

**Second-Order Expansions for a Class of
Quasidifferentiable Functions**

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Foreword

Some results concerning second order expansions for quasidifferentiable functions in the sense of Demyanov and Rubinov whose gradients are quasidifferentiable are represented in this paper. They are similar to those given by Hiriart-Urruty, Strodiot and Nguyen (1984).

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Second-Order Expansions for a Class of Quasidifferentiable Functions

Z.Q. Xia and Zhao Chunjun

1. INTRODUCTION - SOME RESULTS IN [5]

Some results concerning the second-order expansion for $C^{1,1}$ functions which are continuously differentiable and whose gradients are locally Lipschitzian. Several results given in [5] are listed as follows:

Definition 1.1. [Def. 2.1]. Let $f \in C^{1,1}(S)$ and let $x \in S$, where S is an open set in \mathbb{R}^n . The generalized Hessian matrix of f at x is defined by

$$\text{co} \{M \mid \exists \{x_i\}_1^m \rightarrow x \text{ such that } H_f(x_i) \text{ exist and } M = \lim_{i \rightarrow \infty} H_f(x_i)\}$$

where $H_f(x_i)$ denotes the Hessian matrix of f at x_i . By $\partial^2 f(x)$ one denotes the generalized matrix of f at x .

The generalized Hessian matrix of f at x is a nonempty compact convex set of symmetric matrices. The set-valued mapping $x \mapsto \partial^2 f(x)$ is locally bounded, i.e.,

$$\sup \{\|M\| \mid M \in \partial^2 f(V)\} \leq K \quad ,$$

where V is a neighborhood of x and K is a positive constant. The mapping $\partial^2 f(\cdot)$ is an upper semicontinuous (closed) set-valued mapping in the sense of sequences.

Theorem 1.2. [Th. 2.1. and Th. 2.2.]. Let $f \in C^{1,1}(S)$, $F = (f_1, \dots, f_m)^T$ where $f_i \in C^{1,1}(S)$, $\varphi: S' \subset \mathbb{R}^m \rightarrow \mathbb{R}$ where S' is an open set and $F(S) \subset S'$. Then

$$f^{00}(x; u, v) = \limsup_{\substack{x' \rightarrow x \\ \lambda \downarrow 0}} \frac{1}{\lambda} [\langle \nabla f(x' + \lambda u), v \rangle - \langle \nabla f(x), v \rangle], \quad x \in S \quad ; \quad (a)$$

$$(\varphi \circ F)^{00}(x; u, v) \leq \sum_{i=1}^m \frac{\partial \varphi}{\partial y_i}(F(x)) f_i^{00}(x; u, v) \quad (b)$$

$$+ \varphi^{00}(F(x); JF(x)u, JF(x)v) \quad ,$$

and the equality holds whenever either $f_i \in C^2(S)$ for all i , or $\varphi \in C^2(S')$ and $m = 1$;

$$\begin{aligned} \partial^2(\varphi \cdot F)(x)u &\subset \sum_{i=1}^m \frac{\partial \varphi}{\partial y_i}(F(x)) \partial^2 f_i(x)u & (c) \\ &+ J^T F(x) \partial^2 \varphi(F(x)) J F(x)u \end{aligned}$$

for all $u \in \mathbb{R}^n$.

Theorem 1.3. [Th. 2.3]. Let $f \in C^{1,1}$ and $x + \lambda d \in S$ where $\lambda > 0$ and $x \in S$. Then there exists a $C \in (0, \lambda)$ such that

$$f(x + \lambda d) = f(x) + \lambda \langle \nabla f(x), d \rangle + \frac{\lambda^2}{2} \langle M_c d, d \rangle ,$$

where $M_c \in \partial^2 f(x + cd)$.

Theorem 1.4. [Th. 3.1]. Let x^* be a minimum point for

$$\min f(x) \text{ over all } x .$$

Then for each $d \in \mathbb{R}^n$ there exists a matrix $A \in \partial^2 f(x^*)$ such that

$$\langle Ad, d \rangle \geq 0 .$$

In this paper some similar results for quasidifferentiable functions will be derived.

2. QUASI-JACOBIAN MATRICES AND SECOND-ORDER QUASIDIFFERENTIALS

Suppose H is a mapping, a vector-valued function,

$$H: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

defined by

$$x \mapsto H(x) = (h_1(x), h_2(x), \dots, h_m(x))^T ,$$

where $h_i(x), i = 1, \dots, m$, are quasidifferentiable functions defined in \mathbb{R}^n . Define

$$J_D H := [L_D H, \bar{J}_D H] ,$$

where

$$\underline{J}_D H := \left(\bigvee_{i=1}^m \partial h_i \right)^T ,$$

$$\bar{J}_D H := \left(\bigwedge_{i=1}^m \bar{\partial} h_i \right)^T .$$

Proposition 2.1. Suppose H is a quasidifferentiable mapping $(\mathbb{R}^n \rightarrow \mathbb{R}^m)$, [3], [4]. Then $J_D H(x)$ is one of quasidifferentials of the mapping H at $x \in \mathbb{R}^n$.

Proof. In a finite dimensional space \mathbb{R}^m we have

$$\begin{aligned} H'(x; d) &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [H(x + \lambda d) - H(x)] \\ &= \left(\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [h_1(x + \lambda d) - h_1(x)], \dots, \right. \\ &\quad \left. - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [h_m(x + \lambda d) - h_m(x)] \right)^T . \end{aligned}$$

Thus

$$\max_{V \in \underline{\partial} H(x)} \langle V, d \rangle + \min_{W \in \bar{\partial} H(x)} \langle W, d \rangle = \max_{V \in \underline{J}_D H(x)} \langle V, d \rangle + \min_{W \in \bar{J}_D H(x)} \langle W, d \rangle ,$$

i.e., for any $d \in \mathbb{R}^n$ one has

$$\underline{\partial} H(x) - \bar{J}_D H(x) = \underline{J}_D H(x) - \bar{\partial} H(x) .$$

Since $\underline{J}_D H(x)$, $\bar{J}_D H(x)$ are nonempty compact convex sets, it follows from the definition of quasidifferentiable functions that

$$J_D H(x) = [\underline{J}_D H(x), \bar{J}_D H(x)]$$

is a quasidifferential of H at x . \square

It is easy to be seen that

$$DH(x) = \begin{cases} J_D H(x), & \text{if } m > 1 \\ J_D^T H(x), & \text{if } m = 1 \end{cases} ,$$

where

$$J_D^T H(x) := [\underline{J}_D^T H(x), \bar{J}_D^T H(x)] ,$$

[7].

Remark. In the case where H is Lipschitzian the relationship, generally speaking,

$$\begin{aligned} \sum_{i=1}^m \partial_c h_i(x) &= \partial_c H(x) \\ &= J_c H(x) \end{aligned}$$

is not true, but in the case where $H(x)$ is quasidifferentiable the pair

$$\left[\sum_{i=1}^m \underline{\partial} h_i(x), \sum_{i=1}^m \bar{\partial} h_i(x) \right]$$

is really a quasidifferential of H at $x \in \mathbb{R}^n$, where ∂_c denotes the Clarke's generalized matrix or generalized Jacobian matrix, [1], [2].

Definition 2.2. $J_D H(x)$ is called a quasi-Jacobian matrix of the quasidifferentiable mapping H . $\underline{J}_D H(x)$ is called sub-Jacobian matrix, and $\bar{J}_D H(x)$ is called super-Jacobian matrix, [7].

Definition 2.3. Suppose f is a differentiable function defined in \mathbb{R}^n and its gradient is a quasidifferentiable mapping, vector-valued function. A quasi-Jacobian matrix of its gradient $J_D \nabla f(x)$ is called a second-order quasidifferential of f at $x \in \mathbb{R}^n$. The sub-Jacobian matrix is called a second-order sub-quasidifferential. The super-Jacobian matrix is called a second-order super-quasidifferential. They are denoted by $D^2 f(x)$, $\underline{\partial}^2 f(x)$ and $\bar{\partial}^2 f(x)$, respectively, i.e.,

$$\begin{aligned} D^2 f(x) &= [\underline{J}_D \nabla f(x), \bar{J}_D \nabla f(x)] \quad , \\ \underline{\partial}^2 f(x) &= \underline{J}_D \nabla f(x), \quad \bar{\partial}^2 f(x) = \bar{J}_D \nabla f(x) \quad . \end{aligned}$$

Theorem 2.4. Suppose f is a differentiable function defined in \mathbb{R}^n and its gradient is quasidifferentiable. Then

$$\begin{aligned}
 \frac{\partial^2 f(x)}{\partial d_1 \partial d_2} &= f''(x; d_1, d_2) \\
 &= \max_{\substack{V \in \partial^2 f(x) \\ v \in \bar{\partial}^2 f(x)}} (d_1^{+T} V - d_1^{-T} W) d_2 \\
 &+ \min_{\substack{V \in \partial^2 f(x) \\ v \in \bar{\partial}^2 f(x)}} (d_1^{+T} W - d_1^{-T} V) d_2 \\
 &= \max(d_1^{+T} \bar{\partial}^2 f(x) - d_1^{-T} \partial^2 f(x)) d_2 \\
 &+ \min(d_1^{+T} \partial^2 f(x) - d_1^{-T} \bar{\partial}^2 f(x)) d_2 .
 \end{aligned} \tag{2.1}$$

where $d_1 = d_1^+ - d_1^-$, d_1^+ and d_1^- are defined, respectively, by

$$d_1^+ = (d_{1i}^+)_{n \times 1}, \quad d_{1i}^+ = \begin{cases} d_{1i}, & d_{1i} \geq 0, \\ 0, & d_{1i} < 0, \end{cases}$$

and

$$d_1^- = (d_{1i}^-)_{n \times 1}, \quad d_{1i}^- = \begin{cases} 0, & d_{1i} \geq 0 \\ -d_{1i}, & d_{1i} < 0. \end{cases}$$

Proof. Since

$$\begin{aligned}
 \frac{\partial^2 f(x)}{\partial d_1 \partial d_2} &= f''(x; d_1, d_2) \\
 &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [\langle \nabla f(x + \lambda d_2), d_1 \rangle - \langle \nabla f(x), d_1 \rangle] \\
 &= \lim_{\lambda \rightarrow 0} \langle \frac{1}{\lambda} (\nabla f(x + \lambda d_2) - \nabla f(x)), d_1 \rangle \\
 &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n \frac{1}{\lambda} (\nabla f(x + \lambda d_2) - \nabla f(x))_i d_{1i} \\
 &= \sum_{i=1}^n d_{1i} [\max_{v_i \in \partial(\nabla f(x))_i} \langle v_i, d_2 \rangle + \min_{w_i \in \bar{\partial}(\nabla f(x))_i} \langle w_i, d_2 \rangle] .
 \end{aligned}$$

one has

$$f''(x; d_1, d_2) = \left\{ \sum \left[d_{1i}^+ \max_{v_i \in \underline{\partial}(\nabla f(x))_i} \langle v_i, d_2 \rangle + d_{1i}^+ \min_{w_i \in \overline{\partial}(\nabla f(x))_i} \langle w_i, d_2 \rangle \right] \right. \\ \left. + \left[-d_{1i}^- \max_{v_i \in \underline{\partial}(\nabla f(x))_i} \langle v_i, d_2 \rangle - d_{1i}^- \min_{w_i \in \overline{\partial}(\nabla f(x))_i} \langle w_i, d_2 \rangle \right] \right\} .$$

Therefore,

$$f''(x; d_1, d_2) = \max_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \langle V^T d_1^+ - W^T d_1^-, d_2 \rangle \quad (2.2) \\ + \min_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \langle -V^T d_1^- + W^T d_1^+, d_2 \rangle .$$

The (2.1) can be obtained immediately from (2.2). \square

By \diamond we denote an operation as follows:

$$d \diamond [A, B] := [d^T \diamond_1 [A, B], d^T \diamond_2 [A, B]]$$

where

$$d^T \diamond_1 [A, B] := d^+{}^T A - d^-{}^T B$$

and

$$d^T \diamond_2 [A, B] := d^+{}^T B - d^-{}^T A ,$$

here d is a vector in \mathbb{R}^n , A and B are matrices or sets of matrices. Using the notations given above, we may write $f''(x; d_1, d_2)$ as follows:

$$f''(x; d_1, d_2) = \max d_1^T \diamond_1 D^2 f(x) d_2 + \min d_1^T \diamond_2 D^2 f(x) d_2 .$$

Note that generally speaking,

$$d_1^T \diamond_i D^2 f(x) d_2 \neq d_2^T \diamond_i D^2 f(x) d_1 , \quad i = 1, 2 .$$

Proposition 2.5. The following relation holds

$$d_1^T \diamond_1 D^2 f(x) d_2 + d_1^T \diamond_2 D^2 f(x) d_2 = d_1^T (\underline{\partial}^2 f(x) + \overline{\partial}^2 f(x)) d_2^T .$$

Proposition 2.6. Suppose f_1 and f_2 are differentiable functions defined in \mathbb{R}^n , and their gradients are quasidifferentiable. Then

$$D^2(\alpha f_1 + \beta f_2) = \alpha D^2 f_1 + \beta D^2 f_2 . \quad (a)$$

where α and β are any scalars. And

$$\begin{aligned}
 D^2(f_1 \cdot f_2) &= f_1 \cdot J_D \nabla f_2 + f_2 \cdot J_D \nabla f_1 & (b) \\
 &+ \nabla f_1 \otimes J_D f_2 + \nabla f_2 \otimes J_D f_1 \\
 &= f_1 D^2 f_2 + f_2 D^2 f_1 + \nabla f_1 \otimes D^T f_2 + \nabla f_2 \otimes D^T f_1 .
 \end{aligned}$$

where the operation \otimes means that

$$\nabla f_1 \otimes J_D f_2 = \begin{bmatrix} \vdots \\ (\nabla f_1)_i J_D f_2 \\ \vdots \end{bmatrix}$$

Proof. The proof of (2) will be given below. Since

$$\begin{aligned}
 D^2(f_1 \cdot f_2) &= D(f_1 \nabla f_2 + f_2 \nabla f_1) \\
 &= J_D(f_1 \nabla f_2) + J_D(f_2 \nabla f_1) .
 \end{aligned}$$

it is enough to prove that one of the terms in the last expression given above, say, the first term $J_D(f_1 \nabla f_2)$ can be obtained. Since

$$J_D(f_1 \nabla f_2) = J_D \begin{bmatrix} \vdots \\ f_1 (\nabla f_2)_i \\ \vdots \end{bmatrix} ,$$

one has

$$\underline{J}_D(f_1 \nabla f_2) = \sum_{i=1}^n \partial(f_1 (\nabla f_2)_i) , \quad \bar{J}_D(f_1 \nabla f_2) = \sum_{i=1}^n \bar{\partial}(f_1 (\nabla f_2)_i) .$$

Consider the $J_D^T(f_1 (\nabla f_2)_i)$. According to the definition of T we have

$$\begin{aligned}
 J_D^T(f_1 (\nabla f_2)_i) &= D(f_1 (\nabla f_2)_i) \\
 &= (\nabla f_2)_i Df_1 + f_1 D(\nabla f_2)_i \\
 &= (\nabla f_2)_i J_D^T f_1 + f_1 J_D^T(\nabla f_2)_i .
 \end{aligned}$$

From the above one has

$$J_D(f_1 (\nabla f_2)_i) = (\nabla f_2)_i J_D f_1 + f_1 J_D(\nabla f_2)_i . \quad (2.3)$$

Similarly, we can get

$$J_D(f_2 (\nabla f_1)_i) = (\nabla f_1)_i J_D f_2 + f_2 J_D(\nabla f_1)_i . \quad (2.4)$$

The proof of (2) can be completed immediately from (2.3) and (2.4). \square

3. SECOND-ORDER EXPANSION

To begin with, we give the following lemma, *Cauchy's Mean Value Theorem*, in order to deduce second-order expansions.

Lemma 3.1. (Cauchy's Mean Value Theorem). Suppose $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ are quasidifferentiable and for an interval $[x, y] \subset \mathbb{R}^n$

$$0 \notin \langle \underline{\partial}g(\xi) + \bar{\partial}g(\xi), y - x \rangle, \quad \xi \in (x, y) . \quad (3.1)$$

Then there exists an $\eta \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} \in \frac{\langle \underline{\partial}f(\eta) + \bar{\partial}f(\eta), y - x \rangle}{\langle \underline{\partial}g(\eta) + \bar{\partial}g(\eta), y - x \rangle} . \quad (3.2)$$

Proof. The proof can be made along the lines of a process used in the proof of Cauchy's Mean Value Theorem in the elementary calculus. According to [7, Th. 2], the fact $g(x) \neq g(y)$ is true. Thus we can construct an auxiliary function

$$F(\xi) = f(\xi) - \frac{f(y) - f(x)}{g(y) - g(x)}g(\xi) .$$

Without loss of generality, assume that

$$\frac{f(y) - f(x)}{g(y) - g(x)} \leq 0 .$$

It follows from [7, Th. 2] that there exists an $\eta \in (x, y)$ such that

$$F(y) - F(x) \in \langle \underline{\partial}F(\eta) + \bar{\partial}F(\eta), y - x \rangle .$$

We now have to find $DF(\eta)$. From the rules in quasidifferential calculus one has

$$\begin{aligned} DF(\eta) &= Df(\eta) - \frac{f(y) - f(x)}{g(y) - g(x)}Dg(\eta) \\ &= [\underline{\partial}f(\eta) - \frac{f(y) - f(x)}{g(y) - g(x)}\underline{\partial}g(\eta) , \\ &\quad \bar{\partial}f(\eta) - \frac{f(y) - f(x)}{g(y) - g(x)}\bar{\partial}g(\eta)] , \end{aligned}$$

[3], [4]. It is easy to learn by directly calculating F that

$$F(y) - F(x) = 0 .$$

Thus we have

$$0 \in \langle \underline{\partial}f(\eta) + \bar{\partial}f(\eta) - \frac{f(y) - f(x)}{g(y) - g(x)}(\underline{\partial}g(\eta) + \bar{\partial}g(\eta)), y - x \rangle .$$

The proof is completed immediately from the hypothesis of this theorem (3.1).

□

Remark. Let A and B be subsets in \mathbb{R} . The quotient set $A/B = \frac{A}{B}$ is defined as

$$A/B = \left\{ \frac{a}{b} \mid a \in A, b \in B \right\} \quad (3.3)$$

where a/b can be treated as a usual quotient of two real numbers when $b \neq 0$, but as a symbol $a/0$ when $b = 0$. Using the notation (3.3), the hypothesis (3.1) in the theorem given above may be omitted.

Theorem 3.2. Suppose a function f defined in \mathbb{R}^n is differentiable and its gradient is quasidifferentiable. Then for any $x \in \mathbb{R}^n, d \neq 0, d \in \mathbb{R}^n$ and $\lambda > 0$, one has the following second-order expansion (Taylor's expansion of second order)

$$f(x + \lambda d) \in f(x) + \lambda f'(x; d) + \frac{\lambda^2}{2} d^T [\underline{\partial}^2 f(x + \xi d) + \bar{\partial}^2 f(x + \xi d)] d \quad , \quad (3.4)$$

where $\xi \in (0, \lambda)$, or an alternative expression

$$f(x + \lambda d) = f(x) + \lambda f'(x; d) + \frac{\lambda^2}{2} d^T (V + W) d \quad , \quad (3.5)$$

for some V and W , where

$$V \in \underline{\partial}^2 f(x + \xi d)$$

and

$$W \in \bar{\partial}^2 f(x + \xi d) \quad , \quad \xi \in (0, \lambda) \quad .$$

Proof. Making auxiliary functions

$$\varphi(t) = f(x + \lambda d) - f(x + td) - (\lambda - t) f'(x + td; d)$$

and

$$\psi(t) = (\lambda - t)^2 \quad ,$$

We have from the rules of quasidifferential calculus [3] that

$$D\varphi(t) = -D_t f(x + td) - f'(x + td; d) D_t (\lambda - t) \quad (3.6)$$

$$-(\lambda - t)D_t f'(x + td; d)$$

and

$$D\psi(t) = 2(\lambda - t)D(\lambda - t) \quad , \quad (3.7)$$

where $D_t f$ means a quasidifferential of f with respect to t . We now evaluate every term on the right-hand-side of (3.6). For the first term it can be expressed as

$$D_t f(x + td) = [\underline{\partial} f(x + td), d \rangle, \langle \bar{\partial} f(x + td), d \rangle] \quad . \quad (3.8)$$

Since f is differentiable in \mathbb{R}^n , one has

$$D_t f(x + td) = [f'(x + td; d), 0] \quad . \quad (3.9)$$

The second term is

$$f'(x + td; d)[0, 1] \quad . \quad (3.10)$$

In the third term we have from the Proposition 2.5. that

$$D_t f'(x + td; d) = [d^T \diamond_1 D^2 f(x + td) d^T, d^T \diamond_2 D^2 f(x + td) d^T] \quad . \quad (3.11)$$

From the Cauchy's Mean Value Theorem given above it follows that there exists a $\xi \in (0, \lambda)$ such that

$$\frac{\varphi(\lambda) - \varphi(0)}{\psi(\lambda) - \psi(0)} \in \frac{\langle \underline{\partial} \varphi(\xi) + \bar{\partial} \varphi(\xi), \lambda \rangle}{\langle \underline{\partial} \psi(\xi) + \bar{\partial} \psi(\xi), \lambda \rangle} \quad , \quad (3.12)$$

where

$$\varphi(\lambda) = 0, \quad \varphi(0) = f(x + \lambda d) - f(x) - \lambda f'(x; d), \quad \psi(\lambda) = 0, \quad \psi(0) = \lambda^2 \quad .$$

From (3.6), (3.8), (3.9), (3.10) (3.11) and (3.7) one has

$$\underline{\partial} \varphi(\xi) + \bar{\partial} \varphi(\xi) = -d^T (\lambda - \xi) [\underline{\partial}^2 f(x + \xi d) + \bar{\partial}^2 f(x + \xi d)]$$

$$\underline{\partial} \psi(\xi) + \bar{\partial} \psi(\xi) = -2(\lambda - \xi) \quad .$$

It follows from substituting the expressions above into (3.12) that

$$\frac{f(x + \lambda d) - f(x) - \lambda f'(x; d)}{\lambda^2} \in \frac{(\lambda - \xi) d^T [\underline{\partial}^2 f(x + \xi d) + \bar{\partial}^2 f(x + \xi d)] d}{2(\lambda - \xi)} \quad ,$$

and hence

$$f(x + \lambda d) \in f(x) + \lambda f'(x; d) + \frac{\lambda^2}{2} d^T [\underline{\partial}^2 f(x + \xi d)$$

$$+ \bar{\partial}^2 f(x + \xi d)]d \quad ,$$

where $\xi \in (0, \lambda)$. \square

Corollary 3.3. Under the assumptions given in the above theorem, for any $\lambda > 0$ small enough there exists a $\vartheta \in (0, \lambda)$ such that

$$f(x + \lambda d) = f(x) + \lambda f'(x; d) + \lambda \vartheta f''(x; d, d) + o(\lambda^2) \quad . \quad (3.13)$$

Proof. From the above theorem and the Mean Value Theorem we have

$$f'(x + \vartheta d; d) - f'(x; d) = \frac{\lambda}{2} d^T (V + W) d \quad , \quad (3.14)$$

where $\vartheta \in (0, \lambda)$ and

$$V \in \underline{\partial}^2 f(x + \xi d), W \in \bar{\partial}^2 f(x + \xi d), \xi \in (0, \lambda) \quad .$$

Taking limits to (14)/ λ as $\lambda \downarrow 0$, we obtain

$$\lim_{\lambda \downarrow 0} \frac{f'(x + \lambda d; d) - f'(x; d)}{\vartheta} = \lim_{\lambda \downarrow 0} d^T \left[\frac{\lambda}{2\vartheta} (V + W) \right] d \quad .$$

Since $f''(x; d, d)$ exists and $\vartheta \rightarrow 0^+$ as $\lambda \downarrow 0$, the following equalities,

$$\begin{aligned} & \lim_{\lambda \downarrow 0} \frac{1}{\vartheta} [f'(x + \vartheta d; d) - f'(x; d)] \\ &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} [f'(x + \lambda d; d) - f'(x; d)] \\ &= f''(x; d, d) \quad , \end{aligned}$$

hold. Thus

$$\frac{\lambda^2}{2} d^T (V + W) d^T = \lambda \vartheta f''(x; d, d) + \lambda \vartheta \varepsilon \quad , \quad (3.15)$$

where $\varepsilon \rightarrow 0$ as $\lambda \rightarrow 0$, $0 < \vartheta < \lambda$. The (3.13) can be deduced immediately from (15).

\square

EIUO CONDITION (the condition of exchangeability of intersection and union operations) is defined as follows:

Let C be a family of nonempty compact convex set and A be a nonempty compact convex set. We say that C and A satisfy the EIUO condition if the following condition holds

$$\bigcup_{\alpha \in AZ \in C} \bigcap [Z - \alpha] = \bigcap_{Z \in C} \bigcup_{\alpha \in A} [Z - \alpha] . \quad (3.16)$$

Lemma 3.4. If for some $[U_1, U_2] \in D^2f(x)$ the family $\{\underline{\partial}^2f(x) + \bar{\partial}^2f(x)\}$ and set U_2 satisfy the EIUO condition, and also $\{\bar{\partial}^2f(x) - \underline{\partial}^2f(x)\}$ and U_1 satisfy the EIUO condition, where $D^2f(x)$ denotes the class of quasidifferentials of f at x , then

$$\left[\bigcap_{[V_1, V_2] \in D^2f(x)} (V_1 + V_2), \bigcap_{[V_1, V_2] \in D^2f(x)} (V_2 - V_2) \right] \in D^2f(x) .$$

Remark. If $[V_1, V_2] \in D^2f(x)$, then $[V_1 + V_2, V_2 - V_2] \in D^2f(x)$.

Proof of the Lemma. It is clear that

$$\bigcap_{[V_1, V_2] \in D^2f(x)} [U_1 - (V_1 - V_2)]$$

is a nonempty compact convex set as inasmuch $0 \in V_2 - V_2$ and

$$U_1 \subset \bigcap_{[V_1, V_2] \in D^2f(x)} [U_1 - (V_2 - V_2)] .$$

Since $[U_1, U_2], [V_1, V_2] \in D^2f(x)$ and the foregoing remark, we have

$$\bigcap_{[V_1, V_2] \in D^2f(x)} [U_1 - (V_2 - V_2)] = \bigcap_{[V_1, V_2] \in D^2f(x)} [(V_1 + V_2) - U_2] . \quad (3.17)$$

From this the right hand side of the above expression is a nonempty compact convex set. According to the EIUO condition

$$\{\underline{\partial}^2f(x) + \bar{\partial}^2f(x)\}$$

and U_2 satisfy the equality,

$$\bigcap_{[V_1, V_2] \in D^2f(x)} (V_1 + V_2) - U_2 = \bigcap_{[V_1, V_2] \in D^2f(x)} [(V_1 + V_2) - U_2] , \quad (3.18)$$

since

$$\begin{aligned} \bigcap_{[V_1, V_2] \in D^2f(x)} (V_1 + V_2) - U_2 &= \bigcup_{E \in U_2} \left[\bigcap_{[V_1, V_2] \in D^2f(x)} (V_1 + V_2) - E \right] \\ &= \bigcup \bigcap (V_1 + V_2 - E) \end{aligned}$$

and

$$\bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_1 + V_2 - U_2) = \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} \bigcup_{E \in U_2} (V_1 + V_2 - E) .$$

Similarly, one has

$$U_1 - \bigcap_{[V_1, V_2] \in \underline{D} f(x)} (V_1 - V_2) = \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} [U_1 - (V_1 - V_2)] . \quad (3.19)$$

It can be asserted from (17), (18) and (19) that two intersections

$$\bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_1 + V_2)$$

and

$$\bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_2 - V_2)$$

are nonempty compact convex sets, and

$$U_1 - \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_2 - V_2) = \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_1 + V_2) - U_2 .$$

Hence the lemma is true. \square

By $\underline{D}^2 f(x)$ and $\bar{D}^2 f(x)$ we denote the

$$\bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_1 + V_2) \text{ and } \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_2 - V_2) ,$$

respectively, when the conditions in the foregoing lemma are satisfied.

Theorem 3.5. Suppose conditions indicated in the foregoing lemma are satisfied, and assume furthermore that mappings $\underline{D}^2 f(x)$, defined by

$$x \rightrightarrows \underline{D}^2 f(x) = \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_1 + V_2) ,$$

and $\bar{D}^2 f(x)$, defined by

$$x \rightrightarrows \bar{D}^2 f(x) = \bigcap_{[V_1, V_2] \in \underline{D}^2 f(x)} (V_2 - V_2)$$

are upper semi-continuous in a neighborhood in a direction $d \neq 0$. Then for any $\varepsilon > 0$ there exists a $\bar{\lambda}$ small enough such that one has a Taylor's expansion of second-order as follows:

$$f(x + \lambda d) \in f(x) + \lambda f'(x; d) + \frac{\lambda^2}{2} d^T (\underline{D}^2 f(x) + \bar{D}^2 f(x)) d \quad (3.20)$$

$$+ \lambda^2 \varepsilon d^T B$$

where $\lambda \in [0, \bar{\lambda}]$, $B_1^{2n}(0)$ is a unit ball in \mathbb{R}^{2n} .

Proof. The proof of this theorem can be done in terms of Theorem 3.2, Lemma 3.4 and upper semi-continuity of $\underline{D}^2 f(x)$ and $\bar{D}^2 f(x)$ in the sense of the above lemma. \square

4. SECOND-ORDER QUASIDIFFERENTIALS OF A COMPOSITION

We will deduce a formula for second-order quasidifferentials of a composite function in this section.

Definition 4.1. ([4], [8]). Let X and Y be Banach spaces, and H be a mapping: $\Omega \rightarrow Y$ where Ω is an open set in X . The mapping H is called uniformly directionally differentiable at this point and if for any $d \in X$ and $\varepsilon > 0$ there exist numbers $\delta > 0$ and $\alpha_0 > 0$ such that the inequality

$$\|H(x + \alpha v) - H(x) - \alpha H'(x; v)\| < \alpha \varepsilon$$

holds for all $v \in B_\delta^0(d)$, $\alpha \in [0, \alpha_0]$, where

$$B_\delta^0(d) = \{v \mid \|d - v\| < \delta\} .$$

H is referred to as uniformly quasidifferentiable at x if H is uniformly directionally differentiable and quasidifferentiable at x .

Lemma 4.2 [4]. Let X, Y and Z be Banach spaces, and Ω_x, Ω_y and Ω_z be open sets, respectively, in X, Y and Z . A mapping $H: X \rightarrow Y$ is directionally differentiable at a point $x \in \Omega_x$, and a mapping $G: Y \rightarrow Z$ is uniformly directionally differentiable at $H(x)$. Then the mapping $F = G \circ H$ is directionally differentiable at x and

$$\begin{aligned} F'(x; d) &= G'(H(x); H'(x; d)) \\ &= (G'_{H(x)} \circ H'_x)(d) \end{aligned} \tag{4.1}$$

$\forall d \in X .$

Theorem 4.3. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $f = (f_1, \dots, f_m)^T$ and $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$. Suppose f and φ are continuously differentiable in some neighborhoods, respectively, of x and $f(x)$, and assume furthermore that the gradient of $x, \nabla f(x)$, is quasidifferentiable in that neighborhood of x , and the gradient of φ is uniformly quasidif-

ferentiable in that neighborhood of $f(x)$. Then $(\varphi \circ f)$ is twice quasidifferentiable at x and the following formulas hold:

$$(\varphi \circ f)''(x; d_1, d_2) = (\varphi'_{f(x)} \circ f''_x + \varphi''_{f(x)} \circ [f'_x])(d_1, d_2) , \quad (4.2)$$

where

$$\begin{aligned} [f'_x](d_1, d_2) &:= (f'_x(d_1), f'_x(d_2)) \\ &= (f'(x; d_1), f'(x; d_2)) , \end{aligned}$$

and

$$\begin{aligned} D^2(\varphi \circ f)(x) &= \nabla_f^T \varphi(f(x)) \cdot D^2 f(x) \\ &+ J^T f(x) \cdot D_f^2 \varphi(f(x)) J f(x) , \end{aligned} \quad (4.3)$$

where

$$D^2 f := \begin{bmatrix} D^2 f_1 \\ \vdots \\ D^2 f_m \end{bmatrix} ,$$

$$\begin{aligned} J^T f(x) D_f^2 \varphi(f(x)) J f(x) &= [J^T f(x) \cdot \bar{\partial}_f^2 \varphi(f(x)) J f(x) , \\ &J^T f(x) (\bar{\partial}_f^2 \varphi(f(x))) J f(x)] . \end{aligned}$$

Proof. Since

$$(\varphi \circ f)'(x; d_1) = \langle J^T f(x) \cdot \nabla_f \varphi(f(x)), d_1 \rangle ,$$

$\varphi \circ f$ is twice directionally differentiable at x . We now calculate the second-order directional derivative at x . To begin with, it follows from the definition of directional derivative that

$$\begin{aligned} (\varphi \circ f)''(x; d_1, d_2) &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} [\langle \nabla(\varphi \circ f)(x + \lambda d_2), d_1 \rangle \\ &- \langle \nabla(\varphi \circ f)(x), d_1 \rangle] \\ &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \langle [J f(x + \lambda d_2) - J f(x)]^T \nabla_f \varphi(f(x + \lambda d_2)), d_1 \rangle \\ &+ \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \langle J^T f(x) [\nabla_f \varphi(f(x + \lambda d_2)) - \nabla_f \varphi(f(x))], d_1 \rangle . \end{aligned} \quad (4.4)$$

The first limit in (4.4) equals

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \sum_{i=1}^m \left[\left(\frac{\partial \varphi}{\partial f_i} (f(x + \lambda d_2)) \right) \left(\frac{\langle \nabla f_i(x + \lambda d_2) + \nabla f_i(x), d_1 \rangle}{\lambda} \right) \right] \\
 &= \sum_{i=1}^m \frac{\partial \varphi(f(x))}{\partial f_i} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [f'_i(x + \lambda d_2; d_1) - f'_i(x; d)] \\
 &= \sum_{i=1}^m \frac{\partial \varphi(f(x))}{\partial f_i} f''_i(x; d_1, d_2) \\
 &= (\varphi'_{f(x)} \cdot f''_x)(d_1, d_2) .
 \end{aligned}$$

We have obtained the first part of the right hand side of the expression (4.4).

The second limit on the right hand side of (4.4) can be developed as follows. It can be rewritten in the following form:

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} [\varphi'_f(f(x + \lambda d_2); Jf(x)d_1) - \varphi'_f(f(x); Jf(x)d_1)] \\
 &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \langle ((\nabla_f \varphi) \cdot f)(x + \lambda d_2) - ((\nabla_f \varphi) \cdot f)(x), Jf(x)d_1 \rangle .
 \end{aligned}$$

From the lemma 4.2 the above expression becomes

$$\begin{aligned}
 & \langle ((\nabla_f \varphi)'_{f(x)} \cdot f'_x)(d_2), Jf(x)d_1 \rangle \tag{4.5} \\
 &= \langle (\nabla_f \varphi)'(f(x); Jf(x)d_2), Jf(x)d_1 \rangle .
 \end{aligned}$$

Let $v = Jf(x)d_1$ and $u = Jf(x)d_2$. then the expression turns to

$$\begin{aligned}
 \langle (\nabla_f \varphi)'(f(x); u), v \rangle &= (\varphi'_f(x)(v))'_{f(x)}(u) \\
 &= \varphi''_f(f(x); v, u) \tag{4.6} \\
 &= \varphi''_f(f(x); Jf(x)d_1, Jf(x)d_2) \\
 &= (\varphi''_{f(x)} \cdot [f']_x)(d_1, d_2) .
 \end{aligned}$$

This is what we want for the second limit in (4.4).

The demonstration of (4.3) will be done below. Since

$$\begin{aligned}
 (\varphi \cdot f)''(x; d_1, d_2) &= \varphi'(f(x); f''(x; d_1, d_2)) \tag{4.7} \\
 &+ \varphi''(f(x); Jf(x)d_1, Jf(x)d_2) .
 \end{aligned}$$

one has to calculate the two terms on the right hand side of (4.7),

$\varphi'(f(x); f''(x; d_1, d_2))$ and $\varphi''(f(x); Jf(x)d_1, Jf(x)d_2)$, respectively. Let

$$\lambda_i = \frac{\partial \varphi}{\partial f_i}(f(x)) \quad .$$

We have

$$\begin{aligned} \varphi'(f(x); f''(x; d_1, d_2)) &= \langle \nabla_f \varphi(f(x)), f''(x; d_1, d_2) \rangle \\ &= \sum_{i=1}^m \frac{\partial \varphi}{\partial f_i}(f(x)) f''_i(x; d_1, d_2) \\ &= \sum_{i=1}^m (\lambda_i f_i)''(x; d_1, d_2) \\ &= \left(\sum_{i=1}^m \lambda_i f_i \right)''(x; d_1, d_2) \quad . \end{aligned} \tag{4.8}$$

We now calculate the second term on the right hand side of (4.7). Since

$$\begin{aligned} \varphi''(f(x); Jf(x)d_1, Jf(x)d_2) &= \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \langle \nabla_f \varphi(f(x) + \lambda Jf(x)d_2) - \nabla_f \varphi(f(x)), Jf(x)d_1 \rangle \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \langle J^T f(x) \cdot \nabla_f \varphi(f(x) + \lambda Jf(x)d_2) - J^T f(x) \cdot \nabla_f \varphi(f(x)), d_1 \rangle \quad . \end{aligned}$$

one has from the hypotheses in this theorem that

$$\varphi''(f(x); Jf(x)d_1, Jf(x)d_2) = \langle (J^T f(x) \cdot \nabla_f \varphi)'(f(x); Jf(x)d_2, d_1) \rangle \quad .$$

Since

$$\begin{aligned} &(J^T f(x) \cdot \nabla_f \varphi)'(f(x); Jf(x)d_2) \\ &= \max_{V \in \underline{J}^T f(x) \cdot \underline{J}_D \nabla_f \varphi(f(x))} V \cdot Jf(x)d_2 + \min_{W \in \bar{J}^T f(x) \cdot \bar{J}_D \nabla_f \varphi(f(x))} W \cdot Jf(x)d_2 \\ &= \max_{V \in \underline{J}^T f(x) \cdot \underline{J}_D \nabla_f \varphi(f(x))} V d_2 + \min_{W \in \bar{J}^T f(x) \cdot \bar{J}_D \nabla_f \varphi(f(x))} W \cdot d_2 \\ &= \max_{V \in \underline{J}^T f(x) \cdot \underline{J}_D \nabla_f \varphi(f(x))} V d_2 + \min_{W \in \bar{J}^T f(x) \cdot \bar{J}_D \nabla_f \varphi(f(x))} W d_2 \quad , \end{aligned}$$

one has from the proof of Theorem 2.4 that letting

$$\psi''(x; d_1, d_2) = \varphi''(f(x); Jf(x)d_1, Jf(x)d_2) \quad .$$

we have

$$D^2\psi(x) = J^T f(x) \cdot D_f^2 \varphi(f(x)) \cdot Jf(x) \quad . \quad (4.9)$$

Combining (4.8) and (4.9), we obtain (4.3). The proof has been completed.

□

5. NECESSARY CONDITIONS FOR UNCONSTRAINED PROBLEMS

Theorem 5.1. Under the assumptions given in the Theorem 3.9, suppose x is a local minimum point to minimize $f(x), x \in \mathbb{R}^n$. Then for every nonzero $d \in \mathbb{R}^n$ there exist $V \in \underline{\partial}^2 f(x)$ and $W \in \overline{\partial}^2 f(x)$ such that

$$d^T(V+W)d \geq 0 \quad , \quad (5.1)$$

or

$$d^T H d \geq 0 \quad ,$$

where $H \in \underline{\partial}^2 f(x) + \overline{\partial}^2 f(x)$.

Proof. Suppose for any $V, V' \in \underline{\partial}^2 f(x)$ and $W, W' \in \overline{\partial}^2 f(x)$,

$$d^{+T}(V+W')d - d^{-T}(W+V')d < 0 \quad .$$

Then exists a nonzero $d \in \mathbb{R}^n$ such that

$$f''(x;d,d) = \max_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \min_{\substack{V' \in \underline{\partial}^2 f(x) \\ v' \in \underline{\partial}^2 f(x)}} [d^{+T}(V+W')d - d^{-T}(W+V')d] < 0 \quad .$$

Since f is differentiable, the gradient of f at x equals zero, i.e., $\nabla f(x) = 0$. In other words $f'(x;d) = 0$. According to (3.13) we have

$$f(x + \lambda d) - f(x) = \lambda \vartheta f''(x;d,d) + o(\lambda) \quad ,$$

where $\vartheta \in (0, \lambda)$. Thus for $\lambda > 0$ small enough one has

$$f(x + \lambda d) - f(x) < 0 \quad . \quad (5.2)$$

The (5.2) shows that x is not a local minimum point. This leads to that there exist $V \in \underline{\partial}^2 f(x)$ and $W \in \overline{\partial}^2 f(x)$ such that

$$d^{+T}(V+W')d - d^{-T}(W+V')d \geq 0$$

$$\forall (V', W') \in D^2 f(x) = [\underline{\partial}^2 f(x), \overline{\partial}^2 f(x)]$$

holds. Taking $V' = V$ and $W' = W$, we have

$$d^T(V + W)d \geq 0 .$$

The proof is completed. \square

Theorem 5.2. Under the assumptions given in the Theorem 3.2, suppose x is a local minimum point to minimize $f(x), x \in \mathbb{R}^n$. Then for each pair of $V \in \underline{\partial}^2 f(x)$ and $W \in \overline{\partial}^2 f(x)$ there exist vectors $g_1, g_2 \in B_1(0)$ such that

$$g_2^T V - g_1^T W \in g_1^T \underline{\partial}^2 f(x) - g_2^T \overline{\partial}^2 f(x) , \quad (5.3)$$

where

$$g_1 = \sum \lambda_i d_i^{+T}, \quad g_2 = \sum \lambda_i d_i^{-T}$$

$$d_i \in \text{bd}B_1(0), \quad i = 1, \dots, m ,$$

$$\lambda_i \geq 0, \quad i = 1, \dots, m ,$$

$$\sum \lambda_i = 1 .$$

Remark. The expression (5.3) can be rewritten as

$$(\sum \lambda_i d_i^{-T} V - (\sum \lambda_i d_i^{+T}) W \in \quad (5.4)$$

$$(\sum \lambda_i d_i^{+T}) \underline{\partial}^2 f(x) - (\sum \lambda_i d_i^{-T}) \overline{\partial}^2 f(x) ,$$

or

$$-\sum \lambda_i d_i^T \diamond_2(V, W) \in \sum \lambda_i d^T \diamond_1 D^2 f(x) .$$

Proof. From the proof of the above theorem one has that for any $d \in \mathbb{R}^n$, the following holds

$$f''(x; d, d) = f''(x; d)$$

$$= \min_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \max_{\substack{V' \in \underline{\partial}^2 f(x) \\ v' \in \underline{\partial}^2 f(x)}} [d^{+T}(V' + W) - d^{-T}(V + W')] d^T$$

$$= \min_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \max_{\substack{V' \in \underline{\partial}^2 f(x) + W \\ v' \in \underline{\partial}^2 f(x) + V}} (d^{+T} V' - d^{-T} W') d \geq 0 .$$

It follows from this that

$$\min_{\|d\|=1} \min_{\substack{V \in \underline{\partial}^2 f(x) \\ v \in \underline{\partial}^2 f(x)}} \max_{\substack{V' \in \underline{\partial}^2 f(x) \\ v' \in \underline{\partial}^2 f(x)}} (d^{+T} V' - d^{-T} W') d \geq 0 .$$

A closed convex set depending on V and W can be constructed as follows:

$$\begin{aligned} Q(V,W) &:= \text{co}Q'(V,W) \\ &= \text{co} \{ u = d^{+T}V' - d^{-T}W' \mid V' \in \underline{\partial}^2 f(x) + W, \\ &\quad W' \in \bar{\partial}^2 f(x) + V, \|d\| = 1 \} . \end{aligned} \quad (5.5)$$

Hence

$$\min_{\|d\|=1} \min_{\substack{V \in \underline{\partial}^2 f(x) \\ W \in \bar{\partial}^2 f(x)}} \max_{u \in Q(V,W)} \langle u^T, d \rangle \geq 0 ,$$

i.e.,

$$\min_{\|d\|=1} \max_{u \in Q(V,W)} \langle u^T, d \rangle \geq 0 ,$$

for each pair of $V \in \underline{\partial}^2 f(x)$ and $W \in \bar{\partial}^2 f(x)$. According to [6, Lemma 1] or [3], we have the following assertion:

$$0 \in Q(V,W) . \quad (5.6)$$

The (5.6) shows that there exist $d_i \in \text{bd}B_1(0)$, $i = 1, \dots, m$, $\lambda_i \geq 0$, $i = 1, \dots, m$, $\sum \lambda_i = 1$ such that

$$0 \in \sum \lambda_i (d_i^{+T}(\underline{\partial}^2 f(x) + W) - d_i^{-T}(\bar{\partial}^2 f(x) + V)) ,$$

i.e.,

$$\begin{aligned} 0 \in & (\sum \lambda_i d_i^{+T})W - (\sum \lambda_i d_i^{-T})V + \sum (\lambda_i d_i^{+T} \underline{\partial}^2 f(x)) \\ & - \sum (\lambda_i d_i^{-T} \bar{\partial}^2 f(x)) . \end{aligned}$$

Since $\lambda_i d_i^{+T}$ and $\lambda_i d_i^{-T}$ are nonnegative, one has

$$\begin{aligned} (\sum \lambda_i d_i^{-T})V - (\sum \lambda_i d_i^{+T})W \in & (\sum \lambda_i d_i^{+T} \underline{\partial}^2 f(x)) \\ & - (\sum \lambda_i d_i^{-T} \bar{\partial}^2 f(x)) . \end{aligned}$$

Letting

$$g_1 = \sum \lambda_i d_i^+ , \quad g_2 = \sum \lambda_i d_i^- ,$$

we obtain the form (5.3). The proof is completed. \square

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