WORKING PAPER

GRAPHICAL CONVERGENCE OF
SET-VALUED MAPS

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IIASA
International Institute
for Applied Systems Analysis

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FOREWORD

This is an introduction to graphical convergence of set-valued maps and the epigraphical convergence of extended real-valued functions[1].

It is now well established that maps, and more generally, set-valued maps, should be regarded not only as maps from one space to another, but should be characterized in an intrinsic and symmetric way by their graphs.

When dealing with limits of maps, either single-valued or set-valued, it is quite advantageous to overcome the natural reluctance to handle convergence of subsets and to replace pointwise convergence by "graphical convergence": Instead of studying (more or less uniform) limits of the images, one consider the limits of their graphs.

One of the main reasons is that doing so is that a map and its inverse are treated on the same footing . This is quite important in approximation theory and numerical analysis.

The concepts of graphical convergence of set-valued maps are related to the concepts of epigraphical limits of functions, which had recently met an important success to overcome the failure of pointwise convergence in many problems of calculus of variations, optimization, stochastic programming, etc.

Finally, this report provides a first study of the Kuratowski upper and lower limits of tangent cones, which is needed to compute generalized derivatives and epi-derivatives of graphical and epigraphical limits of maps and functions.

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[1]in the development of which IIASA played an important role.

Introduction

Here are some basic notes on graphical limits of single-valued and/or set-valued maps.

To deal with limits is the basis of analysis, or approximation theory, where we have to approximate objects by richer and more familiar ones. And numerical analysis is just taking up this issue for very practical purposes. In particular, solving equations and approximating their solutions is the ultimate task of many mathematicians. It can be regarded in this way:

We want to approximate a solution x_0 to an equation $f(x_0) = y_0$, by approximating both the data y_0 by a sequence of approximate data y_n's, and the map f by a sequence of maps f_n.

Knowing how to find solutions to the approximate problems $f_n(x_n) = y_n$, the problem is how to derive the convergence of the approximate solutions from the convergence of the data y_n and f_n and thus, to pay some attention to the limits of maps.

It is now well established that maps, and more generally, set-valued maps, should be regarded not only as maps from one space to another, but should characterized in an intrinsic and symmetric way by their graphs.

This point of view, which goes back to the protohistory of analysis with Fermat and Descartes dealing with curves rather than functions, has been let aside since for a long time. In particular, as far as limits of functions and maps go, generations of mathematicians have be accustomed to deal with many concepts of convergence of functions, from pointwise to uniform, but all based on the fact that a map is a map, and not a graph.

When dealing with limits of maps, either single-valued or set-valued, it is quite advantageous to overcome the natural reluctance to handle convergence of subsets and to replace pointwise convergence by "graphical convergence": Instead of studying (more or less uniform) limits of the images, we consider the limits of their graphs[2].

One of the main reasons is that doing so is that we treat on the same footing a map and its inverse. This is quite important in approximation

[2]This point of view of regarding maps as graphs has already allowed to build a successful differential calculus of set-valued maps based on "graphical derivatives" rather than set-valued limits of differential quotients.

theory[3], where the problem is to derive pointwise convergence of the inverses from the pointwise convergence of the maps.

Hence, a first problem is to study what are the relations between graphical and pointwise convergence. The main tool for that is the Stability Theorem[4], an outgrowth of the Inverse Function Theorem for set-valued maps[5].

Since graphical limits are limit of graphs, which are subsets, we have to rely on the now classical concepts of Kuratowski upper and lower limits. Hence we begin by a short exposition of these concepts. With the Stability Theorem on one hand, and the concept of proper maps on the other, we are also able to complement the calculus of Kuratowski upper and lower limits of sets, and in particular, to provide criteria for having the natural formulas for direct and inverse images of Kuratowski limits.

After defining and providing the basic properties of graphical convergence, we expose its applications to viability theory, where we show, for instance that the Kuratowski upper limit of viability domains is a viability domain of graphical upper limits.

Finally, we relate the concepts of graphical convergence of set-valued maps to the concepts of epigraphical limits of functions, which has recently met an important success to overcome the failure of pointwise convergence in many problems of calculus of variations, optimization, stochastic programming[6], etc.

The use of this concept is mandatory whenever the order relation of the real line comes into play, as in optimization or Lyapunov stability for instance. In such cases, a real-valued function is replaced by the set-valued map obtained by adding to it the positive cone (for minimization), whose graph is thus the epigraph of the function. Therefore, the convergence of the graphs of such set-valued maps is the convergence of the epigraphs of the associated functions.

We present just a selection of issues dealing with epi-convergence, among

[3] see [12, Stability Theorem 1.1], which adapt to the general case of solving inclusions the principle stating that stability, convergence of the data and "consistency" imply the convergence of the solutions. Consistency is nothing other than graphical lower convergence.

[4] see [12, Proposition 1.1]

[5] see [10, Chapter 7] and [11]

[6] see for instance the book [8] and the bibliography of this book.

which some formulas dealing with the epi-limits of sum and products of functions.

Finally, we relate these concepts of Kuratowski limits with the ones of tangent cones[7], which lay the foundations of the differential calculus of set-valued maps, and which play such an important role in optimization and viability theory.

The problem we begin to consider is to study the Kuratowski upper and lower limits of the tangent cones to subsets K_n in terms of Kuratowski upper and lower limits of sets of the form $(K_n - x_n)/h_n$ when $K_n \ni x_n \to x$ and $h_n \to 0+$, which we call respectively asymptotic paratingent and circatangent cone. One would like to relate them to the tangent cones to the Kuratowski limits of a sequence of subsets K_n, but this seems rather difficult outside the convex realm. Let us just point out that Clarke tangent cones and Bouligand's paratingent cones to a subset K can be regarded as asymptotic circatangent and paratingent cones for constant sequences.

[7]By the way, the various definitions of tangent cones are Kuratowski upper and lower limits of the sets $(K - x)/h$ when $h \to 0+$

Contents

1 Kuratowski Upper and Lower limits

We can also characterize closed set-valued maps and lower semicontinuous set-valued maps through adequate concepts of limits. For that purpose, we introduce the following notations: we associate with a set-valued map $F : X \rightsquigarrow Y$ and $x \in X$ the subsets

$$(1) \quad \begin{cases} i) & \limsup_{x' \to x} F(x') \\ & := \; \{y \in Y \mid \liminf_{x' \to x} d(y, F(x')) = 0\} \\ ii) & \liminf_{x' \to x} F(x') \\ & := \; \{y \in Y \mid \lim_{x' \to x} d(y, F(x')) = 0\} \end{cases}$$

They are obviously **closed**. We also see at once that

$$(2) \quad \liminf_{x' \to x} F(x') \subset \overline{F(x)} \subset \limsup_{x' \to x} F(x')$$

The other advantage of introducing these notions is that we can define kinds of semicontinuity for "discrete" set-valued maps, i.e., "semi" limits of sequences of subsets K_n of a metric space X: we set

$$(3) \quad \begin{cases} i) & \limsup_{n \to \infty} K_n \\ & = \; \{y \in Y \mid \liminf_{n \to \infty} d(y, K_n) = 0\} \\ ii) & \liminf_{n \to \infty} K_n \\ & := \; \{y \in Y \mid \lim_{n \to \infty} d(y, K_n) = 0\} \end{cases}$$

When K_n is a sequence of subsets of a metric space X, we shall also use the following notations:

$$\begin{cases} i) & K^\sharp \; := \; \limsup_{n \to \infty} K_n \\ ii) & K^\flat \; := \; \liminf_{n \to \infty} K_n \end{cases}$$

Definition 1.1 *When $F : X \rightsquigarrow Y$ is a set-valued map, we say that*

$$\limsup_{x' \to x} F(x')$$

is the Kuratowski *(or* Kuratowski-Painlevé *)* **upper limit** *of $F(x')$ when $x' \to x$ and that*

$$\liminf_{x' \to x} F(x')$$

is the Kuratowski *(or* Kuratowski-Painlevé *)* **lower limit** *of* $F(x')$ *when* $x' \to x$.

When K_n *is a sequence of subsets of a metric space* X, *we say that*

$$\limsup_{n\to\infty} K_n \quad \textit{and} \quad \liminf_{n\to\infty} K_n$$

are the **upper and lower Kuratowski limits** *of the sequence* K_n *respectively. A subset* K *is said to be the Kuratowski limit of the sequence* K_n *if*

$$K = \liminf_{n\to\infty} K_n = \limsup_{n\to\infty} K_n =: \lim_{n\to\infty} K_n$$

We observe at once that the Kuratowski upper limits and Kuratowski lower limits of either $F(x)$ or K_n and of either $\overline{F(x)}$ or $\overline{K_n}$ do coincide, since $d(y, K_n) = d(y, \overline{K_n})$.

Any decreasing sequence of subsets K_n has a limit, which is the intersection of their closure:

$$\text{if } K_n \subset K_m \text{ when } n \geq m, \text{ then } \lim_{n\to\infty} K_n = \bigcap_{n\leq 0} \overline{K_n}$$

Remark

The use of the concept of **filter** would avoid to duplicate these definitions in the discrete and continuous cases. We have preferred this longer, may be more pedagogical, solution. □

It is easy to observe that:

Proposition 1.1 *A point* (x, y) *belongs to the* closure of the graph *of a set-valued map* $F : X \rightsquigarrow Y$ *if and only if* $y \in \limsup_{x'\to x} F(x')$ *and* F *is* lower semicontinuous *at* x *if and only if* $F(x) \subset \liminf_{x'\to x} F(x')$

If K_n *is a sequence of subsets of a metric space* X, *then* $\liminf_{n\to\infty} K_n$ *is the* **set of limits of sequences** $x_n \in K_n$ *and* $\limsup_{n\to\infty} K_n$ *is the* **of cluster points of sequences** $x_n \in K_n$ *, i.e., of limits of subsequences* $x_{n'} \in K_{n'}$ *. It is also the subset of cluster points of "approximate" sequences satisfying:*

$$(4) \qquad \forall \epsilon > 0, \quad \exists N(\epsilon) \mid \forall n > N(\epsilon), \quad x_n \in B(K_n, \epsilon)$$

Then we can measure the lack of closedness (of the graph) or the lack of lower semicontinuity by the discrepancy between the values at x of the set-valued maps $F(x)$, $\liminf_{x' \to x} F(x')$and $\limsup_{x' \to x} F(x')$.

Another useful and easy consequence of the Kuratowski limits is the following diagonalization lemma.

Lemma 1.1 *Let us consider a "double" sequence of elements $x_{m,n}$ of a metric space X, such that $\lim_{m \to \infty} \lim_{n \to \infty} x_{m,n}$ does exist. Then there exist sequences $n \to m(n)$ and $m \to n(m)$ such that*

$$\text{(5)} \qquad \begin{cases} \lim_{m \to \infty} \lim_{n \to \infty} x_{m,n} \\ = \lim_{m \to \infty} x_{m,n(m)} \\ = \lim_{n \to \infty} x_{m(n),n} \end{cases}$$

Proof We set $x := \lim_{m \to \infty} \lim_{n \to \infty} x_{m,n}$

1. Let us set $x_m := \lim_{n \to \infty} x_{m,n}$ and $K_n := \{x_{m,n}\}_{m \geq 0}$. Therefore x_m belongs to $\liminf_{n \to \infty} K_n$ for all m. This implies that the limit x of the elements x_m belongs also to $\liminf_{n \to \infty} K_n$, and therefore, is the limit of a sequence of elements y_n belonging to K_n. Such elements can be written $y_n = x_{m(n),n}$.

2. Let us set $L_m := \{x_{m,n}\}_{n \geq 0}$. We observe that each x_m belongs to $\overline{L_m}$. Hence the limit x of the sequence x_m belongs to $\liminf_{m \to \infty} \overline{L_m}$ which is equal to $\liminf_{m \to \infty} L_m$. Consequently, x is the limit of elements $z_m \in L_m$ which can be written $z_m = x_{m,n(m)}$. □

We observe also the quite impressive following equalities:

$$\text{(6)} \qquad \begin{cases} i) & \limsup_{x' \to x} F(x') = \bigcap_{\eta > 0} \overline{\bigcup_{x' \in B(F(x'),\eta)} F(x')} \\ & = \bigcap_{\epsilon > 0} \bigcap_{\eta > 0} \bigcup_{x' \in B(x,\eta)} B(F(x', \epsilon) \\ ii) & \limsup_{n \to \infty} K_n = \bigcap_{N > 0} \overline{\bigcup_{n \geq N} K_n} \\ & = \bigcap_{\epsilon > 0} \bigcap_{N > 0} \bigcup_{n \geq N} B(K_n, \epsilon) \\ iii) & \liminf_{x' \to x} F(x') \\ & = \bigcap_{\epsilon > 0} \bigcup_{\eta > 0} \bigcap_{x' \in B(x,\eta)} B(F(x'), \epsilon) \\ iv) & \liminf_{n \to \infty} K_n \\ & = \bigcap_{\epsilon > 0} \bigcup_{N > 0} \bigcap_{n \geq N} B(K_n, \epsilon) \end{cases}$$

By replacing the balls of a metric space by neighborhoods, we can extend through these formulas the concepts of Kuratowski upper and lower limits to subsets of a topological space.

Many properties of closed and/or lower semicontinuous set-valued maps can be extended to the Kuratowski's limits. For instance,

Theorem 1.1 *Let $K \subset X$ satisfy the following property:*

(7) for all neighborhood $\mathcal{U}$ *of* K, $\exists N \mid \forall n > N, \quad K_n \subset \mathcal{U}$

Then

(8) $$\limsup_{n \longrightarrow \infty} K_n \quad \subset \quad \overline{K}$$

The converse statement is true for any neighborhood $\mathcal{U}$ whose complement is compact.

In particular, if the space X is compact *, then the upper limit $K^\sharp$ enjoys the above property (and thus, is the smallest closed subset satisfying it).*

Proof

The first statement is obvious. For proving the second, let y belong to the complement M of $\mathcal{U}$, which is compact by assumption in the first case or because it is contained in the compact set X in the second case. Then there exist $\epsilon_y > 0$ and N_y such that, for all $n \geq N_y$, y does not belong to $B(K_n, 2\epsilon_y)$. Since M is compact, it can be covered by p balls $B(y_i, \epsilon_{y_i})$. Our claim holds true for all n larger than $N := \max_{i=1,\ldots,p} N_{y_i}$. $\square$

We also remark the following obvious properties:

Proposition 1.2 *Let K_n, L_n be sequences of subsets of a metric space X. Then*

$$
(9)\quad \left\{
\begin{array}{ll}
i) & \limsup_{n\to\infty}(K_n \cap L_n) \\
 & \subset\ \limsup_{n\to\infty} K_n \cap \limsup_{n\to\infty} L_n \\
ii) & \liminf_{n\to\infty}(K_n \cap L_n) \\
 & \subset\ \liminf_{n\to\infty} K_n \cap \liminf_{n\to\infty} L_n \\
iii) & \limsup_{n\to\infty}(K_n \cup L_n) \\
 & =\ \limsup_{n\to\infty} K_n \cup \limsup_{n\to\infty} L_n \\
iv) & \liminf_{n\to\infty}(K_n \cup L_n) \\
 & \supset\ \liminf_{n\to\infty} K_n \cup \liminf_{n\to\infty} L_n \\
v) & \limsup_{n\to\infty} \prod_{i=1}^{n} K_n^i \\
 & \subset\ \prod_{i=1}^{n} \limsup_{n\to\infty} K_n^i \\
vi) & \liminf_{n\to\infty} \prod_{i=1}^{n} K_n^i \\
 & =\ \prod_{i=1}^{n} \liminf_{n\to\infty} K_n^i
\end{array}
\right.
$$

We need also to relate direct and inverse images of Kuratowski upper and lower limits to the Kuratowski upper and lower limits of their direct and inverse images. We mention now the obvious relations and postpone the proofs of criteria which transform the following inclusions to equalities.

Proposition 1.3 *Let K_n be a sequence of subsets of a metric space X, M_n be a sequence of subsets of a metric space Y and $f : X \longmapsto Y$ be a (single-valued) continuous map.*

Then

$$
(10)\quad \left\{
\begin{array}{llll}
i) & f(\limsup_{n\to\infty} K_n) & \subset & \limsup_{n\to\infty} f(K_n) \\
ii) & \limsup_{n\to\infty} f^{-1}(M_n) & \subset & f^{-1}(\limsup_{n\to\infty} M_n) \\
iii) & f(\liminf_{n\to\infty} K_n) & \subset & \liminf_{n\to\infty} f(K_n) \\
iv) & \liminf_{n\to\infty} f^{-1}(M_n) & \subset & f^{-1}(\liminf_{n\to\infty} M_n)
\end{array}
\right.
$$

Kuratowski upper limits and Kuratowski lower limits can be exchanged by duality:

Proposition 1.4 *Let K_n be a sequence of subsets of a Banach space Y. Then*

$$
(11)\qquad \liminf_{n\to\infty} K_n \ \subset\ \limsup_{n\to\infty} K_n^-)^-
$$

The equality holds true when the dimension of X is finite and when the subsets K_n are closed convex cones.

Proof

Let us choose x in $\liminf_{n\to\infty} K_n$ and p in $\limsup_{n\to\infty} K_n^-$. Then there exist a sequence of elements $x_n \in K_n$ converging to x and a subsequence of elements $q_{n'}$ of $K_{n'}^-$ converging to p. Therefore

$$< p, x > \;=\; \lim_{n\to\infty} < q_{n'}, x_{n'} > \;\leq\; 0$$

Then property (11) is checked.

Conversely, let x belong to $(\limsup_{n\to\infty} K_n^-)^-$. We have to prove that the projections $x_n := \pi_{K_n}(x)$ converge to x. But we know that $p_n := x - p_n$ belongs to K_n^- and satisfy $< p_n, x_n >= 0$. Since the dimension of Y is finite, we deduce that subsequences (again denoted) p_n and x_n converge to p and $x-p$ respectively. Since $< p, x-p >= 0$, we deduce that $\|p\|^2 =< p, x >\leq 0$ since p, a cluster point of the sequence p_n, does belong to $\limsup_{n\to\infty} K_n^-$.
□

2 Kuratowski Limits in Lebesgue Spaces

Let (Ω, S, μ) be a measure space and X be a finite dimensional vector-space. Let us consider a sequence of measurable set-valued maps

$$K_n : \omega \in \Omega \rightsquigarrow K_n(\omega) \subset X.$$

We associate to it the subsets $\mathcal{K}_n$ of $L^p(\Omega, X)$ defined by

$$\mathcal{K}_n \;:=\; \{\, x(\cdot) \in L^p(\Omega, X) \;\mid\; \text{for almost all } \omega \in \Omega,\;\; x(\omega) \in K_n(\omega) \,\}$$

The purpose of the next theorem is to compare the Kuratowski limits of the sets $\mathcal{K}_n$ and the sets of selections $x(\cdot)$ of the Kuratowski limits of the sets $K_n(\omega)$.

Theorem 2.1 *Let us assume that the set-valued maps K_n are measurable and that the subsets $\mathcal{K}_n$ are not empty. Then*

$$\begin{cases} \{x(\cdot) \in L^p(\Omega, X) \mid \text{for almost all } \omega,\;\; x(\omega) \in \liminf_{n\to\infty} K_n(\omega)\} \\ \subset\;\; \liminf_{n\to\infty} \mathcal{K}_n \;\;\subset\;\; \limsup_{n\to\infty} \mathcal{K}_n \\ \subset\;\; \{x(\cdot) \in L^p(\Omega, X) \mid \text{for almost all } \omega,\;\; x(\omega) \in \limsup_{n\to\infty} K_n(\omega)\} \end{cases}$$

Proof

1. Let $x(\cdot)$ belong to the first subset. Then the functions $a_n(\cdot)$ defined by

$$a_n(\omega) \ := \ d(x(\omega), K_n(\omega))$$

are measurable and converge to 0 almost everywhere. Let us choose some $y_n(\cdot)$ in $\mathcal{K}_n$, which is not empty by assumption. Since

$$\text{for almost all } \ \omega, \ \ a_n(\omega) \ \|x(\omega) - y_n(\omega)\|$$

and since the right-hand side of this inequality belongs to $L^p(\Omega)$, we deduce from Lebesgue's Theorem that the functions $a_n(\cdot)$ do converge to 0 in $L^p(\Omega)$. Let us introduce now the subsets $L_n(\omega)$ defined by

$$L_n(\omega) \ := \ \{z \in \overline{K_n(\omega)} \ \mid \ \|x(\omega) - z\| = a_n(\omega)\}$$

It is clear that the set-valued map $L_n(\cdot)$ is also measurable. The Measurable Selection Theorem allows us to choose a measurable selection $z_n(\cdot)$ of the set-valued map $L_n(\cdot)$. It belongs to $L^p(\Omega)$ since

$$\text{for almost all } \ \omega, \ \ \|z_n(\omega)\| \leq \|x(\omega)\| + a_n(\omega)$$

Therefore $z_n(\cdot)$ belongs to $\mathcal{K}_n$ and converges to $x(\cdot)$ in $L^p(\Omega)$, i.e., $x(\cdot)$ does belong to the Kuratowski lower limit of the subsets $\mathcal{K}_n$.

2. Let us choose some $x(\cdot)$ in the Kuratowski upper limit of the subsets $\mathcal{K}_n$. Then there exists a subsequence of elements $z_{n'}(\cdot)$ of $\mathcal{K}_{n'}$ converging to $x(\cdot)$ in $L^p(\Omega)$. Then a subsequence (again denoted) $z_{n'}(\cdot)$ converges almost everywhere to $x(\cdot)$ and consequently, for almost all ω, $x(\omega)$ belongs to the Kuratowski upper limit of the subsets $K_n(\omega)$. $\square$

3 Stability Theorem

We shall prove the following Inverse Stability Theorem which has many useful consequences. Let us recall that when $x \in K$, we denote by

$$S_K(x) \ := \ \bigcup_{h>0} \frac{K - x}{h}$$

the cone spanned by $K - x$ and by

$$T_K(x) \ := \ \limsup_{h \to 0+} \frac{K - x}{h}$$

the contingent cone to K at x.

Theorem 3.1 (Inverse Stability Theorem) *Let X and Y be two Banach spaces. We introduce a sequence of continuous linear operators $A_n \in \mathcal{L}(X, Y)$, a sequence of closed subsets $K_n \subset X$.*

Let us consider elements x_n^* of the subsets K_n such that both x_n^* converges to x_0^* and $A_n x_n$ converges to y_0.

We posit the following **stability assumption***: there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$(12) \qquad \begin{cases} \forall x_n \in K_n \cap B(x_0, \eta), \\ A_n S_{K_n}(x_n) \cap B_Y \ \subset \ A_n \left(T_{K_n}(x_n) \cap cB_X\right) + \alpha B_Y \end{cases}$$

Let us set $l := c/(1 - \alpha)$, $\rho < \eta/3l$ and consider elements y_n and x_{0n} satisfying:

$$(13) \qquad \begin{cases} i) \quad x_{0n} \ \in \ K_n \cap B(x_0, \eta/3), \ A_n x_{0n} \ \in \ B(y_0, \rho) \\ ii) \quad y_n \ \in \ A_n(K_n) \cap B(y_0, \eta/3) \end{cases}$$

Then, for any $l' > l$ and $n > 0$, there exist solutions $\widehat{x_n}$ satisfying

$$(14) \qquad \begin{cases} i) \quad \widehat{x_n} \in K_n \ \& \ A_n \widehat{x_n} \ = \ y_n \\ ii) \quad \|\widehat{x_n} - x_{0n}\| \ \leq \ l'\|y_n - A_n x_{0n}\| \end{cases}$$

so that

$$(15) \qquad \begin{cases} d(x_0, K_n \cap A_n^{-1}(y_n)) \ \leq \ l\|y_n - A_n x_{0n}\| \\ \leq \ \|x_0 - x_{0n}\| + l\|y_n - y_0\| + l\|y_0 - A_n x_{0n}\| \end{cases}$$

converges to 0 when x_{0n} converges to x_0 and both $A_n x_{0n}$ and $y_n \in A_n K_n$ converge to y_0.

Proof We choose $\epsilon > 0$ such that

$$(16) \qquad \frac{3\rho}{\eta} \ < \ \epsilon \ < \ \frac{1 - \alpha}{c} \ =: \frac{1}{l}$$

and we consider the elements x_{0n} and y_{0n} satisfying (13).

By Ekeland's Variational Principle (see [23]) , we know that there exists a solution $\widehat{x_n}$ to

$$(17) \quad \begin{cases} i) & \|y_n - A_n\widehat{x_n}\| + \epsilon\|\widehat{x_n} - x_{0n}\| \leq \|y_n - A_n x_{0n}\| \\ ii) & \forall x_n \in K_n, \ \|y_n - A_n\widehat{x_n}\| \leq \|y_n - A_n x_n\| + \epsilon\|x_n - \widehat{x_n}\| \end{cases}$$

We deduce from inequality (17)i) that

$$\|\widehat{x_n} - x_{0n}\| \leq \frac{1}{\epsilon}\|y_n - A_n x_{0n}\| \leq \rho/\epsilon \leq \eta/3$$

so that $\|\widehat{x_n} - x_0\| \leq \eta/3 + \|x_{0n} - x_0\| \leq 2\eta/3$.

Since $y_n - A_n\widehat{x_n} \in A_n(K_n - \widehat{x_n})$, assumption (12) implies that there exist $u_n \in T_{K_n}(\widehat{x_n})$ and $w_n \in Y$ satisfying

$$(18) \quad \begin{cases} i) & y_n - A_n\widehat{x_n} = A_n u_n + w_n \\ ii) & \|u_n\| \leq c\|y_n - A_n\widehat{x_n}\| \ \& \ \|w_n\| \leq \alpha\|y_n - A_n\widehat{x_n}\| \end{cases}$$

By definition of the contingent cone, there exist elements $h > 0$ and $e_h \in X$ converging to $0+$ and 0 respectively such that

$$x_n := \widehat{x_n} + hu_n + he_h \in K_n$$

By taking in inequality (17)ii) such an x_n, by observing that $y_n - A_n x_n = (1-h)(y_n - A_n\widehat{x_n}) + hw_n - he_h$, we deduce that

$$(19) \qquad h\|y_n - A_n\widehat{x_n}\| \leq h\|w_n\| + h\|A_n e_h\| + \epsilon h\|u_n + e_h\|$$

Dividing by $h > 0$ and letting h (and thus, e_h) converge to 0, we get:

$$(20) \qquad \|y_n - A_n\widehat{x_n}\| \leq \|w_n\| + \epsilon\|u_n\| \leq (\alpha + \epsilon c)\|y_n - A_n\widehat{x_n}\|$$

Since we have chosen ϵ such that $\alpha + \epsilon c < 1$, we infer that $\widehat{x_n}$ is a solution to

$$x_n \in K_n \ \& \ A_n\widehat{x_n} = y_n$$

satisfying

$$\|\widehat{x_n} - x_{0n}\| \leq \frac{1}{\epsilon}\|y_n - A_n x_{0n}\| \leq l\|y_n - A_n x_{0n}\| \quad \square$$

As a consequence, we obtain the following important statement.

Theorem 3.2 (Inverse Function Theorem) *Let X and Y be two Banach spaces. We introduce a sequence of continuous linear operators $A_n \in \mathcal{L}(X,Y)$, a sequence of closed subsets $K_n \subset X$.*

Let us consider elements $x_n^\star$ of the subsets K_n such that both $x_n^\star$ converges to $x_0^\star$ and $A_n x_n$ converges to y_0.

We posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$
(21) \qquad \begin{cases} \forall\, x_n \in K_n \cap B(x_0,\eta), \\ B_Y \;\subset\; A_n(T_{K_n}(x_n) \cap cB_X) + \alpha B_Y \end{cases}
$$

Then for any sequence of elements x_{0n} of K_n converging to x_0 and such that $A_n(x_{0n})$ converges to y_0 and any sequence of elements $y_n \in Y$ converging to y_0, we have

$$
d(x_{0n}, K_n \cap f_n^{-1}(y_n)) \;\leq\; l\|y_n - f_n(x_{0n})\|
$$

The stability assumption (21) implies implicitly that x_0 belongs to the lim inf of the subsets K_n. We consider now the lim inf of the contingent cones[8]

$$
T(x_0) := \liminf_{K_n \ni x_n \to x_0} T_{K_n}(x_n) \;=\; \bigcap_{\epsilon>0} \bigcup_{N,\eta} \bigcap_{n \geq N, x_n \in K_n \cap (x+\eta B)} T_{K_n}(x_n) + \epsilon B
$$

and we address the following question: under which conditions does the "pointwise surjectivity assumption"

$$
AT(x_0) \;=\; Y
$$

imply the above stability assumption of the K_n. The next result answers this question when the dimension of Y is finite, unfortunately.

Proposition 3.1 (Pointwise Stability Criterion) *Assume that $T(x_0)$ is convex[9] and that $AT(x_0) = Y$. Then there exists a constant $c > 0$ such that, for all $\alpha \in]0,1[$, there exist $\eta > 0$ and $N \geq 1$ with the following*

[8] which is equal to the asymptotic circatangent cone when the dimension of X is finite. See Proposition 11.2 below.

[9] This is the case when the dimension of X is finite thanks to Proposition 11.2 below.

property: $\forall\, v \in Y,\ \forall\, n \geq N,\ \forall\, x_n \in K_n \cap (x_0 + \eta B)$, *there exist solutions* $u_n \in T_{K_n}(x_n)$ *and* $w_n \in Y$ *to*

$$(22) \qquad Au_n \;=\; v + w_n, \quad \|u_n\|_Z \;\leq c\, \|v\|_Y, \quad \|w_n\|_Y \leq \alpha \|v\|_Y$$

Proof Let S denote the unit sphere of Y, which is compact. Hence there are p elements v_i such that the balls $v_i + \frac{\alpha}{2} B_H$ cover S. Since $T(x_0)$ is convex and $AT(x_0) = Y$, Robinson-Ursescu's Theorem implies the existence of a constant $\lambda > 0$ such that we can associate with any $v_i \in S$ an $u_i \in T(x_0)$ satisfying $\|u_i\|_Z \leq \lambda$. By the very definition of $T(x_0)$, we can associate with $\alpha \in]0, 1[$ integers N_i and $\eta_i > 0$ such that $\forall\, n \geq N_i$, $\forall\, x_n \in K_n \cap (x_0 + eta_i B)$, there exist $u_n^i \in T_{K_n}(x_n)$ satisfying

$$\|u_i - u_n^i\|_Z \;\;\leq\;\; \frac{\alpha}{2} \|A\|_{\mathcal{L}(Z,Y)}$$

Let $N := \max_{l \leq i \leq p} N_i$ and $\eta := \min_{l \leq i \leq p} \eta_i$. We take $n \geq N$ and $x_n \in K_n \cap (x_0 + \eta B)$. Let v belong to Y. There exists $v_i \in S$ such that

$$\|v_i - \frac{v}{\|v\|_Y}\|_Y \;\;\leq\;\; \frac{\alpha}{2}$$

Set $v_n = \|v\|_Y u_n^i$ and $w_n = v - Av_n$. We see that $v_n \in T_{K_n}(x_n)$, that

$$\begin{cases} \|v_n\|_Z \;=\; \|v\|_Y \|u_n i\|_Z \;\leq\; \|v\|_Y(\lambda + \|u_i - u_n^i\|_Z) \\ \leq\; \|v\|_Y(\lambda + \frac{\alpha}{2}\|A\|_{\mathcal{L}(Z,Y)}) \;\leq\; c\|v\|_Y \end{cases}$$

(where $c := \lambda + \|A\|_{\mathcal{L}(Z,Y)}/2$) and that

$$\begin{cases} \|w_n\|_Y \;=\; \|v - A(\|v\|_Y u_n^i)\|_Y \\ =\; \|v\|_Y(\|\frac{v}{\|v\|_Y} - v_i + A(u_i - u_n^i)\|_Y \\ \leq\; \|v\|_Y(\frac{\alpha}{2} + \|A\|_{\mathcal{L}(Z,Y)}\|u_i - u_n^i\|)_Z \;\leq\; \alpha\|v\|_Y \end{cases}$$

This proves our claim. □

4 Kuratowski Limits of Inverse Images

We have seen that inclusion

$$\liminf_{n\to\infty} f^{-1}(M_n) \;\subset\; f^{-1}\left(\liminf_{n\to\infty} M_n\right)$$

is always true. We can provide sufficient conditions for having the equality in the case of lower limits. For instance, the usual Liusternik's Inverse Function Theorem implies the following proposition:

Proposition 4.1 *Let us assume that X and Y are Banach spaces, that the map f is continuously differentiable at some point*

$$x \in f^{-1}\left(\liminf_{n\to\infty} M_n\right)$$

and that $f'(x)$ is surjective. Then

$$x \text{ belongs to } \liminf_{n\to\infty} f^{-1}(M_n) \tag{23}$$

But, before extending this result to more general situations (when X is replaced by a subset K, for instance), let us proceed with the simpler case of convex subsets.

Proposition 4.2 *Let us consider two Banach spaces X and Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$ and two sequences of subsets $L_n \subset X$ and $M_n \subset Y$. We assume that*

$$\begin{cases} i) & L_n \text{ and } M_n \text{ are } \textbf{convex} \\ ii) & L_n \text{ are contained in a bounded set} \\ iii) & \exists \gamma > 0 \mid \gamma B \subset A(L_n) - M_n \end{cases} \tag{24}$$

Then

$$\liminf_{n\to\infty}(L_n \cap A^{-1}(M_n)) = \liminf_{n\to\infty} L_n \cap A^{-1}(\liminf_{n\to\infty} M_n) \tag{25}$$

Proof The inclusion

$$\liminf_{n\to\infty}(L_n \cap A^{-1}(M_n)) \subset A^{-1}(\liminf_{n\to\infty} L_n \cap \liminf_{n\to\infty} M_n)$$

being obvious, let us prove the other one, by checking that any x in

$$\liminf_{n\to\infty} L_n \cap A^{-1}(\liminf_{n\to\infty} M_n)$$

is the limit of a sequence of elements x_n belonging to L_n such that $A(x_n)$ belong to M_n.

We know that x can be approximated by elements $u_n \in L_n$ and that $A(x)$ can be approximated by elements $v_n \in M_n$. Then $\epsilon_n := \|A(u_n) - v_n\|$ converges to 0 and $\theta_n := \frac{\gamma}{\gamma+\epsilon_n}$ converges to 1, belongs to $]0,1[$ and satisfies $\theta_n\epsilon_n = (1-\theta_n)\gamma$. Therefore,

$$(26)\qquad \begin{cases} \theta_n(v_n - A(u_n)) & \in \ \theta_n\epsilon_n B \ = \ (1-\theta_n)\gamma B \\ & \subset \ (1-\theta_n)(A(L_n) - M_n) \end{cases}$$

and consequently, there exist elements $u'_n \in L_n$ and $v'_n \in M_n$ such that

$$(27)\qquad A(\theta_n u_n + (1-\theta_n)u'_n) \ = \ \theta_n v_n + (1-\theta_n)v'_n$$

If we set $x_n := \theta_n u_n + (1-\theta_n)u'_n$, we observe x_n belongs to L_n and that $A(x_n)$ belongs to M_n for these subsets are convex.

Furthermore, $\|x_n - u_n\| = (1-\theta_n)\|u_n - u'_n\|$ converges to 0 since u_n and u'_n remain in a bounded subset by assumption. □

Remark Assumption (24) implies obviously that

$$(28)\qquad 0 \ \in \ \mathrm{Int}(\bigcup_N \bigcap_{n>N}(A(L_n) - M_n)) \ \subset \ \liminf_{n\to\infty}(A(L_n) - M_n) \ \square$$

For non convex subsets L_n and M_n, we obtain the following consequence of the Inverse Stability Theorem 3.2:

Theorem 4.1 *Let X and Y be two Banach spaces. We introduce a continuous linear operator $A \in \mathcal{L}(X,Y)$ and sequences of closed subsets $L_n \subset X$ and $M_n \subset Y$. Let us assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(29)\qquad \begin{cases} \forall\, x_n \in K_n \cap B(x_0,\eta), \ \ y_n \in B(Ax_0,\eta) \\ B_Y \ \subset \ A(T^\flat_{L_n}(x_n) \cap cB_X) - T_{M_n}(y_n) + \alpha B_Y \end{cases}$$

Then the Kuratowski lower limit of $L_n \cap A^{-1}(M_n)$ is equal to the intersection of the Kuratowski lower limit of L_n and the inverse image by A of the Kuratowski lower limit of M_n:

$$(30)\qquad \liminf_{n\to\infty}(L_n \cap A_n^{-1}(M_n)) = \liminf_{n\to\infty} L_n \cap A^{-1}\liminf_{n\to\infty} M_n$$

Proof Since the inclusion

$$\liminf_{n\to\infty}(L_n \cap A_n^{-1}(M_n)) \subset \liminf_{n\to\infty} L_n \cap A^{-1}(\liminf_{n\to\infty} M_n)$$

is obvious, let us take any $x_0 := \lim_{n\to\infty} x_n$ belonging to $\liminf_{n\to\infty} L_n$ such that

$$y_0 := Ax_0 = \lim_{n\to\infty} Ax_{0n} = \lim_{n\to\infty} y_{0n}$$

belongs to $\liminf_{n\to\infty} M_n$.

We then apply Theorem 3.2 to the subsets $L_n \times M_n$ of $X \times Y$ and the continuous linear operators $A \ominus \mathbf{1}$ associating to any (x, y) the element $Ax - y$, since we can write

$$K_n \;:=\; L_n \cap A^{-1}(M_n) \;=\; (A \ominus \mathbf{1})^{-1}(0) \cap (L_n \times M_n)$$

The pair $(x_{0n}, y_{0n}) \in L_n \times M_n$ converges to (x_0, y_0), and $(A \ominus \mathbf{1})(x_{0n}, y_{0n})$ converges to 0.

Furthermore, it is clear that assumption (29) implies the stability assumption (21) of Theorem 3.2.

Therefore, by Theorem 3.2, there exits a solution $(\widehat{x_n}, \widehat{y_n}) \in L_n \times M_n$ to the equation $(A \ominus \mathbf{1})(\widehat{x_n}, \widehat{y_n}) = 0$ such that

$$\|x_{0n} - \widehat{x_n}\| + \|y_{0n} - \widehat{y_n}\| \leq l\|Ax_{0n} - y_{0n} - 0\|$$

This means that $\widehat{x_n}$ belongs to K_n, converges to x_0 and that $A\widehat{x_n}$ converges to y_0.

Remark We can extend this theorem to the case of a sequence of continuous linear operators $A_n \in \mathcal{L}(X, Y)$, where we take

$$(31) \quad \begin{cases} i) & x_0 := \lim_{n\to\infty} x_n \in \liminf_{n\to\infty} L_n \\ ii) & y_0 := \lim_{n\to\infty} A_n x_{0n} = \lim_{n\to\infty} y_{0n} \in \liminf_{n\to\infty} M_n \end{cases}$$

The same proof where A is replaced by A_n implies that

$$(32) \quad \begin{cases} \exists\, \widehat{x_n} \in K_n \quad \text{such that} \\ \begin{cases} i) & \widehat{x_n} \longrightarrow x_0 \\ ii) & A_n\widehat{x_n} \longrightarrow y_0 \end{cases} \end{cases} \quad \square$$

5 Kuratowski Limits of Direct Images

We have seen that inclusion

$$f(\limsup_{n\to\infty} K_n) \subset \limsup_{n\to\infty} f(K_n)$$

is always true. We obtain equalities for the Kuratowski upper limits when f is *proper*[10]

Proposition 5.1 *Let us assume that f is proper, then*

$$f(\limsup_{n\to\infty} K_n) = \limsup_{n\to\infty} f(K_n) \tag{33}$$

and if f is proper and surjective, then

$$\limsup_{n\to\infty} f^{-1}(M_n) = f^{-1}\left(\limsup_{n\to\infty} M_n\right) \tag{34}$$

We can adapt the Closed Range Theorem[11] to obtain the following equality: (We denote by

$$K^{\circ} := \{ p\in X^\star \mid \forall\, x\in K,\ < p,x > \leq 1 \}$$

the polar set of K).

Theorem 5.1 *Let X and Y be reflexive Banach spaces, $K_n \subset X$ be a subsets and $A \in \mathcal{L}(X,Y)$ be a continuous linear operator*[12] *satisfying*

$$0 \in \operatorname{Int}\left(\operatorname{Im}(A^\star) + \bigcup_{N>0}\bigcap_{n>N} K_n^{\circ}\right) \tag{35}$$

[10] We recall that a continuous single-valued map from a metric space X to a metric space Y is **proper** if and only if one of the equivalent statements

If $f(x_n)$converges in Y
then a subsequence of x_n converges in X

or

$$\begin{cases} i) & f \text{ maps closed subsets to closed subsets} \\ ii) & \forall y\in Y,\ f^{-1}(y) \text{ is compact}\end{cases}$$

[11] see [10, Theorem 1.5.5, p28]

[12] Banach's Closed Graph Theorem allows to assume that A is surjective: It is sufficient to decompose A as the product $\tilde{A}\circ\phi$ of the canonical surjection ϕ from X onto its factor space $X/\ker(A)$ and the associated bijective map $\tilde{A}$, which is an isomorphism. Then the properness of A is equivalent to the properness of ϕ.

Let $K_\sigma^\sharp$ denote the Kuratowski upper limit of the subsets K_n when X is supplied with the weak topology. Then

$$\limsup_{n\to\infty} A(K_n) \;\subset\; A(K_\sigma^\sharp) \tag{36}$$

Proof Let us consider a sequence $x_n \in K_n$ such that $A(x_n)$ converges to some y in Y. We shall check that this sequence is weakly bounded, and thus, weakly relatively compact. Let us take for that purpose any $p \in X^\star, \|p\|_\star \leq \gamma$, which can be written, by assumption (35)

$$p := A^\star q + r, \; q \; \in Y^\star, \quad r \in \bigcup_{N>0}\bigcap_{n>N} K_n^\circ \tag{37}$$

Therefore, there exists a N such that $r \in \bigcap_{n>N} K_n^\circ$ and consequently,

$$\left\{\begin{array}{l} \sup_{n>N} < p, x_n >= \sup_{n>N}(< q, Ax_n > + < r, x_n >) \\ \leq \sup_{n>N}(\|q\|\|Ax_n\| + \sigma_{K_n}(r)) \leq \sup_{n>N}(\|q\|\|Ax_n\| + 1 \leq +\infty \end{array}\right. \tag{38}$$

since the converging sequence Ax_n is bounded.

Then a subsequence (again denoted) converges weakly to some x which belongs to $K_\sigma^\sharp$. □

6 Graphical Limits

We shall use these concepts to define *graphical convergence* of set-valued maps.

Definition 6.1 (Graphical Convergence) *Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. We shall say that the set-valued maps $F^\sharp$ and $F^\flat$ from X to Y defined by*

$$\left\{\begin{array}{lll} i) & \mathrm{Graph}F^\sharp & := \limsup_{n\to\infty} \mathrm{Graph}F_n \\ ii) & \mathrm{Graph}F^\flat & := \liminf_{n\to\infty} \mathrm{Graph}F_n \end{array}\right. \tag{39}$$

are the **(graphical) upper and lower limits** *of the set-valued maps F_n respectively.*

We provide a more explicit characterization of these graphical upper and lower limits, which follows immediately from Proposition 1.1.

Proposition 6.1 *Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. Then y belongs to $F^\sharp(x)$ if and only if it is the limit of a subsequence of elements $y_{n'} \in F(x_{n'})$ where $x_{n'}$ converges to x. It belongs to $F^\flat(x)$ if and only if it is the limit of a sequence of elements $y_n \in F(x_n)$ where x_n converges to x.*

Let us point out these useful formulas:

Proposition 6.2 *Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. Then*

$$\begin{cases} F^\flat(x) & \subset & \bigcap_{\epsilon>0} \liminf_{n\to\infty} F_n(B(x,\epsilon)) \\ F^\sharp(x) & \supset & \bigcap_{\epsilon>0} \limsup_{n\to\infty} F_n(B(x,\epsilon)) \end{cases}$$

These formulas can be regarded as relating graphical convergence with some kind of " almost pointwise convergence". But can we compare the graphical convergence of F_n and the "pointwise convergence" of F_n, i.e., the upper and lower Kuratowski's limits of the subsets $F_n(x)$? The following statement provides the easy answers.

Proposition 6.3 *Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. Then the following relations hold true:*

$$\begin{cases} i) & \limsup_{n\to\infty, x_n\to x} F_n(x_n) & = & F^\sharp(x) \\ ii) & \liminf_{n\to\infty, x_n\to x} F_n(x_n) & \subset & F^\flat(x) \end{cases}$$

The missing equality holds true under more assumptions.

Theorem 6.1 *Let X and Y be two Banach spaces. We consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$ and its upper graph limit $F^\flat$ defined by*

$$\text{Graph}(F^\flat) \;=\; \liminf_{n\to\infty} \text{Graph}(F_n)$$

Let us consider $y_0 \in F^\flat(x_0)$ and let us assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$(40) \qquad \begin{cases} \forall\, (x_n, y_n) \in \text{Graph}(F_n) \cap B((x_0,y_0),\eta), \\ \|DF_n(x_n,y_n)\| := \sup_{u\in X} \inf_{v\in DF_n(x_n,y_n)} \|v\|/\|u\| \le c \end{cases}$$

Hence, for any sequence x_n converging to x_0, we have

$$y_0 \;\in\; \liminf_{n\to\infty} F_n(x_n)$$

Proof We apply the Inverse Stability Theorem 3.2 to the case when the subsets K_n are the graphs of the set-valued maps F_n, when M_n are the singletons $\{x_n\}$ and when the continuous linear operator A is the projection π_X from $X \times Y$ onto X, since

$$F_n(x_n) \;= \pi_Y\left(\text{Graph}(F_n) \cap \pi_X^{-1}(x_n)\right)$$

The uniform boundedness of the contingent derivatives on a neighborhood of (x_0, y_0) implies obviously the stability property (21) with $\alpha = 0$. □

We can translate the Inverse Stability Theorem 3.2 into the following useful statement:

Theorem 6.2 *Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$, an element (x_0, y_0) of the graph of its graphical lower limit and let us assume that there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall\ (x_n, y_n) \in \text{Graph}(F_n) \cap B((x_0, y_0), \eta), \\ \forall\, v \in Y,\ \exists\, u_n \in X,\ \exists\, w_n \in Y \quad \textit{such that } v \in DF_n(x_n, y_n)(u_n) + w_n \\ \textit{and } \|u_n\| \;\leq\; c\|v\| \quad \& \quad \|w_n\| \;\leq\; \alpha\|v\| \end{cases}$$

Then, for any sequence (x_{0n}, y_{0n}) converging to (x_0, y_0), for any y_n converging to y_0, we have

$$d(x_{0n}, F_n^{-1}(y_n)) \;\leq\; l\|y_{0n} - y_n\|$$

Proof We apply the Inverse Function Theorem 3.2 with X replaced by $XtimesY$, K_n y $\text{Graph}(F_n)$, A by the projection Π_Y from $X \times Y$ onto Y. We have to prove that assumption (21) of Theorem 3.2is satisfied, i.e., that for all $v \in Y$, there exist (u_n, v_n) in the contingent cone $T_{\text{Graph}(F_n)}(x_n, y_n)$ and $w_n \in Y$ such that $v = v_n + w_n$, $\max(\|u_n\|, \|v_n\|) \leq c\|v\|$ and $\|w_n\| \leq \alpha\|v\|$. These informations are provided by our assumption since the contingent cone to the graph is the graph of the contingent derivative and since $\|v_n = v - w_n\|$ is smaller then or equal to $(1 + \alpha)\|v\|$. □

An important consequence is the Inverse Function Theorem for nonlinear constrained (single-value)d maps.

Theorem 6.3 (Inverse Function Theorem) *Let X and Y be two Banach spaces. We introduce a sequence of continuous single-valued maps f_n*

from X to Y a sequence of closed subsets $K_n \subset X$ and an element $(x_0, y_0$ in the graphical lower limit of the retrictions of f_n to the subsets K_n.

We assume that the functions f_n are differentiable on a neighborhood of x_0 and we posit the following **stability assumption***: there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$(41) \qquad \begin{cases} \forall\, x_n \in K_n \cap B(x_0, \eta), \\ B_Y \;\subset\; f_n'(x_n)(T_{K_n}(x_n) \cap cB_X) + \alpha B_Y \end{cases}$$

Then for any sequence of elements x_{0n} of K_n converging to x_0 such that $f_n(x_{0n})$ converges to y_0 and any sequence of elements $y_n \in Y$ converging to y_0, we have

$$d(x_{0n}, K_n \cap f_n^{-1}(y_n)) \;\leq\; l\|y_n - f_n(x_{0n})\|$$

Proof It is sufficient to recall that the contingent derivative of the restriction $F_n := f_{n|K_n}$ of f_n to K_n is the restiction of the derivative $f_n'(x_n)$ to the contingent cone $T_{K_n}(x_n)$ to K_n at x_n. □

Remark Since the above theorem implies obviously Theorem 3.2, we infer that all these statements are equivalent. □

Monotone and Maximal Monotone Maps do enjoy interesting properties. For instance, it is sufficient to know that the graphical lower limit of a sequence of monotone maps is maximal monotone for deducing that it is actually the graphical limit:

Proposition 6.4 (Graphical Convegence of Monotone Operators) *Let X be a Hilbert space. We suppose that the set-valued maps $F_n : X \rightsquigarrow X^\star$ are monotone and that $F : X \rightsquigarrow X^\star$ is maximal monotone.*

If F is contained in the graphical lower limit $F^\flat$ of the F_n's, then F is actually the graphical limit of the F_n's.

Proof

We have to prove that the graphical upper limit $F^\sharp$ of the set-valued maps F_n is contained in F.

Let p belongs to $F^\sharp(x)$. Hence the pair (x, p) is the limit of a subsequence of elements $(x_{n'}, p_{n'})$ of the graph of F_n.

Take now any pair (y, q) in the graph of F. Since F is contained in the graphical Kuratowski lower limit $F^\flat$ by assumption, we know that there exists a sequence of elements (y_n, q_n) of the graph of F_n converging to (y, q). The monoticity of the set-valued maps F_n implies the inequalities

$$< p_n - q_n \, , \, x_n - y_n > \; \geq \; 0$$

Going to the limit, we deduce that

$$\forall \, (y, q) \in \text{Graph}(F), \;\; < p - q \, , \, x - y > \; \geq \; 0$$

Therefore p belongs to $F(x)$ because of the maximality of the graph of F among monotone graphs. □

7 Stability of Viability Domains and Solution Maps

Let us consider now a sequence of closed viability domains of a set-valued map F[13].

Does the Kuratowski upper limit (see Definition 1.1) of these closed viability domains is still a closed viability domain? The answer is positive.

Theorem 7.1 (Stability of Viability Domains) *Let us consider a nontrivial* upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth . *Let K_n be a sequence of closed viability domains of F. Then the Kuratowski upper limit*

(42)
$$K^\sharp := \bigcap_{\epsilon > 0} \bigcap_{N > 0} \bigcup_{n \geq N} B(K_n, \epsilon)$$

is also a closed viability domain of F.

---

[13] see [13]. A subset $K \subset \text{Dom}(F)$ is a viability domain if and only if

$$\forall \, x \in K, \;\; F(x) \cap T_K(x) \;\neq\; \emptyset$$

The Viability Theorem states that for upper semicontinuous set-valued map with nonempty compact convex images and with linear growth, K is a viability domain if and only if K enjoys the viability property: Foe all initial state in K, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ which is viable in K.

Proof We shall prove that $K^\sharp$ enjoys the viability property. The necessary condition of the Viability Theorem[14] implies that this subset is a viability domain.

Let x belong to $K^\sharp$. It is the limit of a subsequence $x_{n'} \in K_{n'}$. Since the subsets K_n are viability domains, there exist viable solutions $y_{n'}(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at $x_{n'}$. The upper semicontinuity of the solution map implies that a subsequence (again denoted) $y_{n'}(\cdot)$ converges uniformly on compact intervals to a solution $y(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x. Since $y_{n'}(t)$ belongs to $K_{n'}$ for all n', we deduce that $y(t)$ does belong to $K^\sharp$ for all $t > 0$. □

Since we are dealing with Kuratowski upper limits, the question arises whether the Kuratowski upper limit $K^\sharp$ of a sequence of closed viability domains K_n of set-valued maps F_n is a closed viability domain of the closed convex hull of the upper limit $\overline{\text{co}}F^\sharp$ of the set-valued maps F_n defined by

$$\forall\, x \in X,\ \ (\overline{\text{co}}F^\sharp)(x) \ := \ \overline{\text{co}}(F^\sharp(x))$$

Theorem 7.2 (Stability of Solution Maps) *Let us consider a sequence of nontrivial set-valued maps $F_n : X \rightsquigarrow X$ satisfying:*

$$(43) \qquad \exists c > 0 \ \mid \ \forall n > 0,\ \forall x \in \text{Dom}(F_n), \quad \|F_n(x)\| \le c(\|x\| + 1)$$

Then

1. *The Kuratowski upper limit of the solution maps S_{F_n} is contained in the solution map $S_{coF^\sharp}$ of the co-upper limit of the set-valued maps F_n*
2. *If the subsets $K_n \subset \text{Dom}(F_n)$ are closed viability domains of the set-valued maps F_n, then the Kuratowski upper limit $K^\sharp$ is a closed viability domain of $coF^\sharp$.*
3. *The Kuratowski upper limit of the viability kernels $\widehat{K_n}$ of the set-valued maps F_n is contained in the viability kernel of $coF^\sharp$.*

It follows from the adaptation of the Convergence Theorem to limits of set-valued maps.

[14] see [?, Theorem 4.2.1].

Theorem 7.3 (Convergence Theorem) *Let X be a topological vector space, Y be a finite dimensional vector-space and F_n be a sequence of non-trivial set-valued maps from X to Y.*

Let us assume that the set-valued maps F_n are uniformly bounded.

Let I be an interval of $\mathbf{R}$ and let us consider measurable *functions x_m and y_m from I to X and Y respectively, satisfying:*

for almost all $t \in I$ and for all neighborhood $\mathcal{U}$ of 0 in the product space $X \times Y$, there exists $M := M(t, \mathcal{V})$ such that

$$\forall m > M,\ (x_m(t), y_m(t)) \in \text{Graph}(F) + \mathcal{U} \tag{44}$$

If we assume that

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function} x(\cdot) \\ ii) & y_m(\cdot) \in L^1(I, Y; a) \text{ and converges } \textbf{weakly} \text{ in } L^1(I, Y; a) \\ & \textit{to a function } y \in L^1(I, Y; a) \end{cases} \tag{45}$$

then

$$\text{for almost all } t \in I,\ y(t) \in \ \text{co}(F^\sharp(x(t)) \tag{46}$$

Proof The proof is a straightforward extension of the Convergence Theorem and of the following Lemma:

Lemma 7.1 *Let us consider a sequence of subsets K_n contained in a bounded subset of a finite dimensional vector-space X. Then*

$$\overline{\text{co}}(\limsup_{n\to\infty} K_n) \ = \ \bigcap_{N>0} \overline{\text{co}}(\bigcup_{n\geq N} K_n) \tag{47}$$

Proof The closed convex hull of the Kuratowski upper limit is obviously contained in the closed convex subset

$$A \ := \ \bigcap_{N>0} \overline{co} \bigcup_{n\geq N} K_n$$

We have to prove that it is equal to it when the dimension of X is finite and the subsets K_n are contained in a bounded set.

Since an element x of A is the limit of a subsequence of convex combinations v_N of elements of $\bigcup_{n>N} K_n$ and since the dimension of X is an integer p, Carathéodory's Theorem allows to write that

$$v_N \ := \ \sum_{j=0}^{p} a_j^N x_{N_j}$$

where $N_j \geq N$ and where x_{N_j} belongs to K_{N_j}. The vector a^N of $p+1$ components a_j^N contains a converging subsequence (again denoted) a^N which converges to some non negative vector a of $p+1$ components a_j such that $\sum_{j=0}^{p} a_j = 1$.

The subsets K_n being contained in a given compact subset, we can extract successively subsequences (again denoted) x_{N_j} converging to elements x_j, which belong to the Kuratowski upper limit of the subsets K_n. Hence x is equal to the convex combination $\sum_{j=0}^{p} a_j^N x_j$ and the lemma is proved. □

Remark If we dont assume that the set-valued maps F_n are uniformly bounded, then we cannot use the above lemma. However, we can conclude that

$$\begin{cases} \text{for almost all } t \in I, \\ y(t) \in \bigcap_{\eta>0,N>0} \overline{co} \bigcup_{n>N, x_n \in B(x,\eta) \cap \mathrm{Dom} F_n} F_n(x_n) \end{cases} \quad \square$$

8 Epigraphical Limits

Let us consider a sequence of extended real-valued functions

$$V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$$

whose domains

$$\mathrm{Dom}(V_n) \ := \ \{ x \in X \mid V_n(x) < +\infty \}$$

are not empty.

For taking into account the order relation of $\mathbf{R}$, we associate with each extended real-valued function V_n two new set-valued maps $\mathbf{V}_{n\uparrow}$ and $\mathbf{V}_{n\downarrow}$ defined in the following way:

$$(48) \qquad \begin{cases} i) \quad \mathbf{V}_{n\uparrow} := \begin{cases} V_n(x) + \mathbf{R}_+ & \text{if } x \in \mathrm{Dom}(V_n) \\ \emptyset & \text{if } x \notin \mathrm{Dom}(V_n) \end{cases} \\ ii) \quad \mathbf{V}_{n\downarrow} := \begin{cases} V_n(x) - \mathbf{R}_+ & \text{if } x \in \mathrm{Dom}(V_n) \\ \emptyset & \text{if } x \notin \mathrm{Dom}(V_n) \end{cases} \end{cases}$$

We see at once that

$$\begin{cases} i) & \mathrm{Graph}(\mathbf{V}_{n\uparrow}) = \mathrm{Ep}(V_n) \\ ii) & \mathrm{Graph}(\mathbf{V}_{n\downarrow}) = \mathrm{Hp}(V_n) \end{cases}$$

Therefore, by using the concept of graphical upper and lower limit with these two associated set-valued maps, we come up with the concepts of epi and hypo convergence, which are thus obtained by taking Kuratowski upper and lower limits of their epigraphs and hypographs.

Definition 8.1 (Epi-limits) *Let us consider a sequence of extended real-valued functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ whose domains are not empty. We shall say that*

1. *the function $V_\uparrow^\flat$ whose epigraph is the Kuratowski lower limit of the epigraphs of the functions V_n*

$$\mathrm{Ep}(V_\uparrow^\flat) \; := \; \liminf_{n\to\infty} \mathrm{Ep}(V_n) \tag{49}$$

   is the **upper epi-limit** *of the functions V_n*

2. *the function $V_\uparrow^\sharp$ whose epigraph is the Kuratowski upper limit of the epigraphs of the functions V_n*

$$\mathrm{Ep}(V_\uparrow^\sharp) \; := \; \limsup_{n\to\infty} \mathrm{Ep}(V_n) \tag{50}$$

   is the **lower epi-limit** *of the functions V_n*

3. *the function $V_\downarrow^\flat$ whose hypograph is the Kuratowski lower limit of the hypographs of the functions V_n*

$$\mathrm{Hp}(V_\downarrow^\flat) \; := \; \liminf_{n\to\infty} \mathrm{Hp}(V_n) \tag{51}$$

   is the **lower hypo-limit** *of the functions V_n*

4. *the function $V_\downarrow^\sharp$ whose hypograph is the Kuratowski upper limit of the hypographs of the functions V_n*

$$\mathrm{Hp}(V_\downarrow^\sharp) \; := \; \limsup_{n\to\infty} \mathrm{Hp}(V_n) \tag{52}$$

   is the **upper hypo-limit** *of the functions V_n*

If the upper and lower epi-limits coincide, we shall say that the common value

$$V_\uparrow \; := \; V_\uparrow^\sharp \; = \; V_\uparrow^\flat \tag{53}$$

is the **epi-limit** *of the sequence of functions V_n, and we define the* **hypo-limit** *$V_\downarrow$ in the same way.*

The terminology concerning the epi-limits seems at odds with the choice of the Kuratowski semi limits: the **upper** epi-limit is associated with the Kuratowski **lower** limit. However, they are consistent in the case of hypo-limits. This is due to the analytical definitions of these epi-limits, involving the concepts of Γ-convergence and lim sup inf, defined in the following way:

Definition 8.2 (Lim sup inf) *Let L and M be two metric spaces and $\phi : L \times M \mapsto \mathbf{R}$ be a function. We set*

$$\limsup_{x' \to x} \inf_{y' \to y} \varphi(x', y') := \sup_{\epsilon > 0} \inf_{\eta > 0} \sup_{x' \in B(x,\eta)} \inf_{y' \in B(y,\epsilon)} \phi(x', y') \tag{54}$$

The concept of lim inf sup is defined in a symetric way, and the adaptation to sequences (or filters) is straightforward.

Proposition 8.1 *Let us consider a sequence of extended real-valued functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ whose domains are not empty. We obtain the following formulas:*

$$\left\{\begin{array}{lll} i) & V_\uparrow^\flat(x_0) & \\ & = \limsup_{n\to\infty} \inf_{x\to x_0} V_n(x) & \\ ii) & V_\uparrow^\sharp(x_0) & \\ & = \liminf_{n\to\infty, x\to x_0} V_n(x) & \\ iii) & V_\downarrow^\flat(x_0) = -(-V)_\uparrow^\flat(x) & \\ & = \liminf_{n\to\infty} \sup_{x\to x_0} V_n(x) & \\ iv) & V_\downarrow^\sharp(x_0) = -(-V)_\uparrow^\sharp(x) & \\ & = \limsup_{n\to\infty, x\to x_0} V_n(x) & \end{array}\right. \tag{55}$$

Proof We shall check these formulas for epigraphical convergence only.

1. For computing the value of $V_\uparrow^\flat$ at x_0, we use the fact that for every $\lambda \geq V_\uparrow^\flat(x_0)$, there exist sequences of elements x_n converging to x_0 and λ_n to λ such that $\lambda_n \geq V_n(x_n)$.

   Therefore, for all $\epsilon > 0$ and $\eta > 0$, there exists N such that, for all $n \geq N$, we have

$$\inf_{\|x-x_0\|\leq\eta} V_n(x) \leq V_n(x_n) \leq \lambda_n \leq \lambda + \epsilon$$

and thus

$$\sup_{n\geq N}\inf_{\|x-x_0\|\leq\eta} V_n(x) \;\leq\; \lambda+\epsilon$$

from which we deduce that

$$\forall\,\lambda \geq V_\uparrow^\flat(x_0),\quad \limsup_{n\to\infty}\inf_{x\to x_0} V_n(x) \;\leq\; \lambda$$

and thus, that

$$\limsup_{n\to\infty}\inf_{x\to x_0} V_n(x) \;\leq\; V_\uparrow^\flat(x_0)$$

2. Conversely, for proving the other inequality, we have to show that the pair (x_0,λ_0) where $\lambda_0 := \limsup_{n\to\infty}\inf_{x\to x_0} V_n(x)$ belongs to the epigraph of $V_\uparrow^\flat$, i.e., to the Kuratowski lower limit of the epigraphs of the functions V_n. But, by the very definition of the infimum, we deduce that for all $\epsilon > 0$ and $\eta > 0$, there exist N such that, for all $n \geq N$, there exist elements x_n such that

$$V_n(x_n) \leq \inf_{\|x-x_0\|\leq\eta} V_n(x)+\epsilon \leq \sup_{n\geq N}\inf_{\|x-x_0\|\leq\eta} V_n(x)+\epsilon \leq \lambda_0+2\epsilon$$

By taking $\epsilon = \eta = 1/n$ and setting $\lambda_n := \lambda_0 + 2/n$, we have proved that x_n converges to x_0, λ_n to λ_0 and that $V_n(x_n) \leq \lambda_n$ for all n.

3. Let us estimate now any $\lambda \geq V_\uparrow^\sharp(x_0)$. We know that for any $\epsilon > 0$, $\eta > 0$ and $N > 0$, there exist (x_n,λ_n) in the epigraph of V_n satisfying

$$n \;\geq\; N,\;\; \lambda_n \;\leq\; \lambda+\epsilon,\;\; \|x_n - x_0\| \;\leq\; \eta \;\;\&\;\; V_n(x_n) \;\leq\; \lambda_n$$

We deduce that

$$\inf_{n\geq N,\|x-x_0\|\leq\eta} V_n(x) \;\leq\; V_n(x_n) \;\leq\; \lambda_n \;\leq\; \lambda+\epsilon$$

Hence

$$\liminf_{n\to\infty,x\to x_0} V_n(x) \;\leq\; V_\uparrow^\sharp(x_0) \tag{56}$$

4. We conclude by observing that the pair (x_0,λ_1) where

$$\lambda_1 := \liminf_{n\to\infty,x\to x_0} V_n(x)$$

belongs to the epigraph of $V_\uparrow^\sharp$, because, by the very definition of the lim inf, we can construct a subsequence of elements (again denoted) (x_n,λ_n) of the epigraph of V_n converging to (x_0,λ_1).

It may be useful to store these inequalities:

$$(57) \qquad V_{\uparrow}^{\sharp}(x) \leq V_{\uparrow}^{\flat}(x) \leq V_{\downarrow}^{\sharp}(x) \quad \& \quad V_{\uparrow}^{\sharp}(x) \leq V_{\downarrow}^{\flat}(x) \leq V_{\downarrow}^{\sharp}(x)$$

Remark We have defined the concepts of epi-limits from the Kuratowski limits of the epigraphs. Conversely, we can recover the Kuratowski limits of subsets from the epi-limits of their indicators.

Proposition 8.2 *Let us consider a sequence of subsets $K_n \subset X$ and their indicators ψ_{K_n}. Let $K^{\sharp}$ and $K^{\flat}$ denote the Kuratowski upper and lower limits of the K_n's. Then*

$$(58) \qquad \begin{cases} i) & \psi_{K^{\flat}} \textit{ is the lower epi-limit of } \psi_{K_n} \\ ii) & \psi_{K^{\sharp}} \textit{ is the upper epi-limit of } \psi_{K_n} \end{cases} \quad \square$$

Naturally, we can introduce the same definitions for "continuous" parameters $u \in U$, where U is a topological space.

In such a framework, we consider a family of extended real-valued function

$$V(u) : X \mapsto \mathbf{R} \cup \{+\infty\}$$

depending upon the parameter u, to which associate the set-valued maps

$$(59) \qquad \begin{cases} i) & \mathbf{V}(u)_{\uparrow} := \begin{cases} V(u,x) + \mathbf{R}_+ & \text{if } (x,u) \in \mathrm{Dom}(V) \\ \emptyset & \text{if } x \notin \mathrm{Dom}(V) \end{cases} \\ ii) & \mathbf{V}(u)_{\downarrow} := \begin{cases} V(u,x) - \mathbf{R}_+ & \text{if } (x,u) \in \mathrm{Dom}(V) \\ \emptyset & \text{if } x \notin \mathrm{Dom}(V) \end{cases} \end{cases}$$

There is a subtle, but important, difference between the extended real-valued function $V : (u,x) \in U \times X \mapsto \mathbf{R} \cup \{+\infty\}$, for which the variables u and x are on the same footing, and the set-valued maps $\mathbf{V}(\cdot)_{\uparrow} : u \in U \leadsto \mathbf{V}(u)_{\uparrow}$, for which the variable play a different role: u, the role of a parameter, whereas the order relation on $\mathbf{R}$ involves the variable x, when we minimize the function with respect to x, for instance.

To emphasize this difference, the set-valued map $\mathbf{V}(\cdot)_{\uparrow}$ is called a **variational system**.

Definition 8.3 (Variational system) *Let us consider a variational system* $\mathbf{V}(\cdot)_\uparrow$ *defined on* $U \times X$. *We shall say that* $V(\cdot,\cdot)$ *(or* $\mathbf{V}(\cdot)_\uparrow$*), to be precise, is*

1. **upper epi-continuous** *at* u *if the set-valued map* $u' \rightsquigarrow \mathbf{V}(u')_\uparrow$ *is lower semicontinuous, i.e., if and only if*

$$(60) \qquad V(u,x) \;:=\; \limsup_{u' \to u} \inf_{x' \to x} V(u',x')$$

2. **lower epi-continuous** *at* u *if the graph of the set-valued map* $u' \rightsquigarrow \mathbf{V}(u')_\uparrow$ *is closed , i.e., if and only if*

$$(61) \qquad V(u,x) \;=\; \liminf_{u' \to u, x' \to x} V(u,x)$$

3. **epi-continuous** *at* u *if the set-valued map* $u' \rightsquigarrow \mathbf{V}(u')_\uparrow$ *is lower semicontinuous and has a closed graph, i.e., if and only if*

$$(62) \qquad V(u,x) = \limsup_{u' \to u} \inf_{x' \to x} V(u',x') = \liminf_{u' \to u, x' \to x} V(u,x)$$

The definition for hypo-continuity and semicontinuity are naturally symmetric. □

We expect that the infimum of an epi-limit is closely related to the limit of the infima. This is detailed in the following statement:

Proposition 8.3 (Limits of Infima) *Let us consider a sequence of extended real-valued functions* $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ *whose domains are not empty. Then*

$$\limsup_{n \to \infty} (\inf_{x \in X} V_n(x)) \;\leq\; \inf_{x \in X} V_\uparrow^\flat$$

If we assume that the functions V_n *are lower semicontinuous and uniformly* **inf-compact**, *then*

$$\inf_{x \in X} V_\uparrow^\sharp \;\leq\; \liminf_{n \to \infty} (\inf_{x \in X} V_n(x))$$

Consequently, if V *is the epi-limit of a sequence of lower semicontinuous uniformly inf-compact functions* V_n, *then*

$$(63) \qquad \begin{cases} i) & \inf_{x \in X} V(x) \;=\; \lim_{n \to \infty} (\inf_{x \in X} V_n(x)) \\ ii) & \limsup_{n \to \infty} \{x_n | V_n(x_n) = \inf_{x \in X} V_n(x)\} \\ & \subset \;\; \{x_0 | V(x_0) = \inf_{x \in X} V(x)\} \end{cases}$$

Proof

1. Let $\lambda > \inf_{x\in X} V_\uparrow^\flat(x)$ be fixed and x_0 be chosen such that $V_\uparrow^\flat(x_0) < \lambda$. We know that for all $\eta > 0$, there exists $N > 0$ such that

$$\sup_{n\geq N} \inf_{x\in B(x_0,\eta)} V_n(x) \quad \leq \quad V_\uparrow^\flat(x_0)$$

Therefore,

$$\sup_{n\geq N} \inf_{x\in X} V_n(x) \quad \leq \quad \sup_{n\geq N} \inf_{x\in B(x_0,\eta)} V_n(x)$$

so that $\sup_{n\geq N} \inf_{x\in X} V_n(x) \leq \lambda$. Hence it is enough to let N go to ∞ and λ to $V_\uparrow^\flat(x_0)$.

2. Let us consider a subsequence (again denoted) V_n such that

$$\liminf_{n\to\infty}(\inf_{x\in X} V_n(x)) \quad = \quad \lim_{n\to\infty}(\inf_{x\in X} V_n(x))$$

On the other hand, since the functions V_n are inf-compact, the minima are achieved: there exist x_n's such that $\inf_{x\in} V_n(x) = V_n(x_n)$. They remain in a relatively compact subset since the functions V_n are uniformly inf-compact. So a subsequence (again denoted) x_n does converge to some x_0. Therefore,

$$\begin{aligned} &\inf_{x\in X} V_\uparrow^\sharp(x) \quad \leq \quad V_\uparrow^\sharp(x_0) \\ &= \liminf_{n\to\infty, x\to x_0} V_n(x) \quad \leq \quad \inf_{n\to\infty} V_n(x_n) \\ &= \quad \liminf_{n\to\infty} \inf_{x\in X} V_n(x) \end{aligned}$$

3. If we set

$$F_n(x) \quad := \quad \mathbf{V}_{n\uparrow}(x)$$

we see that the level sets of V_n are the inverse images $F_n^{-1}(\lambda)$ of F_n. Since we know that

$$F^{\sharp^{-1}}(\lambda) \quad = \quad \limsup_{n\to\infty, \lambda_n\to\lambda} F_n^{-1}(\lambda_n)$$

we deduce that the level sets of the lower epi-limit are the Kuratowski upper limits of the level sets:

$$\{x \mid V_\uparrow^\sharp(x) \leq \lambda\} \quad = \quad \limsup_{n\to\infty, \lambda_n\to\lambda} \{x \mid V_n(x) \leq \lambda_n\}$$

By taking $\lambda := \inf_{x\in X} V(x)$ and $\lambda_n := \inf_{x\in X} V_n(x)$, which converges to λ by the two first statements of the Proposition, we infer the third one. □

Unfortunately, there are counter-examples for the property that the set of minimizers of the upper epi-limit is the Kuratowski lower limit of the sets of minimizers of the functions V_n. However, the Stability Theorem provides some results about the Kuratowski lower limits of level sets, but which exclude the case when the level sets are set of minimizers.

Proposition 8.4 *Let us consider a sequence of extended real-valued functions $V_n : X \mapsto \mathbf{R}\cup\{+\infty\}$ whose domains are not empty.*

Let us assume that there exist x_0, $N > 0$ and constants $c > 0$, $\eta > 0$ such that , for all $n \geq N$, $x \in B(x_0, \eta)$,

$$(64) \quad \begin{cases} i) & \exists\, u_n^- \in cB_X \quad \text{such that } D_\uparrow(V_n)(x)(u_n^-) = -1 \\ ii) & \exists\, u_n^+ \in cB_X \quad \text{such that } D_\uparrow(V_n)(x)(u_n^+) = +1 \end{cases}$$

Then there exist a constant l such that, for any sequence of elements x_{0n} converging to x_0 and such that $V_n(x_{0n})$ converges to $V_\uparrow^\flat(x_0)$, and for any λ_n converging to $V_\uparrow^\flat(x_0)$, there exist solutions $\widehat{x_n}$ satisfying

$$(65) \quad \begin{cases} i) & V_n(\widehat{x_n}) \leq \lambda_n \\ ii) & \|\widehat{x_n} - x_{0n}\| \leq l|\lambda_n - V_\uparrow^\flat(x_0)| \end{cases}$$

Proof We apply the Stability Theorem 6.1 to the inverses of the set-valued maps $F_n(x) := \mathbf{V}_{n\uparrow}(x)$. We have to check that

$$\|DG_n(\lambda_n, x)(\mu)\| := \sup_{\mu=\pm 1} \inf_{u\in DG_n(\lambda_n,x)(\mu)} \|u\|$$

is bounded by c. It is enough to choose $u = u_n^+$ when $\mu = +1$ and $u = u_n^-$ when $\mu = -1$. Hence assumption (40) of Theorem 6.1 is satisfied, and its conclusions imply the conclusions of the above theorem. □

Remark Observe that these stability assumptions (64) imply that for all $n \geq N$ and all $x_n \in B(x_0, \eta)$,

$$(66) \quad \inf_{x\in X} V_n(x) < V_n(x_n)$$

since the Fermat rule is violated. □

We can also derive from the Inverse Stability Theorem 6.1 criteria which imply some equalities in formulas (57).

Proposition 8.5 *Let us consider a sequence of extended real-valued functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ whose domains are not empty.*

Let us assume that there exist x_0 and constants $c > 0$, $N > 0$ and $\eta > 0$ such that , for all $n \geq N$, $x \in B(x_0, \eta)$, the domains of the contingent epiderivatives $D_\uparrow(V_n)(x)$ are equal to the whole space and

$$(67) \qquad \sup_{\|u\|=1} |D_\uparrow(V_n)(x)(u)| \;\leq\; c$$

Then

$$(68) \qquad V_\uparrow^\flat(\cdot) \;=\; V_\downarrow^\sharp(\cdot)$$

Proof

We apply Theorem 6.1 to the set-valued maps F_n defined by

$$F_n(x) \;:=\; \mathbf{V}_{n\uparrow}(x)$$

It is easy to check that assumption (67) implies Theorem 3.2's stability assumption (21). This is straightforward when $\lambda_n = V(x_n)$, since

$$\|DF_n(x,\lambda_n)\| := \sup_{\|u\|=1} \inf_{\mu \in DF_n(x,\lambda_n)} |\mu| \leq \sup_{\|u\|=1} |D_\uparrow(V_n)(x)(u)| \leq c$$

When $\lambda_n > V(x_n)$, we know that

$$\mathrm{Dom}(D_\uparrow(V_n)(x)) \times \mathbf{R} \;\subset\; \mathrm{Graph}(DF_n(x,\lambda_n))$$

so that

$$\|DF_n(x,\lambda_n)\| := \sup_{\|u\|=1} \inf_{\mu \in DF_n(x,\lambda_n)} |\mu| = 0$$

Hence, there exists a constant c such that, for all $n \geq N$, $x \in \mathrm{Dom}(V_n)$ close to x_0 and $\lambda_n \geq V_n(x)$ close to $V_\uparrow^\flat(x_0)$, we have

$$\|DF_n(x,\lambda_n)\| := \sup_{\|u\|=1} \inf_{\mu \in DF_n(x,\lambda_n)} |\mu| \;\leq\; c$$

Now, to say that $V_\uparrow^\flat(x_0)$ belongs to the Kuratowski lower limit of the $\mathbf{V}_{n\downarrow}(x)$ when $n \to \infty$ and $x \to x_0$ implies that

$$\limsup_{n\to\infty, x\to x_0} V_n(x) \leq V_\uparrow^\flat(x_0) \quad \Box$$

9 Epigraphical limits of Sums of Functions

Let us consider two Banach spaces X and Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$, and two sequences of extended real-valued functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ and $W_n : Y \mapsto \mathbf{R} \cup \{+\infty\}$. We shall compute the upper epi-limit of the functions

$$U_n \ := \ V_n + W_n \circ A$$

in terms of the upper and lower epi-limits of the functions V_n and W_n.

Theorem 9.1 *Let us consider two Banach spaces X and Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$, and two sequences of extended real-valued functions V_n and W_n form X and Y to $\mathbf{R} \cup \{+\infty\}$ respectively.*

1. *The following inequality*

   (69) $$V_\uparrow^\sharp(x_0) + W_\uparrow^\sharp(Ax_0) \ \leq \ U_\uparrow^\sharp(x_0)$$

   holds true.

2.

We posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that, for all n,*

$$\begin{cases} i) & \forall\, x \in \mathrm{Dom}(V_n) \cap B(x_0,\eta), \ \ \forall\, y \in \mathrm{Dom}(W_n) \cap B(Ax_0,\eta) \\ & B_Y \ \subset \ A\left(\mathrm{Dom}(D_\uparrow^\flat(V_n)(x)) \cap cB_X\right) - \mathrm{Dom}(D_\uparrow(W_n)(y)) + \alpha B_Y \\ ii) & \sup_{u \in \mathrm{Dom}(D_\uparrow^\flat(V_n)(x))} |D_\uparrow^\flat(V_n)(x)(u)|/\|u\| \ \leq \ c \\ iii) & \sup_{v \in \mathrm{Dom}(D_\uparrow(W_n)(y))} |D_\uparrow(W_n)(y)(v)|/\|v\| \ \leq \ c \end{cases}$$

Then, the upper epi-limit $U_\uparrow^\flat$ of the sequence of functions $U_n := V_n + W_n \circ A$ satisfies the estimate:

(70) $$U_\uparrow^\flat(x_0) \ \leq \ V_\uparrow^\flat(x_0) + W_\uparrow^\flat(Ax_0)$$

Consequently, if the sequences of functions V_n and W_n have epi-limits V and W respectively, so does the sequence of functions $U_n := V_n + W_n \circ A$ and its epi- limit U satisfies

$$U(x_0) \ = \ V(x_0) + W(Ax_0)$$

Proof

1. Since inequality (69) holds true when one of the values $V_\dagger^\sharp(x_0)$ and $W_\dagger^\sharp(Ax_0)$ is equal to $-\infty$, or $U_\dagger^\sharp(x_0)$ is equal to $+\infty$, we have to check the formula when the two first values are larger than $-\infty$ and the last one smaller than $+\infty$. Then, there are finite number ρ, integer N and positive number η such that, by definition of the lim inf,

$$\forall\, n \geq N, \quad \forall\, x \in B(x_0, \eta), \quad V_n(x) \text{ and } W_n(Ax) \;\geq\; \rho$$

By definition of the lower epi-limit, the pair $(x_0, U_\dagger^\sharp(x_0))$ is the limit of a subsequence (x_n, c_n) satisfying

$$V_n(x_n) + W_n(Ax_n) \;\leq\; c_n$$

The two above inequalities imply that the sequences of real numbers $a_n := V_n(x_n)$ and $b_n := c_n - a_n$ are bounded. Hence, subsequences (again denoted) a_n and b_n do converge to a and b satisfying

$$U_\dagger^\sharp(x_0) = a + b, \quad a \geq V_\dagger^\sharp(x_0), \quad b \geq W_\dagger^\sharp(Ax_0)$$

2. We begin by observing that if we set

$$\begin{cases} i) & K \;:=\; \mathrm{Ep}(V) \times \mathrm{Ep}(W) \times \mathbf{R} \;\subset\; X \times \mathbf{R} \times Y \times \mathbf{R} \times \mathbf{R} \\ ii) & G(x,a,y,b,c) \;:=\; (Ax - y, a + b - c) \\ iii) & H(x,a,y,b,c) \;:=\; (x,c) \end{cases}$$

we can write

$$(71) \qquad \mathrm{Ep}(U) \;=\; H(K \cap G^{-1}(0,0))$$

Therefore, it is sufficient to show that the Kuratowski lower limit of the subsets $K_n \cap G^{-1}(0,0)$ contains the intersection of the Kuratowski lower limit of the subsets K_n with $G^{-1}(0,0)$. For that purpose, we shall use Theorem 3.2.

Let us consider any sequence of elements x_{0n}, a_{0n}, y_{0n}, and b_{0n} converging respectively to x_0, $V_\dagger^\flat(x_0)$, Ax_0 and $W_\dagger^\flat(Ax_0)$ and let us set $c_{0n} := a_{0n} + b_{0n}$.

We observe that the elements $(x_{0n}, a_{0n}, y_{0n}, b_{0n}, c_{0n})$ belong to K_n and that $G(x_{0n}, a_{0n}, y_{0n}, b_{0n}, c_{0n}) = (Ax_{0n} - y_{0n}, 0)$ converges to $(0, 0)$.

We begin by checking that the assumptions of our Theorem imply the stability assumption (21) of Theorem 3.2, i.e., that there exists a constant $c > 0$ such that, for all n, for all

$$(x, a, y, b, c) \in K_n \text{ close to } (x_0, V_\dagger^\flat(x_0), y_0, W_\dagger^\flat(y_0), 0)$$

for all $(z, \lambda) \in X \times \mathbf{R}$, there exist (u, μ, v, ν, δ) and e such that

$$(72) \qquad \begin{cases} i) & z = Au - v + e \ \ \& \ \ \lambda = \mu + \nu - \delta \\ ii) & \|e\| \leq \alpha(\|z\| + |\lambda|) \\ iii) & \|u\| + \|v\| + |\mu| + |\nu| + |\delta| \leq c(\|z\| + |\lambda|) \end{cases}$$

Assumptions (9.1)i) & ii) imply right away that

$$(73) \qquad \begin{cases} i) & z = Au - v + e, \\ ii) & \|e\| \leq \alpha(\|z\| + |\lambda|) \\ iii) & \|u\| + \|v\| + \leq c\|z\| \leq c(\|z\| + |\lambda|) \end{cases}$$

Let us take now $\mu := c\|u\|$, $\nu := c\|v\|$ and $\delta := c(\|u\| + \|v\|) - \lambda$. We deduce from (9.1)iii) that (u, μ) belongs to $\mathrm{Ep}(D_\dagger V_n)(x)$, that (v, ν) belongs to $\mathrm{Ep}(D_\dagger W_n)(y)$

$$D_\dagger(V_n)(x)(u) + D_\dagger(W_n)(y)(v) \leq c(\|u\| + \|v\|) \leq \mu + \nu = \lambda + \delta$$

and that $|\delta| \leq (|\lambda| + c(\|u\| + |v|)) \leq c'(\|z\| + |\lambda|)$.

The conclusion of Theorem 3.2 being available, we then know that there exist elements $(\widehat{x_n}, \widehat{a_n}, \widehat{y_n}, \widehat{b_n}, \widehat{c_n}) \in K_n$ satisfying

$$(74) \begin{cases} i) & G(\widehat{x_n}, \widehat{a_n}, \widehat{y_n}, \widehat{b_n}, \widehat{c_n}) = 0 \\ ii) & \|\widehat{x_n} - x_{0n}\| + \|\widehat{y_n} - y_{0n}\| \\ & +|\widehat{a_n} - a_{0n}| + |\widehat{b_n} - b_{0n}||\widehat{c_n} - c_{0n}| \leq l\|Ax_{0n} - y_{0n}\| \end{cases}$$

Therefore, we infer that

$$(75) \qquad U_n(\widehat{x_n}) \leq \widehat{a_n} + \widehat{b_n} \leq V_\dagger^\flat(x_0) + W_\dagger^\flat(Ax_0) + \epsilon_n$$

since both a_{0n} and $\widehat{a_n}$ converge to $V_\dagger^\flat(x_0)$ and both b_{0n} and $\widehat{b_n}$ converge to $W_\dagger^\flat(Ax_0)$. Such fact implies the inequality (70) we were looking for.
□

10 Attouch's Theorem

Theorem 10.1 (Attouch's Theorem) *Let us consider a sequence of extended real-valued lower semicontinuous convex functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ where X is a Hilbert space. We supply the dual X^* with the weak topology.*

We suppose that the sequence V_n has an epi-limit V.

Let us consider a subgradient $p \in \partial V(x)$ and let us choose sequences of elements x_{0n} and p_{0n} converging to x and p respectively and satisfying

$$(76) \quad \begin{cases} i) & \limsup_{n\to\infty} V_n(x_{0n}) \leq V(x) \\ ii) & \limsup_{n\to\infty} V_n^\star(p_{0n}) \leq V^\star(p) \end{cases}$$

(Such sequences do exist by definition). We introduce the lack of consistency

$$(77) \quad \begin{array}{l} \delta_n := \\ V_n(x_{0n}) - V(x) + V_n^\star(p_{0n}) - V^\star(p_{0n}) + < p, x > - < p_{0n}, x_{0n} > \end{array}$$

Then p belongs to the graphical lower limit $\partial^\flat V := (\partial V)^\flat$ of the subdifferential ∂V_n and we have the estimate

$$(78) \quad d((x_{0n}, p_{0n}), \mathrm{Graph}(\partial V_n)) \leq \max(0, \sqrt{\delta_n})$$

Therefore, ∂V is the graphical limit of the subdifferentials ∂V_n.

Proof We apply Ekeland's Theorem to the lower semicontinuous function $V_n(y) - < p_{0n}, y >$ for

$$\epsilon = \begin{cases} \sqrt{\delta_n} & \text{if } \delta_n > 0 \\ \text{is any positive number} & \text{if } \delta_n \leq 0 \end{cases}$$

Since $\liminf_{n\to\infty} V_n^\star(p_{0n})$ is finite, there exit a constant c and N large enough such that

$$V_n^\star(p_{0n}) \leq c \quad \forall n \geq N$$

This implies that the functions $y \mapsto V_n(y) - < p_{0n}, y >$ are bounded below by $-c$ thanks to the Fenchel inequality.

Hence, we can apply Ekeland's Theorem: there exists a solution x_n satisfying

$$(79)\quad \begin{cases} i) & V_n(x_n) - < p_{0n}, x_n > + \epsilon \ \|x_n - x_{0n}\| \\ & \leq V_n(x_{0n}) - < p_{0n}, x_{0n} > \\ ii) & \forall\, y \in X,\ V_n(x_n) - < p_{0n}, x_n > \\ & \leq V_n(y) - < p_{0n}, y > + \epsilon \|x_n - y\| \end{cases}$$

Inequality (79)ii) tells us that x_n minimizes $y \mapsto V_n(y) - < p_{0n}, y > + \epsilon\|x_n - y\|$. Hence the Fermat Rule implies that 0 belongs to its subdifferential at x_n, which is equal to $\partial V_n(x_n) - p_{0n} + \epsilon B_\star$. This implies that there exists p_n in $\partial V_n(x_n)$ satisfying

$$\|p_{0n} - p_n\| \ \leq \ \epsilon$$

On the other hand, inequality (79)i) yields

$$\|x - x_n\| \ \leq \frac{1}{\epsilon}(V_n(x_{0n}) - V_n(x_n) + < p_{0n}, x_n > - < p, x_{0n} >)$$

Taking into account that $< p, x > = V(x) + V^\star(p)$ because $p \in \partial V(x)$ and that $< p_{0n}, x_n > \leq V_n(x_n) + V_n^\star(p_{0n})$, we obtain

$$\begin{array}{l} \|x - x_n\| \ \leq \frac{1}{\epsilon}(V_n(x_{0n}) - V(x) + V_n^\star(p_{0n}) - V^\star(p_{0n}) \\ + < p, x > - < p_{0n}, x_{0n} >) \ = \ \delta_n/\epsilon \end{array}$$

If the right hand-side of this inequality is non positive, we infer that $x_{0n} = x_n$, and, ϵ being arbitrarely small, that $d(p_{0n}, \partial V_n(x_{0n})) = 0$. If not, we deduce that $\|x - x_n\| \leq \epsilon^2/\epsilon = \epsilon$. Hence the first part of the Theorem is proved.

Since V is the epi-limit of the sequence V_n, we deduce that each pair (x, p) in the graph of ∂V is the limit of sequences (x_{0n}, p_{0n}) satisfying conditions (76). Hence it is also the limit of the sequences (x_n, p_n) of the graph of ∂V_n we have just constructed.

This means that the subdifferential map ∂V of the epi-limit of the functions V_n is contained in the graphical lower limit of the subdifferential maps ∂V_n.

Since ∂V is a maximal monotone set-valued map when X is a Hilbert space, then the second part of the Theorem follows from Proposition ??. □

In particular, we deduce that under the assumptions of Theorem 10.1, the graphical convergence of the subdifferential ∂V_n to ∂V implies

$$\partial V(x) \;=\; \limsup_{n\to\infty} \partial V_n(x_n)$$

However, we need some stability assumptions on the contingent second derivatives of the functions V_n for stating that some $p \in \partial V(x)$ belongs to the Kuratowski lower limit of $\partial V_n(x_n)$.

Naturally, the contingent second derivative $\partial^2 V(x,p)$ of V at some point (x,p) in the graph of ∂V is defined as the contingent derivative of the set-valued map ∂V at (x,p).

Proposition 10.1 *Let X be a Hilbert space. Let us consider a sequence of extended real-valued convex functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ and V whose domains are not empty and sequences x_{0n} and p_{0n} converging to x and p and satisfying propertyies (76)*

Let p belong to $\partial V(x)$. We posit the following stability assumption: There exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that

$$(80) \qquad \begin{cases} \forall\, (x_n, p_n) \in \mathrm{Graph}(\partial V_n) \cap B((x,p),\eta), \\ \|\partial^2 V_n(x_n,p_n)\| := \sup_{u\in X} \inf_{\pi \in \partial^2 V_n(x_n,p_n)(u)} \|\pi\|/\|u\| \leq \epsilon \end{cases}$$

Then

$$p \;\in\; \liminf_{n\to\infty} \partial V_n(x_n)$$

Remark The same proof yields the following result in the non convex case. Recall that the Clarke generalized gradient $\partial V(x)$ of V at x is defined by

$$p \in \partial V(x) \Longleftrightarrow \forall\, v \in X,\; <p,v> \;\leq\; CV(x)(v)$$

and that the subdifferential $\partial^\circ V(x)$ is defined by

$$p \in \partial^\circ V(x) \Longleftrightarrow \forall\, y \in X,\; V(x) - V(y) \;\leq\; <p, x-y>$$

Theorem 10.2 *Let us consider a sequence of extended real-valued lower semicontinuous functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ and V whose domains* *are not empty. Let us consider a subgradient $p \in \partial^\circ V(x)$. If there exist sequences x_{0n} and p_{0n} converging to x and p respectively which satisfy*

$$(81) \qquad \begin{cases} i) & \limsup_{n\to\infty} V_n(x_{0n}) \leq V(x) \\ ii) & \limsup_{n\to\infty} V_n^\star(p_{0n}) \leq V^\star(p) \end{cases}$$

then p belongs to the graphical lower limit $\partial^\flat V := (\partial V)^\flat$ of the Clarke generalized gradients ∂V_n and we have the estimate (78)¿

11 Asymptotic Paratingent and Circatangent Cones

Let X be a topological vector space. We consider a sequence of subsets $K_n \subset X$ of X and their Kuratowski upper and lower limits $K^\sharp$ and $K^\flat$.

Definition 11.1 (Asymptotic Paratingent and Circatangent cones) *If $x \in K^\sharp$, we shall say that*

$$(82) \qquad P_{K^\sharp}^\sharp(x) := \limsup_{n\to\infty, K_n \ni x_n \to x} \frac{K_n - x_n}{h_n}$$

is the **asymptotic paratingent cone** *to the Kuratowski upper limit $K^\sharp$ at x. We shall say that the* **asymptotic circatangent cone** *to the Kuratowski lower limit $K^\flat$ at $x \in K^\flat$ is the set $C_{K^\flat}^\flat(x)$ defined by*

$$(83) \qquad C_{K^\flat}^\flat(x) := \liminf_{n\to\infty, K_n \ni x_n \to x} \frac{K_n - x_n}{h_n}$$

Remark When we consider a **constant sequence** $K_n := K$, we see that the asymptotic paratingent and asymptotic circatangent cones $P_{K^\sharp}^\sharp$ and $C_{K^\flat}^\flat$ coincide with the paratingent $P_K^K(x)$ and the Clarke tangent cone $C_K(x)$ respectively. □

It is easy to observe that the Kuratowski upper limit of the contingent cones to a sequence of K_n is contained in the asymptotic paratingent cone to the Kuratowski upper limit of the K_n's:

Proposition 11.1 *Let us consider a sequence of subsets $K_n \subset X$ of X and their Kuratowski upper limit $K^\sharp$. Then*

$$(84) \qquad \limsup_{x_n \to x} T_{K_n}(x_n) \subset P_{K^\sharp}^\sharp(x)$$

The asymptotic circatangent cone to a Kuratowski lower limit is always a closed convex cone, and is actually equal to the Kuratowski lower limit of the contingent cones:

Proposition 11.2 *Let us consider a sequence of subsets $K_n \subset X$ of X and their Kuratowski lower limit $K^\flat$.*

1. *The asymptotic circatangent cone $C^\flat_{K^\flat}(x)$ is always a closed convex cone*
2. *If X is reflexive and is supplied with the weak topology, and if the subsets K_n are weakly closed, then*

$$C^\flat_{K^\flat(x)} = \liminf_{n\to\infty, K_n\ni x_n \to x} T_{K_n}(x_n) \tag{85}$$

Proof

1. Let v_1 and v_2 belong to $C^\flat_{K^\flat}(x)$. For proving that $v_1 + v_2$ belongs to this cone, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $x_n \in K_n$ converging to x. There exists a sequence of elements v_{1n} converging to v_1 such that the elements $x_{1n} := x_n + h_n v_{1n}$ do belong to K_n for all n. But since x_{1n} does also converge to x, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall\, n,\ \ x_{1n} + h_n v_{2n} \ = \ x_n + h_n(v_{1n} + v_{2n}) \ \in \ K_n$$

This implies that $v_1 + v_2$ belongs to $C^\flat_{K^\flat}(x)$ because the sequence of elements $v_{1n} + v_{2n}$ converges to $v_1 + v_2$.

2. Let us take $x \in K^\flat$ and $v \in \liminf_{x_n \to x} T_{K_n}(x_n)$. We infer that, for all $\epsilon > 0$, $x_n \in K_n$ close to x and t small enough, inequalities

$$\forall\, z_n \in \pi_{K_n}(x_n + tv),\ \|z - x\| \ \le \ 2\|x_n + tv - x\|$$

imply that z_n remains close to x, that there exists $v_n \in T_{K_n}(z_n)$ in a neighborhood of v, so that

$$\begin{cases} d(v, T_K(\pi_{K_n}(x_n + \tau v))) & \le \ \|v - v_n\| \\ & = \ \epsilon \end{cases}$$

for large enough n's. Let us set $g_n(t) := d_{K_n}(x_n + tv)$. Since $g(\cdot)$ is locally lipschitzean, it is almost everywhere differentiable. It implies[15] implies that $g'_n(\tau) \leq d(v, T_{K_n}(\pi_{K_n}(x_n + \tau)v)) \leq \epsilon$. We then integrate from 0 to t and get that, for all $x_n \in K_n$ close to x and for all $t \in]0, h]$ for some small enough positive h,

$$d_{K_n}(x_n + tv) \leq \int_0^t d(v, T_{K_n}(\pi_{K_n}(x_n + \tau v)))d\tau \leq t\epsilon$$

We have proved that v belongs to $C^\flat_{K^\flat}(x)$.

3. Let v belong to $C^\flat_{K^\flat}(x)$. Then, for all $\epsilon > 0$, there exist $\eta > 0$, N and $\beta > 0$ such that, for all $h \leq \beta$, $n \geq N$ and $x_n \in K_n \cap B(x, \eta)$,

$$d_{K_n}(x_n + hv) \leq h\epsilon$$

Then we can associate with such x_n elements $y_n^h \in K_n$ such that

$$\|x_n - y_n^h + hv\| \leq 2h\epsilon$$

We set $v_n^h := (y_n^h - x_n)/h$. Since $\|v_n^h - v\| \leq 2\epsilon$ and since the space is reflexive, a subsequence (again denoted) v_n^h converges weakly to some $v_n \in v + 2\epsilon B$. Such a v_n belongs to the contingent cone $T_{K_n}(x_n)$ (when X is supplied with the weak topology) and converges to v. □

When the subsets K_n are convex, we obtain the following telations:

Proposition 11.3 *Let us consider a sequence of convex subsets $K_n \subset X$ and its Kuratowski convex upper and lower limits $K^\sharp$ and $K^\flat$. Then:*

$$(86) \quad \begin{cases} i) & T_{K^\flat}(x) \subset \liminf_{x_n \to x} T_{K_n}(x_n) \\ ii) & T_{K^\sharp}(x) \subset \limsup_{x_n \to x} T_{K_n}(x_n) \end{cases}$$

Proof

1. Let us take $x \in K^\flat$ and $v \in S_{K^\flat}(x)$, the cone spanned by $K^\flat - x$. Hence there exists $h > 0$ such that $x + hv \in K^\flat$

[15] see [?, Proposition 4.1.3]

Let us consider any sequence of elements x_n belonging to K_n and converging to x and let y_n denote the projection of $x + hv$ onto the closure of K_n. Since y_n converges to $x + hv$, the sequence of elements $v_n := (y_n - x_n)/h$ converges to v. Since

$$x_n + hv_n = (1 - \frac{h_n}{h})x_n + \frac{h_n}{h}y_n$$

belongs to K_n, we infer that v_n belongs to the contingent cone $T_{K_n}(x_n)$. Hence v, the limit of the v_n's, belongs to the Kuratowski lower limit of the tangent cones $T_{K_n}(x_n)$

2. Let x belong to $K^\sharp$ and v be chosen arbitrarily in the cone spanned by $K^\sharp - x$. By denoting by x_n and y_n the projections onto K_n of x and $x + hv$ respectively, we infer that $v_n := (y_n - x_n)/h$ belongs to the tangent cone $T_{K_n}(x_n)$ and converges to v. □

Remark We deduce from Attouch's Theorem 10.1 that if the sequence of convex subsets K_n has a limit K, then

$$(87) \qquad N_K(x) \;=\; \limsup_{n\to\infty, x_n \to x} N_{K_n}(x_n)$$

(We take for function V_n the indicators ψ_{K_n} of the K_n, which epi-converges to the indicator of K. We use then the fact that the subdifferential of an indicator is the normal cone.)

When the dimension of X is finite, we deduce by transposition from Proposition 1.3 that if the sequence of convex subsets K_n has a limit K, then

$$\liminf_{n\to\infty, x_n\to x} T_{K_n}(x_n) \;=\; T_K(x) \quad \square$$

We shall deduce from the Stability Theorem a formula for the asymptotic circatangent cone of an inverse image.

Theorem 11.1 *Let X and Y be two Banach spaces. We introduce a continuous linear operator $A \in \mathcal{L}(X,Y)$ and sequences of closed subsets $L_n \subset X$ and $M_n \subset Y$. Let us assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$(88) \qquad \begin{cases} \forall\, x_n \in K_n \cap B(x_0,\eta), \;\; y_n \in M_n \cap B(Ax_0,\eta) \\ B_Y \;\subset\; A(T^\flat_{L_n}(x_n) \cap cB_X) - T_{M_n}(y_n) + \alpha B_Y \end{cases}$$

Let x belong to the Kuratowski lower limit of the sequence $L_n \cap A^{-1}(M_n)$ (which is equal to the intersection of the Kuratowski lower limit of L_n and the inverse image by A of the Kuratowski lower limit of M_n). Then

$$C^{\flat}_{L^{\flat}}(x) \cap A^{-1}C^{\flat}_{M^{\flat}}(Ax) \subset C^{\flat}_{L^{\flat} \cap A^{-1}M^{\flat}}(x) \tag{89}$$

Proof Take any sequence of elements $x_n \in L_n \cap A^{-1}M_n$ which converges to x. (We already know that x is the limit of such a sequence). Let us take any $u \in C^{\flat}_{L^{\flat}}(x)$ such that $Au \in C^{\flat}_{M^{\flat}}(Ax)$. Hence for any sequence $h_n > 0$, there exist sequences u_n and v_n converging to u and Au respectively such that, for all $n \geq 0$,

$$x_n + h_n u_n \in L_n \ \ \& \ \ Ax_n + h_n v_n \in M_n$$

We apply now Theorem 3.2 to the subsets $L_n \times M_n$ of $X \times Y$ and the continuous linear operators $A \ominus \mathbf{1}$ associating to any (x, y) the element $Ax - y$, since we can write

$$K_n \ := \ L_n \cap A^{-1}(M_n) \ = \ (A \ominus \mathbf{1})^{-1}(0) \cap (L_n \times M_n)$$

The stability assumptions of this Theorem are obviously satisfied. The pair $(x_n + h_n u_n, Ax_n + h_n v_n)$ belongs to $L_n \times M_n$ and

$$(A \ominus \mathbf{1})(x_n + h_n u_n, Ax_n + h_n v_n) \ \text{converges to} \ 0$$

Therefore, by Theorem 3.2, there exits a solution $(\widehat{x_n}, \widehat{y_n}) \in L_n \times M_n$ to the equation $(A \ominus \mathbf{1})(\widehat{x_n}, \widehat{y_n}) = 0$ such that

$$\|x_n + h_n u_n - \widehat{x_n}\| + \|Ax_n + h_n v_n - \widehat{y_n}\| \leq l h_n \|Au_n - v_n - 0\|$$

Hence $\widehat{u_n} := (x_n - \widehat{x_n})/h_n$ converges to u, and we oobserve that for all $n \leq 0$, $x_n + h_n u_n$ belongs to $L_n \cap A^{-1}(M_n)$ because $x_n + h_n \widehat{u_n} = \widehat{x_n}$ belongs to L_n and $A(x_n + h_n \widehat{u_n}) = \widehat{y_n}$ belongs to M. □

We consider now the asymptotic paratingent cones to direct images.

Theorem 11.2 *Let X and Y be two Banach spaces. We introduce a continuous linear operator $A \in \mathcal{L}(X, Y)$, a sequence of closed subsets $K_n \subset L \subset X$ and an element x_0 in the Kuratowski upper limit $K^{\sharp}$ of the K_n's.*

We assume that the restriction of A to L is proper from L to Y, so that $A(K^\sharp) = (A(K))^\sharp$.

We posit the following **stability assumption**: *for all $x_0 \in K \cap A^{-1}(y_0)$, there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that*

$$(90) \qquad \begin{cases} \forall\, x_n \in K_n \cap B(x_0, \eta), \\ AS_{K_n}(x_n) \cap B_Y \;\subset\; A\left(T_{K_n}(x_n) \cap cB_X\right) + \alpha B_Y \end{cases}$$

Then, if X is reflexive and supplied with the weak topology, we have:

$$(91) \qquad \overline{\bigcup_{x \in K \cap A^{-1}(y)} AP^\sharp_{K^\sharp}(x)} \;=\; P^\sharp_{A(K^\sharp)}(Ax)$$

Proof We always have the inclusion

$$(92) \qquad \forall\, x \cap K^\sharp \cap A^{-1} \cap y_0,\;\; A(P^\sharp_{K^\sharp}(x)) \;\subset\; P^\sharp_{A(K^\sharp)}(y_0)$$

Conversely, let us take $v \in P^\sharp_{K^\natural}(y_0)$. Then there exist sequences of elements $h_n > 0$, $y_n \in A(K_n)$ and $v_n \in Y$ converging to 0, y_0 and v respectively such that

$$(93) \qquad \forall\, n \geq 0,\;\; y_n + h_n v_n \;\in\; A(K_n)$$

We can write $y_n := Ax_{0n}$ where, A being proper, a subsequence (again denoted) x_{0n} converges to some $x_0 \in K \cap A^{-1}(y_0)$.

By Theorem 3.1, there exist a constant l' and solutions $x_n \in K$ to the equation $Ax_n = y_n + h_n v_n$ such that

$$(94) \qquad \|x_{0n} - x_n\| \;\leq\; \|Ax_{0n} - y_n - h_n v_n\| \;=\; h_n \|v_n\|$$

Therefore, the sequence of elements u_n is bounded, so that a subsequence (again denoted) u_n converges (weakly when the dimension of X is infinite) to some u. □

Remark We can adapt the properness criterion given by Theorem 5.1 for obtaining the following result:

Proposition 11.4 *Let X and Y be two Banach spaces. We introduce a continuous linear operator $A \in \mathcal{L}(X, Y)$, a sequence of closed subsets $K_n \subset$*

X and an element x_0 in the Kuratowski lower limit $K^\flat$ of the K_n's. We posit the following assumption:

$$(95)\quad 0 \in \mathrm{Int}\left(\mathrm{Im}(A^\star) + \bigcup_{N>0,\eta>0}\ \bigcap_{n\geq N, x_n\in K_n\cap(x_0+\eta B)} K_n^\circ \cap N_{K_n}^0(x_n)\right)$$

Then, if X is reflexive and supplied with the weak topology, we have:

$$(96)\qquad \overline{\bigcup_{x\in K\cap A^{-1}(y)} AP_{K^\sharp}^\sharp} = P_{A(K^\sharp)}^\sharp(Ax)$$

Proof let us take $v \in P_{K^\flat}^\sharp(y_0)$. Then there exist sequences of elements $h_n > 0$, $y_n \in A(K_n)$ and $v_n \in Y$ converging to 0, y_0 and v respectively such that

$$(97)\qquad \forall\, n \geq 0,\ y_n + h_n v_n = A(x_n) \in A(K_n)$$

Let us consider solutions $x_{0n} \in K_n$ to the equation $Ax_{0n} = y_n$ and set $u_n := (x_n - x_{0n})/h_n$. We shall prove that the sequences x_{0n} and u_n are pointwise bounded. Since the space X is reflexive, this will imply that subsequences (again denoted) x_{0n} and u_n converge to some $x_0 \in K \cap A^{-1}(y_0)$ and u respectively, so that u is an element of $P_{K^\sharp}^\sharp(x_0)$.

For proving our claim, we associate with any $p \in X^\star$ an element $q \in Y^\star$, an integer N such that, for all $n \geq N$, there exist $r_n \in K_n^\circ \cap N_{K_n}^0(x_n)$ satisfying $p = A^\star q + r_n$.

Then

$$(98)\quad \begin{cases} i) & <p, x_{0n}>=<q, y_n> + <r_n, x_{0n}> \leq \|q\|\|y_n\| + 1 \\ & \text{because } r_n \in K_n^\circ \\ ii) & <p, u_n>=<q, v_n> + <r_n, \frac{x_{0n}-x_n}{h_n}> \leq \|q\|\|v_n\| + 0 \\ & \text{because } r_n \in N_{K_n}^0(x_n) \end{cases}$$

Therefore, our sequences x_{0n} and u_n are bounded. □

12 Asymptotic Paratingent and Circatangent Epiderivatives

We are now able to define asymptotic epi-derivatives of a sequence of functions V_n, by taking the asymptotic tangent cones to their epigraphs.

Definition 12.1 (Asymptotic Epiderivatives) *Let us consider a sequence of extended real-valued functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ whose domains are not empty and an element x_0 in the Kuratowski lower limit of the domains of the functions V_n. We shall say that the function $C_\dagger^\flat V(x_0)$ defined by*

$$(99) \quad C_\dagger^\flat V(x_0)(u) := \limsup_{n\to\infty, x\to x_0, V_n(x)\le\lambda\to V_\dagger^\flat(x_0), h\to 0+} (V_n(x+hu')-\lambda)/h$$

is the **asymptotic circatangent epi-derivative** *of the sequence of functions V_n at x_0 in the direction u.*

We see at once that the epigraph of $C_\dagger^\flat V(x_0)$ is the asymptotic circatangent cone to the epigraphs of the functions V_n at $(x_0, V_\dagger^\flat(x_0))$, or, equivalently, that $C_\dagger^\flat V(x_0)$ is the upper epi-limit of the difference quotients

$$u' \mapsto (V_n(x+hu')-\lambda)/h$$

when

$$n \to \infty \;\; \& \;\; (x,\lambda,h) \in \mathrm{Ep}(V_n) \times \mathbf{R}_+ \text{ converges to } (x_0, V_\dagger^\flat(x_0), 0)$$

We deduce that the asymptotic circatangent epiderivative is a positively homogeneous, lower semicontinuous and convex function form X to $\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$.

We shall estimate the asymptotic circatangent epiderivative of a family of functions $U_n := V_n + W_n \circ A$.

Theorem 12.1 *Let us consider two Banach spaces X and Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$, and two sequences of extended real-valued functions V_n and W_n form X and Y to $\mathbf{R} \cup \{+\infty\}$ respectively. Let x_0 belong to the Kuratowski lower limit of the domains of the functions $U_n := V_n + W_n \circ A$.*

We posit the following **stability assumption**: *there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that, for all n,*

$$(100) \begin{cases} i) & \forall\, x \in \mathrm{Dom}(V_n) \cap B(x_0,\eta), \;\; \forall\, y \in \mathrm{Dom}(W_n) \cap B(Ax_0,\eta) \\ & B_Y \subset A\left(\mathrm{Dom}(D_\dagger^\flat(V_n)(x)) \cap cB_X\right) \\ & -\mathrm{Dom}(D_\dagger(W_n)(y)) + \alpha B_Y \\ ii) & \sup_{u\in\mathrm{Dom}(D_\dagger^\flat(V_n)(x))} |D_\dagger^\flat(V_n)(x)(u)|/\|u\| \;\le\; c \\ iii) & \sup_{v\in\mathrm{Dom}(D_\dagger(W_n)(y))} |D_\dagger(W_n)(y)(v)|/\|v\| \;\le\; c \end{cases}$$

Then, the asymptotic circatangent epiderivative of the sequence of functions $U_n := V_n + W_n \circ A$ *satisfies the estimate:*

$$(101) \qquad C_\dagger^\flat(U)(x_0)(u) \;\leq\; C_\dagger^\flat(V)(x_0)(u) + C_\dagger^\flat(W)(Ax_0)(Au)$$

Proof We apply Theorem 3.2 since we have seen that if we set

$$(102) \left\{ \begin{array}{lll} i) & K & := \mathrm{Ep}(V) \times \mathrm{Ep}(W) \times \mathbf{R} \subset X \times \mathbf{R} \times Y \times \mathbf{R} \times \mathbf{R} \\ ii) & G(x,a,y,b,c) & := (Ax - y, a + b - c) \\ iii) & H(x,a,y,b,c) & := (x,c) \end{array} \right.$$

we can write

$$(103) \qquad \mathrm{Ep}(U) \;=\; H(K \cap G^{-1}(0,0))$$

The stability assumption of Theorem 3.2 being the same that the ones of Theorem 70, we already know that they can be derived from assumptions (21).

Hence, we deduce that

$$\begin{array}{l} C_{K^\flat}^\flat(x_0, V_\dagger^\flat(x_0), Ax_0, W_\dagger^\flat(Ax_0), U_\dagger^\flat(x_0)) \cap G^{-1}(0,0) \\ \subset \;\; C_{K^\flat \cap G^{-1}(0)}^\flat(x_0, V_\dagger^\flat(x_0), Ax_0, W_\dagger^\flat(Ax_0), U_\dagger^\flat(x_0)) \end{array}$$

It remains to show that this inclusion implies inequality (101).

Let us set $\lambda = C_\dagger^\flat(V)(x_0)(u)$, $\mu = C_\dagger^\flat(W)(Ax_0)(Au)$ and $\nu := \lambda + \mu$. Hence the element $(u, \lambda, Au, \mu, \nu)$ belongs to

$$C_{K^\flat}^\flat(x_0, V_\dagger^\flat(x_0), Ax_0, W_\dagger^\flat(Ax_0), U_\dagger^\flat(x_0)) \cap G^{-1}(0,0)$$

By Theorem 70, it then belongs to the asymptotic circatangent cone to the subsets $K_n \cap G^{-1}(0,0)$. Then, for all sequence $h_n > 0$, there exist elements (u_n, λ_n, μ_n) converging to (u, λ, μ) such that, for all $n \geq N$,

$$(x_n + h_n u_n, a_n + h_n \lambda_n, Ax_n + h_n Au_n, b_n + h_n \mu_n, a_n + b_n + h_n \nu_n) \in K_n$$

Therefore, the pairs $(x_n + h_n u_n, a_n + b_n + h_n(\lambda_n + \mu_n))$ belong to the epigraph of U_n. Since $(u_n, \lambda_n + \mu_n)$ converges to (u, ν), we deduce that

$$C_\dagger^\flat(U)(x_0)(u) \;\leq\; \nu \;=\; C_\dagger^\flat(V)(x_0)(u) + C_\dagger^\flat(W)(Ax_0)(Au) \;\; \square$$

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