Special Conditions on Tradeoffs*

Charles M. Harvey

May 1987
WP-87-47

*This research is supported in part by the National Science Foundation under grant no. SES84-10665.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

Foreword

This paper discusses prescriptive models for an individual's or society's tradeoffs between different objectives. These objectives may refer to different attributes, different time-periods, or different individuals. Conditions on tradeoffs are shown to imply additive value functions that are sufficiently structured to be tractable in applications and are sufficiently general to represent preference issues concerning equity between the objectives and the dependence of tradeoffs on status quo positions.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

1. Introduction

This paper is a systematic discussion of prescriptive models concerning tradeoffs between different objectives. The purpose of these models is to show the implications of preferences between consequences that are relatively simple to compare for preferences between the actual alternatives being examined in a public policy evaluation or in a corporate planning study.

It is assumed that appropriate objectives have been identified and that appropriate variables x_i, $i = 1,...,n$, have been selected to describe the objectives (see, e.g., Keeney, 1982). These variables may measure the outcomes of different attributes, the outcomes in different time-periods, the outcomes for different individuals, or a combination of these types of outcomes. It is assumed that the consequences of the actual alternatives are included in a product set of consequences described by vectors $(x_1,...,x_n)$ of amounts of the variables x_i, $i = 1,...,n$. Roughly speaking, the results in this paper are concerned with the implications of tradeoffs between two of the variables, e.g., x_1 and another x_j, for preferences between the multivariable consequences $(x_1,...,x_n)$.

There are important analogies between these models of tradeoffs attitudes for multivariable consequences and the (better-known) models of risk attitudes for single-variable lotteries. In particular, the following two analogies will be emphasized here.

(a) The preference conditions of an expected-utility model (e.g., the substitution principle) are analogous to the preference conditions of an additive-value model (e.g., preferential independence). Therefore, a general single-variable utility function is analogous to a multivariable value function of the additive form:

$$V(x_1,...,x_n) = v_1(x_1) + v_2(x_2) + \cdots + v_n(x_n) \quad .$$

(b) The preference condition of risk neutrality is analogous to the preference conditions of inequity neutrality and tradeoffs independence (see Section 3) that are assumed in cost-benefit models of tradeoffs between different individuals, in discounting models of tradeoffs between different periods, and in willingness-to-pay models of tradeoffs between different attributes. Therefore, a linear utility function is analogous to a value function having one of the following forms:

$$V(x_1,\ldots,x_n) = x_1 + x_2 + \cdots + x_n \qquad \text{(cost-benefit)} \qquad (1.1)$$

$$V(x_1,\ldots,x_n) = x_1 + a_2 x_2 + \cdots + a_n x_n \qquad \text{(discounting)} \qquad (1.2)$$

$$V(x_1,\ldots,x_n) = x_1 + f_2(x_2) + \cdots + f_n(x_n) \qquad \text{(willingness-to-pay)} \qquad (1.3)$$

Typically, the coefficients in (1.2) are assumed to correspond to a fixed discount rate and the functions in (1.3) are assumed to be linear. For surveys of advantages and limitations, cost-benefit models are discussed in Bentkover et al. (1986), Fischhoff et al. (1981), Mishan (1976), and Stokey and Zeckhausen (1978); discounting models are discussed in Arrow (1976), Lind et al. (1982), and Page (1977); and willingness-to-pay models are discussed in Brown et al. (1974), Cummings et al. (1986), Keeney and Raiffa (1976), and Jones-Lee (1982).

The analogies (a) and (b) are proposed as being natural but not historical. During the past few decades, there has been an extensive interest in the theory and use of expected-utility models (see, e.g., the recent survey article by Farquhar, 1984 with 190 references). There has not, however, been a comparable interest in additive-value models; instead, interest has been primarily in the special types of models having the value functions (1.1) - (1.3).

The issue of concern for risk is excluded in a model that assumes risk neutrality since the model cannot consider the effects of such non-neutral risk attitudes as risk aversion. There are two preference issues that similarly are excluded in a cost-benefit model, a discounting model, or a willingness-to-pay model.

The first issue is that of *concern for equity* between the variables $x_i, i = 1,\ldots,n$. Suppose, for example that the variables x_i denote net-benefits to different individuals. In such a model, it might be important to examine preferences such that a consequence $(x_1,\ldots,x_n)$ is preferred to a consequence $(x_1',\ldots,x_n')$ with $\Sigma x_i' = \Sigma x_i$ provided that the net-benefits x_i are more equally distributed than are the net-benefits x_i'.

The second issue is that of the *dependence of tradeoffs* between variables on the base amounts of the variables. Suppose, for example, that the variables denote benefits in different periods. In such a model, it might be important to consider preferences such that there is a willingness to incur a greater cost in one period in order to obtain a specified benefit in another period if the base cost in the first period is low than if the base cost in the first period is high. This issue also can be expressed as a discrepancy between the amount that an individual would be ***willing to pay*** to acquire a benefit and the amount that an individual would be ***willing to accept*** to give up the benefit.

This paper discusses a number of models that can include these two preference issues. In summary, conditions on tradeoffs are used to structure these models so that the effects on preferences of a concern for equity or of a dependence of tradeoffs can be examined in a tractable manner. Thus a prescriptive theory for these preference issues is developed that is analogous to the theory for the preference issue of concern for risk.

The paper is organized as follows. First, the additive-value model due to Debreu (1960) is described. In this context, the preference issues of concern for equity and of tradeoffs dependence are discussed. Then, three simplified versions of the additive-value model are described that are appropriate for tradeoffs between individuals, for tradeoffs between periods, and for tradeoffs between attributes. The primary purpose of this material is to establish a framework for the discussion of more specific models.

The paper then discusses conditions on tradeoffs that are ***special*** in that they imply special forms of the additive value function. First, a family of conditions of ***tradeoffs independence*** is discussed. Each of these conditions implies that the additive value function is of a different specific type. Second, a family of conditions of ***tradeoffs constancy*** is discussed. Each of these conditions is weaker than the corresponding condition of tradeoffs independence, and implies that the value function belongs to a parametric family of functions. Third, a condition of linear tradeoffs is discussed. This condition implies that the value function is of the form (1.1)-(1.3) or is of a logarithmic form.

These special conditions on preferences among multivariable consequences are analogous to well-studied conditions on preferences among single-variable lotteries that imply special forms of a utility function. Moreover, the models presented in this paper are very similar to the models on risk attitudes that are presented in Harvey (1987); the similarity is stressed here in the choice of terminology and in the organization of the material.

A total of sixteen special conditions (Definitions 5.1, 5.3, 6.1, 6.3) are discussed in this paper; two of these conditions are part of the willingness-to-pay "folklore," two are described in Kirkwood and Sarin (1980), four are described in Harvey (1985a, b, c), (1986a, b), and eight appear to be new. The paper by Kirkwood and Sarin provides proofs of the results presented whereas the five papers by Harvey refer directly or indirectly to the working paper, Harvey (1981). In all these papers, moreover, it is assumed that the value functions have continuous second derivatives and positive first derivatives. The proofs presented here build upon arguments in Harvey (1987) that do not require such extra assumptions.

This paper also discusses methods by which the special conditions of tradeoffs independence and tradeoffs constancy can be applied to represent the issues of concern for equity and tradeoffs dependence. Two contrasting methods are described. In the first method, a special condition on tradeoffs is used with specific assessments of the degree of concern for equity or of the degree of tradeoffs dependence in order to evaluate which of the alternative consequences is preferred. In the second method, a special condition on tradeoffs is used to show which degrees of concern for equity or of tradeoffs dependence imply which of the alternative consequences is preferred.

The significance of this paper for applications, e.g., benefits assessment studies, is as follows. To date, the primary decision analysis approach for modeling expressed-preference information has been that of multiattribute utility theory, i.e., the use of an additive or a multiplicative utility function. Then, the evaluation of tradeoffs is partially replaced by the evaluation of risk attitudes. This paper discusses an analogous approach: the use of an additive value function that is sufficiently structured so that the evaluation of tradeoffs can be considered directly.

2. Additive-Value Models

This section describes the additive-value model that is developed in Debreu (1960). Other versions of Debreu's model are discussed in Fishburn (1970), Koopmans (1972), and Pfanzagl (1971), and other types of additive-value models are developed in Fishburn (1969), Krantz et al. (1971), Luce and Tukey (1964), and Scott (1964). Expository treatments are given, for example, in Keeney and Raiffa (1976) and Roberts (1979).

Consider a decision problem in which $n \geq 2$ variables, $i = 1,\ldots,n$, have been chosen, and the amounts x_i of these variables are in specified non-point intervals I_i, $i = 1,\ldots,n$. Let $c = (x_1,\ldots,x_n)$ denote a *consequence* having amounts x_i in I_i, $i = 1,\ldots,n$, and let C denote the product set of all such consequences.

The following notation will be used. For any variable (to be denoted either by i or by x_i), let x_i^c denote the amounts of the variables other than x_i, i.e., the complementary amounts. For any variables x_i and x_j, $i \neq j$, let x_{ij}^c denote the amounts of the variables other than x_i and x_j. Then, a consequence also can be denoted by $c = (x_i, x_i^c)$ and by $c = (x_i, x_j, x_{ij}^c)$ where it is assumed that the n variables have been put into the usual order $x_1,\ldots,x_n$. When convenient, the complementary amounts x_i^c and x_{ij}^c will be omitted so that $c = (x_i)$ and $c = (x_i, x_j)$.

Suppose that preferences between the consequences in C are denoted by the preference relation $c \succsim c'$ (c is at least as preferred as c'), and that the preference relations $c \sim c'$ (c is indifferent to c') and $c \succ c'$ (c is preferred to c') are related to $c \succsim c'$ by: $c \sim c'$ provided that $c \succsim c'$ and $c' \succsim c$, and $c \succ c'$ provided that $c \succsim c'$ and not $c' \succsim c$.

2.1 Tradeoffs and value functions

A pair $(C, \succsim)$ as described above will be called a ***tradeoffs model***. A real-valued function V defined on C such that $c \succsim c'$ if and only if $V(c) \geq V(c')$ for any c, c' in C will be called a *value function* for $(C, \succsim)$.

Consider the following conditions on a tradeoffs model $(C, \succsim)$:

(A) $\succsim$ is *transitive* and *complete*.

(B) $\succsim$ is *monotone* in each variable, $i = 1,\ldots,n$, that is, for any consequences (x_i, x_i^c) and (x_i', x_i^c) that differ only in the amounts x_i and x_i' of the i-th variable,

$$(x_i, x_i^c) \succsim (x_i', x_i^c) \text{ iff } x_i \geq x_i' \quad .$$

(C) $\succsim$ is *continuous* in each variable, $i = 1,\ldots,n$, that is, for any consequences $(x_i, x_i^c) \succ (y_i, y_i^c)$, there exists changes w, $w' > 0$ such that,

$$(x_i - w, x_i^c) \succ (y_i, y_i^c) \text{ and } (x_i, x_i^c) \succ (y_i + w', y_i^c)$$

The following result is implied by results in Debreu (1954, 1964), Fishburn (1970, Theorems 3.3 and 3.6), and Koopmans (1972, Result A in Chapter 3).

Theorem 2.1 A tradeoffs model $(C, \succsim)$ satisfies the preference conditions (A) - (C) if and only if there exists a value function V for $(C, \succsim)$ that is strictly increasing in each variable and is jointly continuous.

We will be concerned with such value functions of the following type.

Definition 2.1 A value function of the form,

$$V(x_1,\ldots,x_n) = v_1(x_1) + \cdots + v_n(x_n) \quad , \tag{2.1}$$

where for each $i = 1,\ldots,n$, the function v_i is strictly increasing and continuous on the interval I_i will be called an ***additive value function***. Each function v_i, $i = 1,\ldots,n$, will be called a ***component function***. If a tradeoffs model $(C, \succsim)$ has such a value function, then $(C, \succsim)$ will be called an ***additive-value model***.

An additive value function $V(x_1,\ldots,x_n)$ with $n \geq 2$ is "ordinal" in the sense that it does not represent degrees of preference between consequences, and is "cardinal" in the sense that it is unique up to a positive linear transformation (see, e.g., Fishburn, 1976).

Conditions on tradeoffs that imply an additive-value model can be stated in terms of the following concept. Consider any two variable i and j, any two pairs of amounts x_i^0, x_i and x_j^0, x_j of these variables, and any amounts x_{ij}^c of the other variables. If

$$(x_i^0, x_j, x_{ij}^c) \sim (x_i, x_j^0, x_{ij}^c) \quad ,$$

then x_i^0, x_i will be called a ***tradeoffs pair*** corresponding to x_j^0, x_j conditional on x_{ij}^c. See Figure 2.1 for diagrams of tradeoffs pairs.

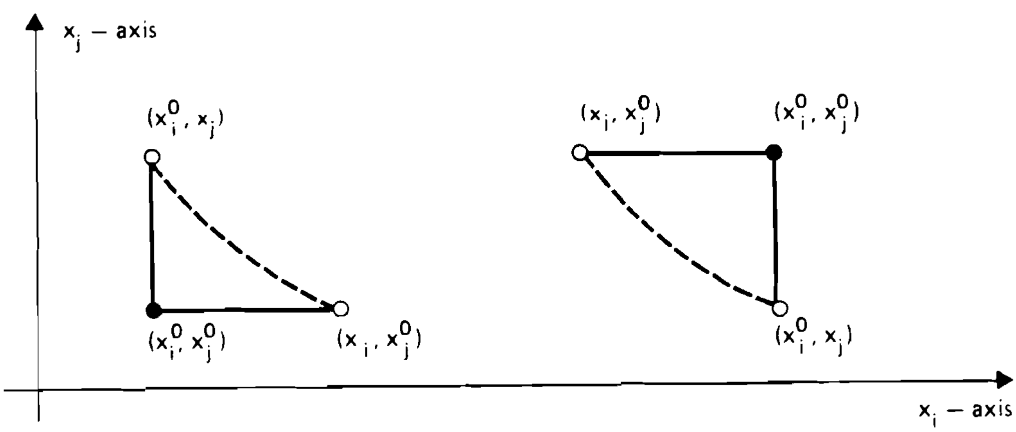


Figure 2.1 Tradeoffs pairs x_i^0, x_i and x_j^0, x_j

2.2 Tradeoffs midvalues

Suppose that for two variables i, j and two amounts x_i^0, x_i of the variable i, there is third amount $\hat{x}_i$ of the variable i such that both of the pairs x_i^0, $\hat{x}_i$ and $\hat{x}_i$, x_i are tradeoffs pairs for a common pair x_j^0, x_j of the variable j and common amounts x_{ij}^c of any other variables. Then $\hat{x}_i$ will be called a *tradeoffs midvalue* of x_i^0 and x_i conditional on x_j^0, x_j and x_{ij}^c. Conditions (A)-(C) imply that a tradeoffs midvalue $\hat{x}_i$ of x_i^0 and x_i conditional on x_j^0, x_j and x_{ij}^c is unique. The existence of a tradeoffs midvalue of two amounts x_i^0, x_i may depend on which variable j is considered.

Definition 2.2 A tradeoffs model $(C, \succsim)$ will be said to have *tradeoffs midvalues independence* provided that for any two variables i and j, if an amount $\hat{x}_i$ is the tradeoffs midvalue of two amounts x_i^0 and x_i conditional on some amounts x_j^0, x_j and x_{ij}^c, then $\hat{x}_i$ is the tradeoffs midvalue of x_i^0 and x_i conditional on any amounts y_j^0, y_j and y_{ij}^c such that either x_i^0, $\hat{x}_i$ or $\hat{x}_i$, x_i is a tradeoffs pair corresponding to y_j^0, y_j.

Theorem 2.2 (Debreu, 1960). For any $n \geq 2$, a tradeoffs model $(C, \succsim)$ satisfies conditions (A)-(C) and the condition of tradeoffs midvalues independence if and only if $(C, \succsim)$ can be represented by an additive value function.

2.3 Tradeoffs amounts

When there are three or more variables, then the condition of tradeoffs midvalues independence is equivalent to the simplier condition below.

Consider two variables i, j and two pairs of amounts x_i^0, x_i and x_j^0, x_j where the amounts x_i^0 and x_j^0 are regarded as fixed. Then, x_i^0 and x_j^0 will be called *base amounts*. If x_i^0, x_i is a tradeoffs pair corresponding to x_j^0, x_j conditional on complementary amounts x_{ij}^c, then x_i will be called a *tradeoffs amount* corresponding to x_j conditional on x_{ij}^c. Conditions (A)-(C) imply that a tradeoffs amount x_i corresponding to x_j conditional on x_{ij}^c is unique. Such a tradeoffs amount x_i may or may not exist.

Definition 2.3 A tradeoffs model $(C, \succsim)$ with $n \geq 3$ will be said to have *tradeoffs amounts independence* provided that for any two variables i and j, if an amount x_i is the tradeoffs amount corresponding to an amount x_j conditional on base amounts x_i^0, x_j^0 and complementary amounts x_{ij}^c, then x_i is the tradeoffs amount corresponding to x_j conditional on x_i^0, x_j^0 and any complementary amounts y_{ij}^c.

Theorem 2.3 (Debreu, 1960). For any $n \geq 3$, a tradeoffs model $(C, \succsim)$ satisfies conditions (A)-(C) and the condition of tradeoffs amounts independence if and only if $(C, \succsim)$ can be represented by an additive value function.

The conditions of tradeoffs midvalues independence and tradeoffs amounts independence do not involve any arithmetic operations on the variables, and are invariant for any strictly increasing transformations of the variables. By contrast, these properties are not true for the conditions of tradeoffs independence and tradeoffs constancy that are discussed in Sections 5 and 6.

3. Tradeoffs Dependence and Concern for Equity

This section discusses the two preference issues of the dependence of tradeoffs on base amounts and of concern for equity. The issues can be described by considering a selected variable, here to be labeled x_1, and another variable x_j, $j \neq 1$. Typically but not necessarily, the variable x_1 will measure a monetary objective.

As an illustration, consider the tradeoffs of a society between energy costs and air quality. Suppose that: (1) the variable x_1 measures the consequent financial positions of society (and larger amounts x_1 corresponding to less cost are preferred), and (2) the variable x_j measures a consequent effect of air quality on the environment or on health (and larger amounts x_j corresponding to less pollution are preferred). Suppose, moreover, that current policy will result in a financial position of x_1^0 and an air quality level of x_j, i.e., a consequence (x_1^0, x_j), whereas an alternative policy will result in a financial position of x_1 and an air quality level of x_j^0, i.e., a consequence (x_1, x_j^0). Does society wish to move from (x_1^0, x_j) to (x_1, x_j^0)?

The change in financial position from x_1^0 to x_1 can be emphasized by the notation $x_1 = x_1^0 + h$. Social tradeoffs between energy costs and air quality may be such that the consequence (x_1^0, x_j), is indifferent to an alternative $(x_1^0 + h, x_j^0)$

with an increase in cost, i.e., $h < 0$, and an improved air quality, i.e., $x_j^0 > x_j$, or is indifferent to an alternative $(x_1^0 + h, x_j^0)$ with a decrease in cost, i.e., $h > 0$, and a worsened air quality, i.e., $x_j^0 < x_j$.

The issue of ***tradeoffs dependence*** can be stated as follows: If the change h is held fixed, then will the tradeoffs pair x_j^0, x_j for a pair $x_1^0, x_1^0 + h$ depend on the base amount x_1^0? For example, if social tradeoffs between financial position and air quality are such that $(x_1^0 + h, x_j^0) \sim (x_1^0, x_j)$ for some amounts $x_j^0 > x_j$, i.e., air quality is improved by an extra cost of $-h > 0$, then will people also be indifferent if they have a larger base financial position y_1^0. A typical attitude is that a more prosperous society would be better able to afford the extra cost and would therefore prefer the alternative with improved air quality, i.e., that $y_1^0 > x_1^0$ implies $(y_1^0 + h, x_j^0) \succ (y_1^0, x_j)$. See Figure 3.1a for a diagram of such preferences. Tradeoffs dependence viewed as "income effects" is discussed in Randall and Stoll (1980) and Willig (1976).

Definition 3.1 For two variables x_1 and x_j in an additive-value model, consider any two pairs of amounts $x_1^0, x_1^0 + h$ and x_j^0, x_j with $h \neq 0$.

(a) Preferences will be called *tradeoffs independent* provided that: If $(x_1^0 + h, x_j^0) \sim (x_1^0, x_j)$ for some amount x_1^0, then $(y_1^0 + h, x_j^0) \sim (y_1^0, x_j)$ for any larger or smaller amount y_1^0.

(b) Preferences will be called *tradeoffs decreasing* provided that: If $(x_1^0 + h, x_j^0) \sim (x_1^0, x_i)$ for some amount x_1^0, then (1) $(y_1^0 + h, x_j^0) \succ (y_1^0, x_j)$ for any larger amount y_1^0 when $h < 0$ and for any smaller amount y_1^0 when $h > 0$, and (2) $(y_1^0 + h, x_j^0) \prec (y_1^0, x_j)$ for any smaller amount y_1^0 when $h < 0$ and for any larger amount y_1^0 when $h > 0$.

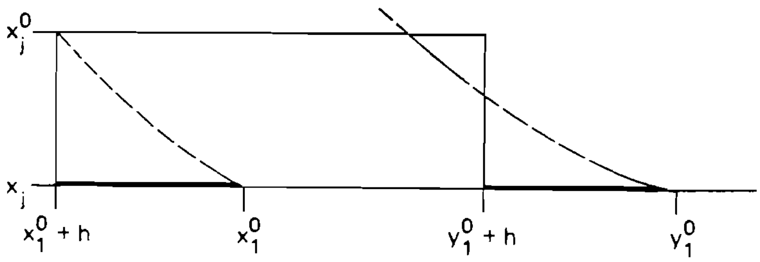
The issue of tradeoffs dependence often is seen as a discrepancy between willingness to pay (WTP) and willingness to accept (WTA). In the above example, if people at a position (x_1, x_j) are willing to pay a maximum amount of $p = -h$ in order to improve air quality from x_j to x_j', that is, $(x_1 - p, x_j') \sim (x_1, x_j)$, and people at the position (x_1, x_j') are willing to accept a minimum amount of a for a worsening of air quality from x_j' to x_j, that is, $(x_1 + a, x_j) \sim (x_1, x_j')$, then is $a = p$? See Figure 5.1b. Empirical studies suggest that for many individuals a is much larger than p (see, e.g., Loehman, 1985 and the studies cited in Cummings et al., 1986, p. 35).

The issue of *concern for equity* can be stated as follows: If two consequences (x_1^0, x_j) and (x_1, x_j^0) having extreme amounts of the variables are indifferent, then will the consequence having "averages" of x_1^0 and x_1 and of x_j^0 and x_j be preferred to (x_1^0, x_j) and (x_1, x_j^0) ? For example, suppose that a consequence (x_1^0, x_j) with a low financial position x_1^0 and a high air quality x_j is indifferent to a consequence (x_1, x_j^0) with a high financial position x_1 and a low air quality x_j^0. Consider the ordinary average $\bar{x}_1 = \frac{1}{2}(x_1^0 + x_1)$ of the financial positions and the corresponding amount $\tilde{x}_j$ such that $(x_1^0, \tilde{x}_j) \sim (\bar{x}_1, x_j^0)$. The average consequence $(\bar{x}_1, \tilde{x}_j)$ may be preferred to the extreme consequences (x_1^0, x_j) and (x_1, x_j^0) as having more equity or balance between energy costs and air quality. See Figure 3.1b for a diagram of such preferences.

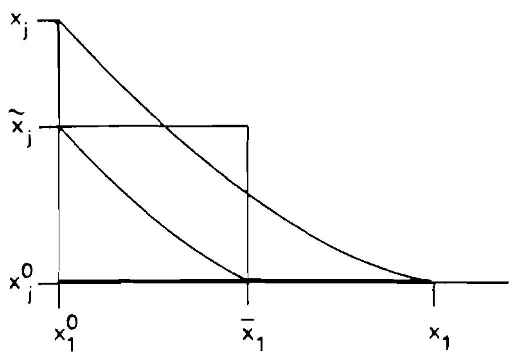
Definition 3.2 For two variables x_1 and x_j in an additive-value model, consider any amounts x_1^0, x_1, and $\bar{x}_1 = \frac{1}{2}(x_1^0 + x_1)$, and any amounts x_j^0, x_j, and $\tilde{x}_j$ such that $(x_1^0, x_j) \sim (x_1, x_j^0)$ and $(x_1^0, \tilde{x}_j) \sim (\bar{x}_1, x_j^0)$.

(a) Preferences will be called *inequity neutral* provided that $(\bar{x}_1, \tilde{x}_j)$ is indifferent to (x_1^0, x_j) and (x_1, x_j^0).

(b) Preferences will be called *inequity averse* provided that $(\bar{x}_1, \tilde{x}_j)$ is preferred to (x_1^0, x_j) and (x_1, x_j^0) whenever $x_1^0 \neq x_1$ and $x_j^0 \neq x_j$.



(a) Tradeoffs decreasing preferences where $x_j < x_j^0$



(b) Inequity averse preferences where $x_j^0 < x_j$

Figure 3.1 Illustrations of tradeoffs dependence and concern for equity

The preference conditions in Definitions 3.1 and 3.2 are equivalent to properties of the component function $v_1(x_1)$ in an additive value function. These results are analogous to results relating risk attitudes to properties of a single-variable utility function.

Theorem 3.1 Suppose that preferences satisfy the conditions of an additive-value model. Then, the properties within each of the following parts are equivalent.

Part I:

(a) Preferences are tradeoffs independent for the variable x_1.

(b) Preferences are inequity neutral for the variable x_1.

(c) The component function $v_1(x_1)$ in (2.1) is linear.

Part II:

(a) Preferences are tradeoffs decreasing for the variable x_1.

(b) Preferences are inequity averse for the variable x_1.

(c) The component function $v_1(x_1)$ in (2.1) is strictly concave.

Conditions of tradeoffs increasing preferences and inequity prone preferences also can be defined; such preferences occur if and only if $v_1(x_1)$ is strictly convex. A corollary of Theorem 3.1 is that the preference conditions in Definitions 3.1 and 3.2 are independent of which variable x_j, $j \neq 1$, is considered.

There are important preference issues distinct from tradeoffs dependence and concern for equity that are not discussed in this paper. First, there are different types of preference effects that may be present under various circumstances. For example, there are the effects of a person's reference point (e.g., the WTP amount to *obtain* a benefit may differ from the WTP amount to *keep* the benefit), process effects (e.g., a person's WTP amount may depend on whether other people will have a similar obligation), compensation rights, and the heterogeneity of preferences in a group. Second, there are different types of preference biases, e.g., cognitive dissonance, information bias, strategic mispresentation, and hypothetical bias. Cummings et al. (1986) discusses these and other preference issues.

4. Simplified Additive Value Functions

This section discusses several types of additive-value models in which the value function (2.1) is simplified to a value function that can model the issues of concern for equity and of tradeoffs dependence separately from the basic issue of tradeoffs. The three value functions (4.1)-(4.3) discussed below are intended to model tradeoffs between individuals, tradeoffs between periods, and tradeoffs between attributes, respectively.

4.1 Equal tradeoffs amounts models

The following condition is intended for models in which the variables x_i measure the same quantity for different individuals. This condition is equivalent to the condition of *equal individuals* in Harvey (1985b).

Definition 4.1 An additive-value model will be said to have *equal tradeoffs amounts* provided that the intervals I_i, $i = 1,\dots,n$, are equal and that for any two variables i, j and any pairs of amounts x^0, x and y^0, y, if x is a tradeoffs amount corresponding to y conditional on x^0, y^0 when x^0, x amounts of the variable i and y^0, y are amounts of the variable j, then x also is a tradeoffs amount corresponding to y conditional on x^0, y^0 when x^0, x are amounts of the variable j and y^0, y are amounts of the variable i.

Theorem 4.1 (Harvey, 1985b). An additive-value model has equal tradeoffs amounts if and only if preferences are represented by a value function of the form

$$V(x_1,\dots,x_n) = v(x_1) + v(x_2) + \cdots + v(x_n) \tag{4.1}$$

where v is a common component function.

4.2 Equal tradeoffs midvalues models

The following condition is intended for models in which the variables x_i measure the same quantity in different periods. An equivalent condition of *equal tradeoffs comparisons* is defined in Harvey (1986a).

Definition 4.2 An additive-value model will be said to have *equal tradeoffs midvalues* provided that the intervals I_i, $i = 1,\dots,n$, are equal and that for any two variables i, j and any amounts $x_0, \hat{x}, x$, if $\hat{x}$ is a tradeoffs midvalue of x^0, x when $x^0, \hat{x}, x$ are amounts of the variable i, then $\hat{x}$ also is a tradeoffs midvalue of x^0, x when $x^0, \hat{x}, x$ are amounts of the variable j.

Theorem 4.2 (Harvey, 1986a) An additive-value model has equal tradeoffs mid-values if and only if preferences are represented by a value function of the form

$$V(x_1,\ldots,x_n) = v(x_1) + a_2 v(x_2) + \cdots + a_n v(x_n) \tag{4.2}$$

where v is a common component function and $a_2,\ldots,a_n$ are positive coefficients.

4.3 Standard additive-value models

The discussion in this subsection is intended for models in which the variables x_i measure different quantities, i.e., different attributes. Then, the assessment of an additive value function often can be facilitated by using tradeoffs between a selected variable (for example, x_1) and the other variables (then, $x_2,\ldots,x_n$) in order to rescale $x_2,\ldots,x_n$ in accord with x_1. The willingness-to-pay method uses this idea in the special case of tradeoffs independence (see, e.g., Brown et al., 1974, Chapters 9, 10 and Keeney and Raiffa, 1976, pages 125-127). The idea can be described in general as follows (see also Harvey, 1985a).

Suppose that amounts x_1^*, $i = 1,\ldots,n$, of the variables have been chosen that are especially convenient to consider. The amounts x_i^* will be called *standard amounts*. In many applications, the model will be framed so that the variables x_i, $i = 1,\ldots,n$, represent changes from the status quo, and it will be appropriate to choose standard amounts $x_i^* = 0$ that denote the no-cost or no-effect amounts. In other applications, the variables x_i may be defined as consequent positions, and it will be appropriate to choose non-zero standard amounts. For example, a variable x_i may be defined as an asset position or as a level of risk, and it then may be appropriate to choose x_i^* as a non-zero status quo amount.

Definition 4.3 In an additive-value model, suppose that an amount x_j of a variable $j = 2,\ldots,n$ has a tradeoffs amount x_1 conditional on the standard base amounts $x_j^0 = x_j^*$ and $x_1^0 = x_1^*$, that is, $(x_1,x_j^*) \sim (x_1^*,x_j)$. Then, x_1 will be called the *standard tradeoffs amount* for x_j. The functions $x_1 = f_j(x_j)$, $j = 2,\ldots,n$ defined in this manner will be called *standard tradeoffs functions*.

The conditions of an additive-value model imply that for each $j = 2,\ldots,n$, the function $f_j(x_j)$ is defined, continuous, and strictly increasing on a subinterval of I_j that contains x_j^* and that $f_j(x_j^*) = x_1^*$.

The functions $f_j(x_j)$ as specified in Definition 4.3 are related by the formula, $f_j(x_j) = x_j^* + g_j(x_j)$, $j = 2,\ldots,n$, to the "standard pricing-out functions" $g_j(x_j)$

defined in Harvey (1985a). The reason for the present definition is to avoid an involvement with the arithmetic operation +.

Since the component functions v_i, $i = 1,...,n$, are continuous and strictly increasing, it always is possible to include endpoints or $\pm\infty$ in an interval I_i so that the image interval $v_i(I_i)$ is closed. As a matter of convenience, it will be assumed that the intervals $v_i(I_i)$ are closed. Then, standard tradeoffs amounts always exist in the following sense.

Proposition 4.1 For any additive-value model, there exists a variable x_i (which may be labeled x_1) and standard amounts $x_1^*,...,x_n^*$ such that a standard tradeoffs amount $x_1 = f_j(x_j)$ exists for any amount x_j of any variable, $j = 2,...,n$.

We wish to require the existence of standard tradeoffs amounts in a different sense, namely, that they exist for a specified variable x_1 and specified standard amounts $x_1^*,...,x_n^*$. This requirement does not appear to be restrictive in practice. An additive-value model of this type will be called a *standard additive-value model.*

Any standard additive-value model simplifies to a model having a value function (4.1) if the objectives other than the first are rescaled so that they are measured by the variables $y_j = f_j(x_j)$, $j = 2,...,n$. Then, the single component function v can be regarded as representing the preference issues of concern for equity and tradeoffs dependence.

Theorem 4.3 (Harvey, 1985a, b, 1986a). An additive-value model is a standard additive-value model with respect to the variable x_1 and standard amounts $x_1^*,...,x_n^*$ if and only if preferences are represented by a value function of the form

$$V(x_1,...,x_n) = v(x_1) + v(f_2(x_2)) + \cdots + v(f_n(x_n)) \tag{4.3}$$

where v is a component function and $f_j(x_j)$, $j = 2,...,n$ are the standard tradeoffs functions for the variables x_j, $j = 2,...,n$. For such a model:

(a) If the condition of equal tradeoffs amounts is satisfied, then (4.3) simplifies to (4.1), and the standard tradeoffs functions are $f_j(x_j) = x_j$, $j = 2,...,n$.

(b) If the condition of equal tradeoffs midvalues is satisfied, then (4.3) simplifies to (4.2), and the standard tradeoffs functions are the generalized averages

$$f_j(x_j) = v^{-1}(a_j v(x_j) + (1-a_j)v(x_j^*)),\ j = 2,\ldots,n \quad . \tag{4.4}$$

where v is any component function in (4.2).

For a standard additive-value model, the value function (4.3) provides a means of separating the basic issue of tradeoffs from the issues of concern for equity and tradeoffs dependence. Here, the standard tradeoffs functions f_j, $j = 2,\ldots,n$, represent the issue of tradeoffs and the component function v represents the issues of concern for equity and tradeoffs dependence. For the more restricted standard additive-value models in (a) and (b) above, the standard tradeoffs functions f_j, $j = 2,\ldots,n$, are restricted as described.

5. Conditions of Tradeoffs Independence

This section discusses a family of conditions on tradeoffs that includes the conditions of tradeoffs independence and inequity neutrality discussed in Section 3. The interest here is not, as in Section 3, to identify important preference issues. Rather, it is to define conditions that will be as simple as possible to verify, e.g., by defining a condition that considers only a small class of consequences.

For any specific measurement scale for the variable x_1, i.e., any specific arithmetic operation on x_1, four equivalent conditions are defined. If any one of these conditions is satisfied, then the component function v in (4.1)-(4.3) has a corresponding special form.

Analogous conditions on types of risk neutrality for single-variable lotteries are discussed in Harvey (1987).

5.1 Absolute tradeoffs independence

The conditions described here are equivalent to the conditions of tradeoffs independence and inequity neutrality defined in Section 3. Conditions (a) and (c) appear to be part of the willingness-to-pay "folklore" and are discussed, for example, in Harvey (1985a).

Definition 5.1 Four conditions of *absolute tradeoffs independence* between x_1 and another variable x_j are as follows:

(a) *Absolute tradeoffs amounts independence.* If an amount x_j has a tradeoffs amount of x_1 for base amounts $x_1^0 = x_1^* + h$ and $x_j^0 = x_j^*$, then $x_1 = f_j(x_j) + h$, that is, $(x_1^* + h, x_j) \sim (f_j(x_j) + h, x_j^*)$.

(b) *Absolute tradeoffs willingness independence.* If a pair of consequences (x_1, x_j) and $(x_1 - p, x_j')$ are indifferent and a pair of consequences (x_1, x_j') and $(x_1 + a, x_j)$ are indifferent, then $p = a$.

(c) *Absolute tradeoffs midvalues independence.* If two amounts x_1 and x_1' have a tradeoffs midvalue of $\hat{x}_1$ with respect to the variable x_j, then $\hat{x}_1 = x_1 + h$ and $x_1' = \hat{x}_1 + h$ for the same amount h, that is $\hat{x}_1 = \frac{1}{2}(x_1 + x_1')$.

(d) *Absolute tradeoffs changes independence.* If a pair of consequences $c_1 = (x_1, x_j')$ and $c_2 = (x_1', x_j^*)$ are indifferent and a pair of consequences $c_3 = (x_1^* + h, x_j^*)$ and $c_4 = (x_1^*, x_j)$ are indifferent, then the combination of c_1 and c_3 is indifferent to the combination of c_2 and c_4, that is, $(x_1 + h, x_j') \sim (x_1', x_j)$.

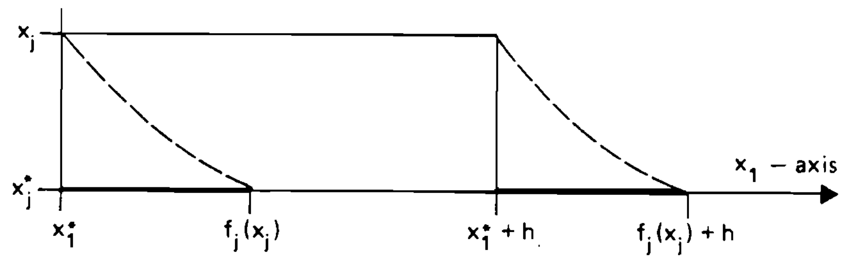
These conditions are illustrated in Figure 5.1 with indifferences denoted by dashed lines.

The condition (d) can be viewed as a preference condition on sums of amounts in consequences: If $(x_1\ x_j') \sim (x_1', x_j^*)$ and $(x_1^* + h, x_j^*) \sim (x_1^*, x_j)$, then there is indifference between the sum of (x_1, x_j') and the change from x_1 to $x_1 + h$ and the sum of (x_1', x_j^*) and the change from x_j^* to x_j. An analogous issue concerning sums of lotteries is discussed in Harvey (1986c) and Tversky and Kahneman (1981).

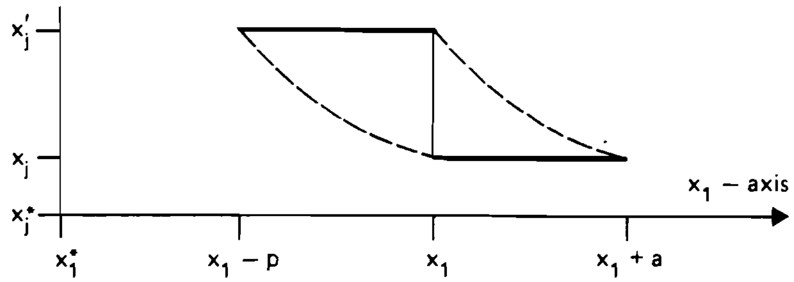
Other preference conditions can be defined that are equivalent to those of Definition 5.1. In particular, there are equivalent conditions such as the following that involve more than two variables.

Absolute tradeoffs joint independence. If amounts x_j and x_k of two variables have absolute tradeoffs changes of h_j and h_k, that is, $(x_1^* + h_j, x_j^*) \sim (x_1^*, x_j)$ and $(x_1^* + h_k, x_k^*) \sim (x_1^*, x_k)$, then the combination of x_j and x_k has an absolute tradeoffs change of $h_j + h_k$, that is $(x_1^* + h_j + h_k, x_j^*, x_k^*) \sim (x_1^*, x_j, x_k)$.

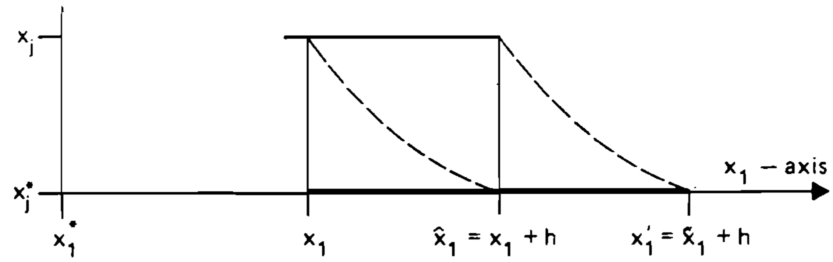
Weaker versions of parts of the following result are well known. For example, Keeney and Raiffa (1976, pages 125-127) show the equivalence of properties (a) and (e) under the assumption that tradeoffs amounts exist for any base amounts,



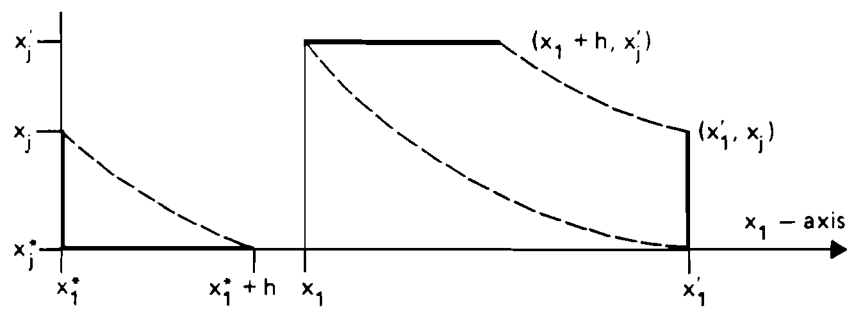
(a) Absolute tradeoffs amounts independence



(b) Absolute tradeoffs willingness independence



(c) Absolute tradeoffs midvalues independence



(d) Absolute tradeoffs changes independence

Figure 5.1. Illustrations of absolute tradeoffs independence

and Harvey (1985a) states, but does not prove, the equivalence of (a), (c), and (e) under differentiability assumptions regarding $v(x_1)$.

Theorem 5.1 For a standard additive-value model, the conditions (a)-(d) of absolute tradeoffs independence are equivalent to each other and to the property:

(e) A value function for preferences is determined as (1.3).

An immediate corollary of Theorem 5.1 is that if tradeoffs between x_1 and another variable x_j satisfy the conditions of absolute tradeoffs independence, then tradeoffs between x_1 and any variable x_k, $k = 2,\dots,n$, satisfy the conditions of absolute tradeoffs independence.

For a tradeoffs problem in which x_1 measure the outcomes of a monetary attribute and the other variables x_j, $j = 2,\dots,n$, measure the outcomes of non-monetary attributes, the value function typically is chosen to be of the type (1.3), i.e., a willingness-to-pay model is used. The conditions in Theorem 5.1 provide a means of determining whether the preferences involved can be adequately represented by such a model.

For a tradeoffs problem in which the variables $x_1,\dots,x_n$ measure the outcomes for different individuals or the outcomes in different periods, the value function typically is chosen to be of the type (1.1) or of the type (1.2), i.e., a cost-benefit model or a discounting model is used. The following result states conditions under which such models are appropriate. Weaker versions of parts of this result are in Harvey (1985b), (1986a).

Corollary 5.1 For a standard additive-value model:

(a) The conditions of equal tradeoffs amounts and absolute tradeoffs independence are satisfied if and only if a value function for preferences is determined as (1.1). (In this case, the standard tradeoffs functions are $f_j(x_j) = x_j$, $j = 2,\dots,n$, as specified in Theorem 4.3.).

(b) The conditions of equal tradeoffs midvalues and absolute tradeoffs independence are satisfied if and only if a value function for preferences is determined as (1.2). In this case, the standard tradeoffs functions are

$$f_j(x_j) = x_1^* + a_j(x_j - x_j^*),\ j = 2,\dots,n \quad . \tag{5.1}$$

where the positive coefficients a_j, $j = 2,\dots,n$, are as in (1.2).

5.2 $\circ$-tradeoffs independence

A family of conditions of tradeoffs independence can be defined as follows. Suppose that the interval I_1 of amounts of the variable x_1 is contained in (possibly is equal to) an open interval I on which there is a continuous group operation $x \circ x'$. Then, there exists a real-valued, continuous, and strictly increasing function g defined on I such that $g(x \circ x') = g(x) + g(x')$ for all x, x' in I (see, e.g., Aczél, 1966, p. 254). Such a function g will be called a *scaling function*. If x_1 is replaced by the variable $z_1 = g(x_1)$, then the arithmetic operation $x \circ x'$ is replaced by ordinary addition $z + z'$. Such rescaling of a variable also is described in Harvey (1987). It is not required that x_1^* is the identity for the operation $x \circ x'$.

Definition 5.2 Four conditions of *$\circ$-tradeoffs independence* between x_1 and another variable x_j can be obtained by replacing the + operations in parts (a)-(d) of Definition 5.1 by $\circ$ operations. (In (b), the amount $x_1 - p$ is replaced by $x_1 \circ p^{-1}$ where p^{-1} denotes the inverse of p.) The resulting conditions will be called: (a) $\circ$-tradeoffs amounts independence, (b) $\circ$-tradeoffs willingness independence, (c) $\circ$-tradeoffs midvalues independence, and (d) $\circ$-tradeoffs changes independence.

These conditions can be illustrated by replacing the + operations in Figure 5.1 by $\circ$ operations.

Theorem 5.2 For a standard additive-value model, the conditions (a)-(d) of $\circ$-tradeoffs independence are equivalent to each other and to the property:

(e) A value function for preferences is determined as

$$V(x_1,\ldots,x_n) = g(x_1) + g(f_2(x_2)) + \cdots + g(f_n(x_n)) \tag{5.2}$$

where g is any scaling function for the operation $x \circ x'$.

5.3 Relative tradeoffs independence

A primary type of $\circ$-tradeoffs independence is that in which the group operation $x \circ x'$ is multiplication. This subsection discusses independence conditions for the operation of multiplication and for a more general class of operations called shift multiplication. Independence conditions also can be discussed for the group operations described in Harvey (1987).

Suppose that the variable x_1 has been defined to measure *changes* from a status quo position; for example, x_1 may measure net gains or losses from a initial asset position. The status quo position will be denoted by a constant a (which may be either specified or unspecified). The standard change x_1^* will often but not necessary be chosen as $x_1^* = 0$.

Suppose, moreover, that the variable $y_1 = a + x_1$ associated with x_1 can be interpreted as measuring positions resulting from the changes x_1, e.g., final asset positions. The amounts y_1 will be referred to as *consequent positions*. Let $y_1^* = a + x_1^*$ denote the standard consequent position. Assume that $y_1 = a + x_1 > 0$ for all x_1 in the interval I_1.

Independence conditions can be defined in terms of relative changes in the variable y_1. For example, imagine that the tradeoffs midvalue of two amounts y_1 and y_1' is that amount $\hat{y}_1$ such that $\hat{y}_1 = hy_1$ and $y_1' = h\hat{y}_1$ for the same multiple $h > 0$. Then, for example, tradeoffs x_j^0, x_j are the same from a base position of half $\hat{y}_1$ to $\hat{y}_1$ as from a base position of $\hat{y}_1$ to twice $\hat{y}_1$. In terms of percent changes, this condition states that an amount $\hat{y}_1$ is the tradeoffs midvalue of two amounts y_1 and y_1' provided that $\hat{y}_1 = y_1 + my_1$ and $y_1' = \hat{y}_1 + m\hat{y}_1$ for the same *percent* $m = h - 1 > -1$.

Suppose that preferences $\succsim_y$ regarding consequent positions $y_1 = a + x_1$ are framing consistent (Harvey, 1986c) with preferences $\succsim$ regarding changes x_1. Then, the above condition for the variable y_1 is equivalent to the condition (c) in Definition 5.3 below for the variable x_1.

Conditions (a) and (c) below are discussed in Harvey (1985a).

Definition 5.3 Four conditions of *relative tradeoffs independence* between x_1 and another variable x_j are as follows:

(a) *Relative tradeoffs amounts independence.* If an amount x_j has a tradeoffs amount of x_1 for the base amounts $x_1^0 = x_1^* + m(a + x_1^*)$ and $x_j^* = x_j^*$ with a percent $m > -1$, then $x_1 = f_j(x_j) + m(a + f_j(x_j))$.

(b) *Relative tradeoffs willingness independence.* If a pair of consequences (x_1, x_j) and $(x_1 - \frac{m}{m+1}(a + x_1), x_j')$ are indifferent and a pair of consequences (x_1, x_j') and $(x_1 + m'(a + x_1), x_j)$ are indifferent, then $m = m'$.

(c) *Relative tradeoffs midvalues independence.* If two amounts x_1 and x_1' have a tradeoffs midvalue of $\hat{x}_1$ with respect to the variable x_j, then $\hat{x}_1 = x_1 + m(a + x_1)$ and $x_1' = \hat{x}_1 + m(a + \hat{x}_1)$ for the same percent $m > -1$.

(d) *Relative tradeoffs changes independence.* If two consequences $c_1 = (x_1, x_j')$ and $c_2 = (x_1', x_j^{\bullet})$ are indifferent and two consequences $c_3 = (x_1^{\bullet} + my_1^{\bullet}, x_j^{\bullet})$ and $c_4 = (x_1^{\bullet}, x_j)$ are indifferent, then the combination of c_1 and c_3 is indifferent to the combination of c_2 and c_4, that is, $(x_1 + m(a + x_1), x_j') \sim (x_1', x_j)$.

The above conditions involve the operation $x_1 \circ x_1' = (a + x_1)(a + x_1') - a$ defined on the interval $x_1 > -a$. This operation is referred to in Harvey (1987) as a *shift multiplication*. The corresponding operation on consequent positions $y_1 = a + x_1$ is that of ordinary multiplication defined on the interval $y_1 > 0$.

The condition (d) can be viewed as a preference condition on products of amounts in consequences: If $(y_1, x_j') \sim_y (y_1' \; x_j^{\bullet})$ and $(hy_1^{\bullet}, x_j^{\bullet}) \sim_y (y_1^{\bullet}, x_j)$, then there is indifference between the product of (y_1, x_j') and the change from y_1 to hy_1 and the product of $(y_1', x_j^{\bullet})$ and the change from $x_j^{\bullet}$ to x_j.

Theorem 5.3 For a standard additive-value model, the conditions (a)-(d) of relative tradeoffs are independence equivalent to each other and to the property:

(e) A value function for preferences is determined as

$$V(x_1,\dots,x_n) = \log(a + x_1) + \log(a + f_2(x_2)) + \cdots + \log(a + f_n(x_n)) \quad . \tag{5.3}$$

Since a (general) value function for a preference relation is unique only up to a strictly increasing transformation, the function $\exp V$ represents the same preferences as does the function V in (5.3). Thus, the additive value function (5.3) can be "rewritten" in multiplicative form as

$$V_m(x_1,\dots,x_n) = (a + x_1)(a + f_2(x_2)) \cdots (a + f_n(x_n)) \quad . \tag{5.4}$$

For a model of tradeoffs between different individuals or of tradeoffs between different periods, it may be appropriate to simplify the multiplicative value function (5.4) as follows.

(a) The conditions of equal tradeoffs amounts and relative tradeoffs independence are satisfied if and only if a value function for preferences is determined as

$$V_m(x_1,\dots,x_n) = (a + x_1)(a + x_2) \cdots (a + x_n) \quad . \tag{5.5}$$

(b) The conditions of equal tradeoffs midvalues and relative tradeoffs independence are satisfied if and only is a value function for preferences is determined as

$$V_m(x_1,\ldots,x_n) = (a + x_1)(a + x_2)^{a_2}\cdots(a + x_n)^{a_n} \tag{5.6}$$

where $a_2,\ldots,a_n$ are positive constants.

5.4 Linear tradeoffs functions

Consider the tradeoffs paradigm in this paper, that amounts x_1^0, x_j^0 are fixed and amounts x_1, x_j vary so that $(x_1, x_j^0) \sim (x_1^0, x_j)$. Then, x_1 is a continuous, strictly increasing function of x_j. As an extension of previous terminology, such a function will be denoted by $x_1 = f_j^0(x_j)$ and will be called a ***tradeoffs function***.

An especially simple form for a tradeoffs function is that of a linear function. Benefits assessment studies typically assume linearity of tradeoffs between monetary attributes (cost) and non-monetary attributes (effects). According to the following result, the use of linear tradeoffs functions is possible only in certain special types of models.

Theorem 5.4 For an additive-value model, suppose that for two variables x_1 and x_j and for any base amounts x_1^0, x_j^0, the tradeoffs function is linear, that is, $f_j^0(x_j) = x_1^0 + r^0(x_j - x_j^0)$ with possibly different r^0 for $x_j > x_j^0$ and $x_j < x_j^0$. Then, the model has one of the following special forms:

(I) Both variables x_1 and x_j satisfy the conditions of absolute tradeoffs independence. Preferences are represented by a value function of the form

$$V(x_1,\ldots,x_n) = x_1 + \cdots + rx_j + \cdots \tag{5.7}$$

for some constant $r > 0$. The tradeoffs function for base amounts x_1^0, x_j^0 is $f_j^0(x_j) = x_1^0 + r(x_j - x_j^0)$.

(II) Both variables x_1 and x_j satisfy the conditions of relative tradeoffs independence. Preferences are represented by a value function of the form

$$V(x_1,\ldots,x_n) = \log(x_1 - x_1^c) + \cdots + \log(x_j - x_j^c) + \cdots \tag{5.8}$$

for some constants x_1^c, x_j^c. The tradeoffs function for base amounts $x_1^0 x_j^0$ is $f_j^0(x_j) = x_1^0 + ((x_1^0 - x_1^c)/(x_j^0 - x_j^c))(x_j - x_j^0)$.

(III) Both variables x_1 and x_j satisfy the conditions of $\circ$-tradeoffs independence with respect to operations $x \circ x' = x^d - (x^d - x)(x^d - x')$. Preferences are represented by a value function of the form

$$V(x_1,\ldots,x_n) = -\log(x_1^d - x_1) + \cdots -\log(x_j^d - x_j) + \cdots \qquad (5.9)$$

for some constants x_1^d, x_j^d. The tradeoffs function for base amounts x_1^0, x_j^0 is $f_j^0(x_j) = x_1^0 + ((x_1^d - x_1^0)/(x_j^d - x_j^0))(x_j - x_j^0)$.

Note that, with respect to each of the variables x_1 and x_j as the primary variable, preferences in the model (I) are tradeoffs independent and inequity neutral, preferences in the model (II) are tradeoffs decreasing and inequity averse, and preferences in the model (III) are tradeoffs increasing and inequity prone.

The value function (5.7) for the model (I) typically has been assumed in models of tradeoffs between attributes, periods, or individuals. An important consideration for this choice has been that of tractability. For a study in which tradeoffs dependence or a concern for equity are important preference issues, the value function (5.8) should be viewed as an alternative choice that is similarly tractable. The implementation of a model (II) is discussed in Section 7. The case of tradeoffs between monetary position and risk of fatality due to a specified cause is discussed in Harvey (1985b).

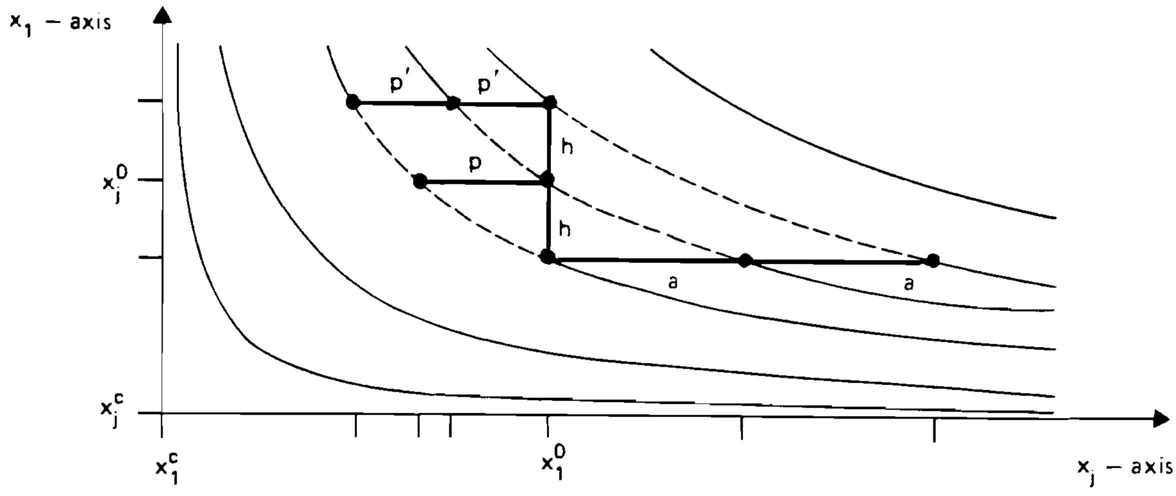


Figure 5.2 Illustration of a logarithmic value function for a model (II)

Preferences that are in accord with the tradeoffs model (II) can be characterised by a variety of lists of conditions. For an application to a benefits assessment study in which the variable x_1 measures a monetary attribute and the variable x_j measures a non-monetary attribute, the following list might be useful to consider. These conditions are illustrated in Figure 5.2.

Conditions characterising a tradeoffs model (II):

(i) For some amounts $x_j < x_j^0$ of the non-monetary variable and a base amount x_1^0 of the monetary variable, the willingness-to-pay amount p to improve from x_j to x_j^0 is less than the willingness-to-accept amount a to worsen from x_j^0 to x_j.

(ii) For some base amount x_1^0 of the monetary variable, and a fixed increase $h = x_j^0 - x_j$ in the non-monetary variable, the willingness-to-pay amount p to improve from x_j to x_j^0 is larger for smaller (i.e., more serious) amounts x_j.

(iii) For any base amounts x_1^0 and x_j^0, the tradeoffs function $f_j^0(x_j)$ is linear.

6. Conditions of Tradeoffs Constancy

This section discusses a family of conditions on tradeoffs, each of which implies that the component function v belongs to a parametric family of functions. Each condition is a weakening of a corresponding condition of tradeoffs independence.

The implications in this section depend upon the following mathematical result concerning a functional equation (Harvey, 1987, Appendix: proof of Theorem 5).

Proposition 6.1 Suppose that a real-valued function v is continuous and strictly increasing on a non-point interval I. Then, the following are equivalent:

(a) There exists an amount $\delta > 0$ such that for any amounts $x < y < z$ in I, if $v(y) = \frac{1}{2}(v(x) + v(z))$ and $v(z) - v(x) \leq \delta$, then

$$v(y + h) = \tfrac{1}{2}(v(x + h) + v(z + h))$$

for any change h such that $x + h$, $z + h$ are in I and $v(z + h) - v(x + h) \leq \delta$.

(b) The function v is of the linear-exponential form

$$v(x) = \begin{cases} ae^{rx} + b, & r > 0 \\ ax + b, & r = 0 \\ -ae^{rx} + b, & r < 0 \end{cases}$$

for some parameter value r and some constants $a > 0$ and b.

6.1 Absolute tradeoffs constancy

For tradeoffs between attributes, the condition (c) below is the *delta property* introduced in Kirkwood and Sarin (1980) and the condition (a) is that of *absolute pricing-out amounts* introduced in Harvey (1985a). Both conditions are discussed for tradeoffs between individuals in Harvey (1985b, c), and are discussed for tradeoffs between periods in Harvey (1986a, b). Each of the definitions (a) and (c) given here differs slightly from previous definitions in that the class of consequences to be considered is somewhat smaller and thus the task of verifying the condition is somewhat simpler.

For the preference issue of attitude toward risk, Harvey (1987) discusses two conditions called *c.* absolute risk constancy and *g.* absolute risk constancy that are analogous to the conditions (a) and (c) respectively.

Definition 6.1 Four conditions of *absolute tradeoffs constancy* between x_1 and another variable x_j are as follows:

(a) *Absolute tradeoffs amounts constancy.* Suppose that two pairs of amounts x_1^*, x_1' and x_1^0, x_1 have a common tradeoffs pair of the form x_j^*, x_j. Then, for any amount h the two pairs of amounts $x_1^* + h$, $x_1' + h$ and $x_1^0 + h$, $x_1 + h$ have a common tradeoffs pair x_j, x_j' (whenever either has a tradeoffs pair).

(b) *Absolute tradeoffs willingness constancy.* Suppose that a pair of consequences (x_1^0, x_j) and $(x_1^0 - p, x_j^0)$ are indifferent and a pair of consequences (x_1^0, x_j^0) and $(x_1^0 + a, x_j)$ are indifferent. Then, for any amounts x_1 and x_j', x_j'', $(x_1, x_j') \sim (x_1 - p, x_j'')$ if and only if $(x_1, x_j'') \sim (x_1 + a, x_j')$.

(c) *Absolute tradeoffs midvalues constancy.* Suppose that two amounts x_1 and x_1' have a tradeoffs midvalue of $\hat{x}_1$ with respect to a tradeoffs pair of the form x_j^*, x_j. Then, for any amount h, the two amounts $x_1 + h$ and $x_1' + h$ have the tradeoffs midvalue $\hat{x}_1 + h$ (whenever there is a tradeoffs midvalue).

(d) *Absolute tradeoffs changes constancy.* Suppose that a pair of consequences (x_1^0, x_j) and (x_1, x_j^0) are indifferent. If $\tilde{x}_j$ and $\tilde{x}_j^0$ are two amounts such that $f_j(\tilde{x}_j) = f_j(x_j) + h$ and $f_j(\tilde{x}_j^0) = f_j(x_j^0) + h$ for the same absolute change h, then the pair of consequences $(x_1^0 + h, \tilde{x}_j)$ and $(x_1 + h, \tilde{x}_j^0)$ are indifferent.

These conditions are illustrated in Figure 6.1 with indifferences denoted by dashed lines. Each condition is a weakening of the corresponding condition of absolute tradeoffs independence illustrated in Figure 5.1.

The condition (d) can be viewed as a preference condition on equal absolute changes in amounts: If $(x_1^0, x_j) \sim (x_1, x_j^0)$, then there is indifference between the consequences obtained by changing all four amounts by the same absolute change h measured according to the first variable. For example, suppose that the variables x_i, $i = 1,\ldots,n$, measure net-benefits to different individuals and the condition of equal tradeoffs amounts is satisfied. Then, condition (d) states that if $(x_i^0, x_j) \sim (x_i, x_j^0)$, then there is indifference also for any additional net-benefit h to the i and j individuals, that is, $(x_i^0 + h, x_j + h) \sim (x_i + h, x_j^0 + h)$.

In Theorem 6.1 below, the equivalence of (c) and (e) is established in Kirkwood and Sarin (1980), and the equivalence of (a) and (e) is stated in Harvey (1985a) and proved in Harvey (1981). The above papers make the unneeded assumptions that the component function v is twice continuously differentiable and that the first derivative v' is positive.

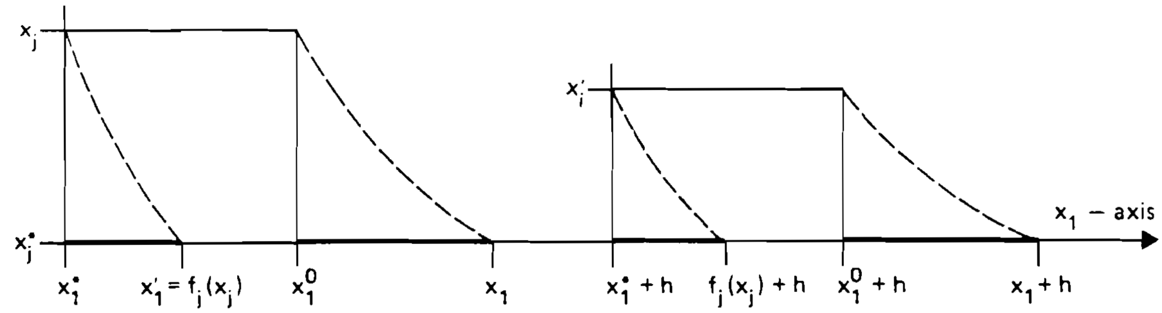
Theorem 6.1 For a standard additive-value model, the conditions (a)-(d) of absolute tradeoffs constancy are equivalent to each other and to the property:

(e) There exists a value function of the form

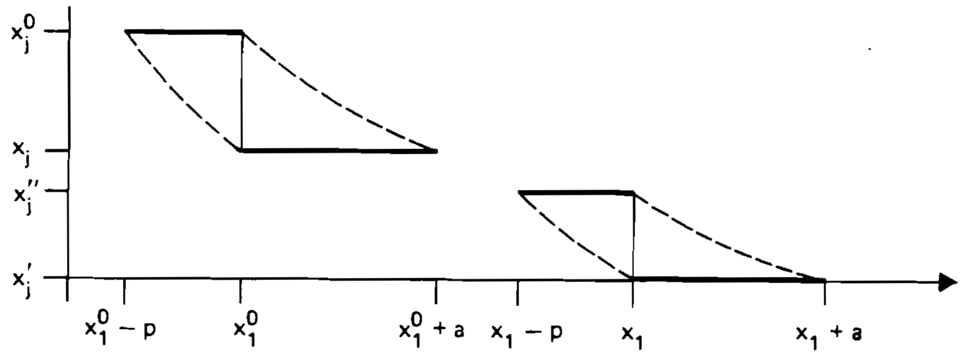
$$V(x_1,\ldots,x_n) = \begin{cases} \exp(rx_1) + \Sigma_{k=2}^{n}\exp(rf_k(x_k)), & r > 0 \\ x_1 + \Sigma_{k=2}^{n} f_k(x_k), & r = 0 \\ -\exp(rx_1) - \Sigma_{k=2}^{n}\exp(rf_k(x_k)), & r < 0 \end{cases} \qquad (6.1)$$

for some amount of the parameter r.

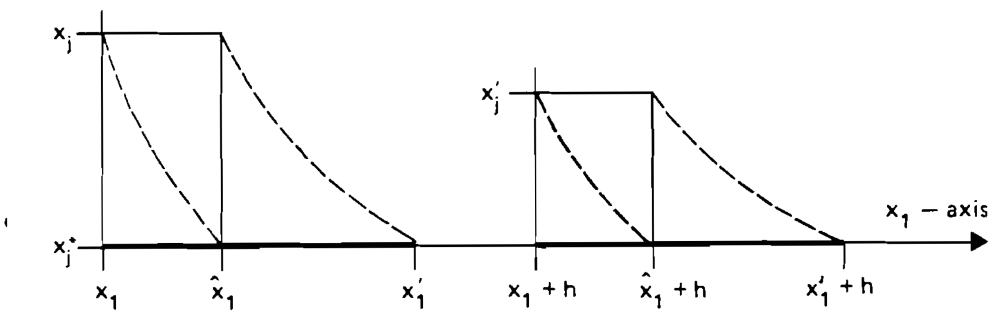
An immediate corollary of Theorem 6.1 is that if tradeoffs between x_1 and another variable x_j satisfy the conditions of absolute tradeoffs constancy, then tradeoffs between x_1 and any variable x_k, $k = 2,\ldots,n$, satisfy the conditions of absolute tradeoffs constancy.



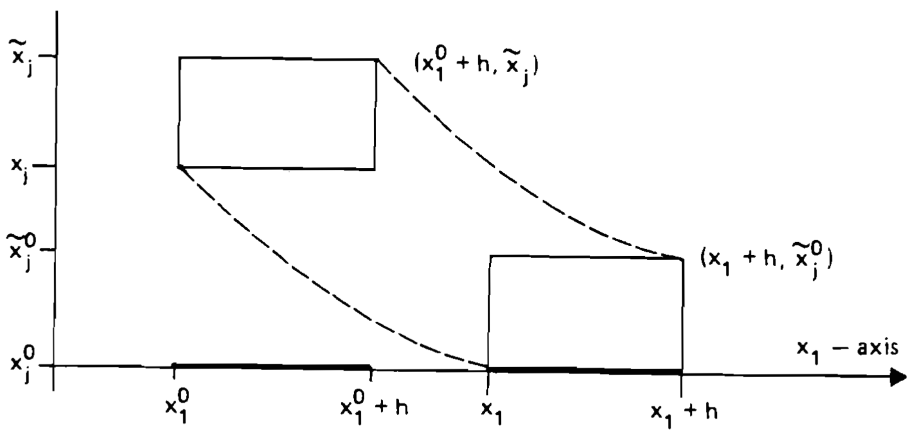
(a) Absolute tradeoffs amounts constancy



(b) Absolute tradeoffs willingness constancy



(c) Absolute tradeoffs midvalues constancy



(d) Absolute tradeoffs changes constancy

Figure 6.1 Illustrations of absolute tradeoffs constancy

For a model of tradeoffs between individuals or between periods, it may be appropriate to simplify the value function (6.1) as follows. Weaker versions of parts of this result are stated without proof in Harvey (1985b, c), (1986a, b).

Corollary 6.1 For a standard additive-value model:

(a) The conditions of equal tradeoffs amounts and absolute tradeoffs constancy are satisfied if and only if v in (4.1) has the linear-exponential form in (6.1).

(b) The conditions of equal tradeoffs midvalues and absolute tradeoffs constancy are satisfied if and only if v in (4.2) has the linear-exponential form in (6.1).

6.2 $\circ$-tradeoffs constancy

Suppose that $x \circ x'$ denotes an arithmetic operation as described in Section 5.2.

Definition 6.2 Four conditions of $\circ$-*tradeoffs constancy* between x_1 and another variable x_j can be obtained by replacing the + operations in parts (a)-(d) of Definition 6.1 by $\circ$ operations. The resulting conditions will be called: (a) $\circ$-*tradeoffs amounts constancy*, (b) $\circ$-*tradeoffs willingness constancy*, (c) $\circ$-*tradeoffs midvalues constancy*, and (d) $\circ$-*tradeoffs changes constancy*.

These conditions can be illustrated by replacing the + operations in Figure 6.1 by $\circ$ operations.

Theorem 6.2 For a standard additive-value model, the conditions (a)-(d) of $\circ$-tradeoffs constancy are equivalent to each other and to the property:

(e) There exists a value function of the form

$$V(x_1,\ldots,x_n) = \begin{cases} \exp(rg(x_1)) + \Sigma_{k=2}^{n}\exp(rg(f_k(x_k))), & r > 0 \\ g(x_1) + \Sigma_{k=2}^{n} g(f_k(x_k)), & r = 0 \\ -\exp(rg(x_1)) - \Sigma_{k=2}^{n}\exp(rg(f_k(x_k))), & r < 0 \end{cases} \qquad (6.2)$$

where g is any scaling function for $x \circ x'$ and r is a parameter amount.

6.3 Relative Tradeoffs constancy

A primary type of o-tradeoffs constancy is that which corresponds to the condition of relative tradeoffs independence discussed in Section 5.3. The interpretation there of x_1 as a measure of changes and of $y_1 = a + x_1$ as a measure of consequent positions is used in the following discussion.

The condition (c) below is the ***proportional delta property*** introduced in Kirkwood and Sarin (1980) and the condition (a) below is that of ***relative pricing-out amounts*** introduced in Harvey (1985a). Both conditions are also discussed in Harvey (1985b, c), (1986a, b). For the preference issue of attitudes toward risk, Harvey (1987) discusses conditions called *c.* relative risk constancy and *g.* relative risk constancy that are analogous to the conditions (a) and (c) respectively.

Definition 6.3 Four conditions of *relative tradeoffs constancy* between x_1 and another variable x_j are as follows:

(a) *Relative tradeoffs amounts constancy.* Suppose that two pairs of amounts x_1^*, x_1' and x_1^0, x_1 have a common tradeoffs pair of the form x_j^*, x_j. Then, for any percent $m > -1$, the two pairs of amounts $x_1^* + m(a + x_1^*)$, $x_1' + m(a + x_1')$ and $x_1^0 + m(a + x_1^0)$, $x_1 + m(a + x_1)$ have a common tradeoffs pair x_j', x_j'' whenever either has a tradeoffs pair.

(b) *Relative tradeoffs willingness constancy.* Suppose that a pair of consequences (x_1^0, x_j) and $(x_1^0 - \frac{m}{m+1}(a + x_1^0), x_j^0)$ are indifferent and a pair of consequences (x_1^0, x_j^0) and $(x_1^0 + m'(a + x_1^0), x_j)$ are indifferent. Then, for any amounts x_1 and x_j', x_j'', $(x_1, x_j') \sim (x_1 - \frac{m}{m+1}(a + x_1), x_j'')$ if and only if $(x_1, x_j'') \sim (x_1 + m'(a + m_1), x_j')$.

(c) *Relative tradeoffs midvalues constancy.* Suppose that two amounts x_1 and x_1' have a tradeoffs midvalue of $\hat{x}_1$ with respect to a tradeoffs pair of the form x_j^*, x_j. Then, for any percent $m > -1$, the two amounts $x_1 + m(a + x_1)$ and $x_1' + m(a + x_1')$ have the tradeoffs midvalue $\hat{x}_1 + m(a + \hat{x}_1)$ whenever there is a tradeoffs midvalue.

(d) *Relative tradeoffs changes constancy.* Suppose that a pair of consequences (x_1^0, x_j) and (x_1, x_j^0) are indifferent. If $\tilde{x}_j$ and $\tilde{x}_j^0$ are two amounts such that $f_j(\tilde{x}_j) = f_j(x_j) + m(a + f_j(x_j))$ and $f_j(\tilde{x}_j^0) = f_j(x_j^0) + m(a + f_j(x_j^0))$ for the same percent $m > -1$, then the pair of consequences $(x_1^0 + m(a + x_1^0), \tilde{x}_j)$ and $(x_1 + m(a + x_1), \tilde{x}_j^0)$ are indifferent.

The condition (d) can be viewed as a preference condition on equal relative changes in amounts. In particular, if the condition of equal tradeoffs amounts is satisfied, then (d) states that whenever two consequences (x_1^0, x_j) and (x_i, x_j^0) are indifferent, then so are any consequences obtained by modifying the four amounts by the same percent m.

In Theorem 6.3 below, the equivalence of (c) and (e) is established in Kirkwood and Sarin (1980), and the equivalence of (a) and (e) is stated in Harvey (1985a) and proved in Harvey (1981). These papers make unneeded assumptions as noted for Theorem 6.1 regarding the derivatives of the component function v.

Theorem 6.3 For a standard additive-value model, the conditions (a)-(d) of relative tradeoffs constancy equivalent to each other and to the property:

(e) There exists a value function of the form

$$V(x_1,\ldots,x_{}) = \begin{cases} (a + x_1)^r + \Sigma_{k=2}^{n}(a + f_k(x_k))^r , & r > 0 \\ \log(a + x_1) + \Sigma_{k=2}^{n}\log(a + f_k(x_k)) , & r = 0 \\ -(a + x_1)^r - \Sigma_{k=2}^{n}(a + f_k(x_k))^r , & r < 0 \end{cases} \quad (6.3)$$

for some amount of the parameter r.

For a model of tradeoffs between individuals or between periods, it may be appropriate to simplify the value function (6.3) as follows. Weaker versions of parts of this result are stated without proof in Harvey (1985b, c), (1986a, b).

(a) The conditions of equal tradeoffs amounts and relative tradeoffs constancy are satisfied if and only if v in (4.1) has the logarithm-power form in (6.3).

(b) The conditions of equal tradeoffs midvalues and relative tradeoffs constancy are satisfied if and only if v in (4.2) has the logarithm-power form in (6.3).

7. Suggestions for Implementation

This section describes two contrasting procedures for the application of a model of tradeoffs between different attributes. In practice, a combination of these procedures can be used. Similar procedures are possible for tradeoffs between different periods and for tradeoffs between different individuals.

It is assumed that the decision problem has been bounded, that the alternative plans or policies have been specified, that the criteria or objectives for comparing the alternatives have been identified, and that the effects of each alternative on the objectives have been quantified as a vector of variable amounts (see, e.g., Keeney, 1982 and Cox, 1986). The question to be discussed is how a modeling of tradeoffs between the variables can provide insight into the preference side of the decision problem.

As an illustration, imagine that a model is to be developed as part of a planning study for air pollution control. Suppose that the tradeoffs between monetary position as measured by a variable x_1 and air quality or one of its effects as measured by a variable x_j depend crucially on the status quo position of x_1. In the terminology of Definition 3.1, suppose that preferences are tradeoffs decreasing. For example, tradeoffs may depend on the current strength of the economy in the region.

Outlines of two procedures for modeling tradeoffs between a variable x_1 and variables x_j, $j = 2,\dots,n$, are as follows.

7.1 Direct Method

The preferred alternative consequence can be determined by steps (i)-(iv) below. Each of steps (i)-(iii) is to involve an appropriate sample of indifference assessments.

(i) Verify the conditions of an additive-value model, in particular, one of conditions of independence and the condition of framing consistency.

(ii) Choose a standard amount x_1^* and standard amount(s) x_j^* that are convenient to consider, and evaluate the standard tradeoffs function(s) $f_j(x_j)$.

(iii) Verify one of the special conditions of Definitions 5.3, 6.1, and 6.3, and evaluate the parameter(s) in the resulting value function. Here, it may be appropriate to assess both willingness-to-pay and willingness-to-accept amounts for a sample of base monetary amounts.

(iv) Use the value function thus determined to compare the alternative consequences.

The procedures below consider only the relative tradeoffs independence model (II). Similar, somewhat less simple, procedures are possible for the other special tradeoffs models.

7.2 Indirect Method

The implications of different tradeoffs and different degrees of tradeoffs dependency can be determined by steps (i)-(iv) below. Each of steps (i), (ii) involves a judgment that a preference condition to be used does not exclude an important preference issue.

(i) Verify the appropriateness of the preference conditions of an additive-value model.

(ii) Verify the appropriateness of linear tradeoffs function(s). Ask questions of an "if-then" type: for example, if $(x_1^0, x_j^0 - h) \sim (x_1^0 - p, x_j^0)$, then is it reasonable that also $(x_1^0, x_j^0 - 2h) \sim (x_1^0 - 2p, x_j^0)$?

(iii) Choose a standard amount x_1^* and standard amount(s) x_j^*. For each variable $j = 2,...n$, choose an amount $x_j \neq x_j^*$ and choose (but do not assess) an indifference comparison such as $(x_1^* - p_j, x_j^*) \sim (x_1^*, x_j)$. The comparison is to represent tradeoffs between x_1 and x_j. In addition, choose (but do not assess) a single indifference comparison such as $(x_1^*, x_k^*) \sim (x_1^* + a, x_k)$. This comparison is to represent the degree of tradeoffs dependence.

(iv) For any hypothetical assessment of the above comparisons, i.e., for amounts p_j and a, calculate first the resulting value function and then the preferred alternative consequence. Report in a convenient format which ranges of the tradeoffs amounts p_j and a imply which of the consequences is preferred.

Appendix: Proofs of Results

Proof of Theorem 2.1. This result is a corollary of, for example, Theorems 3.3 and 3.6 in Fishburn (1970). Conditions (A)-(C) can be shown to imply conditions 1.-3. in Theorem 3.3. Thus, it follows from Theorems 3.3 and 3.6 that there exists a jointly continuous value function v. Moreover, condition (B) implies that v is strictly increasing in each variable. The converse implications are straightforward to verify.

Proof of Theorem 2.2. This result is a part of Theorem 3 in Debreu (1960) since the first part of the proof of that result is to establish the condition of complementary tradeoffs independence and the second part of the proof of that result is to use complementary tradeoffs independence to construct an additive value function.

Proof of Theorem 2.3. This result is a restatement of Theorem 3 in Debreu (1960) for the special case in which each variable x_i is defined on a non-point interval I_i and larger values of each variable are preferred.

Proof of Theorem 3.1. Consider a value function (2.1) for an additive-value model. Assume that $h \neq 0$ and $x_1^0 \neq x_1$.

Preferences are tradeoffs independent if and only if for any fixed h sufficiently near to zero the difference $v_1(x_1^0 + h) - v_1(x_1^0)$ is constant for all x_1^0 such that x_1^0 and $x_1^0 + h$ are in the interval I_1. This property holds if and only if v_1 is linear on I_1. Preferences are tradeoffs decreasing if and only if for any fixed h sufficiently near to zero the difference $|v_1(x_1^0 + h) - v_1(x_1^0)|$ is strictly decreasing for all x_1^0 such that x_1^0 and $x_1^0 + h$ are in I_1. This property holds if and only if v_1 is strictly concave on I_1.

Preferences are inequity neutral if and only if for any x_1^0 and x_1 sufficiently near to each other the average $\bar{x}_1$ is the tradeoffs midvalue of x_1^0 and x_1. Then, $v_1(x_1) = \frac{1}{2}v_1(x_1^0) + \frac{1}{2}v_1(x_1)$. This property holds if and only if v_1 is linear on I_1. Preferences are inequity averse if and only if for any x_1^0 and x_1 sufficiently near to each other $\bar{x}_1$ is greater than the tradeoffs midvalue of x_1^0 and x_1. Then, $v_1(\bar{x}_1) > \frac{1}{2}v_1(x_1^0) + \frac{1}{2}v_1(x_1)$. This property holds if and only if v_1 is strictly concave on I_1.

Proof of Proposition 4.1 For an additive-value model, consider the ranges $v_i(I_i)$ of the component functions v_i, $i = 1,\ldots,n$. An interval $v_i(I_i)$ will be called as *large* as another interval $v_j(I_j)$ provided that $v_j(I_j) + c \subset v_i(I_i)$ for some constant c. Since the intervals $v_i(I_i)$ are closed, this ordering is transitive and complete. Thus, it is possible to relabel the variables if necessary so that

$$v_n(I_n) + b_n \subset v_{n-1}(I_{n-1}) + b_{n-1} \subset \cdots \subset v_1(I_1)$$

for some constants b_j, $j \neq 1$. Choose $x_1^* = v_1^{-1}(v_1^*)$ where v_1^* is any point in $v_n(I_n) + b_n$, and choose $x_j^* = v_j^{-1}(v_1^* - b_j)$, $j \neq 1$. Then, $v_j(x_j^*) = v_1(x_1^*) - b_j$, $j \neq 1$. For any variable $j \neq 1$ and any amount x_j in I_j, the point $v_j(x_j) + b_j$ is in $v_1(I_1)$, and thus $v_j(x_j) + b_j = v_1(x_1)$ for some x_1 in I_1. Thus, $v_j(x_j) - v_j(x_j^*) = v_1(x_1) - v_1(x_1^*)$, and so x_j has the standard tradeoffs amount x_1.

Proof of Theorem 4.3. Suppose that an additive-value model is a standard additive-value model with respect to the variable x_1 and the standard amounts $x_1^*,\ldots,x_n^*$. Then, for any variable $j \neq 1$ and any amount x_j in the interval I_j, there exists a standard tradeoffs amount $x_1 = f_j(x_j)$. Thus, $(x_1^*, x_j) \sim (f_j(x_j), x_j^*)$, and so $v_1(x_1^*) + v_j(x_j) = v_1(f_j(x_j)) + v_j(x_j^*)$. Then, $v_j(x_j) = v_1(f_j(x_j)) + b_j$ where $b_j = v_j(x_j^*) - v_1(x_1^*)$. Therefore, (4.3) with $v = v_1$ is a value function for the model. It is straightforward to verify that, conversely, if (4.3) is a value function as described, then the model is a standard additive-value model.

Suppose that in a standard additive-value model the standard amounts are equal, $x_i^* = x_0^*$, $i = 1,\ldots,n$, and the intervals I_i are equal, $I_i = I_0$, $i = 1,\ldots,n$. Then, (2.1) implies that $v_1(x_0^*) + v_j(x_j) = v_1(f_j(x_j)) + v_j(x_0^*)$ for any variable $j \neq 1$ and any amount x_j in I_0. If there are equal tradeoffs amounts, then choose $v_j = v_1 = v$, $j \neq 1$, as in (4.1). Thus, $f_j(x_j) = x_j$, $j \neq 1$. If there are equal tradeoffs midvalues, then choose $v_j = a_j v_1 = a_j v$, $j \neq 1$, as in (4.2). Thus, the functions $f_j(x_j)$, $j \neq 1$, are as specified in (4.4.).

Proof of Theorem 5.1. Suppose that the component function v in (4.3) has been normalized so that $v(x_1^*) = 0$. Define the function w by $w(y) = v(x_1^* + y)$.

Assume condition (a). Then, $v(f_j(x_j) + h) = v(x_1^* + h) + v(f_j(x_j))$ for any amounts h and x_j such that the functions v are defined. Thus, $v(x_1^* + h + h') = v(x_1^* + h) + v(x_1^* + j')$ for $h' = f_j(x_j) - x_1^*$. Therefore, $w(h + h') = w(h) + w(h')$ for any amounts h in $I_1 - x_1^*$ and h' in $f_j(I_j) - x_1^*$ such that $h + h'$ is in $I_1 - x_1^*$. This implies Cauchy's functional equation for the interval $I_1 - x_1^*$. Since w is continuous and strictly increasing, it follows that $w(y) = ay$ for some constant $a > 0$ (see, e.g., Aczél, 1966, p. 46). Then, $v(x_1) = a(x_1 - x_1^*)$, and thus by a positive linear transformation, (1.3) is a value function.

Condition (b) implies condition (c) as follows. If an amount $\hat{x}_1$ is the tradeoffs midvalue of two amounts x_1 and x_1', then by (b) $x_1 = \hat{x}_1 - p$ and $x_1' = \hat{x}_1 + a$ where $p = a$, and hence $\hat{x}_1 = x_1 + h$, $x_1' = \hat{x}_1 + h$ for the same amount h.

Assume condition (c). Then, $v(\hat{x}_1 - h) + v(f_j(x_j)) = v(\hat{x}_1)$ and $v(\hat{x}_1) + v(f_j(x_j)) = v(\hat{x}_1 + h)$ for any amounts $\hat{x}_1, h$, and x_j such that the functions v are defined. Thus, $2v(\hat{x}_1) = v(\hat{x}_1 - h) + v(\hat{x}_1 + h)$ for any amounts $\hat{x}_1$ and h such that $\hat{x}_1 - h$ and $\hat{x}_1 + h$ are in I_1 and $v(\hat{x}_1 + h) - v(\hat{x}_1)$ and

$v(\hat{x}_1) - v(\hat{x}_1 - h)$ are in $v(f_j(I_j))$. This implies Jensen's functional equation for the interval I_1. Since v is continuous, it follows that v is linear (see, e.g., Aczél, 1966, p. 46). Then, $v(x_1^*) = 0$ and v strictly increasing implies that $v(x_1) = a(x_1 - x_1^*)$ for some constant $a > 0$, and thus (1.3) is a value function.

Condition (d) implies condition (a) as follows. If an amount x_j has a tradeoffs amount of y_1 with respect to base amounts $x_1^0 = x_1^* + h$ and $x_j^0 = x_j^*$, then in (d) choose $c_1 = c_2 = (x_1^* = h, x_j^*)$ and $c_3 = (x_1^* + k, x_j^*) \sim c_4 = (x_1^*, x_j)$. Then by (d), $(x_1^* + h + k, x_j^*) \sim (x_1^* + h, x_j)$, and so $x_1^* + k = f_j(x_j)$ implies $y_1 = f_j(x_j) + h$.

It is straightforward to verify that, conversely, if (1.3) is a value function, then each of the conditions (a)-(d) is satisfied.

Proof of Corollary 5.1. The conditions of absolute tradeoffs independence imply an additive value function (2.1) with $v_1(x_1) = x_1$. In part (a), the condition of equal tradeoffs amounts implies that there is a value function (4.1) with $v(x_1) = v_1(x_1) = x_1$ and hence a value function (1.1). In part (b), the condition of equal tradeoffs midvalues implies that there is a value function (4.2) with $v(x_1) = v_1(x_1) = x_1$ and hence a value function (1.2). Then, formula (4.4), with v normalized so that $v(x_j) = x_j - x_j^*$, implies (5.1). The converse implications are immediate.

Proof of Theorem 5.2. Let $y = g(x)$ denote a scaling function for the group operation $x_1 \circ x_1'$, and let C_y denote the set of vectors $(y_1, x_2, \ldots, x_n)$ such that $y_1 = g(x_1)$ is in the interval $g(I_1)$ and $x_2, \ldots, x_n$ are in the intervals $I_2, \ldots, I_n$. The preference relation $\succsim$ on the set C of consequences $(x_1, \ldots, x_n)$ defines a preference relation $\succsim_y$ on the set C_y by: $(y_1, x_2, \ldots, x_n) \succsim_y (y_1', x_2', \ldots, x_n')$ if and only if $(x_1, x_2, \ldots, x_n) \succsim (x_1', x_2', \ldots, x_n')$ where $y_1 = g(x_1)$ and $y_1' = g(x_1')$. Suppose that $V(x_1, \ldots, x_n)$ is a value function of the form (4.3) for the preference relation $\succsim$. Then, the function

$$V_y(y_1, x_2, \ldots, x_n) = v(g^{-1}(y_1)) + v(f_2(x_2)) + \cdots + v(f_n(x_n))$$

is a value function for the preference relation $\succsim_y$. It is straightforward to verify that C_y and $\succsim_y$ form a standard additive-value model with standard amounts $y_1^* = g(x_1^*)$ and $x_2^*, \ldots, x_n^*$.

Each of four conditions of $\circ$-tradeoffs independence holds for $(C,\succsim)$ if and only if the corresponding condition of absolute tradeoffs independence holds for $(C_y,\succsim_g)$. By applying Theorem 5.1 to $(C_y,\succsim_y)$, each of the conditions of absolute tradeoffs independence holds for $(C_y,\succsim_y)$ if and only if $\succsim_y$ is represented by an additive value function with $v_1(y_1) = y_1$.

If $\succsim_y$ has such a value function, then there exists a value function V_y above such that $v(g^{-1}(y_1)) = v_1(y_1) = y_1$. Then, $v(x) = g(x)$, and thus the corresponding value function V is of the form (5.2). Conversely, if there exists a value function V of the form (5.2), and hence with $v(x) = g(x)$, then the component function $v_1(y_1)$ in the corresponding value function V_y is $v_1(y_1) = v(g^{-1}(y_1)) = y_1$.

Proof of Theorem 5.3. Let $y_1 = \alpha + x_1$ with x_1 in I_1 define a change of variable for $(C,\succsim)$. As in the proof of Theorem 5.2, a set C_y of vectors $(y_1,x_2,\ldots,x_n)$ can be defined from C, and a preference relation $\succsim_y$ on these vectors can be defined from $\succsim$. If $V(x_1,\ldots,x_n)$ is a value function of the form (4.3) for $(C,\succsim)$, then

$$V_y(y_1, x_2,\ldots,x_n) = v(y_1 - \alpha) + v(f_2(x_2)) + \cdots + v(f_n(x_n))$$

is a value function for $(C_y,\succsim_y)$.

Each of the conditions (a)-(d) of relative tradeoffs independence holds for $(C,\succsim)$ if and only if the corresponding condition of $\circ$-tradeoffs independence, where the group operation $y \circ y'$ is that of multiplication, holds for $(C_y,\succsim_y)$. By applying Theorem 5.2 with the scaling function $g(y) = \log y$ to $(C_y, \succsim_y)$, each of the conditions of $\circ$-tradeoffs independence holds if and only if $\succsim_y$ is represented by an additive value function with $v_1(y_1) = \log y_1$.

If $\succsim_y$ has such a value function, then there exists a value function V_y above such that $v(y_1 - \alpha) = v_1(y_1) = \log y_1$. Then, $v(x) = \log(\alpha + x)$, and thus the corresponding value function V is of the form (5.3). Conversely, if there exists a value function V of the form (5.3), and hence with $v(x) = \log(\alpha + x)$, then the component function $v_1(y_1)$ in the corresponding value function V_y is $v_1(y_1) = v(y_1 - \alpha) = \log y_1$.

Proof of Theorem 5.4 Assume that the variables x_1, x_j are exchanged if necessary so that there exist standard amounts x_1^*, x_j^* with $f_j(I_j) \subset I_1$ (see proof of Proposition 4.1). By assumption, the standard tradeoffs function f_j is linear, $f_j(x_j) = x_1^* + r(x_j - x_j^*)$. The variable x_j will be replaced by the variable $y_j = f_j(x_j)$. Then, $v_j(x_j) = v(y_j)$ and $v_1(x_1) = v(x_1)$ for a common component function v. The linearity of tradeoffs between x_1 and x_j implies the linearity of tradeoffs between x_1 and y_j.

Choose amounts $x_1 = y_j = x_{(0)}$ and $x_1' = y_1' = x_{(1)}$ in $f_j(I_j)$ such that $x_{(0)} < x_{(1)}$. Then, $[x_1, x_1'] \times [y_j\, y_j']$ is a "preference square" in that $(x_1, y_j') \sim (x_1', y_j)$. Since the variables x_1 and y_j have the same component function, the tradeoffs midvalues of x_1, x_1' and of y_j, y_j' are equal. Let $x_{(\frac{1}{2})}$ denote this common amount. Then, $x_{(0)} < x_{(\frac{1}{2})} < x_{(1)}$.

In a similar fashion, we can consider the tradeoffs midvalue $x_{(\frac{1}{4})}$ of $x_{(0)}$, $x_{(\frac{1}{2})}$ and the tradeoffs midvalue $x_{(\frac{3}{4})}$ of $x_{(\frac{1}{2})}$, $x_{(1)}$. The linearity of the tradeoffs function for base amounts $x_1^0 = x_{(\frac{1}{2})}$ and $y_j^0 = x_{(0)}$ implies that

$$\frac{x_{(\frac{1}{4})} - x_{(0)}}{x_{(\frac{1}{2})} - x_{(0)}} = \frac{x_{(\frac{3}{4})} - x_{(\frac{1}{2})}}{x_{(1)} - x_{(\frac{1}{2})}} \quad . \tag{A1}$$

Moreover, the linearity of the tradeoffs function for base amounts $x_1^0 = x_{(\frac{1}{4})}$ and $x_j^0 = x_{(0)}$ implies that

$$\frac{x_{(\frac{1}{4})} - x_{(0)}}{x_{(\frac{1}{2})} - x_{(0)}} = \frac{x_{(\frac{1}{2})} - x_{(\frac{1}{4})}}{x_{(\frac{3}{4})} - x_{(\frac{1}{4})}} \quad . \tag{A2}$$

The equations (A1) and (A2) will be shown to determine $x_{(\frac{1}{4})}$ and $x_{(\frac{3}{4})}$.

The parts (I)-(III) of Theorem 5.4 correspond to the three cases $x_{(\frac{1}{2})} = \bar{x}$, $x_{(\frac{1}{2})} < \bar{x}$, and $x_{(\frac{1}{2})} > \bar{x}$ where $\bar{x} = \frac{1}{2}(x_{(0)} + x_{(1)})$. First, suppose that $x_{(\frac{1}{2})} = \bar{x}$. Then, (A1) implies that $x_{(\frac{3}{4})} - x_{(\frac{1}{4})} = x_{(\frac{1}{2})} - x_{(0)}$, and therefore (A2) implies that $x_{(\frac{1}{4})} - x_{(0)} = x_{(\frac{1}{2})} - x_{(\frac{1}{4})}$. Thus, $x_{(\frac{1}{4})} = \frac{1}{2}(x_{(0)} + x_{(\frac{1}{2})})$ and $x_{(\frac{3}{4})} = \frac{1}{2}(x_{(\frac{1}{2})} + x_{(1)})$.

By iteration of this argument, a midvalue $x_{(p)}$ satisfies the formula

$$x_{(p)} = p x_{(1)} + (1-p) x_{(0)} \tag{A3}$$

for any dyadic number $0 \le p \le 1$. If v is normalized so that $v(x_{(0)}) = 0$ and $v(x_{(1)}) = 1$, then $v(x_{(p)}) = p$ for any dyadic number $0 \le p \le 1$ with $x_{(p)}$ as specified by (A3). Thus,

$$v(x) = x_{(0)} + \frac{x - x_{(0)}}{x_{(1)} - x_{(0)}} \quad \text{for } x_{(0)} \le x \le x_{(1)} \ . \tag{A4}$$

If the interval $[x_{(0)}, x_{(1)}]$ is a proper subinterval of I_1, then by similar arguments the form (A4) of $v(x)$ can be shown to hold for all x in I_1. Now, consider the component function, $v_j(x_j) = v(y_j) = v(x_1^* + r(x_j - x_j^*))$. By renormalization of $v_1(x_1)$ and $v_j(x_j)$, there exist component functions $v_1(x_1) = x_1$ and $v_j(x_j) = r x_j$ as in (5.7). The value function (5.7) implies that the tradeoffs function for base amounts x_1^0, x_j^0 is $f_j^0(x_j) = x_1^0 + r(x_j - x_j^0)$.

Second, suppose that $x_{(\frac{1}{2})} < \bar{x}$. Then, (A1) implies that $x_{(\frac{3}{4})} = x_{(\frac{1}{2})} + (x_{(\frac{1}{4})} - x_{(0)})(x_{(1)} - x_{(\frac{1}{2})}) / (x_{(\frac{1}{2})} - x_{(0)})$, and therefore (A2) implies that

$$(x_{(\frac{1}{2})} - x_{(\frac{1}{4})}) + (x_{(\frac{1}{4})} - x_{(0)}) \frac{x_{(1)} - x_{(\frac{1}{2})}}{x_{(\frac{1}{2})} - x_{(0)}} = (x_{(\frac{1}{2})} - x_{(0)}) \frac{x_{(\frac{1}{2})} - x_{(\frac{1}{4})}}{x_{(\frac{1}{4})} - x_{(0)}} \ .$$

Thus, $(x_{(\frac{1}{2})} - x_{(0)})^{\frac{1}{2}} (x_{(\frac{1}{2})} - x_{(\frac{1}{4})}) = (x_{(1)} - x_{(\frac{1}{2})})^{\frac{1}{2}} (x_{(\frac{1}{4})} - x_{(0)})$. Since the solution $x_{(\frac{1}{4})}$ of this equation is unique, the solution $x_{(\frac{1}{4})}$, $x_{(\frac{3}{4})}$ of (A1), (A2) is unique.

Now observe that since $x_{(0)} < x_{(\frac{1}{2})} < \bar{x}$, there exists a unique number $x^c < x_{(0)}$ such that $x_{(\frac{1}{2})}$ satisfies the formula,

$$x_{(p)} = x^c + (x_{(1)} - x^c)^p (x_{(0)} - x^c)^{1-p} \ . \tag{A5}$$

It may be verified that $x_{(\frac{1}{4})}$ and $x_{(\frac{3}{4})}$ as defined by (A5) satisfy (A1), (A2), and hence are the amounts determined by (A1), (A2). By iteration of this argument, a midvalue $x_{(p)}$ satisfies (A5) for any dyadic number $0 \le p \le 1$. If v is normalized so that $v(x_{(0)}) = 0$ and $v(x_{(1)}) = 1$, then $v(x_{(p)}) = p$ for any dyadic number

$0 \leq p \leq 1$. Then, by the continuity of v, $v(x_{(p)}) = p$ for any real number $0 \leq p \leq 1$ with $x_{(p)}$ as specified by (A5). Thus, for $x_{(0)} \leq x \leq x_{(1)}$,

$$v(x) = \log\left[\frac{x - x^c}{x_{(0)} - x^c}\right] / \log\left[\frac{x_{(1)} - x^c}{x_{(0)} - x^c}\right] . \tag{A6}$$

If the interval $[x_{(0)}, x_{(1)}]$ is a proper subinterval of I_1, then by similar arguments the form (A6) of $v(x)$ can be shown to hold for all x in I_1. Now, consider the component function, $v_j(x_j) = v(y_j) = v(x_1^* + r(x_j - x_j^*))$. By renormalization of $v_1(x_1)$ and $v_j(x_j)$, there exist component functions $v_1(x_1)$ and $v_j(x_j)$ as described in (5.8). The value function (5.8) implies that the tradeoffs function for base amounts x_1^0, x_j^0 is $f_j^0(x_j) = x_1^0 + ((x_1^0 - x_1^c) / (x_j^0 - x_j^c))(x_j - x_j^c)$.

The third case is that in which $x_{(\frac{1}{2})} > \bar{x}$. The arguments are similar to those for the second case above, and hence are omitted.

Proof of Theorem 6.1. Consider a standard additive-value model with a value function (2.1) that is of the form (4.3).

Assume condition (a) of absolute tradeoffs amounts constancy. We will show that condition (c) then is satisfied. Consider two amounts x_1 and x_1' that have a tradeoffs midvalue of $\hat{x}_1$ with respect to a tradeoffs pair x_j^*, x_j. It can be assumed that x_1 and x_1' are labeled so that x_j^*, x_j is a tradeoffs pair for x_1, $\hat{x}_1$ and for $\hat{x}_1$, x_1'.

Now, consider a change H such that $x_1 + H$, $\hat{x}_1 + H$, and $x_1' + H$ are in I_1. It will be shown that $v(\hat{x}_1 + H) - v(x_1 + H) = v(x_1' + H) - v(\hat{x}_1 + H)$. This implies that if there is a tradeoffs midvalue of the pair $x_1 + H$ and $x_1' + H$, then $\hat{x}_1 + H$ is the tradeoffs midvalue of $x_1 + H$ and $x_1' + H$.

As a division of the argument into cases, first assume the following: (i) $x_1 < \hat{x}_1 < x_1'$ and $x_j^* < x_j$, (ii) $H > 0$, and (iii) There exists an $\tilde{x}_1$ such that $v(\tilde{x}_1) - v(x_1^*) > M$ where

$$M = \max_{0 \leq p \leq 1} v(\hat{x}_1 + pH) - v(x_1 + pH), v(x_1' + pH) - v(\hat{x}_1 + pH) .$$

Then, $\tilde{x}_1 > f_j(x_j)$. Choose an integer m sufficiently large such that

$h = H/m < \tilde{x}_1 - f_j(x_j)$. Then, $x_1^* + h$ and $f_j(x_j) + h$ are between x_1^* and $\tilde{x}_1$ and thus are in I_1. For $k = 0,\ldots,m$, define $x_1(k) = x_1 + kh$, $\hat{x}_1(k) = \hat{x}_1 + kh$, and $x_1'(k) = x_1' + kh$. Then, $x_1(0) = x_1$, $\hat{x}_1(0) = \hat{x}_1$, $x_1'(0) = x_1'$ and $x_1(m) = x_1 + H$, $\hat{x}_1(m) = \hat{x}_1 + H$, $x_1'(m) = x_1' + H$. When $k = 0$, then $v(\hat{x}_1) - v(x_1) = v(f_j(x_j)) - v(x_1^*)$ implies by (a) that $v(\hat{x}_1 + h) - v(x_1 + h) = v(f_j(x_j) + h) - v(x_1^* + h)$, and $v(x_1') - v(\hat{x}_1) = v(f_j(x_j)) - v(x_1^*)$ implies by (a) that $v(x_1' + h) - v(\hat{x}_1 + h) = v(f_j(x_j) + h) - v(x_1^* + h)$. Thus, $v(\hat{x}_1 + h) - v(x_1 + h) = v(x_1' + h) - v(\hat{x}_1 + h)$. By iteraction of this argument, we may conclude after m steps that $v(\hat{x}_1 + H) - v(x_1 + h) = v(x_1' + H) - v(\hat{x}_1 + H)$.

If assumption (iii) is not satisfied, then subdivide the interval $[x_1, x_1']$ into $2l$ subintervals $[y^{(k-1)}, y^{(k)}]$, $k = 1,\ldots,2l$, such that $v(y^{(k)}) - v(y^{(k-1)}) = \frac{1}{2l}(v(x_1') - v(x_1))$. Then, $y^{(0)} = x_1$, $y^{(l)} = \hat{x}_1$, and $y^{(2l)} = x_1'$. Choose l sufficiently large so that there exists an $\tilde{x}_1$ with $v(\tilde{x}_1) - v(x_1^*) > M/2l$. For each $k = 1,\ldots,2l - 1$, the previous argument can be applied to show that $v(y^{(k)} + H) - v(y^{(k-1)} + H) = v(y^{(k)} + H) - v(y^{(k)} + H)$. By adding over k, it follows that $v(x_1' + H) - v(\hat{x}_1 + H) = v(\hat{x}_1 + H) - v(x_1 + H)$.

If assumption (ii) is not satisfied, i.e., $H < 0$, then begin with the pair $x_1 + H$, $x_1' + H$ in I_1. Suppose that $x_1 + H$, $x_1' + H$ has a tradeoffs midvalue of $\hat{x}(H)$. Subdivide the interval $[x_1 + H, x_1' + H]$ if necessary so that the subintervals have a common tradeoffs pair of the form x_j^*, x_j'. Then, by the above result, $x_1 = (x_1 + H) - H$ and $x_1' = (x_1' + H) - H$ have the tradeoffs midvalue $\hat{x}_1(H) - H$. Therefore, $\hat{x}_1(H) - H = \hat{x}_1$, and thus $\hat{x}_1(H) = \hat{x}_1 + H$.

If assumption (i) is not satisfied, i.e., $x_1 > \hat{x}_1 > x_1'$ and $x_j^* > x_j$, then arguments parallel to the above can be used.

Condition (b) of absolute tradeoffs willingness constancy implies condition (c) by the following argument. Suppose that $\hat{x}_1$ is the tradeoffs midvalue of x_1 and x_1' with respect to a tradeoffs pair x_j^*, x_j. Then, $(\hat{x}_1 - p, x_j) \sim (\hat{x}_1, x_j^*)$ with $p = \hat{x}_1 - x_1$ and $(\hat{x}_1 + a, x_j^*) \sim (\hat{x}_1, x_j)$ with $a = x_1' - \hat{x}_1$. Thus, (b) implies that for any change h and amount x_j', $(x_1 + h, x_j') = (\hat{x}_1 + h - p, x_j') \sim (\hat{x}_1 + h, x_j^*)$ if and only if $(x_1' + h, x_j^*) = (\hat{x}_1 + h + a, x_j^*) \sim (\hat{x}_1 + h, x_j')$.

Assume condition (c) of absolute tradeoffs midvalues constancy. Choose two amounts x_j^*, x_j in I_j and define $\delta = |v_j(x_j) - v_j(x_j^*)|$. If $v(\hat{x}_1) = \frac{1}{2}(v(x_1) + v(x_1'))$ for some amounts $x_1 < \hat{x}_1 < x_1'$ in I_1 with $v(x_1') - v(x_1) \leq \delta$, then $\hat{x}_1$ is the tradeoffs midvalue of x_1 and x_1' with respect to a tradeoffs pair x_j^*, x_j. By condition (c), this implies that $\hat{x}_1 + h$ is the tradeoffs midvalue of $x_1 + h$ and $x_1' + h$ for any h such that $x_1 + h$ and $x_1' + h$ have a tradeoffs midvalue. Thus, $v(\hat{x}_1 + h) = \frac{1}{2}(v(x_1 + h) + v(x_1' + h))$ for any amounts $x_1 + h < \hat{x}_1 + h < x_1' + h$ in I_1 with $v(x_1' + h) - v(x_1 + h) \leq \delta$. It follows by Proposition 6.1 that v has a linear-exponential form. Thus, there exists a value function of the form (6.1).

The argument that condition (d) implies condition (a) is similar to the argument that (a) implies (c), and hence for reasons of brevity is omitted.

It is straightforward to verify that if there exists a value function of the form (6.1), then each of the conditions (a)-(d) is satisfied.

Proof of Corollary 6.1. The conditions of absolute tradeoffs constancy imply that the component function $v(x)$ in (4.1) and (4.2) has a linear-exponential form. Thus, arguments similar to those for Corollary 5.1 can be used.

Proof of Theorem 6.2. Let $y = g(x)$ denote a scaling function for the operation $x \circ x'$. A tradeoffs model $(C_y, \succsim_y)$ can be defined from the given tradeoffs model $(C, \succsim)$ as in the proof of Theorem 5.2. Each of the four conditions of $\circ$-tradeoffs constancy holds for $(C, \succsim)$ if and only if the corresponding condition of absolute tradeoffs constancy holds for $(C_y, \succsim_y)$. Thus, arguments similar to those for Theorem 5.2 can be used to show that each of the conditions of $\circ$-tradeoffs constancy is equivalent to the existence of a value function of the form (6.2).

Proof of Theorem 6.3. Let $y_1 = a + x_1$ denote the consequent positions corresponding to changes x_1. A tradeoffs model $(C_y, \succsim_y)$ can be defined from the given tradeoffs model $(C, \succsim)$ as in the proof of Theorem 5.3. Each of the four conditions of relative tradeoffs constancy holds for $(C, \succsim)$ if and only if the corresponding condition of $\circ$-tradeoffs constancy, where the operation $y \circ y'$ is that of multiplication, holds for $(C_y, \succsim_y)$. Thus, arguments similar to those for Theorem 5.3 can be used to show that each of the conditions of relative tradeoffs constancy is equivalent to the existence of a value function of the form (6.3).

References

Aczél, J. 1966. *Functional Equations and Their Applications*. Academic Press, New York.

Arrow, K.J. 1976. The Rate of Discount for Long-Term Public Investment. In *Energy and the Environment: A Risk-Benefit Approach*, H. Ashkey, R.L. Rudman and C. Whipple (eds.). Pergamon Press, NY.

Bentkover, J.D., V.T. Covello, and J. Mumpower (eds.). 1986. *Benefits Assessment: The State of the Art*. D. Reidel, Dordrecht, Holland.

Brown, R.V., A.S. Kahr and C. Peterson. 1974. *Decision Analysis for the Manager*. Holt, Rinehart & Winston, New York.

Cox, L.A. 1986. Theory of Regulatory Benefits Assessment: Econometric and Expressed Preference Approaches. In *Benefits Assessment: The State of the Art*. J.D. Bentkover, V.T. Covello, and J. Mumpower (eds.), D. Reidel, Dordrecht, Holland.

Cummings, R.G., D.S. Brookshire, and W.D. Schulze. 1986. *Valuing Public Goods: An Assessment of the Contingent Valuation Method*. Rowan and Albenhead.

Debreu, G. 1954. Representation of a Preference Ordering by a Numerical Function. In *Decision Processes*, R.M. Thrall, C.H. Coombs, and R.L. Davis (eds.). Wiley, New York.

Debreu, G. 1960. Topological Methods in Cardinal Utility Theory. In *Mathematical Methods in the Social Science*, K.J. Arrow, S. Karlin, and P. Suppes (eds.). Stanford University Press, Stanford, California.

Debreu, G. 1964. Continuity Properties of Partian Utility *International Economic Review* 5, 285-293.

Farquhar, P.H. 1984. Utility Assessment Methods. *Management Science* 30, 1283-1300.

Fischhoff, B., S. Lichtenstein, P. Slovic, S.L. Derby and R. Keeney. 1981. *Acceptable Risk*. Cambridge University Press, Cambridge.

Fishburn, P.C. 1969. Preferences, Summation, and Social Welfare Functions. *Management Science* 16, 179-186.

Fishburn, P.C. 1970. *Utility Theory for Decision Making*. Wiley, New York.

Fishburn, P.C. 1976. Cardinal Utility: An Interpretative Essay. *Rivista Internazionale di Scienze Economiche e Commercialle* 13, 1102-1114.

Harvey, C.M. 1981. Assessment of Value Tradeoffs When the Pricing Out Amounts Depend on Financial Position. Working Paper, Dickinson College, Carlisle, Pa.

Harvey, C.M. 1985a. Assessment of Preferences by Conditions on Pricing-Out Amounts. *Operations Research* 33, 443-454.

Harvey, C.M. 1985b. Decision Analysis Models for Social Attitudes Toward Inequity. *Management Science* 31, 1199-1212.

Harvey, C.M. 1985c. Preference Functions for Catastrophe and Risk Inequity. *Large Scale Systems* 8, 131-146.

Harvey, C.M. 1986a. Value Functions for Infinite-Period Planning. *Management Science* 32, 1123-1139.

Harvey, C.M. 1986b. Utility Functions for Infinite-Period Planning. Working Paper, IIASA, Laxenburg, Austria.

Harvey, C.M. 1986c. A Prescriptive Model for Averse-Prone Risk Attitudes, Working Paper, IIASA, Laxenburg, Austria.

Harvey, C.M. 1987. Special Conditions on Risk Attitudes. Working Paper, IIASA, Laxenburg, Austria.

Jones-Lee, M.W. (ed.). 1982. *The Value of Life and Safety*. Praeger Publishers, New York.

Keeney, R.L. 1982. Decision Analysis: An Overview. *Operations Research* 30, 803-838.

Keeney, R.L. and H. Raiffa. 1976. *Decisions with Multiple Objectives; Preferences and Value Tradeoffs*. Wiley, New York.

Koopmans, T.C. 1972. Representation of Preference Orderings over Time. In *Decision and Organization*, C.B. McGuire and R. Radner (Eds.), North Holland, Amsterdam.

Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky. 1971. *Foundations of Measurement, Volume 1*. Academic Press. New York.

Lind, R.C., K.A. Arrow, G.R. Corey, P. Dasgupta, A.K. Sen, T. Stauffer, J.E. Stiglitz, J.A. Stockfisch, and R. Wilson. 1982. *Discounting for Time and Risk in Energy Policy*, Resources for the Future, Inc., Washington, D.C.

Loehman, E.T. 1985. Measuring Benefit Functions for Air Quality from Survey Data. Working Paper, Purdue University, West Lafayette, Indiana.

Luce, R.D. and Tukey, J.W. 1964. Simultaneous Conjoint Measurement: A New Type of Fundamental Measurement. *Journal of Mathematical Psychology* 1, 1-27.

Mishan, E.J. 1976. *Cost-Benefit Analysis*, Praeger Publishers, New York.

Page, T. 1977. *Conservation And Economic Efficiency*, John Hopkins University Press, Baltimore, MD.

Pfanzagl, J. 1971. *Theory of Measurement*. Physica-Verlag, Würzburg, F.R.G.

Randall, A. and J.R. Stoll. 1980. Consumer's Surplus in Commodity Space. *American Economic Review* 70, 449-454.

Roberts, F.S. 1979. *Measurement Theory with Applications to Decisionmaking, Utility and the Social Sciences*. Addison-Wesley. Reading, Mass.

Scott, D. 1964. Measurement Structures and Linear Inequalities. *Journal of Mathematical Psychology* 1, 233-247.

Stokey, E. and R. Zeckhausen. 1978. *A Primer for Policy Analysis*. W.W. Norton, New York.

Willig, R.D. 1976. Consumer's Surplus Without Apology. *American Economic Review* 66, 89-97.

Tversky, A. and D. Kahneman. 1981. The Framing of Decisions and the Psychology of Choice, *Science*, Vol. 211, 453-458.