HERMITE POLYNOMIALS EXPANSIONS FOR DISCRETE-TIME NONLINEAR FILTERING

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FOREWORD

The paper deals with a method for the approximation of rather general discrete-time nonlinear filtering problems, which allows the evaluation of suitably chosen approximation errors. The particular Hermite polynomials expansion used for the approximation provide error bounds which may often prove better than those obtained by similar techniques.

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ABSTRACT

A finite-dimensional approximation to general discrete-time nonlinear filtering problems is provided. It consists in a direct approximation to the recursive Bayes formula, based on a Hermite polynomials expansion of the transition density of the signal process. The approximation is in the sense of convergence, in a suitable weighted norm, to the conditional density of the signal process given the observations. The choice of the norm is in turn made so as to guarantee also the convergence of the conditional moments as well as to allow the evaluation of an upper bound for the approximation error.

CONTENTS

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1. INTRODUCTION

We consider the following discrete-time partially observable process (x_t, y_t), x_t, $y_t \in \mathbf{R}$, with x_t the unobservable and y_t the observable components, given for $t = 0$ $1, \ldots, T$ on some probability space $(\Omega, \mathcal{F}, P)$ by

$$x_{t+1} = a(x_t) + v_{t+1}; \qquad x_0 = v_0 \tag{1.a}$$

$$y_t = c(x_t) + w_t; \qquad y_0 = w_0 \tag{1.b}$$

where $\{v_t\}$ and $\{w_t\}$ are independent standard white Gaussian noises.

Given a measurable function f, we shall be concerned with the solution to the filtering problem, namely the computation for each $t = 1, \ldots, T$, assuming it exists, of the least squares estimate of $f(x_t)$ given the observations up to time t, namely

$$E\{f(x_t) \mid \mathcal{F}_t^y\} \tag{2}$$

where $\mathcal{F}_t^y := \sigma\{y_s \mid s \leq t\}$.

The filtering problem can be more generally described in terms of conditional distributions as follows. Given a Markov process x_t with known transition densities $p(x_t \mid x_{t-1})$ and an observable process y_t, characterized by a known conditional density $p(y_t \mid x_t)$, it is desired to compute for each $t = 1, \ldots, T$ the filtering density $p(x_t \mid y^t)$ where $y^t := \{y_0, y_1, \ldots, y_t\}$.

A solution to this problem can be obtained by means of the recursive Bayes formula

$$p(x_t \mid y^t) = \frac{p(y_t \mid x_t)\, p(x_t \mid y^{t-1})}{\int p(y_t \mid x_t)\, p(x_t \mid y^{t-1})\, \mathrm{d}x_t} =$$

$$= \frac{p(y_t \mid x_t) \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid y^{t-1})\, \mathrm{d}x_{t-1}}{\int p(y_t \mid x_t) \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid y^{t-1})\, \mathrm{d}x_{t-1}\, \mathrm{d}x_t} \tag{3}$$

However, there is an inherent computational difficulty with this formula due to the fact that the integral

$$\int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid y^{t-1})\, dx_{t-1}$$

is parametrized by $x_t \in \mathbf{R}$.

As it will be briefly reviewed in the next section (see also [5]), this difficulty disappears in all those situations when $p(x_t \mid x_{t-1})$ is a combination of functions separated in the two variables, i.e.

$$p(x_t \mid x_{t-1}) = \sum_{i=0}^{n} \varphi_i(x_t)\, \psi_i(x_{t-1}) \tag{4}$$

and for such situations an explicit finite-dimensional filter can be provided.

In [5] the computational advantage resulting from (4) was exploited in order to approximate $p(x_t \mid y^t)$ by means of approximating densities $p_n(x_t \mid y^t)$, $n \geq 1$, that could be explicitly computed in a recursive way. Such $p_n(x_t \mid y^t)$ were obtained by means of the recursive Bayes formula (3) using approximations to $p(x_t \mid x_{t-1})$ given by suitable nonnegative functions $p_n(x_t \mid x_{t-1})$ of the form (4). Furthermore the approximation was such that an explicitly computable bound could be obtained for an appropriate approximation error. In addition, if $f(\cdot)$ does not grow more than exponentially, then also $E\{f(x_t) \mid \mathcal{F}_t^y\}$ could be approximated by $\int f(x_t)\, p_n(x_t \mid y^t)\, dx_t$ with a corresponding error bound.

The practically important problem of deriving explicit error bounds for the nonlinear filtering problem was also studied in [3] for discrete-time problems and later [4] the results were extended to continuous-time problems (see also [2, 6] for different techniques that however do not lead to explicit error bounds). While in [3] the approximation is obtained by approximating the model (1), the method followed in [5] consists in a direct approximation to the solution to the recursive Bayes formula.

The aim of this paper is to study a particular case of the technique described in [5], consisting in a Hermite polynomial expansion of $p(x_t \mid x_{t-1})$. This method provides an approximation to the nonlinear filtering problem with improved error bounds with respect to those given in [5].

In the next Section 2 we shall review the results in [5] that will be needed in the sequel, while in the following Section 3 the Hermite polynomial approximation will be examined in detail.

2. GENERAL APPROXIMATION RESULTS

As mentioned in the introduction, the computational difficulty due to the parametization of the integrals in (3) disappears for transition densities of the form (4).

In fact, letting $\propto$ denote proportionality, it is easily seen that when (4) holds $p(x_t \mid y^t)$ can actually be computed by means of (3) resulting in

$$p(x_t \mid y^t) \propto \sum_{i=0}^{n} d_i(y^{t-1})\, p(y_t \mid x_t)\, \varphi_i(x_t); \quad t = 1, \ldots, T \tag{5}$$

where the vector $d(y^{t-1})$ of the coefficients in the combination can be recursively obtained as

$$d_i(y^{\circ}) = \int \psi_i(x_0)\, p(x_0)\, \mathrm{d}x_0; \quad i = 0, \ldots, n \tag{6.a}$$

$$d(y^t) = d(y^{t-1})\, B(y_t); \quad t \geq 1 \tag{6.b}$$

with $B(y_t) = \{b_{ij}(y_t)\}_{i,j=0,\ldots,n}$ where

$$b_{ij}(y_t) = \int \psi_j(x_t)\, p(y_t \mid x_t)\, \varphi_i(x_t)\, \mathrm{d}x_t \; . \tag{6.c}$$

In this section we shall show how a suitably chosen positive approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ produces, through the recursive Bayes formula (3), an approximation to the filtering density $p(x_t \mid y^t)$ as well as to the corresponding filter $E\{f(x_t) \mid y^t\}$, for which explicit upper bounds to the approximation error can be evaluated.

To this end it will be convenient to provide approximations to $p(x_t \mid y^t)$ in a suitable weighted norm of the type.

$$\| g \|_{\alpha} := \int \alpha(x) \mid g(x) \mid \mathrm{d}x \tag{7}$$

In what follows we shall choose $\alpha(x) = \exp[\alpha \mid x \mid]$, $(\alpha > 0)$, as this will enable us to approximate $E\{f(x_t) \mid \mathcal{F}_t^y\}$ for all those $f(\cdot)$ for which $\mid \exp[-\alpha \mid x \mid] f(x) \mid < +\infty$; in particular, it will allow the approximation of all the conditional moments, as long as they exist.

The general approximation results are given in [5] and summarized in Theorem 1 below, which is based on the following assumptions.

There exist a function $V(y_t)$ and constants U, Q, Z, Z_n such that for all t:

A.1: $\inf_{x_t} p(y_t \mid x_t) \geq V(y_t) > 0$

$\sup_{x_t} p(y_t \mid x_t) \leq U$

A.2: $\int \inf_{x_{t-1}} p_n(x_t \mid x_{t-1})\, dx_t \geq W > 0$

A.3: $\sup_{x_{t-1}} \| p_n(x_t \mid x_{t-1}) \|_\alpha \leq Z$

A.4: $\sup_{x_{t-1}} \| p(x_t \mid x_{t-1}) - p_n(x_t \mid x_{t-1}) \|_\alpha \leq Z_n$

with $\lim_{n \to \infty} Z_n = 0$ ■

We then have

THEOREM 1 *Under A.1–A.4 we have for all* $t \geq 1$

a) $\| p(x_t \mid y^t) - p_n(x_t \mid y^t) \|_\alpha \leq Z_n \sum_{s=1}^{t} (2U^2 V^{-2}(y_t)\, W^{-1} Z^2)^s$

b) $| E\{f(x_t) \mid \mathcal{F}_t^y\} - \int f(x_t)\, p_n(x_t \mid y^t)\, dx_t | \leq M Z_n \sum_{s=1}^{t} (2U^2 V^{-2}(y_t)\, W^{-1} Z^2)^s$.

where $M > 0$ *is such that* $| f(x) e^{-\alpha |x|} | \leq M$. ■

3. HERMITE POLYNOMIALS APPROXIMATION

In this section we shall provide an approximate solution to the nonlinear filtering problem (1), (2), based on a nonnegative approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ of type (4) and given in terms of a Hermite polynomials expansion of $p(x_t \mid x_{t-1})$. For the validity of the results of the previous section it is necessary to show that assumptions A.1–A.4 are satisfied. To this end we shall need the following additional assumption on model (1):

A.5: There exist constants A and C such that

$$\sup_x | a(x) | \leq A \tag{8}$$

$$\sup_x | c(x) | \leq C \tag{9}$$

■

Taking into account that, due to the normalization in (3), we can take $p(y_t \mid x_t) = \exp - (y_t - c(x_t))^2/2$ we have that (9) implies A.1 with $U = 1$ and $V(y_t) = \exp(| y_t | + C)^2/2$.

We now recall some properties of Hermite polynomials that will be needed in the sequel. Denoting by $H_k(x) := (-1)^k e^{x^2/2}(d^k/dx^k)\, e^{-x^2/2}$ the k-th Hermite polynomial we have [1], [7]

P.1: For all t and x

$$
\begin{aligned}
e^{tx - t^2/2} &= \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} e^{x^2/2} \frac{(-t)^k}{k!} \frac{d^k}{dx^k} e^{-x^2/2}
\end{aligned}
\tag{10}
$$

P 2: For all x and positive integer k

$$
H_k(x) = k! \sum_{m=0}^{[k/2]} \frac{(-1)^m}{2^m m!} \frac{x^{k-2m}}{(k-2m)!} \tag{11}
$$

where $[k/2]$ is the maximum integer not greater than $k/2$.

P.3: For all x and positive integer k

$$
| H_k(x) | \leq l \sqrt{k!}\, e^{x^2/4} \tag{12}
$$

where l is known as Charlier's constant and $l \simeq 1.0864$.

P.4: For all x, y and positive integer k

$$
H_k(x+y) = \sum_{m=0}^{k} \binom{k}{m} H_{k-m}(x) y^m \tag{13}
$$

■

The expansion in (10) suggests the following approximation

$$
e^{tx - t^2/2} \simeq \sum_{k=0}^{n-1} H_k(x) \frac{t^k}{k!} \tag{14}
$$

with corresponding absolute error

$$
\begin{aligned}
R_n &= \Big| e^{tx - t^2/2} - \sum_{k=0}^{n-1} H_k(x) \frac{t^k}{k!} \Big| = \\
&= \frac{|t|^n}{n!} \Big| \frac{d^n}{dt^n} e^{tx - t^2/2} \Big|_{t=\theta} \qquad \text{with } \theta \in (0, t)
\end{aligned}
\tag{15}
$$

Taking into account that

$$\frac{d^n}{dt^n}\, e^{tx - t^2/2} = \sum_{j=0}^{n} \binom{n}{j} x^j e^{tx} H_{n-j}(t)(-1)^{j-n} \cdot e^{-t^2/2} =$$

$$= (-1)^{-n} H_n(t - x) e^{tx - t^2/2}$$

where in the last equality P.4 has been used, we have, using P.3, the following upper bound for the approximation error (15)

$$\begin{aligned} R_n &\le \frac{|t|^n}{n!}\, e^{\theta x - \theta^2/2} l \sqrt{n!}\, e^{(\theta - x)^2/4} = \\ &= \frac{|t|^n}{\sqrt{n!}}\, l\, e^{\theta x/2 - \theta^2/4 + x^2/4} \qquad \text{with } \theta \in [0, t] \end{aligned} \tag{16}$$

With the notation introduced above, the approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ which will be used in the sequel is given by

$$p_n(x_t \mid x_{t-1}) = e^{-x_t^2/2} \sum_{k=0}^{n-1} H_k(x_t)\, \frac{a^k(x_{t-1})}{k!} \tag{17}$$

where n is an odd integer. The reason for the choice of n odd is that in this case $p_n(x_t \mid x_{t-1})$ turns out to be positive. It is apparent from (17) that $p_n(x_t \mid x_{t-1})$ is of type (4). Furthermore, as it can be easily seen, we have using (10)

$$\sum_{k=0}^{n-1} H_k(x_t) \frac{a^k(x_{t-1})}{k!} \ge e^{-A^2/2 - A|x_t|} \tag{18}$$

so that

$$p_n(x_t \mid x_{t-1}) \ge e^{-(A + |x_t|)^2/2} \tag{19}$$

and consequently assumption A.2 is satisfied with $W = 2(1 - \Phi(A))$, where Φ is the standard normal distribution function. It remains now to show that assumptions A.3 and A.4 are satisfied. This will be done in Propositions 3 and 4 below, for which we need some preliminary results.

LEMMA 1 *For any real α and positive integer n*

$$\int_0^{\alpha^2/2} \frac{\left[\alpha - \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}}\, e^y \mathrm{d}y \le \lambda_1(n, \alpha) \tag{20}$$

where

$$\lambda_1(n, \alpha) = e^{\alpha^2/2} \sum_{j=0}^{n} \binom{n}{j} \alpha^j (-1)^{n-j} |\alpha|^{n-j+1} / (n - j + 1) \tag{21}$$

PROOF Denoting by I the integral in the l.h.s. of (20) and using the change of variable $t = \alpha^2 - 2y$ we have

$$I \leq \frac{e^{\alpha^2/2}}{2} \int_0^{\alpha^2} \frac{(\alpha - \sqrt{t})^n}{\sqrt{t}} \mathrm{d}t =$$

$$= \frac{e^{\alpha^2/2}}{2} \sum_{j=0}^{n} \binom{n}{j} \alpha^j (-1)^{n-j} \int_0^{\alpha^2} t^{(n-j-1)/2} \mathrm{d}t =$$

$$= e^{\alpha^2/2} \sum_{j=0}^{n} \binom{n}{j} \alpha^j (-1)^{n-j} |\alpha|^{n-j+1} / (n - j + 1)$$

■

LEMMA 2 *For any real α and positive integer n*

$$\int_{-\infty}^{\alpha^2/2} \frac{\left[\alpha + \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}} e^y \mathrm{d}y = \lambda_2(n, \alpha) \tag{22}$$

where

$$\lambda_2(n, \alpha) = e^{\alpha^2/2} \sum_{j=0}^{n} \binom{n}{j} 2^{\frac{n-j-1}{2}} \alpha^j \Gamma\left[\frac{n - j - 1}{2} + 1\right] \tag{23}$$

PROOF Denoting by I the integral in the l.h.s. of (22) and by the change of variables $\alpha^2 - 2y = 2u$ we have

$$I = e^{\alpha^2/2} \int_0^{+\infty} \frac{(\alpha + \sqrt{2u})^n}{\sqrt{2u}} e^{-u} \mathrm{d}u =$$

$$= e^{\alpha^2/2} \sum_{j=0}^{n} \binom{n}{j} \alpha^j \int_0^{+\infty} (2u)^{\frac{n-j-1}{2}} e^{-u} \mathrm{d}u =$$

$$= e^{\alpha^2/2} \sum_{j=0}^{n} \binom{n}{j} \alpha^j \left[\sqrt{2}\right]^{n-j-1} \Gamma\left[\frac{n - j - 1}{2} + 1\right]$$

■

LEMMA 3 *For any real α and positive integer n*

$$\int_0^{+\infty} x^n e^{\alpha x - x^2/2} \mathrm{d}x \leq \lambda(n, \alpha) \tag{24}$$

where

$$\lambda(n, \alpha) = \lambda_1(n, \alpha) + \lambda_2(n, \alpha) \tag{25}$$

with $\lambda_1(n, \alpha)$, $\lambda_2(n, \alpha)$ *given by (21), (23).*

PROOF Denoting by I the integral in the l.h.s. of (24) we have

$$I = \int_0^{\alpha} x^n e^{\alpha x - x^2/2} \mathrm{d}x + \int_{\alpha}^{+\infty} x^n e^{\alpha x - x^2/2} \mathrm{d}x$$

and by the change of variable $y = \alpha x - x^2/2$ we obtain

$$I = \int_0^{\alpha^2/2} \frac{\left[\alpha - \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}} e^y \mathrm{d}y + \int_{-\infty}^{\alpha^2/2} \frac{\left[\alpha + \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}} e^y \mathrm{d}y$$

The result then follows from Lemmas 1,2. ∎

As an immediate consequence of Lemma 3 we have

COROLLARY 1 *For any real* α *and positive integer* n

$$\int_{-\infty}^{+\infty} |x|^n e^{\alpha|x| - x^2/2} \mathrm{d}x \le 2\lambda(n, \alpha) \tag{26}$$

where $\lambda(n, \alpha)$ *is defined in (25).* ∎

LEMMA 4 *For any real* $\alpha > 0$, β, γ

$$\int_{-\infty}^{+\infty} e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = \sqrt{\pi/\alpha}\, e^{(\beta^2 - 4\alpha\gamma)/4\alpha} \tag{27}$$

PROOF Completing the square in the exponent and using the change of variable $y = x + \beta/2\alpha$ we have

$$\int_{-\infty}^{+\infty} e^{-(\alpha x^2 + \beta x + \gamma)} \mathrm{d}x = e^{(\beta^2 - 4\alpha\gamma)/4\alpha} \int_{-\infty}^{+\infty} e^{-\alpha y^2} \mathrm{d}y$$

from which the result follows immediately. ∎

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$$\lambda(n, \alpha) = \lambda_1(n, \alpha) + \lambda_2(n, \alpha) \tag{25}$$

with $\lambda_1(n, \alpha)$, $\lambda_2(n, \alpha)$ *given by (21), (23).*

PROOF Denoting by I the integral in the l.h.s. of (24) we have

$$I = \int_0^{\alpha} x^n e^{\alpha x - x^2/2} \mathrm{d}x + \int_{\alpha}^{+\infty} x^n e^{\alpha x - x^2/2} \mathrm{d}x$$

and by the change of variable $y = \alpha x - x^2/2$ we obtain

$$I = \int_0^{\alpha^2/2} \frac{\left[\alpha - \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}} e^y \mathrm{d}y + \int_{-\infty}^{\alpha^2/2} \frac{\left[\alpha + \sqrt{\alpha^2 - 2y}\right]^n}{\sqrt{\alpha^2 - 2y}} e^y \mathrm{d}y$$

The result then follows from Lemmas 1,2. ∎

As an immediate consequence of Lemma 3 we have

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from which the result follows immediately. ∎

COROLLARY 2 *For any real α and β*

$$\int_{-\infty}^{+\infty} e^{\alpha|x| + \beta x - x^2/2} \mathrm{d}x \leq \mu(\alpha, \beta) \tag{28}$$

where

$$\mu(\alpha, \beta) = \sqrt{2\pi}\,[e^{(\beta+\alpha)^2/2} + e^{(\beta-\alpha)^2/2}] \tag{29}$$

PROOF

$$\int_{-\infty}^{+\infty} e^{\alpha|x| + \beta x - x^2/2} \mathrm{d}x = \int_{0}^{+\infty} e^{-(\beta-\alpha)x - x^2/2} \mathrm{d}x + \int_{0}^{+\infty} e^{(\beta+\alpha)x - x^2/2} \mathrm{d}x$$

The result then follows from Lemma 4. ■

We are now in the position to prove that with the choice made for $p_n(x_t \,|\, x_{t-1})$, given by (17), the assumptions required by Theorem 1 are satisfied so that it is possible to evaluate the error bound provided there.

PROPOSITION 1 *For any real α and positive integer n*

$$\| p_n(x_t \,|\, x_{t-1}) \|_\alpha \leq Z \tag{30}$$

where

$$Z = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} A^k \sum_{m=0}^{[k/2]} \frac{\lambda(k-2m, \alpha)}{2^{m-1} m!(k-2m)!} \tag{31}$$

with λ defined in (25).

PROOF With the notation $x = x_t$, $t = a(x_{t-1})$ and using (17) and (11) we have

$$\| p_n(x_t | \, x_{t-1}) \|_\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha|x|} \left| \sum_{k=0}^{n} e^{-x^2/2} H_k(x) \frac{t^k}{k!} \right| \mathrm{d}x$$

$$\leq \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} |t|^k \sum_{m=0}^{[k/2]} \frac{1}{2^m m!(k-2m)!} \int_{-\infty}^{+\infty} e^{\alpha|x| - x^2/2} |x|^{k-2m} \mathrm{d}x$$

Using Corollary 1 and (8) we obtain the final result. ■

PROPOSITION 2 *For any real α and positive integer n*

$$\| p(x_t | x_{t-1}) - p_n(x_t | x_{t-1}) \|_\alpha \leq Z_n$$

where

$$Z_n = \sqrt{2}\, l \frac{A^n}{n!} e^{A^2/4} \left[e^{(A/2-\alpha)^2} + e^{\alpha^2} \right]$$

with l as in (12).

PROOF Using as before the notation $x = x_t$, $t = a(x_{t-1})$ as well as (16) we have

$$\| p(x_t | x_{t-1}) - p_n(x_t | x_{t-1}) \| \alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha|x| - x^2/2} | R_n | \, dx \leq$$

$$\leq \frac{|t|^n}{\sqrt{n!}} \frac{l}{\sqrt{2\pi}} e^{-\theta^2 t^2/4} \int_{-\infty}^{+\infty} e^{-\theta t x/2 + x^2/4 + \alpha|x| - x^2/2}$$

Using Corollary 2 and (8) we obtain the final result. ∎

The results of this section allow the evaluation of the error bounds given in Theorem 1. It could be easily verified that these bounds are in many instances better than those obtained in [5].

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