires a highly special path-dependense where the probability of adding a ball of type j exactly equals the proportion of type j. For a much wider set of applications we would want to consider a more general situation where the probability of an addition to type j is an arbitrary function of the proportions at all types. Moreover, to allow for realistic applications we would want more than two dimensions - two colors - and functions that may change with time. To describe our new process will proceed a little more formally, drawing on our previous work [3], [4] and that of Hill, Lane and Sudderth [2].

We now take an urn of infinite capacity that my contain balls of N possible colors and allow new units to be added at each time with probabilities that are not necessarily equal to but a function of the proportions. Let the vector $X_n = (X_n^1, X_n^2, \ldots, X_n^N)$ describe the proportions of balls of types 1 to N respectively, at time n (after n balls have been added). Let $\{q_n(x)\}$ be a sequence of vector-functions (urn functions) mapping the proportions (of colors) into the probabilities (of an addition to each color) at time n. Thus, starting at time 1 with an initial vector of balls $b_1 = (b_1^1, b_1^2, \ldots, b_1^N)$, one ball is added to the urn at each time; and at time n it is of color i with probability $q_n^i(X_n)$, $\sum_n q_n^i(x_n) = 1$. The scheme is iterated to yield the proportions vectors X_1, X_2, $X_3, \ldots$. Of interest is whether $\{X_n\}$ tends to a random limit vector X, with probability one, where X is selected from some set of possible limit vectors.

Let the total number of balls initially be $\gamma = \sum_i b_1^i$. At time n, define the random variable with one nonzero component

$$\beta_n^i(x) = \begin{cases} 1 \text{ with probability } q_n^i(x) \\ 0 \text{ with probability } 1 - q_n^i(x),\ i = 1, \ldots, N \end{cases} .$$

Then additions of i-type balls to the urn follow the dynamics

$$b_{n+1}^i = b_n^i + \beta_n^i(X_n),\ n \geq 1 \ .$$

Dividing through by the total ball number $(\gamma + n - 1)$, the evolution of the proportion of i-types, $X_n^i = b_n^i/(\gamma + n - 1)$, is described by

$$X_{n+1}^i = X_n^i + \frac{1}{\gamma + n} [\beta_n^i(X_n) - X_n^i], \; n \geq 1$$

with $X_1^i = b_1^i / \gamma$, $i = 1, 2, \ldots, N$.

We can rewrite it in the form

$$X_{n+1}^i = X_n^i + \frac{1}{\gamma + n} [q_n^i(X_n) - X_n^i] + \frac{1}{\gamma + n} \eta_n^i, \; n \geq 1 \tag{1}$$

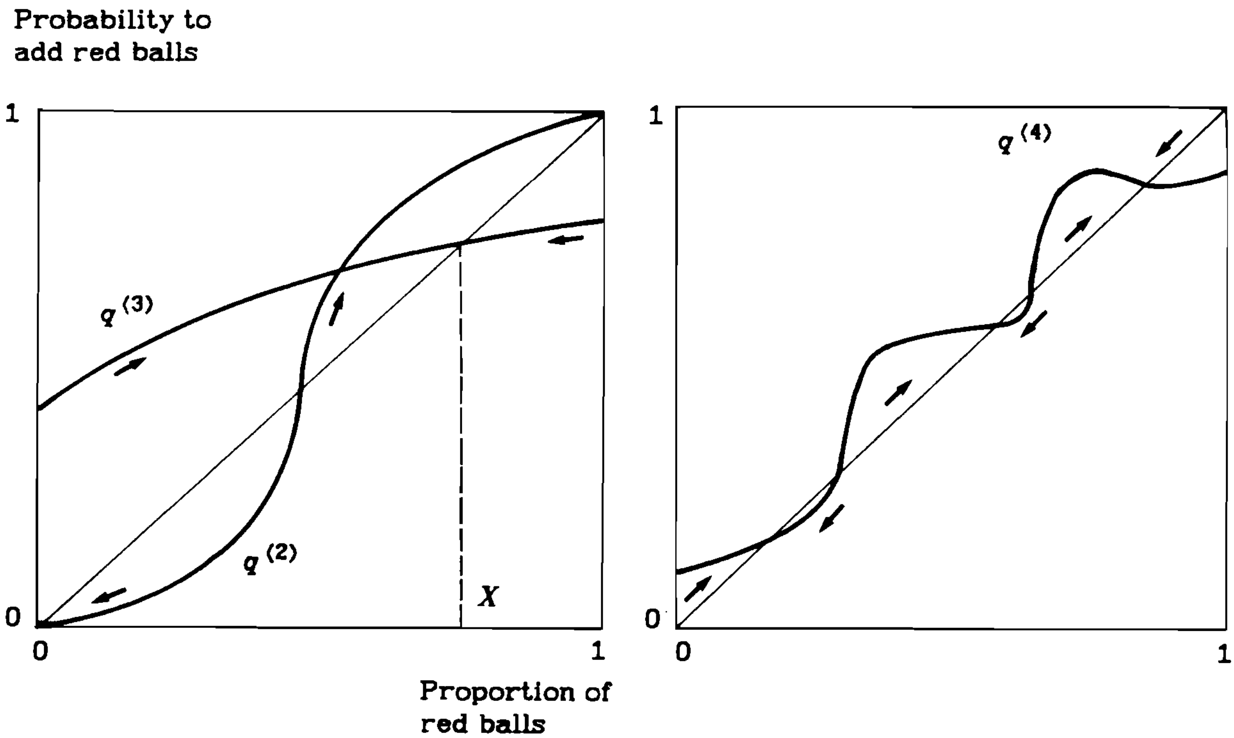
$$X_1^i = b_1^i / \gamma, \; i = 1, 2, \ldots, N$$

where $\eta_n^i = \beta_n^i(X_n) - q_n^i(X_n)$.

Equation (1) consists of a deterministic "driving" part and a perturbational part (the η-term in (1)). Notice that the conditional expectation of η_n^i with respect to X_n is zero, so that the expected motion of X_{n+1} is given by the "driving" part of (1) as

$$E\{X_{n+1}^i \mid X_n\} = X_n^i + \frac{1}{\gamma + n} [q_n^i(X_n) - X_n^i] \;, \tag{2}$$

Thus on the average the motion tends to be directed by the term $q_n(X_n) - X_n$.



FIGURES 3a and 3b

In Figures 3a, for example, urn function 2 shows a tendency toward 0 or 1. Urn function 3 shows a tendency toward X. In each case there is an "attraction" toward certain fixed points of q. Figure 3b shows a more complicated urn function with attractions toward several fixed points.

The standard Polya process discussed earlier is represented by an urn function that is identically equal to x, and so has no expected motion driving it.

Notice that for full determination of the urn process we need not N-dimensional dynamic system, but only $N - 1$ dimensional, because of equations

$$\sum_{i=1}^{N} X_n^i = 1, \ \sum_{i=1}^{N} q_n^i = 1, \ n \geq 1 \tag{3}$$

Consequently, the urn functions can be defined only in the space R^{N-1} as $q_n^i(X^1, X^2, \ldots, 1 - \sum_{i=1}^{N-1} X^i)$. According to that fact let us consider

$$T_{N-1} = \left\{ x \in R^{N-1} : x^i \leq 0, \ \sum_{i=1}^{N-1} x^i \leq 1 \right\}$$

and let L_{N-1} be the set of inner points T_{N-1} whose coordinates are the rational numbers. We consider only $N - 1$ dimensional vectors $X_n = (X_n^1, X_n^2, \ldots, X_n^{N-1})$ and a sequence of urn functions $\{q_n(x)\}$, $n \geq 1$ mapping R^{N-1} into L_{N-1}.

The basic dynamic equation (with such addition) is similar to equation (1)

$$X_{n+1}^i = X_n^i + (\gamma + n)^{-1}[\beta_n(X_n) - X_n^i] = \tag{4}$$

$$= X_n^i + (\gamma + n)^{-1}[q_n^i(X_n) - X_n^i] + (\gamma + n)^{-1}\eta_n^i, \ n \leq 1$$

$$X_1^i = b_1^i/\gamma, \ i = 1, 2, \ldots, N - 1 \ ,$$

Where $\eta_n^i = \beta_n^i(X_n) - q_n^i(X_n)$.

4. PRELIMINARY REMARKS

It is tempting to conjecture from Figure 3 that the dynamics must tend toward a fixed point of the urn function q, which is the limit of the urn functions q_n. We might conjecture further from Figures 3a and 3b that not any fixed point will do. Some fixed points appear to be stable ones (they attract) - while others are unstable (they repel). Notice from (2), that the attraction toward fixed points diminishes at a rate $1/n$ - so that the process may not have sufficient motion to be able to arrive at attracting fixed points.

However this is not the case.

The system actually converges to stable fixed points. For two-dimensional processes with stationary urn functions independent of n, this was first proved in 1980 in the elegant article of Hill, Lane, and Suddeth [2]. In articales [3], [4] we investigated the convergence of the N-dimensional processes with non-stationary urn functions described here. Let us mention some details of these articles.

The functions $q_n(x)$ had to be converging to a function $q(x)$ faster than $\{1/n\}$ converges to zero.

Under existence of twice continuously differentiable Lyapunov function $v(x)$ the non-linear Polya process X_n, $n \geq 1$ have to converge to the set

$$B = \{x \in T_{N-1} : < q(x) - x, \nabla v(x) > = 0\} ,$$

If the set B contains a finite number of connected components and

$$\sup_{x \in L_{n-1}} \| q_n(x) - q(x) \| = \alpha_n, \sum_{n \geq 1} \alpha_n n^{-1} < \infty .$$

Generally speaking the set B is wider than the set of fixed points of q. For example, with $N = 2$ and the urn function of Figure 4, the Lyapunov function $v(x) = (x - \varphi)^2$ and $\varphi \in B$, but φ is not a fixed point of q. If q is a continuous function, then B coincides with the fixed points of q.

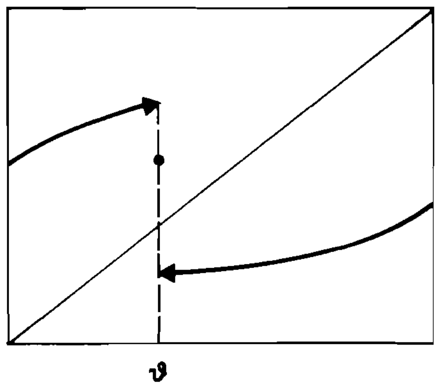


FIGURE 4

Here are two applications of results outlined above.

EXAMPLE 3 A second Duel Autocatalytic Reaction. Consider a slightly different version of the chemical reaction given earlier:

$$S + 2R \rightarrow 3R + \text{Waste Molecule } E$$

$$S + 2B \rightarrow 3B + \text{Waste Molecule } F \ .$$

In this case a single substrate molecule S is converted into either B or R-form (with waste molecules E and F) according to whether it encounters two B-molecules before two R-molecules. We may think of the process of "sampling" the next three B- or R-molecules encountered and adding one to B or R according as two out of the tree molecules sampled are B or R. Now, the probability than an R-molecule is added is

$$q_n = \sum_{k=2}^{3} H(k;\, n,\, n_R,\, 3)$$

where H is the Hypergeometric distribution parametrized by n and n_R. In this urn scheme the urn function is non-stationary. The sequence $\{q_n\}$ satisfies mentioned above conditions. The limit urn function has an S-shape as in q^2 Figure 3a. There are stable fixed points at 0 and 1 and an unstable one at 0,5. Therefore, in contrast to the previous example where any intermediate concentration between 0 and 100% could emerge, only extreme 0 or 100% concentrations of R or of B can emerge.

EXAMPLE 4 There are N regions in a country. As before, firms are added to an industry by spinning off from parent firms. Now suppose that a firm in region j spins off a new firm that settles in region i with some positive probability $q(i,\, j)$, where $\sum_{i=1}^{N} q(i,\, j) = 1$ for all j. In this case

$$q_n(x) = Qx$$

where Q is the matrix $Q = \{g(i,\, j)\}$. The regional structure that will emerge corresponds to a fixed point $z = Qz$. And from the condition that all $q(i,\, j)$ elements are greater than zero, there is a unique, stable point. Therefore, in this case the regional shapes of the industry must converge to a unique pre-determined structure.

EXAMPLE 5 Adoption of technologies that compete can be usefully modeled as a non-linear Polya process. A unit increment – an individual adoption – is added, each time of choice to a given technology with a probability that depends on the numbers (or proportions) holding each technology at that particular time. We can use our strong-law theorems to show circumstances under which increasing returns to adoption (the probability of adoption rises with the share of the market) may drive the adopter "market" to a single dominant technology, with small events early on "selecting" the technology that takes over [21], [22].

The main purpose of this article is to investigate more general non-linear Polya processes with random additions of balls at each time, with discontinuous urn functions requiring nondifferentiable Lyapunov functions. We also discuss results on the convergence of one-dimensional processes ($N = 2$) and processes with so-called separable urn functions without making use of Lyapunov functions. Up to a certain extent results of this paper are concerned with the convergence of stochastic procedures for solving nonlinear equations with discontinuous functions. This also coincides with the convergence of gradient-type procedures of nondifferentiable (stochastic) optimization and path-dependent branching processes.

5. NON-LINEAR PROCESSES WITH GENERAL INCREMENTS

Let the number of balls added to the urn at each step be random and may take values 0, 1,... . In this case the generalized Polya processes are much similar to the branching processes with path-dependent increments. Such a case is inconvinient for the investigation by the conventional theory of branching processes.

Consider the urn of the infinite capacity with vector β_n of n color balls at time n.

Let Z_+^N be the set of N-dimensional vectors with non-negative coordinates, i.e., for any $i \in Z_+^N$ we have $i^j \geq 0$, $j = 1, 2, \ldots, N$. Suppose that $]i[= \sum_{j=1}^{N} i^j$ and $\beta_n(x)$ is a random vector such that

$$P\{\beta_n(x) = i\} = q_n(i, x), \sum_{i \in Z_+^N} q_n(i, x) = 1, x \in L_{N-1} \ .$$

The i-the coordinate of $\beta_n(x)$ is equal to the number of i-type balls added to the urn at time n. The total number of balls γ_{t+1} in the urn at time $t + 1$ can be calculated as

$$\gamma_{t+1} = \gamma_t +]\beta_t(X_t)[, \gamma_1 = \sum_{i=1}^{N} \beta_1^i \ , \tag{5}$$

and the number of i-color balls and their proportion are calculated as the following

$$\beta_{t+1}^i = \beta_t^i + \beta_t^i(X_t), \ t \geq 1 \ ,$$

$$X_{t+1}^i = X_t^i + \gamma_t^{-1} \varsigma_t^i(X_t, \gamma_t), \ t \geq 1 \ , \tag{6}$$

where

$$\varsigma_t^i(X_t, \gamma_t) = (\beta_t^i(X_t) - X_t^i]\beta_t(X_t)[)(1 + \gamma_t^{-1}]\beta_t(X_t)[)^{-1} \ ,$$

$$X_1^i = \beta_1^i/\gamma_1,\ i = 1, 2, \ldots, N-1,\ X_1^N = 1 - \sum_{i=1}^{N-1} X_t^i \ .$$

Consider the conditions which allow for representing the relations (6) and the form similar to (4). The constants will be designated by the letter C. Henceforth the relations with random variables should be understood almost for sure.

LEMMA 1 *Suppose that*

1 *there exists* $r \geq 2$ *such that for all* $x \in Z_{N-1}$, $n \geq 1$ $\sum_{i \in Z_t^N}]i[^r q_n(i, x) \leq C_0$. *Then*

a)

$$E(\varsigma_n(X_n, \gamma_n) \mid X_n, \gamma_n) = f_n(X_n) + \delta_n(X_n, \gamma_n) \ ,$$

where

$$f_n^j(x) = \sum_{i \in Z_t^N} (i^j - x^j]i[) q_n(i, x),\ x \in Z_{N-1},\ |\delta_n(X_n, \gamma_n)| \leq$$

$$\leq C_1 \gamma_n^{-1},\ j = 1, 2, \ldots, N-1 \ .$$

If besides this

2 *there exist* $q(i, x)$, $i \in Z_t^N$, $x \in L_{N-1}$ *such that*

$$\sum_{i \in Z_t^N} q(i, x) = 1,\ \sum_{i \in Z_t^N}]i[^l q(i, x) \leq C_2$$

for some $l \geq 2$, $x \in Z_{N-1}$ and $|q_n(i, x) - q(i, x)| \leq$

$$\leq \sigma_n \text{ for all } i \in Z_t^N,\ x \in Z_{N-1},\ n \geq 1 \ ;$$

then

b) $f_n(x) = f(x) + \sigma_n(x)$, *where*

$$f^j(x) = \sum_{i \in Z_t^N} (i^j - x_j]i[) q(i, x),\ |\sigma_n^j(x)| \leq C_3 \sigma_n^{-H/_{N+H}},\ H =$$

$$= \min(z, l) - 1,\ x \in Z_{N-1},\ j = 1, 2, \ldots, N-1 \ .$$

If along with the condition 1 the following is true

3 *for all* $x \in L_{N-1}$, $q_n(0, x) \leq L < 1/2$, $n \geq 1$ *then*

c) *with probability 1*

$$1 - \alpha \leq \varliminf_{n \to \infty} \gamma_n/n \leq \varlimsup_{n \to \infty} \gamma_n/n \leq C_0^{1/r} ,$$

d) *with probability 1*

$$\sum_{k=n}^{\infty} \gamma_k^{-1} \eta_k(X_k, \gamma_k) \to 0$$

for $n \to \infty$, *where*

$$\eta_k(X_k, \gamma_k) = \varsigma_k(X_k, \gamma_k) - E(\varsigma_k(X_k, \gamma_k) \mid X_k, \gamma_k),\ k \geq 1 .$$

PROOF Note that

$$|\varsigma_n^j(X_n, \gamma_n)| \leq 2]\beta_n(X_n)[,\ n \geq 1 , \tag{7}$$

$$|i^j - X_n^j]i[| \leq 2]i[,\ n \geq 1 , \tag{8}$$

where $j = 1, 2, \ldots, N-1$, i is a random vector from Z_t^N. From the condition 1 and Gölder inequality for any $p \in (0, l]$ we obtain

$$\sup_{x \in Z_{N-1}} \sup_{n \geq 1} \sum_{i \in Z_t^N}]i[^p q_n(i, x) \leq C_0^{p/r} . \tag{9}$$

From the estimates (7), (8) and (9) with $p = 1$ it follows that $E(\beta_n(X_n) \mid X_n)$, $E(\varsigma_n(X, \gamma_n) \mid X, \gamma_n)|$ and $f_n(X_n)$ are correctly determined. It is not difficult to see that

$$E(\varsigma_n^j(X_n, \gamma_n) \mid X_n, \gamma_n) = f_n^j(X_n) + \delta_n^j(X_n, \gamma_n) ,$$

$$\delta_n^j(X_n, \gamma_n) = \gamma_n^{-1} \sum_{i \in Z_t^n} \frac{]i[}{1 + \gamma_n^{-1}]i[} (i^j - X_n^j]i[) q_n(i, X_n) .$$

From the estimates (8) and (9) with $p = 2$ we shall obtain

$$|\delta_n^j(X_n, \gamma_n)| \leq 2\gamma_n^{-1} \sum_{i \in Z_t^N}]i[^2 q_n(i, X_n) \leq 2\gamma_n^{-1} C_0^{2/r} ,$$

i.e., the statement a) is proved.

Suppose $Z_+^N(L) = \{i \in Z_+^N :]i[\leq L\}$. Using the inequality (8) and the condition 2, we have

$$|\sigma_n^j(X_n)| \leq \sum_{i \in Z_+^N} | i^j - X_n^j | i[\| q_n(i, X_n) - q(i, X_n) | \leq$$

$$\leq 2\left\{\sigma_n \sum_{i \in Z_+^N(L)}]i[+ \sum_{i \in Z_+^N / Z_+^N(L)}]i[(q_n(i, X_n) + + q(i, X_n))\right\} . \tag{10}$$

The set $Z_+^N(L)$ contains not more than $(L + 1)^{N-1}$ elements and so

$$\sum_{i \in Z_+^N(L)}]i[\leq (L + 1)^N . \tag{11}$$

For an arbitrary non-negative random variable ξ with the moment of the order $\mu > 0$ and number $\delta \in [0, \mu)$, the following inequality is correct

$$E\xi^\delta \chi_{\{\xi > L\}} \leq L^{\delta - \mu} E\,\xi^\mu , \tag{12}$$

where $L > 0$, $\chi_{\{\xi > L\}}$ is the indicator of the event $\{\xi > L\}$ Hence, for $\mu = r$ or $\mu = q$, $\delta = 1$ taking into account conditions 1, 2 we have

$$\sum_{i \in Z_+^N / Z_+^N(L)}]i[(q_n(i, X_n) + q(i, X_n)) \leq (L + + 1)^{-r+1} C_0 + (L + 1)^{-l+1} C_2 \leq (L + 1)^{-H} C_3 , \tag{13}$$

where $C_3 = 2 \max(C_0, C_2)$. Substituting the estimates (11), (13) with $L = L_n = [\sigma_n^{-1/N+H}]$ into the inequality (10), we obtain the statement b). Here $[a]$ is an integral part of the real number a.

From conditions 1, 3 and the estimate (9) with $p = 1$ for $h \geq 2$, we have

$$1 - \alpha \leq \frac{1}{n - 1} \sum_{i=1}^{n-1} r_i(X_i) \leq C_0^{1/r} , \tag{14}$$

where $r_i(X_i) = E(]\beta_i(X_i)[\,|\,X_i)$, $i \geq 1$. On the basis of inequality (9) with $p = 1$ and $p = 2$, we obtain

$$Ez_n^2 \leq 2C_0^{2/r}/n - 1 \tag{15}$$

Due to Chebyshev inequality

$$z_n = \frac{1}{n-1} \sum_{i=1}^{n-1} (|\beta_i(X_i)| - r_i(X_i)) \to 0 \quad \text{for } n \to \infty \ , \tag{16}$$

in probability.

Let τ_n − be an σ − algebra generated by $X_1, X_2, \ldots, X_n$, $n \geq 2$ Using the estimate (9) with $p = 1$ and $p = 2$, we have

$$E(z_{n+1}^2 \mid \tau_n) \leq z_n^2 + 2n^{-1} C_0^{2/r}, \ n \geq 2 \ .$$

From the inequality (15) it follows that the pair $\{f_n = z_n^2 + 2C_0^{2/r} \sum_{i \geq n+1} i^{-2}; \ \tau_n\}$, $n \geq 2$ is non-negative supermartingale. Hence there exists a random variable f, such that $f_n \to f$ with probability 1 for $n \to \infty$. On the basis of the relation (16) $f = 0$, i.e. with probability 1

$$z_n \to 0 \quad \text{for } n \to \infty \ . \tag{17}$$

Since

$$\frac{\gamma_n}{n-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} r_i(X_i) + z_n, \ n \geq 2 \ , \tag{18}$$

then with regard to the relations (14) and (17) we obtain the statement c).

Note that

$$\sum_{l=0}^{n-1} P\{A_n^l\} = 1 \ , \tag{19}$$

and due to the condition 3

$$P\{A_n^l\} \leq C_{n-1}^l \alpha^l \ , \tag{20}$$

where $A_n^l = \{\exists n_i, \ 1 \leq n_i \leq n-1, \ 1 \leq i \leq l : |\beta_{n_i}(X_{n_i})| = 0\}$, C_{n-1}^l is the number of combinations form $n - 1$ to l. If an event A_n^l occurs, then

$$\gamma_n \geq \gamma_1 + n - 1 - l \ . \tag{21}$$

Let

$$k_n = \begin{cases} n/2 & \text{if } n \text{ is an even number,} \\ n - 1/2 & \text{if } n \text{ is an odd number} \ . \end{cases}$$

Then for $l = 0, 1, \ldots, n$ $C_n^l \leq C_n^{k_n}$ and on the basis of the relations (19)–(20), we obtain

$$E\gamma_n^{-2} \leq \sum_{i=0}^{n-1} P\{A_n^i\}(\gamma_1 + n - 1 - i)^{-2} \leq$$

$$\leq (\gamma_1 + n - 1 - k)^{-2} + \sum_{i=k+1}^{n-1} P\{A_n^i\} \leq \tag{22}$$

$$\leq (\gamma_1 + n - 1 - k)^{-2} + (n - 1 - k) C_{n-1}^{k_n - 1} \alpha^{k+1} ,$$

where $1 < k < n - 1$. From Stirling formula

$$\lim_{n \to \infty} \frac{C_n^{k_n} \sqrt{\pi n / 2}}{2^n} = 1 . \tag{23}$$

From condition 3 for all sufficiently small $\epsilon > 0$, we have $\alpha^{1-\epsilon} < 1/2$. Let us fix one of those ϵ. Let n_ϵ be the least integer, larger than $(n - 1)(1 - \epsilon)$. Since the exponential curve decreases faster than any degree function, then from (12) for $k = n_\epsilon$ we have

$$E\gamma_n^{-2} \leq C_\epsilon n^{-2} \tag{24}$$

and, consequently,

$$\sum_{n \geq 1} E\gamma_n^{-2} = C_4 \tag{25}$$

If

$$v_k = \sum_{i=1}^{k-1} \gamma_i^{-1} \eta_i(X_i, \gamma_i),\ k \geq 2,\ v_\infty = \sum_{i=1}^{\infty} \gamma_i^{-1} \eta_i(X_i, \gamma_i) ,$$

then from the estimates (24), (25) and the inequality (9) with $p = 1$, $p = 2$ we have

$$E \| v_k \|^2 = \sum_{i=1}^{k-1} E\gamma_i^{-2} E(\| \eta_i(X_i, \gamma_i) \|^2 | \tilde{r}_i) \leq$$

$$\leq 8NC_0^{2/r},\ k \geq 2 ,$$

i.e., v_k, $k \geq 2$ and v_∞ are nonsingular random vectors. Here $\tilde{r}_k$ – is the σ – algebra generated by $X_1, X_2, \ldots, X_k, \gamma_1, \gamma_2, \ldots, \gamma_k$. Since $E(v_{k+1} | \tilde{r}_k) = v_k$, then the pair $\{v_k; \tilde{r}_k\}$, $k \geq 2$ is quadratically integrable martingale and, consequently, $v_k \to v_\infty$ with probability 1 for $k \to \infty$.

The lemma is proved.

COROLLARY 1 *From the statement c) under conditions 1, 3, $\sum_{n \geq 1} \gamma_n^{-1} = \infty$ with probability 1.*

REMARK 1 Let for all $i \in Z_t^N$ the functions $q(i, x)$ be continuous in $x \in L_{N-1}$. If $\sum_{i \in Z_+^N}]i[^q q(i, x)$ is uniformly bounded for $x \in L_{N-1}$ (it is guaranteed by the condition 2 of Lemma 1), then the sequence determining the function f (due to estimates (8) and (12) with $\mu = q$, $\delta = 1$) is uniformly convergent and the function f is continuous.

REMARK 2 If there is $L > 0$ such that $q_n(i, x) = 0$, $n \geq 1$, $q(i, x) = 0$ for all $i \in Z_+^N$, $]i[> L$, $x \in L_{N-1}$, then $\| \sigma_n(x) \| \leq C\sigma_n$, $n \geq 1$, $x \in L_{N-1}$.

6. EQUATIONS WITH DISCONTINUOUS FUNCTIONS

Let $\bar{A}$ be the set of the limit points A, where A is an arbitrary bounded set in R^M. Denote by X a connected compact in R^M and let Y be a dense subset of X. Sets X, Y of the paragraph correspond to sets T_{n-1}, L_{N-1} defined in paragraph 3. Consider a function φ on Y with values in R^N, such that

$$\sup_{y \in Y} \| \varphi(y) \| = C < \infty \ . \tag{26}$$

For any $x \in X$ define the following sets

$$A_\varphi(x) = \{g : \exists y_k,\ y_k \neq x,\ y_k \in Y,\ k \geq 1,\ \lim_{k \to \infty} y_k = x,\ \lim_{k \to \infty} \varphi(y_k) = g\} \ ,$$

$$B_\varphi(x) = \begin{cases} A_\varphi(x) & \text{for } x \in X \setminus Y \ , \\ A_\varphi(x) \cup \{\varphi(x)\} & \text{for } x \in Y \ ; \end{cases}$$

$$A^\varphi(x) = co A_\varphi(x),\ B^\varphi(x) = co\, B_\varphi(x) \ .$$

Let us call φ ***weakly continuous*** on Y at the point y if there is a sequence $\{y_k\}$, $k \geq 1$, such that $y_k \in Y \setminus \{y\}$ for all k, $\lim_{k \to \infty} y_k = y$ and $\lim_{k \to \infty} \varphi(y_k) = \varphi(y)$. The function φ is ***weakly continuous on*** Y if it is weakly continuous at each point of this set.

LEMMA 2 *If φ is weakly continuous on Y, then $A^\varphi(x) = B^\varphi(x)$ for any $x \in X$.*

Let $A^\varphi(X) = \{x \in X : A^\varphi(x) \ni 0\}$, $B^\varphi(X) = \{x \in X : A^\varphi(x) \ni 0\}$. Obviously, $B^\varphi(X) \supseteq A^\varphi(X)$ and these sets represent sets of solutions of equation $\varphi(x) = 0$, $x \in X$ with discontinuous function φ. The difference between set $A^\varphi(X)$ and $B^\varphi(X)$ is shown in Figure 5, where $X = Y = [a, b]$, $\theta \notin A^\varphi[a, b]$, but $\theta \in B^\varphi[a, b]$. Introduce multifunctions $A^\varphi : x \to A^\varphi(x)$; $B^\varphi : x \to B^\varphi(x)$.

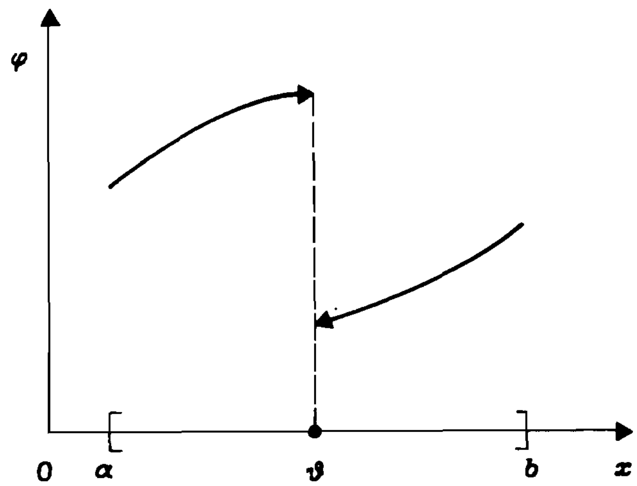


FIGURE 5

LEMMA 3 *Let* $x_k \in X$, $g_k \in B^{\varphi}(x_k)$, $x_k \neq x$, $k \geq 1$,

$$\lim_{k \to \infty} x_k = x, \quad \lim_{k \to \infty} g_k = g \ .$$

Then $g \in A^{\varphi}(x)$.

PROOF Since $g_k \in B^{\varphi}(x_k)$, then due to Karateodori theorem, there exist non-negative members $\lambda_i^{(k)}$, such that

$$\sum_{i=1}^{N+1} \lambda_i^{(k)} = 1 \ ,$$

$$g_k = \sum_{i=1}^{N+1} \lambda_i^{(k)} \lim_{p \to \infty} \varphi(y_{ip}^{(k)}) \ ,$$

where $\lim_{p \to \infty} y_{ip}^{(k)} = x_k$, $y_{ip}^{(k)} \in Y$. There are subsequences $\{p_j\}$, $\{k_j\}$ such that

$$\lim_{j \to \infty} \lambda_i^{k_j} = \lambda_i \geq 0, \ y_{ip_j}^{k_j} \neq x \ \text{ for all } j \geq 1 \ ,$$

$$\lim_{j \to \infty} y_{ip_j}^{k_j} = x, \ \lim_{j \to \infty} \varphi(y_{ip_j}^{k_j}) = r_i \ ,$$

where $i = 1, 2, \ldots, N+1$, $\sum_{i=1}^{N+1} \lambda_i = 1$. By the definition of the set $A^{\varphi}(x)$, we obtain $\sum_{i=1}^{N+1} \lambda_i r_i \in A^{\varphi}(x)$ and $\lim_{j \to \infty} g_{k_j} = \sum_{i=1}^{N+1} \lambda_i r_i$. Then $\lim_{j \to \infty} g_{k_j} = g$ and thus $g \in A^{\varphi}(x)$.

COROLLARY 2 *The mappings A^{φ} and B^{φ} are upper semicontinuous (closed).*

COROLLARY 3 *The sets $A^{\varphi}(X)$ and $B^{\varphi}(X)$ are closed.*

COROLLARY 4 $B^{\varphi}(X) \leq A^{\varphi}(X)$, *i.e., all connected components of the set $B^{\varphi}(X) \backslash A^{\varphi}(X)$ are single-point components.*

Let $p_A(y, a)$ be vector the projection of a onto the convex compact $A^{\varphi}(y)$,

$$A(y, \epsilon) = co\{\varphi(x) : x \in Y \cap V^M(y, \epsilon)\},\ t^y(\epsilon) =$$

$$= \sup_{x \in Y \cap V^M(y, \epsilon)} \| p_A(y, \varphi(x)) - \varphi(x) \| \ ,$$

where

$$\epsilon > 0,\ V^M(y, \epsilon) = \{x \in R^M : 0 < \| x - y \| < \epsilon\},\ y \in X \ .$$

LEMMA 4 *For any vector $v \in A(y, \epsilon)$ there exists $\tilde{v} \in A^{\varphi}(y)$, such that $\| v - \tilde{v} \| \leq t^y(\epsilon)$, $\lim_{\epsilon \to 0} t^y(\epsilon) = 0$.*

PROOF Let $v \in A(y, \epsilon)$. From the Karateodori theorem: $v = \sum_{i=1}^{N+1} \lambda_i \varphi(y_i)$, where $\sum_{i=1}^{N+1} \lambda_i = 1$, $\lambda_i \geq 0$, $y_i \in Y \cap V^M(y,\epsilon)$. Assuming that $\tilde{v} = \sum_{i=1}^{N+1} \lambda_i\, p_A(y, \varphi(y_i))$, we obtain

$$\| v - \tilde{v} \| \leq \sum_{i=1}^{N+1} \lambda_i \| \varphi(y_i) - p_A(y, \varphi(y_i)) \| \leq t^y(\epsilon)$$

and with regard to the convexity of $A^{\varphi}(y)$ we have $\tilde{v} \in A^{\varphi}(y)$, i.e., the first statement is true.

Suppose that the second statement is incorrect. Then there is a sequence of positive numbers $\{\epsilon_k\}$ and $\beta > 0$, such that

$$\lim_{k \to \infty} \epsilon_k = 0,\ t^y(\epsilon_k) > 2\beta \ \text{for} \ k \geq 1 \ .$$

By the definition of the precise upper bound there are $x_k \in Y \cap V^M(y, \epsilon_k)$ such that

$$\| p_A(y, \varphi(x_k)) - \varphi(x_k) \| \geq \beta > 0 \tag{27}$$

We can suppose that $\lim_{k \to \infty} \varphi(x_k) = g$. Then from the Lemma 3, $g \in A^{\varphi}(y)$ and from the continuity of the projection operator, we have $\lim_{k \to \infty} p_A(y, \varphi(x_k)) = g$. Therefore

$$\lim_{k \to \infty} \| p_A(y, \varphi(x_k)) - \varphi(x_k) \| = 0 \ . \tag{28}$$

The relations (27) and (28) are contradictory.

The lemma is proved.

REMARK 3 The statement of Lemma 4 will remain true if $A^{\varphi}(y)$ is substituted by $B^{\varphi}(y)$, $p_A(y, \cdot)$ by $p_B(y, \cdot)$ and $A(y, \epsilon)$ by $B(y, \epsilon)$ where

$$B(y, \epsilon) = co\{\varphi(x) : x \in Y \cap V^M(y, \epsilon) = \{x \in R^M : \| x - y \| < \epsilon\} \ .$$

7. ITERATIVE PROCEDURES FOR SOLVING EQUATIONS WITH DISCONTINUOUS FUNCTIONS

Consider the sequence $\{z_s\} \subseteq Y$ defined by the rule

$$z_{s+1} = z_s + \rho_s[\varphi(z_s) + \mu_s + \delta_s], \ s \geq 1 \ , \tag{29}$$

where ρ_s is a positive number

$$\lim_{s \to \infty} \rho_s = 0 \ , \tag{30}$$

$$\sum_{s \geq 1} \rho_s = \infty \ ; \tag{31}$$

μ_s and δ_s are vectors, such that

$$\lim_{s \to \infty} \mu_s = 0 \ , \tag{32}$$

$$\lim_{n \to \infty} \| \sum_{s \geq n} \rho_s \delta_s \| = 0 \ . \tag{33}$$

The following lemma is close to results of articles [10] and [11].

LEMMA 5 *Let the relations (26), (29)-(33) be fulfilled. Then* $\lim_{s \to \infty} \| z_{s+1} - z_s \| = 0$ *and the sequence* $\{z_s\}$ *does not converge to any point of* $X \backslash B^{\varphi}(X)$.

PROOF From the relations (26), (29)-(33) follows the first statement:

$$\| z_{s+1} - z_s \| \leq \rho_s(C + \| \mu_s \| + \| \delta_s \|) \to 0 \text{ for } s \to \infty \ ,$$

Suppose that the second statement is incorrect; i.e. $\lim_{s\to\infty} z_s = z$, $z \in X \backslash B^{\varphi}(X)$. Using the equality (29), we obtain for $m > n$

$$z_{m+1} = z_n + \rho_{nm}(g_{nm} + \mu_{nm}) + \delta_{nm} \ , \tag{34}$$

where

$$\rho_{nm} = \sum_{i=n}^{m} \rho_i, \; g_{nm} = \sum_{i=n}^{m} \rho_i \rho_{nm}^{-1} \varphi(z_i), \; \mu_{nm} = \sum_{i=n}^{m} \rho_i \rho_{nm}^{-1} \mu_i, \; \delta_{nm} = \sum_{i=n}^{m} \rho_i \delta_i \ .$$

On the basis of the Remark 3 and the convergence of the sequence $\{z_s\}$ to z, we will find vectors $\tilde{g}_{nm} \in A^{\varphi}(z)$ such that

$$\lim_{n\to\infty} \| g_{nm} - \tilde{g}_{nm} \| = 0 \ . \tag{35}$$

Since $z \notin B^{\varphi}(X)$, then

$$\| \tilde{g}_{nm} \| \geq \gamma(z) > 0 \ , \tag{36}$$

where $\gamma(z) = \min_{g \in B^{\varphi}(z)} \| g \|$. From relations (32)-36) for all sufficiently large n, we have

$$\| z_{m+1} - z_n \| \geq \frac{1}{2}\gamma(z)\rho_{nm} \to \infty$$

for $m \to \infty$. This contradicts the convergence of the $\{z_s\}$.

COROLLARY 5 *Let the conditions of Lemma 5 be satisfied and there is a subsequence* $\{z_{p_k}\}$ *such that* $\lim_{k\to\infty} z_{p_k} = z$, $z \in X \backslash B^{\varphi}(X)$. *Then there exists* $\epsilon(z) > 0$ *such that for* $\epsilon \in (0, \epsilon(z))$ *and all sufficiently large* k *we have* $n_k(\epsilon) < \infty$, $m_k(\epsilon) < \infty$, *where*

$$n_k(\epsilon) = \max\{s : s < p_k, \|z_s - z_{p_k}\| \geq \epsilon\}, \; m_k(\epsilon) = \min\{s : s > p_k, \| z_s - z_{p_k} \| \geq \epsilon\} \ .$$

COROLLARY 6 *Under conditions of Lemma 5, the set* $Z = \{z_s\}$ *is a connected compact.*

It seems to be true that the sequence $\{z_s\}$ converges to the set $B^{\varphi}(X)$. However, the mentioned above assumptions are not sufficient to prove this fact for the case of $M > 1$. In this case the traditional way of arguing is to introduce a Lyapunov function which behaves, in a sense, monotonously over the sequence $\{z_s\}$. Let us confine ourselves to the functions satisfying Lipschitz condition $\sigma(X)$ where $\sigma(X)$ is some open set containing X. For any such a function Φ and any direction $v \in R^M$ there exists

$$\Phi^-(x, v) = \lim_{t \downarrow 0} t^{-1}[\Phi(x + tv) - \Phi(x)] \ .$$

at each point $x \in X$.

LEMMA 6 *Let the conditions of Lemma 5 be fulfilled and there is a Lipschitz function* Φ *on* $\sigma(X)$ *such that for any* $z \in X \backslash B^{\varphi}(X)$

$$\inf_{g \in A^{\varphi}(z)} \Phi^-(z, \| g \|^{-1} g) = \mu(z) > 0 \ . \tag{37}$$

Then for each a subsequence $\{z_{p_k}\}$, $\lim z_{p_k} = z$, $z \in X \backslash B^{\varphi}(X)$, *and* $\epsilon \in (0, \epsilon(z))$ *we have*

$$\overline{\lim_{k \to \infty}} \max_{p_k \le s \le m_k(\epsilon)} \Phi(z_s) > \Phi(z) \ ,$$

where $\epsilon(z)$ *and* $m_k(\epsilon)$ *are defined in Corollary 5.*

PROOF For all sufficiently large k, determine the numbers $n_k = \min\{s : p_k \le s < m_k(\epsilon), z_s \ne z\}$. Consider the equality (34) with $n = n_k$, $m_k = m_k(\epsilon) - 1$. Taking into account the estimate (26), we have

$$\| g_{n_k} m_k(\epsilon) - 1 \| \le \sum_{i = n_k}^{m_k(\epsilon) - 1} \rho_i \rho_{n_k}^{-1} m_k(\epsilon) - 1 \| \varphi(z_i) \| \le C \ . \tag{38}$$

If the $k \ge k(\epsilon)$, $\| z_{p_k} - z \| < \epsilon$, then from Lemma 4 there exist the vectors $g_k(\epsilon) \in A^{\varphi}(z)$ such that

$$\| g_{n_k m_k(\epsilon) - 1} - g_k(\epsilon) \| \le t^z(2\epsilon) \ . \tag{39}$$

Since $z \in A^{\varphi}(X)$, then from Lemma 4 for all sufficiently small ϵ and $k \ge k(\epsilon)$ we obtain

$$\| g_{n_k m_k(\epsilon) - 1} \| \ge \gamma(z) > 0 \ , \tag{40}$$

and due to (32), (33)

$$\lim_{k \to \infty} (\mu_{n_k m_k(\epsilon) - 1} + \delta_{n_k m_k(\epsilon) - 1}) = 0 \ . \tag{41}$$

Since $m_k(\epsilon)$ is the first after p_k moment of deviation of the sequence $\{z_s\}$ from the subsequence $\{z_{p_k}\}$ over the distance ϵ, then with regard to (30), (34), (38) and (40) we obtain

$$\epsilon / C \le \underline{\lim}_{k \to \infty} \rho_{n_k m_k(\epsilon) - 1} \le \overline{\lim_{k \to \infty}} \rho_{n_k m_k(\epsilon) - 1} \le \epsilon / \gamma(z)$$

Sine $A^{\varphi}(z)$ is a compact, then we can find the subsequence $\{n_{k'}\}$ such that

$$\lim_{k' \to \infty} g_k(\epsilon) = g(\epsilon), \quad \lim_{k' \to \infty} \rho_{n_k m_k(\epsilon) - 1} = \rho(\epsilon) \ .$$

for some vector $g(\epsilon) \in A^{\varphi}(z)$ and a number $\rho(\epsilon) \in [1/C, 1/\gamma(z)]$ The subsequence $\{n_{k'}\}$ depends on ϵ. From relations (34) and (41) we have

$$z_{m_{k'}(\epsilon)} = z + \epsilon\rho(\epsilon)[g(\epsilon) + o(1)] \ , \tag{42}$$

$$\lim_{\epsilon \to \infty} \lim_{k' \to \infty} o(1) = 0 \ . \tag{43}$$

Using once again the compactness of $A^{\varphi}(z)$ we can find the sequence of positive numbers $\{\epsilon_l\}$, such that

$$\lim_{l \to \infty} \epsilon_l = 0, \quad \lim_{l \to \infty} g(\epsilon_l) = g \in A^{\varphi}(z) \ ,$$

$$\lim_{l \to \infty} \rho(\epsilon_l) = \rho \in [1/C, 1/\gamma(z)] \ .$$

From relations (37), (42) and (43) since the function Φ satisfies the Lipschitz condition, we have

$$\underline{\lim}_{l \to \infty} \ \underline{\lim}_{k' \to \infty} \frac{\Phi(z_{m_{k'}(\epsilon_l)}) - \Phi(z)}{\epsilon_l} \geq \rho\mu(z) > 0 \ .$$

Values $m_k(\epsilon)$ do not increase monotonously with respect to ϵ, therefore from the latter inequality the required result follows.

REMARK 4 Instead of the Lipschitz condition for Φ on an open set containing X we can require the Lipschitz conditions for Φ on X and the existence of $\Phi(x, v)$ for all $x \in X$.

REMARK 5 The class of Lipschitz on $\sigma(X)$ functions satisfying the inequality (37) represents the convex cone, closed with respect to the operations of taking the minimum and maximum among a finite set of functions.

The statement given below is useful for the convergence study of the scheme (29).

LEMMA 7 *Let the conditions of Lemma 5 be satisfied and there is a Lipschitz on $\sigma(X)$ function Φ, such that for any $z \in X \backslash B^{\varphi}(X)$ the inequality (37) is valid. Suppose that $\{n_k\}$ and $\{\tau_k\}$, $n_k < \tau_k$, $k \geq 1$ are the subsequences for which $\lim_{k \to \infty} z_{\tau_k} = z'$, $D \cap B^{\varphi}(X) = \emptyset$, where $D = \overline{\{z_s, \, n_k \leq s \leq \tau_k, \, k \geq 1\}}$. Then $\Phi(z) < \Phi(z')$ for any $z \in D/\{z'\}$.*

PROOF D is a compact set, i.e., the continuous function Φ obtains its maximal and minimal values on D. Let the statement of the lemma be incorrect. The $\max_{z \in D} \Phi(z) = \Phi(\bar{z})$, $\Phi(\bar{z}) > \Phi(z')$. Since $\bar{z} \notin B^{\varphi}(X)$, then due to the Corollary 5 for any $\epsilon \in (0, \min(\epsilon(\bar{z}), \| \bar{z} - z' \|))$ and the subsequence $\{z_{p_{k'}}\}$, $\lim_{k' \to \infty} z_{p_{k'}} = \bar{z}$, $n_{k'} < p_{k'} < r_{k'}$, $k' \leq 1$ the numbers $m_{k'}(\epsilon) < r_{k'}$ is found, such that by Lemma 6

$$\lim_{k' \to \infty} \max_{p_{k'} \leq s \leq m_{k'}(\epsilon)} \Phi(z_s) > \Phi(z) \ .$$

It contradicts the fact that $\bar{z}$ is a point of maximum on D. Thus $\Phi(z') \geq \Phi(z)$, $z \in D$. If $z \in D \backslash \{z'\}$, then taking a subsequence $\{z_{p_{k''}}\}$ such that $\lim_{k'' \to \infty} z_{p_{k''}} = z$ and $\epsilon \in (0, \min(\epsilon(z), \| z - z' \|))$ we obtain in a similar way

$$\overline{\lim_{k'' \to \infty}} \max_{p_{k''} \leq s \leq m_{k''}(\epsilon)} \Phi(z_s) > \Phi(z) \ .$$

Hence it follows that there can be found the point $\tilde{z}$ with $\Phi(\tilde{z}) > \Phi(z)$ and $\Phi(z') \geq \Phi(\tilde{z}) > \Phi(z)$.

The lemma is proved.

REMARK 6 If in addition to the conditions of Lemma 7 $\lim_{k \to \infty} z_{n_k} = z''$, then we obtain, in a similar way, that for any $z \in D \backslash \{z'\} \backslash \{z''\}$ the inequalities $\Phi(z'') < \Phi(z) < \Phi(z')$ are true.

LEMMA 8 *Let the conditions of Lemma 6 be satisfied and the set $B^{\varphi}(X)$ consists of a finite number of the connected components B_i, and $\Phi(z) = \Phi_i$ (const.) for any $z \in B_i$. Then the sequence $\{z_r\}$ converges to the set $B^{\varphi}(X)$.*

PROOF Let $Z \backslash B^{\varphi}(X) \neq \emptyset$. According to Corollary 6 Z is compact, i.e., for a certain $\bar{z} \in X$ and all $z \in X$ we have $\Phi(\bar{z}) \geq \Phi(z)$. Assuming $\bar{z} \notin B^{\varphi}(X)$ we can find (on the basis of Lemma 6) $\tilde{z} \in X$, such that $\Phi(\tilde{z}) > \Phi(\bar{z})$. So $\bar{z} \in B^{\varphi}(X)$ and for any $z \in Z \backslash B^{\varphi}(X)$ we obtain $\Phi(z) < \Phi(\bar{z})$.

Let B_j be a connected component containing $\bar{z}$, and $z' \in Z \backslash B^{\varphi}(X)$, $\epsilon > 0$ be smaller than the smallest distance between connected components $B^{\varphi}(X)$ and the distance form z' to $B^{\varphi}(X)$. For an arbitrary subsequence $\{z_{p_k}\}$ converging to B_j there is the first moment $n_{k'}^{\epsilon}$ when the sequence (29) leaves the ϵ-neighborhood of the set B_j. For an arbitrary $r \in (0, \epsilon)$ there also exist moments $\{n_{k'}^{r}\}$ of the last before $n_{k'}^{\epsilon}$, leaving of the r-neighborhood of B_j. We can assume that $\lim_{k' \to \infty} z_{n_{k'}^{\epsilon}} = z^{\epsilon}$, $\lim_{k' \to \infty} z_{n_{k'}^{r}} = z^{r}$. Then from the

Remark 6 follows $\Phi(z^{\tau}) < \Phi(z^{\epsilon})$. If τ tends to 0, then we obtain $\Phi_j \leq \Phi(z^{\epsilon})$, what is impossible, since $z^{\epsilon} \notin B^{\varphi}(X)$.

The lemma is proved.

Consider now one more variant of reasoning which makes it possible to prove the convergence in the most general case. It is based on the fact that if we can find a limit point of the sequence (29) at which the value $\Phi \bar{\in} \Phi(B^{\varphi}(X))$, then this point does not belong $B^{\varphi}(X)$ and the sequence (29) "pass through" this point with increasing the functional.

LEMMA 9 *If the conditions of the Lemma 6 are fulfilled and the set $\Phi(B^{\varphi}(X))$ does not contain non-degenerate segments, then the sequence $\{z_s\}$ converges to the set $B^{\varphi}(X)$.*

PROOF Let $Z \backslash B^{\varphi}(X) \neq \emptyset$, i.e., the lemma's statment is incorrect. According to corollary 6 and continuity Φ, the set $\Phi(Z)$ is a connected compact. If $Z \backslash B^{\varphi}(X) \neq \emptyset$, then from Lemma 6, the set $\Phi(Z)$ is a non-degenerate segment. From condition of lemma $\Phi(B^{\varphi}(X))$ and, consequently, $\Phi(Z \cap B^{\varphi}(X)) \subseteq \Phi(Z)$ does not contain non-degenerate segments. Therefore, the set $Q = \Phi(Z) \backslash \Phi(Z \cap B^{\varphi}(X))$ is not empty. The set $\Phi(Z \cap B^{\varphi}(X))$ is closed, hence Q is an open set in $\Phi(Z)$. The segment $\Phi(Z)$ is crossed by the sequence $\{\Phi(z_s)\}$ the infinite number of times. Then from Lemma 5 for any interval $(\gamma', \gamma'') \subset Q$ the numbers q_k and p_k can be found, $k \geq 1$, with the following properties: $q_k < p_k$, $\Phi(z_{q_k} \geq \gamma''$, $\Phi(z_{p_k}) \leq \gamma'$, $\Phi(z_s) \in (\gamma', \gamma'')$ for $q_k < s < p_k$. We can assume that $\lim_{k \to \infty} z_{q_k} = z$. Then $\Phi(z) = \gamma''$ and $z \in B^{\varphi}(X)$. The function Φ is uniformly continuous on X, i.e., for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$, such that for all $x', x'' \in X$ the value $|\Phi(x') - \Phi(x'')| < \epsilon$ as soon as $\|x' - x''\| < \delta(\epsilon)$. The moments of leaving $m_k(\delta)$ which correspond to the subsequence $\{z_{q_k}\}$, for any $\delta < \delta(\gamma'' - \gamma')$ can be found among the numbers $q_k, q_k + 1, \ldots, p_k$ So for such δ

$$\overline{\lim_{k \to \infty}} \max_{q_k \leq s \leq m_k(\delta)} \Phi(z_s) \leq \Phi(z)$$

which contradicts the statement of Lemma 6.

The lemma is proved.

COROLLARY 7 *If all connected components $B^{\varphi}(X)$ are single-point components, then under conditions of Lemma 9, the sequence $\{z_s\}$ converges.*

REMARK 7 Requirement that $\Phi(B^{\varphi}(X))$ does not contain non-degenerate segments can be formulated in another way [13]: for any interval (a, b), the set $(a, b)\backslash\Phi(B^{\varphi}(X)) \neq \emptyset$, or the set $\Phi(B^{\varphi}(X))$ is nowhere dense [10].

REMARK 8 Lemma 9 can be obtained from Lemma 3, of the paper [14].

Consider in more detail the case of $M = 1$. Then $X = [a, b]$, $-\infty < a < b < \infty$. The corresponding Lyapunov function is easily constructed. Noting that $A^{\varphi}[a, b] = [\underline{a}(x), \bar{a}(x)]$, take

$$\Phi'(x) = \begin{cases} \alpha, & \bar{a}(x) < 0, \\ 0, & [\underline{a}(x), \bar{a}(x)] \ni 0, \\ \beta, & \underline{a}(x) > 0, \end{cases}$$

$$\Phi(x) = \int_a^x \Phi'(y)dy \quad ,$$

where $\alpha < 0$, $\beta > 0$. Then Φ satisfies the Lipschitz condition on $[a, b]$ and inequality (37). Thus in order to apply Lemma 9 it remains to show that $\Phi(B^{\varphi}[a, b])$ does not contain non-degenerate segments. It can be done assuming for example, that $B^{\varphi}[a, b]$ contains not more than the countable number of the connected components. However, in the case $M = 1$ it is possible also to prove the convergence without using the Lyapunov function. Let us consider such an approach.

LEMMA 10 *If $M = 1$ and the conditions of Lemma 5 are fulfilled, then the sequence $\{z_s\}$ converges to the set $B^{\varphi}(X)$.*

PROOF According to Corollary 6, the set Z is a connected compact, i.e., $Z = [\gamma, \delta]$, $a \leq \gamma \leq \delta \leq b$. Let $Z \cap Q \neq \emptyset$, where $Q = [a, b]\backslash B^{\varphi}[s, b]$. Set Q is open in $[a, b]$, i.e., representable in the form of the summation of no more than countable number of intervals (α_i, β_i), $i \geq 1$ and, possibly, half-intervals $[a, \alpha_0)$ and $(\beta_0, b]$. There may be two cases: $Z \cap Q$ contains no segments and $Z \cap Q$ contains non-degenerate segment $[\gamma', \delta']$. In the first case Z consists of an unique point a or b. But since it does not belong to $B^{\varphi}[a, b]$ then such a situation is impossible due to the second statement of Lemma 5. In the second case for any point $z \in [\gamma', \delta']$ one of the inequalities $\underline{b}(z) > 0$ or $\bar{b}(z) < 0$ is valid, where $B^{\varphi}(z) = [\underline{b}(z), \bar{b}(z)]$. Let the first inequality take place (arguments for the second one are similar). Since $Z \supset (\gamma', \delta')$, then for any $\epsilon > 0$ such that $(z - \epsilon, z + \epsilon) \subset (\gamma', \delta')$, the segment $[z - \epsilon, z + \epsilon]$ is intersected by the sequence $\{z_s\}$ the infinite number of times. So numbers $p_k < q_k$, $k \geq 1$, can be found, such that $z_{p_k} \geq z + \epsilon$, $z_{q_k} \leq z - \epsilon$,

$z_s \in (z - \epsilon,\ z + \epsilon)$ for $p_k < s < q_k$. Then from equality (34) and Remark 3 we have

$$z_{q_k} - z_{p_k} \geq \rho_{p_k q_k - 1}[\underline{b}(z) - t^z(\epsilon) - \mu_{p_k q_k - 1}] + \delta_{p_k q_k - 1} \ .$$

If ϵ is so small that $t^z(\epsilon) < \underline{b}(z)$, then from relations (32) and (33) we obtain $\underline{\lim}_{k \to \infty} (z_{q_k} - z_{p_k}) \geq 0$. However, according to the construction: $z_{q_k} - z_{p_k} \leq -2\epsilon < 0$ for $k \geq 1$, that contradicts the previous inequality. Therefore, $Z \cap Q = \phi$, i.e., $Z \subseteq B^{\varphi}[a,\ b]$.

The lemma is proved.

Let us characterize those segments which can be entirely completed by the limit points of the sequence $\{z_s\}$. We take $D^{\varphi}[a,\ b] = \{x \in A^{\varphi}[a,\ b]$ and for any small $\epsilon > 0$ within one of the intervals $(x - \epsilon,\ x)$ or $(x,\ x + \epsilon)$ there can be found at least two points x_ϵ^- and x_ϵ^+ from Y, such that $\varphi(x_\epsilon^-) < 0$ and $\varphi(x_\epsilon^+) > 0\}$.

LEMMA 11 *The set $D^{\varphi}[a,\ b]$ is closed.*

PROOF Let $\{x_k\} \subset D^{\varphi}[a,\ b]$, $x_k \neq x$, $k \geq 1$, and $\lim_{k \to \infty} x_k = x$. We will show that $x \in D^{\varphi}[a,\ b]$. For any $\epsilon > 0$ there is a number $k(\epsilon)$, such that for $k \geq k(\epsilon)$, we have $x_k \in (x - \epsilon,\ x + \epsilon)$. Take $\epsilon_k = \min(|x - x_k|,\ |x_k - x - \epsilon|,\ |x_k - x + \epsilon|)$. Since $x_k \in D^{\varphi}[a,\ b]$, then within one of the intervals $(x_k - \epsilon_k,\ x_k)$ or $(x_k,\ x_k + \epsilon_k)$ there will be found at least two points x_k^- and x_k^+ from Y, such that $\varphi(x_k^-) < 0$ and $\varphi(x_k^+) > 0$. Since $\epsilon > 0$ is arbitrary, and by construction, the set $(x_k - \epsilon_k,\ x_k) \cup (x_k,\ x_k + \epsilon_k)$ is contained either in the interval $(x - \epsilon,\ x)$, or in $(x,\ x + \epsilon)$, then the point x belongs to $D^{\varphi}[a,\ b]$.

The lemma is proved.

LEMMA 12 *Let $M = 1$, the relations (26), (29), (30), (31) and (33) are fulfilled, $\mu_s \equiv 0$ and the set $D^{\varphi}[a,\ b]$ is nowhere dense. Then the sequence $\{z_s\}$ converges to points of the set $B^{\varphi}[a,\ b]$.*

PROOF On the basis of Corollary 6, the set Z is a connected compact in R^1, i.e., $Z = [\gamma,\ \delta]$, $a \leq \gamma \leq \delta \leq b$. If $\gamma = \delta$, then the sequence $\{z_s\}$ converges and on the basis of Lemma 5 $\gamma \in B^{\varphi}[a,\ b]$. Suppose that $\gamma < \delta$. The set $D^{\varphi}]a,\ b]$ is nowhere dense, therefore we can find a non-degenerate segment $[\gamma',\ \delta'] \subseteq [\gamma,\ \delta]$ which does not contain the set $D^{\varphi}[a,\ b]$. For any $z \in (\gamma',\ \delta')$ it is possible to find $\epsilon > 0$, such that $(z,\ z + \epsilon) \subset (\gamma',\ \delta')$ and for all $x \in (z,\ z + \epsilon) \cap Y$ we obtain $\varphi(x) \geq 0$ or $\varphi(x) \leq 0$. Suppose that $\varphi(x) \geq 0$ (in other cases the proof is similar). Numbers $p_k < q_k$, $k \geq 1$ exist such that $z_{p_k} \geq z + 3\epsilon/4$, $z_{q_k} \leq z + \epsilon/4$, $z_s \in (z + \epsilon/4,\ \ z + 3\epsilon/4)$ for $p_k < s < q_k$. Then $\overline{\lim}_{k \to \infty} (z_{q_k} - z_{p_k})$

$\leq -\epsilon/2 < 0$. On the other hand, from relations (33) and (34) and the fact that $\mu_s \equiv 0$ we obtain

$$z_{q_k} - z_{p_k} \geq \delta_{p_k q_k - 1} \to 0$$

for $k \to \infty$ i.e., $\underline{\lim}_{k \to \infty} (z_{q_k} - z_{p_k}) \geq 0$.

Obtained inequalities are contradictory. Thus Z consists of the unique point and the sequence $\{z_s\}$ converges to the point of $B^{\varphi}[a, b]$.

REMARK 9 In the non-explicite form, the particular case of Lemma 12 is treated i the paper [7].

The analysis of the arguments, used here for proving the convergence of one-dimensional sequences of type (27) can be used in a more general case.

Let us call M-dimensional function φ as a separable function on Y, if $\varphi^j(y) = g_j(y^j)r^j(y)$ where r^j preserves the sign on Y, i.e., $r^j(y) \geq 0$ or $r^j(y) \leq 0$ for all $y \in Y$, $j = 1, 2, \ldots, M$. Denote the projection of X on j-th coordinate axis as X. Then, due to compactness and connectedness of X and also continuity of projection operator, the set X^j can be either a non-degenerate segment, or a point.

LEMMA 13 *Let condition of Lemma 5 be fulfilled and function φ is separable. Then the sequene $\{z_s\}$ converges to the set $B^{\varphi}(X)$. Moreover, if all connected components $B^{\varphi}(X)$ are single-point ones or $\mu_s \equiv 0$ and set $D^{g_j}(X^j)$ is nowhere dense for those j for which X^j is a non-degenerate segment, then this sequence is convergent.*

To prove this lemma it is sufficient to (almost) repeat proof so Lemmas 10 and 12 for each coordinate of the sequence $\{z_s\}$, and to recollect Corollary 7.

8. CONVERGENCE WITH PROBABILITY 1

Let us use obtained results to prove the convergence with probability 1 of the balls' proportions in the generalized urn scheme to the set $B^f(T_{N-1})$. Let the distance from point t to a compact T be denoted by $\rho(t, T)$.

THEOEREM 1 *Let the following conditions be fulfilled*

1 $\beta_1^i \geq 1$, $i = 1, 2, \ldots, N$;

2 *for all* $x \in L_{N-1}$, $n \geq 1$ $q_n(0, x) \leq \alpha < 1/2$, $\sum_{i \in Z_+^N} q_n(i, x) = 1$ *and for some* $r \geq 2$ $\sum_{i \in Z_+^N} |i|^r q_n(i, x) \leq c < \infty$;

3 *there are functions* $q(i, x)$, $i \in Z_+^N$, $x \in Z_{N-1}$ *such that: for all* $x \in Z_{N-1}$ $\sum_{i \in Z_+^N} q(i, x) = 1$ *for some* $q \geq 2$ $\sum_{i \in Z_+^N} |i|^q q(i, x) \leq c < \infty$ *and for all* $i \in Z_+^N$, $| q(i, x) - q_n(,, x) | \leq \sigma_n$, $\lim_{n \to \infty} \sigma_n = 0$;

4 *there is a Lipschitz on* $\sigma(T_{N-1})$ *function* F *such that for any* $z \in T_{N-1} \backslash B^f(T_{N-1})$ *the inequality* $F^-(z, \| g \|^{-1} g) \leq \mu(z) > 0$ *takes place uniformly for* $g \in A^f(z)$, *and the set* $F(B^f(T_{N-1}))$ *does not contain non-degenerate segments. Then,* $\rho(X_n, B^f(T_{N-1})) \to 0$ *with probability 1, for* $n \to \infty$. *What is more, if all the connected components of the set* $B^f(T_{N-1})$ *are single-point components, then* X_n *converges with probability 1 (to some random vector* X_0, *whose distribution is concentrated on* $B^f(T_{N-1})$*).*

PROOF On the basis of the statement a) – d) of Lemma 1, Corollary 1 and relation (6) we have

$$X_{t+1} = X_t + \gamma_t^{-1} [f(X_t) + \sigma_t(X_t) + \\ + \delta(X_t, \gamma_t) + \eta_t(X_t, \gamma_t)], \; X_t \in L_{N-1}, \; t \geq 1 \quad , \tag{44}$$

$$P\left\{ \overline{\lim_{t \to \infty}} \, \gamma_t^{-1} t < \infty \right\} = 1 \quad , \tag{45}$$

$$P\left\{ \sum_{t \geq 1} \gamma_t^{-1} = \infty \right\} = 1 \quad , \tag{46}$$

$$\| \sigma_t(X_t) \| \leq C \sigma_t^{H/N+H} \to 0 \text{ for } t \to \infty \quad , \tag{47}$$

$$P\left\{ \lim_{s \to \infty} \| \sum_{t \geq s} \gamma_t^{-1} \delta_t(X_t, \gamma_t) \| = 0 \right\} \geq \\ \geq P\left\{ \lim_{s \to \infty} \sum_{t \geq s} C \gamma_t^{-2} = 0 \right\} = 1 \quad , \tag{48}$$

$$P\left\{ \lim_{s \to \infty} \sum_{t \geq s} \gamma_t^{-1} \eta_t(X_t, \gamma_t) = 0 \right\} = 1 \quad . \tag{49}$$

Let

$$\Omega_0 = \left\{ \overline{\lim_{t \to \infty}} \gamma_t^{-1} t < \infty \right\} \cap \left\{ \sum_{t \geq 1} \gamma_t^{-1} = \infty \right\}$$

$$\cap \{ \lim_{s \to \infty} \| \sum_{t \geq s} \gamma_t^{-1} \delta_t(X_t, \gamma_t) \| = 0 \} \cap \left\{ \lim_{s \to \infty} \sum_{t \geq s} \gamma_t^{-1} \eta_t(X_t, \gamma_t) = 0 \right\} .$$

Proceeding from the relations (45), (46), (48) and (49) we obtain

$$P\{\Omega_0\} = 1 \ . \tag{50}$$

Fix an artitrary elementary event $\omega \in \Omega_0$ and consider the following correspondence among terms of the relations (29) and (44):

$$z_s \leftrightarrow X_s, \rho_s \leftrightarrow \gamma_s^{-1}, \varphi \leftrightarrow f, \mu_s \leftrightarrow \sigma_s(X_s) \ ,$$

$$\delta_s \leftrightarrow \delta_s(X_s, \gamma_s) + \eta_s(X_s, \gamma_s), X \leftrightarrow T_{N-1}, Y \leftrightarrow L_{N-1} \ .$$

With regard to the condition 4 and Lemma 9, $\rho(X_t, B^f(T_{N-1})) \to 0$ for $t \to \infty$. Now the first statement of the theorem follows from equality (50). The second statement is obtained using Corollary 7.

REMARK 10 Let $f(x) = 0$ for all $x \in L_{N-1}$. This is the case, for example, for the traditional Polya urn scheme. As it follows from arguments of the Theorem 1 the convergence with probability 1 takes place under conditions 1–3 and the convergence of the series $\sum_{n \geq 1} n^{-1} \sigma_n^{H/N+H}$. As a simple consequence of the proved statement and the Remark 2 we shall formulate the appropriate result for the generalized urn scheme with balls added one at a time.

THEOREM 2 *Let the following conditions be fulfilled*

1. $\beta_1^i \geq 0, i = 1, 2, \dots, N$;
2. *there exists the vector-function* q *such that uniformly for* $x \in Z_{N-1}$, $\| q_n(x) - q(x) \| \leq \sigma_n \to 0$ *for* $n \to \infty$;
3. *there is a Lipschitz on* $\sigma(T_{N-1})$ *function* F, *such that for any* $z \in T_{N-1}\ B^\tau(T_{N-1})$ *the inequality* $F^-(z, \| g \|^{-1} g) \geq \mu(z) > 0$ *takes place uniformly for* $g \in A^\tau(z)$ *and the set* $F(B^\tau(T_{N-1}))$ *does not contain non-degenerate segments, where* $\tau(x) = q(x) - x$.

   Then $\rho(X_n, B^\tau(T_{N-1})) \to 0$, *with the probability 1* $n \to \infty$. *What is more, if all connected components* $B^\tau(T_{N-1})$ *are single-pointed components, then* X_n *converges with probability 1.*

REMARK 11 This theorem generalizes the results known earlier [3]. In particular, it requires less smooth Lyapunov functions, slower decreasing values σ_n and the terminal sets containing the infinite number of connected components.

In the urn scheme with balls added one at a time, we will call a vector-function of the urn q a separable function, if $q^j(x) = x^j + g_j(x^j)\vartheta_j(x)$, where ϑ_j preserves the sign on L_{N-1}, $j = 1, 2, \ldots, N-1$.

Such urn's functions occur in the traditional Polya scheme in the urn scheme with balls of two colors. For such urn functions, Lemma 13, Remark 2 and relations (44)-(49) make it possible to prove the theorem of convergence without using Lyapunov function.

THEOREM 3 *Let conditions 1 and 2 of Theorem 2 be fulfilled, and the vector-function q is separable. Then, with probability 1 $\rho(X_n, B^{\tau}(T_{N-1})) \to 0$ for $n \to \infty$. What is more, if all the connected components of the set $B^{\tau}(T_{N-1})$ are single-point components or $\sum_{n \geq 1} n^{-1}\sigma_n < \infty$ and the sets $D^{g_j}[0, 1]$, $j = 1, 2, \ldots, N-1$, are nowhere dense, then X_n converges with probability 1.*

THEOREM 4 *Let the following conditions be fulfilled*

1 $\beta_1 \geq 1, \beta_2 \geq 1$;

2 *for all $x \in R(0, 1)$, $n \geq 1$, $q_n(0, x) \leq \alpha < 1/2$, $\sum_{i \in Z_+^2} q_n(i, x) = 1$ and $\sum_{i \in Z_+^2} (i^1 + i^2)^{\tau} q_n(i, x) \leq c < \infty$ for some $\tau \geq 2$;*

3 *there are functions $q(i, x)$, $i \in Z_+^2$, $x \in R(0, 1)$ such that for all $x \in R(0, 1)$ $\sum_{i \in Z_+^2} q(i, x) = 1$, $\sum_{i \in Z_+^2} (i^1 + i^1)^q q(i, x) \leq C < \infty$, $q \geq 2$ and for all $i \in Z_+^2$ $|q(i, x) - q_n(i, x)| \leq \sigma_n$, $\lim_{n \to \infty} \sigma_n = 0$. Then with probability 1 $\rho(X_n, B^f[0, 1]) \to 0$, $n \to \infty$. What is more, if all connected components $B^f[0, 1]$ are single-point or $\sum_{n \geq 1} n^{-1}\sigma_n^{H/2+H} < \infty$, $H = \min(\tau, q) - 1$ and the set $D^f[0, 1]$ is nowhere dense, then X_n converges with probability 1.*

Let us give an example, showing that in the case of an urn with balls of two colors added one at a time, the sequence $\{X_n\}$, $n \geq 1$, may have no limit with probability 1, and for each elementary event $\omega \in \tilde{\Omega}$, $P\{\tilde{\Omega}\} = 1$, the limit points of this sequence cover a non-degenerate segment.

Consider such an urn scheme. On iterating the relations (4) for $1 \leq i \leq n$ we have

$$X_{n+1} = \Pi_{i=1}^{n}\left[1 - \frac{1}{\gamma + i}\right] + \sum_{i=1}^{n} \frac{1}{\gamma + i} \Pi_{j=i+1}^{n}\left[1 - \frac{1}{\gamma + j}\right] q_i(X_i) +$$

$$+ \sum_{i=1}^{n} \frac{1}{\gamma + i} \Pi_{j=i+1}^{n}\left[1 - \frac{1}{\gamma + j}\right]\eta_i = \tag{51}$$

$$= I_1(n) + I_2(n) + I_3(n) \ .$$

Uniformly for $n > k$

$$\lim_{k \to \infty} \Pi_{i=k}^{n}\left[1 - \frac{1}{\gamma + i}\right]\frac{h + \gamma}{k + \gamma} = 1 \ . \tag{52}$$

Since $X_1 \in R(0, 1)$ then

$$I_1(n) \to 0 \ \text{ for } n \to \infty \tag{53}$$

Let $I(n) = 1/(\gamma + n)\sum_{i=1}^{n} q_i(X_i)$, $n \geq 1$. On the basis of equality

$$\Pi_{i=k}^{n}\left[1 - \frac{1}{\gamma + i}\right] = \frac{k + \gamma}{n + \gamma}(1 + \epsilon_{kn}) \ ,$$

where $\lim_{k \to \infty} \sup_{n \geq k} |\epsilon_{kn}| = 0$. Since for all $n \geq 1$ $q_n(X_n) \in [0, 1]$, then

$$| I_2(n) - I(n) | \leq \frac{1}{\gamma + n} \sum_{i=1}^{n} | \epsilon_{in} | \to 0 \ \text{ for } n \to \infty \tag{54}$$

Note, that the term $I_3(n)$ satisfies the recurrent relation

$$I_3(n + 1) = \left[1 - \frac{1}{\gamma + n + 1}\right] I_3(n) + \frac{1}{\gamma + n + 1} \eta_{n+1}$$

Therefore, since

$$E(\eta_n | T_n) = 0, \ E(\eta_n^2 | T_n) = (1 - q_n(X_n))q_n(X_n)$$

then

$$E[I_3(n + 1)^2 | T_{n+1}] = \left[1 - \frac{1}{\gamma + n + 1}\right]^2 I_3(n)^2 +$$

$$+ \frac{1}{(\gamma + n + 1)^2} E(\eta_{n+1}^2 | T_{n+1}) \leq \left[1 - \frac{1}{\gamma + n + 1}\right]^2 I_3(n)^2 + \tag{55}$$

$$+ \frac{1}{(\gamma + n + 1)^2} E(\eta_{n+1}^2 | T_{n+1}) \leq \left[1 - \frac{1}{\gamma + n + 1}\right]^2 I_3(n)^2 +$$

$$+ (\gamma + n + 1)^{-2}, \ n \geq 1 \ ,$$

where T_n is σ-algebra, generated by $X_1, X_2, \ldots, X_n$, Strengthening the inequality (55), we have

$$E[I_3(n+1)^2 \mid T_{n+1}] \le I_3(n)^2 + (\gamma + n + 1)^{-2},\ n \ge 1 \ .$$

Hence the pair $\{\tilde{I}_3(n) = I_3(n) + \sum_{i \ge n}(\gamma + i)^{-2},\ T_{n+1}\}$ forms non-negative supermartingal and, consequently, there is a random variable I, such that with probability 1

$$\tilde{I}_3(n) \to I,\ n \to \infty \ .$$

Since $\sum_{i \ge n}(\gamma + i)^{-2} \to 0$ for $n \to \infty$, then for $n \to \infty$

$$I_3(n) \to I \text{ with probability 1} \tag{56}$$

Proceeding from inequality (55) for any $\Delta \in (1, 2)$, $n \ge n(\Delta)$ we obtain

$$EI_3(n+1)^2 \le EI_3(n)^2\left[1 - \frac{\Delta}{\gamma + n + 1}\right] + (\gamma + n + 1)^{-2} \ ,$$

and

$$EI_3(n+1)^2 \le \Pi_{i=1}^{n+1}\left[1 - \frac{\Delta}{\gamma + n + 1}\right]EI_3(n)^2 +$$

$$+ \sum_{i=1}^{n+1}(\gamma + i + 1)^{-2}\Pi_{j=i+1}^{n+1}\left[1 - \frac{\Delta}{\gamma + j + 1}\right] \ .$$

Using equality (52) and the last estimate, we obtain

$$\overline{\lim_{n \to \infty}}\ EI_3(n)^2(\gamma + n) < \infty \ .$$

Hence, on the basis of Chebyshev inequality we have

$$I_3(n) \to 0 \text{ with probability 1 for } n \to \infty \ . \tag{57}$$

With regard to the relations (56) and (57) $I = 0$, and, consequently with probability 1

$$I_3(n) \to 0,\ n \to \infty \tag{58}$$

From relations (51), (53), (54) and (58) we obtain that with probability 1

$$|X_n - I(n)| \to 0,\ n \to \infty \tag{59}$$

Thus, for $n \to \infty$ the limiting behavior X_n coincides with limiting behavior of $I(n)$. Now it si clear that choosing appropriate functions q_n, $n \ge 1$, it is possible to obtain that X_n,

$n \geq 1$ would not converge to the limit. Restrict ourselves to q_n, $n \geq 1$, which are constant over $R(0, 1)$.

Let $0 < \delta < \alpha < \beta < \sigma < 1$ and suppose that $\beta_1 = \beta_2 = \cdots = \beta_{n_0} = \alpha$, $n_0 = \max(1, [(\beta - \alpha)^{-1} \max(\sigma - \alpha, \beta - \delta)] - \gamma + 2)$, where square brackets denote the integer part of the number;

$$\beta_{n_0+1} = \beta_{n_0+2} = \cdots = \beta_{n_1} = \sigma,\ n_1 = \max\{n : n > n_0, \frac{n_0\alpha + (n - n_0)\sigma}{n + \gamma} \leq \beta\}$$

$$\beta_{n_{1+1}} = \beta_{n_1+2} = \cdots = \beta_{n_2} = \delta,\ n_2 = \max\{n : n > n_1 ,$$

$$\frac{n_0\alpha + (n_1 - n_0)\sigma + (n - n_1)\delta}{n + \gamma} \geq \alpha\} ,$$

$$\beta_{n_2+1} = \beta_{n_2+2} = \cdots = \beta_{n_3} = \sigma,\ n_3 = \max\{n : n > n_2 ,$$

$$\frac{n_0\alpha + (n - n_2 + n_1 - n_0)\sigma + (n_2 - n_1)\delta}{n + \gamma} \leq \beta\} \text{ and so on } .$$

Due to the construction of the sequence $\{\beta_n\}$ we have $\beta_n \in [\delta, \sigma]$ for all n and $\{1/(\gamma + n)\sum_{i=1}^{n}\beta_i\} = [\alpha, \beta]$. Suppose that $q_n(x) = \beta_n$ $x \in R(0, 1)$, $n \geq 1$. Denoting by $\tilde{\Omega}$ the event when relation (53) occurs we have $\{\tilde{\Omega}\} = 1$ and for any elementary event the set of limit points $\{X_n\}$, $n \geq 1$ is $[\alpha, \beta]$.

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