INCREMENT-DECREMENT LIFE TABLES: A COMMENT

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INCREMENT-DECREMENT LIFE TABLES: A COMMENT*

Andrei Rogers** and Jacques Ledent***

In a recent useful paper published in this journal, Schoen (1975) poses the problem of constructing a set of k interrelated increment-decrement life tables but presents explicit solutions only for two-table and three-table models. These are derived by solving, in each instance, a set of simultaneous linear equations whose algebraic solution, though straightforward, is as Schoen rightly observes "a bit complicated." Inasmuch as this complexity increases exponentially with larger values of k, the computational economy of a matrix solution becomes especially desirable. A matrix solution also is useful in that it more clearly identifies the correspondence between single-table and k-table formulations. For example, as we show below, the single-table formula (Schoen's Eq. 11):

$$ {}_nP_x = \frac{1 - \frac{n}{2}\,{}_nM_x}{1 + \frac{n}{2}\,{}_nM_x} = \left(1 + \frac{n}{2}\,{}_nM_x\right)^{-1} \left(1 - \frac{n}{2}\,{}_nM_x\right) \qquad [1] $$

has as its k-table analogue the matrix expression

$$ {}_n\underset{\sim}{P}_x = \left(\underset{\sim}{I} + \frac{n}{2}\,{}_n\underset{\sim}{M}_x\right)^{-1} \left(\underset{\sim}{I} - \frac{n}{2}\,{}_n\underset{\sim}{M}_x\right) \ . \qquad [2] $$

This note derives Eq. 2 and illustrates its use in a problem area not included in Schoen's list of potential

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applications, namely, multiregional life table construction. In this particular application, increments are due to in-migration and decrements result from out-migration.

The General Model in Matrix Form

Schoen's three sets of algebraic equations for the linear model (Eqs. 1, 2, and 4 in his paper) may be reduced to two equations by substitution and then compactly expressed in matrix form as

$$\{\underset{\sim}{\ell}_{x+n}\} = \{\underset{\sim}{\ell}_x\} - {}_n\underset{\sim}{M}_x \{{}_n\underset{\sim}{L}_x\} \qquad [3]$$

and

$$\{{}_n\underset{\sim}{L}_x\} = \frac{n}{2}\left(\{\underset{\sim}{\ell}_x\} + \{\underset{\sim}{\ell}_{x+n}\}\right) \quad , \qquad [4]$$

where

$$\{\underset{\sim}{\ell}_x\} = \begin{bmatrix} {}^1\ell_x \\ {}^2\ell_x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad \{{}_n\underset{\sim}{L}_x\} = \begin{bmatrix} {}^1_nL_x \\ {}^2_nL_x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

and

$$ {}_n\underset{\sim}{M}_x = \begin{bmatrix} \left({}^1_nM^d_x + \sum_{j\neq 1}^{k} {}^1_nM^j_x\right) & -{}^2_nM^1_x & \cdots \\ -{}^1_nM^2_x & \left({}^2_nM^d_x + \sum_{j\neq 2}^{k} {}^2_nM^j_x\right) & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

Note that the components of ${}_n\underset{\sim}{M}_x$ are the death and out-migration rates defined in Schoen's Eq. 2.

Substituting [4] into [3] yields

$$\{\underset{\sim}{\ell}_{x+n}\} = \{\underset{\sim}{\ell}_x\} - \frac{n}{2}\, {}_n\underset{\sim}{M}_x \left(\{\underset{\sim}{\ell}_x\} + \{\underset{\sim}{\ell}_{x+n}\}\right) \quad ,$$

and solving algebraically for $\{\underset{\sim}{\ell}_{x+n}\}$ gives

$$\{\underset{\sim}{\ell}_{x+n}\} = \left[\left(\underset{\sim}{I} + \frac{n}{2}\, {}_n\underset{\sim}{M}_x\right)^{-1} \left(\underset{\sim}{I} - \frac{n}{2}\, {}_n\underset{\sim}{M}_x\right)\right] \{\underset{\sim}{\ell}_x\}$$

$$= {}_n\underset{\sim}{P}_x \{\underset{\sim}{\ell}_x\} \quad , \qquad [5]$$

where ${}_n\underset{\sim}{P}_x$ is a matrix of interstate transition probabilities arranged in transposed order:

$${}_n\underset{\sim}{P}_x = \begin{bmatrix} {}^1_n p^1_x & {}^2_n p^1_x & \cdots \\ {}^1_n p^2_x & {}^2_n p^2_x & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Note that the definition of ${}_n\underset{\sim}{P}_x$ in Eq. 5 is the one set out earlier in Eq. 2, and observe that a more general expression for [5] is

$$\{\underset{\sim}{\ell}_{x+n}\} = \left[\left[\underset{\sim}{I} + {}_n\underset{\sim}{M}_x \left(n\underset{\sim}{I} - \underset{\sim}{A}_x\right)\right]^{-1} \left(\underset{\sim}{I} - {}_n\underset{\sim}{M}_x\, \underset{\sim}{A}_x\right)\right] \{\underset{\sim}{\ell}_x\}$$

where $\underset{\sim}{A}_x$ is a diagonal matrix with diagonal elements ${}^i a_x$. When we choose $\underset{\sim}{A}_x$ to be $\frac{n}{2}\, \underset{\sim}{I}$ we obtain Eq. 5.

The Multiregional Life Table

Consider a multiregional system of k regions, each with an observed set of age-specific death rates and a corresponding set of age-destination-specific out-migration rates. Assemble these rates to form the matrices ${}_n\underset{\sim}{M}_x$ and derive the matrices ${}_n\underset{\sim}{P}_x$ using Eq. 2. Assign to each region a radix equal to 100,000, say, and apply Eq. 5 recursively to trace through the life-residence history of each regional birth cohort (i.e., radix). Let ${}^{ij}\ell_x$ denote the number of individuals residing in region j at exact age x who were born in region i. Then

$$\left\{{}^{i\cdot}\underset{\sim}{\ell}_x\right\} = \begin{bmatrix} {}^{i1}\ell_x \\ {}^{i2}\ell_x \\ \vdots \end{bmatrix} \qquad [6]$$

is the vector describing the distribution of the i-born population by place of residence at exact age x, and the equations needed to compute the life-residence history of the (100,000)k births that constitute the total radix of the multiregional life table are given by Eqs. 2, 3, 4, and 5 once $\{\underset{\sim}{\ell}_x\}$ is replaced by $\left\{{}^{i\cdot}\underset{\sim}{\ell}_x\right\}$. Note that Eq. 4 in this case defines the vector $\left\{{}^{i\cdot}_{n}\underset{\sim}{L}_x\right\}$, which lists, by region of residence, the number of individuals born in region i who are alive in the life table stationary population between the ages of x and x + n, or the number of person-years lived by this life table cohort between those ages in each of the regions of residence.

Upon obtaining values for the various ${}^{ij}\ell_x$ and ${}^{ij}_{n}L_x$ one may calculate the expectation of life beyond age x for each regional birth cohort ${}^{i\cdot}e_x$ and disaggregate this total by region of residence to find ${}^{ij}e_x$. For example, for x = 0, we have that

$$ {}^{i\cdot}e_0 = \sum_{j=1}^{k} \left[\sum_{y=0}^{z} \frac{{}^{ij}_{n}L_y}{100{,}000} \right] = \sum_{j=1}^{k} {}^{ij}e_0 \quad , \qquad [7] $$

where z denotes the starting age of the terminal age interval (85 years and over, say). Thus a baby just born in region i may be said to have a life expectancy of ${}^{i\cdot}e_0$ years out of which it is expected that ${}^{ij}e_0$ years will be lived in region j.

By way of illustration, Table 1 presents sex-specific regional expectations of life at birth by place of residence for the four-region population system comprised of the U.S. Census Regions of the United States: Northeast, North Central, South, and West. (Included also are corresponding expectations derived using an alternative method to be discussed in the conclusion of this note.)

Table 1

Regional Expectations of Life at Birth by Region of Residence $\left({}^{ij}e_0\right)$: United States Males and Females, 1968*

A. Males

Region of Birth	Region of Residence 1.	2.	3.	4.	Total
1. Northeast	47.24 (47.15)	5.00 (5.05)	9.56 (9.77)	5.12 (5.18)	66.92 (67.15)
2. North Central	3.53 (3.55)	46.32 (46.19)	9.80 (9.99)	7.38 (7.54)	67.03 (67.28)
3. South	4.52 (4.60)	6.98 (7.14)	48.19 (48.02)	6.41 (6.54)	66.09 (66.30)
4. West	3.68 (3.70)	7.11 (7.25)	10.35 (10.57)	46.31 (46.18)	67.45 (67.70)

B. Females

Region of Birth	Region of Residence 1.	2.	3.	4.	Total
1. Northeast	54.21 (54.13)	5.05 (5.08)	9.94 (10.11)	5.20 (5.25)	74.40 (74.56)
2. North Central	3.75 (3.76)	52.28 (52.14)	10.33 (10.48)	7.90 (8.05)	74.26 (74.44)
3. South	4.99 (5.06)	7.74 (7.88)	54.69 (54.53)	6.82 (6.93)	74.24 (74.40)
4. West	3.89 (3.90)	7.82 (7.94)	11.13 (11.32)	52.55 (52.41)	75.38 (75.57)

*Expectations in parentheses were derived using an alternative method to Schoen's (Rogers, 1975, pp. 82-83).

Life Table Death and Migration Rates

In his discussion of "The General Model," Schoen considers the case where several radix values are known. His formula for life table rates in Eq. 2, however, is correct only for the special case of a single radix. The more general case of several positive radices, such as is found in multiregional life table construction, leads to the formula

$$_{n}\underset{\sim}{M}_{x} = \left(\underset{\sim}{\ell}_{x} - \underset{\sim}{\ell}_{x+n}\right) {}_{n}\underset{\sim}{L}_{x}^{-1} \quad , \qquad [8]$$

where the matrix $\underset{\sim}{\ell}_{x}$ is composed of the vectors $\left\{{}^{i\cdot}\underset{\sim}{\ell}_{x}\right\}$ defined in Eq. 6:

$$\underset{\sim}{\ell}_{x} = \left[\left\{{}^{1\cdot}\underset{\sim}{\ell}_{x}\right\} \ \left\{{}^{2\cdot}\underset{\sim}{\ell}_{x}\right\} \cdots \left\{{}^{k\cdot}\underset{\sim}{\ell}_{x}\right\}\right] \quad ,$$

and $_{n}\underset{\sim}{L}_{x}$ is formed analogously. For example, in the two-region case one finds that:

$$_{n}^{1}M_{x}^{2} = \frac{\dfrac{\left({}^{22}\ell_{x} - {}^{22}\ell_{x+n}\right)}{{}^{22}_{n}L_{x}} - \dfrac{\left({}^{12}\ell_{x} - {}^{12}\ell_{x+n}\right)}{{}^{12}_{n}L_{x}}}{\dfrac{{}^{11}_{n}L_{x}}{{}^{12}_{n}L_{x}} - \dfrac{{}^{21}_{n}L_{x}}{{}^{22}_{n}L_{x}}} \quad ,$$

and not

$$_{n}^{1}M_{x}^{2} = \frac{{}_{n}^{1}d_{x}^{2}}{{}_{n}^{1}L_{x}}$$

as defined by Schoen's Eq. 2.

Conclusion

Schoen's paper is a valuable addition to the scant literature on increment-decrement life tables. The application of his method to multiregional life table construction eliminates an important restrictive assumption present in the procedure described by Rogers (1975), namely, the assumed absence of multiple transitions during a unit age interval. In most numerical applications, however, the two approaches yield similar results, with Schoen's method producing higher values for the diagonal elements of ${}_n\underset{\sim}{P}_x$. (Compare the expectations in Table 1 with those in the parentheses.) For example, in the two-region case one can establish that:

$$
{}_n^1p_x^1 = {}_n^1\hat{p}_x^1 + \left[\frac{\left(\frac{n}{2}\right)^2 \, {}_n^1M_x^2 \, {}_n^2M_x^1}{{}^1Q_x \, {}^2Q_x - \left(\frac{n}{2}\right)^2 \, {}_n^1M_x^2 \, {}_n^2M_x^1} \right] \left({}_n^1\hat{p}_x^1 + 1 \right) \quad , \qquad [9]
$$

where ${}^1Q_x$ and ${}^2Q_x$ are defined as in Schoen's Eq. 9, with the difference that a superscript has been added to distinguish the region to which the Q refers. Thus

$$
{}^iQ_x = 1 + \frac{n}{2} \, {}_n^iM_x^d + \frac{n}{2} \, {}_n^iM_x^j \qquad (i,j = 1,2;\ i \neq j) \quad .
$$

The "caret" (or hat) over the p in [9] differentiates numerical estimates derived using Rogers' procedure from those obtained with Schoen's method. In the Rogers procedure (Rogers, 1975, pp. 82-83):

$$ {}_{n}^{1}\hat{p}_{x}^{1} = \frac{1 - \frac{n}{2}\,{}_{n}^{1}M_{x}^{d} - \frac{n}{2}\,{}_{n}^{1}M_{x}^{2}}{1 + \frac{n}{2}\,{}_{n}^{1}M_{x}^{d} + \frac{n}{2}\,{}_{n}^{1}M_{x}^{2}} $$

Referring to Schoen's Eq. 9 we see that his formula for the same quantity is:

$$ {}_{n}^{1}p_{x}^{1} = \frac{1 - \frac{n}{2}\,{}_{n}^{1}M_{x}^{d} - \frac{n}{2}\,{}_{n}^{1}M_{x}^{2}\left(\frac{{}^{2}P_{x}}{{}^{2}Q_{x}}\right)}{1 + \frac{n}{2}\,{}_{n}^{1}M_{x}^{d} + \frac{n}{2}\,{}_{n}^{1}M_{x}^{2}\left(\frac{{}^{2}P_{x}}{{}^{2}Q_{x}}\right)} $$

where ${}^{2}P_{x} = 1 + \frac{n}{2}\,{}_{n}^{2}M_{x}^{d}$.

Since the quantity in the square brackets in [9] is always non-negative, we may conclude that ${}_{n}^{1}p_{x}^{1} \geq {}_{n}^{1}\hat{p}_{x}^{1}$ and, therefore, that ${}_{n}^{1}p_{x}^{2} \leq {}_{n}^{1}\hat{p}_{x}^{2}$.

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