# Working Paper

SPECIAL CONDITIONS ON RISK ATTITUDES *

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International Institute for Applied Systems Analysis
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## FOREWORD

This paper is concerned with a traditional problem in decision analysis, that of developing simple prescriptive models of preferences between lotteries. A general expected-utility model is assumed throughout. First, the condition of risk neutrality is shown to belong to a family of conditions, each of which determines the decision maker's utility function. Second, the condition of a constant risk attitude is shown to belong to an analogous family of conditions, each of which determines the decision maker's utility function except for a single parameter. Assumptions of the utility function's differentiability, and often of its continuity, are not needed in these models. Two contrasting methods are discussed by which the models can be used in applications.

Subject classification:

851. expected utility, risk attitude

## SPECIAL CONDITIONS ON RISK ATTITUDES

### Introduction

Prescriptive decision analysis models of individual and social preferences require conditions on preferences that structure the model into a tractable form. Such conditions should be sufficiently inclusive to allow for a modeling of the preference issues that are judged important and yet sufficiently restrictive to allow for an analysis of the implications of relatively simple value judgments on the relatively complex choices between the actual alternatives.

This paper considers an important and well-studied type of preference, that of risk attitudes when the consequences are described by a single variable. Preliminary material describes two versions of the expected-utility model; the first implies that the utility function is strictly increasing and the second implies that the utility function is also continuous.

The first part of this paper discusses a family of conditions on risk attitude, one of which is the condition of risk neutrality, here distinguished as "absolute risk neutrality." Each of the conditions in this family is shown to determine a different utility function.

This family of conditions includes the condition, here called "relative risk neutrality," that is introduced in Harvey (1981) and that is shown here to imply, without any assumptions of differentiability or continuity of the utility function $u$, that $u$ is a generalized logarithmic function, that is, $u(x) = \log(a + x)$. The constant $a$ can be interpreted as an initial asset position. If $a$ is regarded as an unspecified parameter, so that the condition of relative risk neutrality implies a one-parameter family of utility functions, then this condition can be used in the same manner as the condition of constant risk aversion. Since relative risk neutrality represents an attitude of decreasing risk aversion, it may be the more appropriate condition for use in simple, prescriptive models of risk attitudes.

The second part of this paper discusses another family of conditions on risk attitude, one of which is the condition of a constant risk attitude, here distinguished as "absolute risk constancy," that is introduced in different forms in Arrow (1971), Pfanzagl (1959), and Pratt (1964). This family of conditions is shown to include the condition, here called "relative risk constancy," that is introduced in different forms in Pratt (1964, "constant proportional risk aversion") and Harvey (1981, "linear risk attitude"). Each of the conditions in this family is shown to imply, without any assumptions of differentiability or continuity of the utility function u , that u belongs to an associated parametric family of functions.

Each of the conditions in the two families mentioned above is a "special condition" in that it either determines the utility function or implies a parametric form for the utility function. Two different methods are described for taking advantage of the resulting simplicity in the preference model. In the first method, a special condition is used to evaluate a specific utility function; in the second method, a special condition is used to evaluate the implications of differing degrees of risk aversion for preferences among the decision maker's actual alternatives.

For reasons of convenience, the proofs of the results in this paper are placed in an appendix. However, the proofs are an important part of the results being presented.

## 1. Expected-Utility Models

This section describes two expected-utility models that are obtained by specialization of the model developed in Herstein and Milnor (1953). An expected-utility model was first developed in von Neumann and Morgenstern (1944). Other models are developed in Debreu (1960), deGroot (1970), Fishburn (1970), (1975), (1982), Jensen (1967), Ledyard (1971), Luce and Raiffa (1957), Marschak (1950), Nielsen (1984), Pfanzagl (1959), Raiffa (1968), Roberts (1979, pp.354-360), Savage (1954), and Toulet (1986).

Suppose that consequences are described by the amounts $x$ in an interval $C$ that contains more than one point. Let $\ell = \langle x_i , p_i \rangle$ denote a lottery having a finite number of consequences $x_i$ in $C$ with probabilities $p_i$, $i = 1, \ldots, m$, and let $L$ denote the set of such lotteries.

In particular, let $\ell_x$ denote a lottery having the consequence $x$ with probability 1, and let $\ell_{x,x'}$ denote a lottery having the consequences $x$ and $x'$ each with probability ½. Assume that any consequence $x$ is identical to the lottery $\ell_x$ in $L$, and any two-stage lottery having a lottery $\ell$ with probability $p$ and a lottery $\ell'$ with probability $1-p$ is identical to the one-stage lottery $p\ell + (1-p)\ell'$ in $L$.

Assume that a preference relation $\succsim$, "is at least as preferred as," is defined on the set $L$ of lotteries. Preference relations $\sim$, "is indifferent to," and $\succ$, "is preferred to," can be specified in terms of $\succsim$ by: $\ell \sim \ell'$ provided that $\ell \succsim \ell'$ and $\ell' \succsim \ell$, and $\ell \succ \ell'$ provided that $\ell \succsim \ell'$ and not $\ell' \succsim \ell$.

Consider the following preference conditions on the lottery space $(L, C, \succsim)$:

(A) The preference relation $\succsim$ on $L$ is transitive and complete.

(B) Monotonicity in consequences. For any $x, x'$, in $C$,

$$\ell_x \succsim \ell_{x'} \text{ iff } x \geqslant x' .$$

(C) Continuity in probabilities. For any $\ell$, $\ell'$, $\ell''$ in L, the sets $\{p : p\ell + (1-p)\ell' \succsim \ell''\}$ and $\{p : p\ell + (1-p)\ell' \precsim \ell''\}$ are closed.

(D) Substitution principle. For any $\ell$, $\ell'$, $\ell''$ in L, $\ell \sim \ell'$ implies $\frac{1}{2}\ell + \frac{1}{2}\ell'' \sim \frac{1}{2}\ell' + \frac{1}{2}\ell''$.

Condition (D) is also called an independence axiom. It implies, in the presence of (A) and (C), the corresponding condition with probabilities $p$, $1-p$ between 0 and 1 in place of $\frac{1}{2}$.

The term increasing function will mean a strictly increasing function.

Theorem 1. (Herstein and Milnor) A lottery space $(L, C, \succsim)$ satisfies the above conditions (A) - (D) if and only if there exists a real-valued function u that is defined and increasing on the interval C such that

$$\ell \succsim \ell' \text{ iff } \Sigma_{i=1}^{m} p_i u(x_i) \geq \Sigma_{i=1}^{m'} p'_i u(x'_i) \quad (1)$$

for any lotteries $\ell = \langle x_i, p_i \rangle$ and $\ell' = \langle x'_i, p'_i \rangle$ in L.

A lottery space satisfying (A) - (D) and a utility function u as described in Theorem 1 will be called an expected-utility model and will be denoted by $(L, C, \succsim, u)$. Note that this definition is more restrictive than usual in that it includes condition (B) and the resulting property that u is increasing.

As is well-known, the utility function u in an expected-utility model is unique up to a positive linear transformation. When no confusion can result, a utility function u for a preference relation $\succsim$ will be referred to as the utility function corresponding to $\succsim$. A condition on preferences that determines the utility function (in this sense) or implies that it belongs to a parametric family of functions will be called a special preference condition. Any other condition (e.g., (A)-(D) above) will be called a general preference condition.

The continuity of a utility function is implied by each of the following general preference conditions:

(E1) Continuity in consequences. For any $\ell$ in L, the sets $\{x \text{ in } C : x \succsim \ell\}$ and $\{x \text{ in } C : x \precsim \ell\}$ are closed in C.

(E2) Existence of certainty equivalents. For any $\ell$ in L, there exists an x in C such that $x \sim \ell$.

(E3) Equal-chance continuity in consequences. For any $\ell_{y,y'}$ in L, the sets $\{x \text{ in } C : x \succsim \ell_{y,y'}\}$ and $\{x \text{ in } C : x \precsim \ell_{y,y'}\}$ are closed in C.

(E4) Existence of equal-chance certainty equivalents. For any $\ell_{y,y'}$ in L, there exists an x in C such that $x \sim \ell_{y,y'}$.

Theorem 2. In an expected-utility model, the conditions (E1)-(E4) are equivalent to each other and are satisfied if and only if the utility function u is continuous. In such a model, if a function f is defined and increasing on C such that

$$x \sim \ell_{y,y'} \text{ implies } f(x) = \tfrac{1}{2}f(y) + \tfrac{1}{2}f(y'), \tag{2}$$

for any x, y, y' in C, then f is a utility function for $\succsim$.

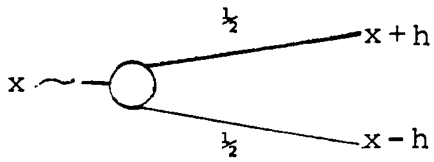
A model $(L, C, \succsim, u)$ as in Theorem 2 will be called a continuous expected-utility model; then, $\succsim$ will be called a continuous preference relation.

Conditions similar to (A)-(D) and (E2) that are necessary and sufficient for a continuous, increasing function u that represents $\succsim$ as in (1) were established in a different context by de Finetti (1931). (See Hardy et al. (1934, pp. 158-163) for an exposition in English.)

## 2. Conditions of Risk Neutrality

This section discusses a family of conditions of risk neutrality, each of which corresponds to a type of scale for the consequences $x$.

The usual condition of risk neutrality for a preference relation $\succsim$ can be defined as,




(3)

for any $x+h$ and $x-h$ in $C$. This condition will be called <u>absolute risk neutrality</u> to emphasize that it is defined in terms of absolute changes in the variable $x$.

A preference relation $\succsim$ satisfies the indifference equation (3) if and only if the utility function $u$ satisfies Jensen's functional equation,

$$u(x) = \tfrac{1}{2}u(x+h) + \tfrac{1}{2}u(x-h) \tag{3)'$$

for any $x+h$ and $x-h$ in $C$. In general, (3)' does not imply that u is linear; however, if $u$ is increasing as in the version of the expected-utility model described in Theorem 1, then (3)' implies that $u$ is linear (see, e.g., Darboux, 1875 and Hamel, 1905).

A family of conditions of risk neutrality can be defined as follows. Suppose that $C$ is contained in (possibly is equal to) an open interval $I$ on which there is a continuous group operation $x \circ x'$. Then, there exists a <u>scaling function</u> that associates $\circ$ with the ordinary addition of real numbers; that is, there exists a continuous, increasing function $g$ as in Figure 1 with domain I and range $(-\infty, \infty)$ such that $g(x \circ x') = g(x) + g(x')$ for all $x, x'$ in I (Aczel, 1966, p.254). Moreover, a scaling function is unique up to multiplication by a positive number, i.e., $\hat{g}(x) = a\,g(x)$, $a > 0$, for any two scaling functions $g(x)$ and $\hat{g}(x)$. It follows immediately that the group $(I, \circ)$ is commutative, that $e = g^{-1}(0)$ is the identity, and that any $x \neq e$ has inverse $x^{-1} = g^{-1}(-g(x))$ with e strictly between x and $x^{-1}$.

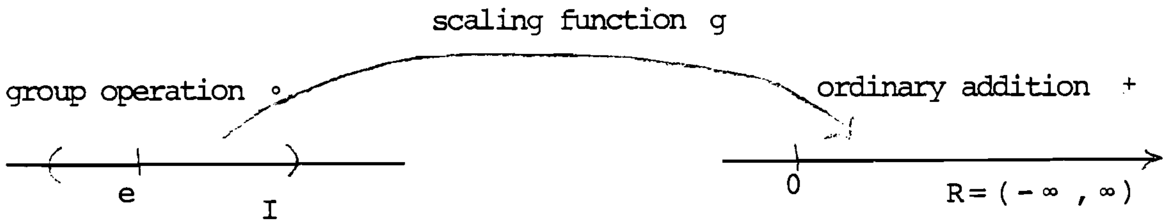
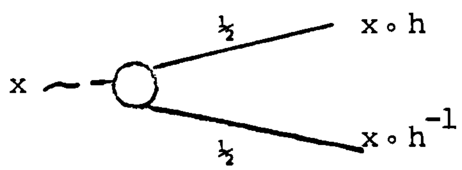



Figure 1. A scaling function g for a group operation ∘

<u>Definition 1.</u> A preference relation $\succsim$ will be called <u>∘ - risk neutral</u> with respect to a group operation ∘ provided that




(4)

for any x in C and h in I with $x \circ h$ and $x \circ h^{-1}$ in C.

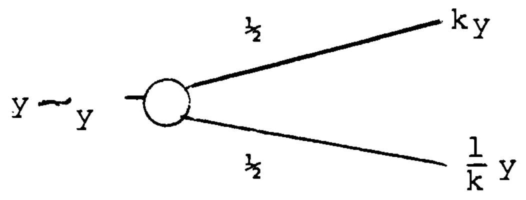
The condition (4) can be interpreted as stating that for risk-taking purposes the changes from a fixed amount x to the amounts $x \circ h$ and $x \circ h^{-1}$ are equally serious.

<u>Theorem 3.</u> The preference relation $\succsim$ in an expected-utility model is ∘ - risk neutral with respect to a group operation ∘ if and only if the utility function u for $\succsim$ is determined as any scaling function g for ∘ .

Several types of ∘-risk neutrality are discussed in Section 3 below and in Section 8.

## 3. Relative Risk Neutrality

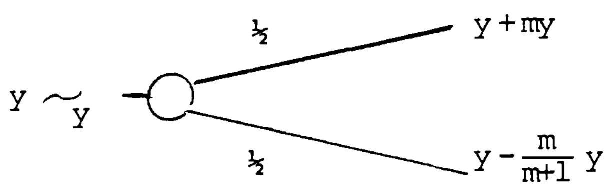
Suppose that the consequences $x$ are possible gains and losses, and the interval $C$ is the range of such financial changes. Let $a$ denote the decision maker's initial asset position (known or unknown) as somehow defined, and let $y = a + x$, $x$ in $C$, measure his or her final asset positions. Assume that the range $a + C$ of possible final asset positions is contained in the interval $(0, \infty)$. It may be appropriate to evaluate a decision maker's risk attitude by considering relative changes in the final asset positions $y = a + x$, $x$ in $C$. For the preference relation $\succsim_y$ on lotteries with final asset positions, consider the condition that




(5)

for any $y$ in $a + C$ and any $k > 1$ with $ky$ and $k^{-1}y$ in $a + C$. For example, when $k = 2$ this condition states that the decision maker is indifferent between having a final asset position $y$ for certain and having an equal-chance lottery in which $y$ is either doubled or halved.

Condition (5) can also be written in terms of percent changes. It then states that




(5)'

for any $y$ in $a + C$ where the quantities $m = k - 1 > 0$ and $\frac{m}{m+1} = 1 - \frac{1}{k} > 0$ can be interpreted as percents. For example, when $m = 1$ this condition states that the decision maker is indifferent between a final asset position y and an equal-chance lottery in which y is either increased by 100% or is decreased by 50%.

Condition (5)' can be written in terms of net gains x as:

$$a+x \sim_y \begin{cases} \tfrac{1}{2} & a+x+m\,(a+x) \\ \tfrac{1}{2} & a+x-\dfrac{m}{m+1}\,(a+x)\,. \end{cases}$$

Assuming that the decision maker's preferences concerning final asset positions are "framing consistent" (Harvey, 1986b) with his preferences concerning net gains, it follows that (5) equivalent to the following condition.

Definition 2. (Harvey, 1981) A preference relation $\succsim$ will be called relative risk neutral provided that

$$x \sim \begin{cases} \tfrac{1}{2} & x+m\,(a+x) \\ \tfrac{1}{2} & x-\dfrac{m}{m+1}\,(a+x) \end{cases} \qquad (6)$$

for any x in C and any $m>0$ with $x+m\,(a+x)$ and $x-\frac{m}{m+1}\,(a+x)$ in C.

The group operation used in (6) is that of $x \circ x' = (x+a)(x'+a)-a$ defined on the interval $I_a = (0,\infty)-a$. Here, $e=1-a$ and $x^{-1} = (1/(x+a))-a$. The operation $\circ$ on $I_a$ will be referred to as a shift multiplication.

Theorem 4. The preference relation $\succsim$ in an expected utility model is relative risk neutral with respect to an initial asset position a if and only if

$$u(x) = \log\,(a+x)\,,\quad x \text{ in } C\,, \qquad (7)$$

is a utility function for $\succsim$.

Theorem 4 strengthens a result in Harvey (1981, Theorem 5 and Erratum) in that its hypotheses are weaker. Here, the utility function u is not assumed to be twice continuously differentiable with u' positive; indeed, u is not even assumed to be continuous.

The utility functions (7) are themselves well-known (see, e.g., Grayson, 1960 and Rubenstein, 1977), and are called generalized logarithmic functions. As shown in Pratt (1964, p. 133), any generalized logarithmic function represents an attitude of decreasing risk aversion.

## 4. Assessment Methods

To determine whether a decision maker's preferences are absolute risk neutral or are relative risk neutral with respect to a known amount a, simply ask whether the indifferences (3) or (6) are true for a representative selection of consequences. Since a condition of risk neutrality determines the utility function, no further steps are needed.

When the initial asset position a is undefined or is defined but unknown, the utility functions (7) may be regarded as a one-parameter family of functions, namely, the generalized logarithmic functions. Then, two possible (and very different) assessment methods are as follows.

Method 1. To verify the condition of relative risk neutrality with respect to an unknown amount a, assess indifference comparisons (6) for several different ranges of consequences in C, and ask whether the values of the parameter a calculated from these assessments are approximately equal. If so, then relative risk neutrality is an appropriate condition for the decision maker's preferences, and a generalized logarithmic function (7) is determined in terms of the common value of a; if not, then relative risk neutrality is not an appropriate condition for the decision maker's preferences.

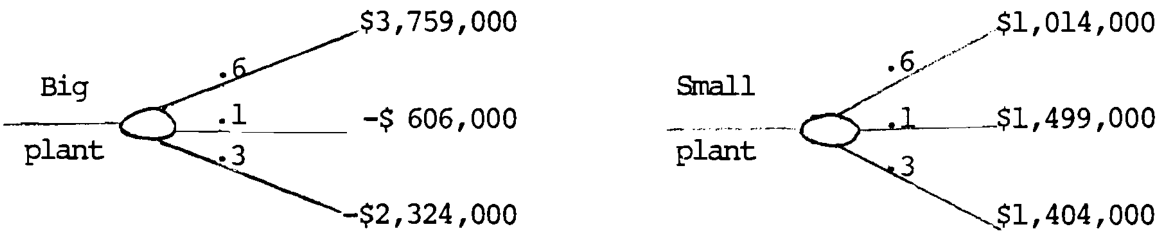
<u>Method 2</u>. Determine that the issue of risk is important and that preferences can be qualitatively described by the property of decreasing risk aversion. Then, select a single even-chance lottery $\ell_{x',x''}$ such that the consequences $x', x''$ span much of the interval C and are relatively convenient to consider. For any possible certainty equivalent x of the lottery $\ell_{x',x''}$ i.e., any consequence x between $x'$ and $x''$, calculate first the corresponding value of the parameter a and then, by using $u(x) = \log(a+x)$, the resulting preferences between the decision maker's alternative choices. Report this information by showing for which intervals of certainty equivalents x each of the alternative choices is most preferred.

Method 1 with sensitivity testing follows the usual approach to preference modeling in decision analysis; that is, a person in a decision making role is asked to make specific lottery comparisons and a utility function is calculated from this information. Discussions and references may be found, for example, in Farquhar (1984) and Keeney (1982).

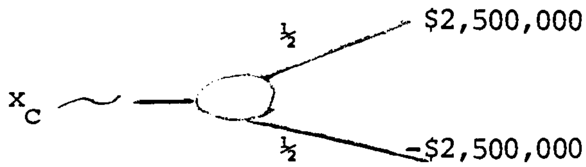
Method 2 follows an approach that has not been much used in decision analysis. An application of this approach (involving a different preference issue) is presented in Harvey (1983), and a related approach is discussed in Hammond (1974).

As an illustration of Method 2, we will apply it to a case study presented in Magee (1964). Here, a manufacturing firm called Stygian Chemical Industries, Ltd. "must decide whether to build a small plant or a large one with an expected market life of ten years. The decision hinges on what size the market for the product will be."

If each consequence is described by its net present value using a discount rate of 10%, and an attitude of risk aversion is assumed, then the decision tree for the choices faced by Stygian Chemical Industries can be reduced to a comparison of the following two lotteries:




Suppose that we introduce, for example, the following simpler comparison:




For each hypothetical certainty equivalent $x_c$ between -$2,500,000 and $0, it is possible to calculate the corresponding utility function

$$u(x) = \log\ (a+x)\ ,\quad a = \frac{x_c^2 + (2{,}500{,}000)^2}{2\ (-x_c)}$$

and hence a corresponding preference for one of the two plant sizes. These implications are shown in Figure 2.

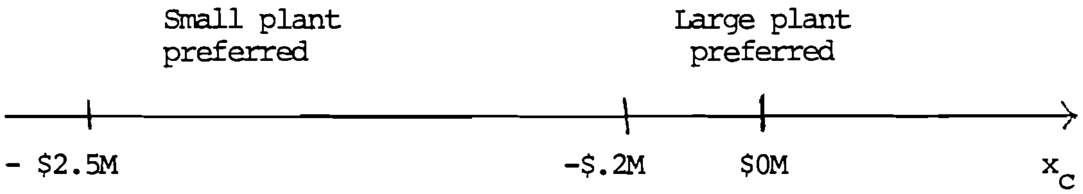



Figure 2. Preferred plant size as a function of risk attitude

Analyses similar to that shown in Figure 2 are also possible for more than two alternative plant sizes. If there is a finite number of plant sizes, then a diagram like that in Figure 2 can be reported showing a finite number of intervals. If there is a continuum of plant sizes, then a graph can be reported showing for each amount $x_c$ the corresponding most preferred plant size.

The above example can also be viewed as a "what-if" analysis having the two-step structure shown in Figure 3.

If: Various modeling assumptions are satisfied (e.g., consequences can be adequately described by their net present values); the general conditions of expected-utility are satisfied; and the specific condition of relative risk neutrality is satisfied.

Then:

If $x_c > -\$200{,}000$, then the alternative of building a large plant is preferred.

If $x_c < -\$200{,}000$, then the alternative of building a small plant is preferred.

**Figure 3.** A "what-if" analysis of preferred plant sizes

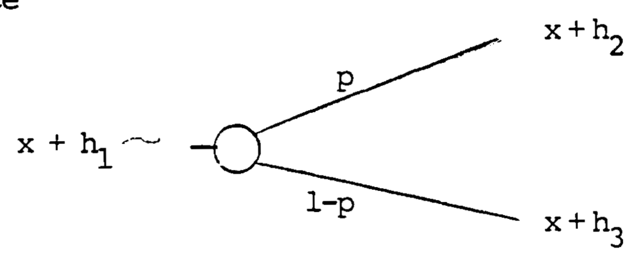
The distinctive feature of Method 2 is that it does not depend upon specific assessments by an identified decision maker. Instead, it reports the implications of preferences between relatively simple outcomes to preferences between the relatively complex actual choices. This approach may be useful for applications in which it is felt that an assumption of risk neutrality is not appropriate and that the issue of risk aversion should be modeled as simply as possible. In particular, this approach may be useful for applications to public policy evaluation as a means of clarifying the impact of different possible attitudes toward risk on the part of the public.

## 5. Conditions of Risk Constancy

This section first discusses several specializations of the condition of a constant risk attitude in the form specified in Harvey (1981) and Pfanzagl (1959). Then, a family of conditions of risk constancy is defined that corresponds to the family of conditions of risk neutrality in Section 2. Other constant risk properties have been discussed in Arrow (1971), Pratt (1964) and, more recently, in Bell (1984), Dybvig and Lippman (1983), Epstein (1985), Farquhar and Nakamura (1985), Kihlstrom et al. (1981), Machina (1982), Raiffa (1986, p. 90), Ross (1981), and Rothblum (1975).

The term "constant risk aversion" is due to Pratt (1964). In this paper, there will be no requirement of a risk averse attitude, and the adjective "absolute" will be used to emphasize that preferences are constant for absolute changes in the decision maker's financial position.

A preference relation $\succsim$ will be called absolute risk constant provided that, for any amounts $h_1, h_2, h_3$ and any probability $0 < p < 1$, if the indifference

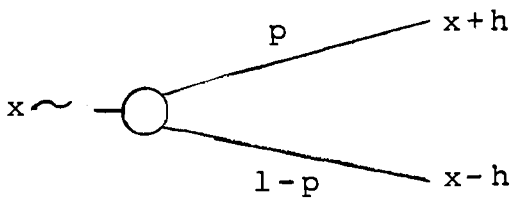



is satisfied for some $x$ with $x+h_1$, $x+h_2$, and $x+h_3$ in $C$, then it is satisfied for any $x$ with $x+h_1$, $x+h_2$, and $x+h_3$ in $C$. This definition is that in Harvey (1981, "general $\phi(x)$ risk attitude" with $\phi(x) = 1$) and is a restatement of that in Pfanzagl (1959, "consistency axiom"). For reasons of brevity, any phase of the form "if ... for some x , then ... for any x" as in the above definition will be stated as "... uniformly in x ."

In this paper, the condition of absolute risk constancy is weakened by reducing the class of lotteries on which it must be verified.

Definition 3. Consider a lottery space $(L, C, \succsim)$:

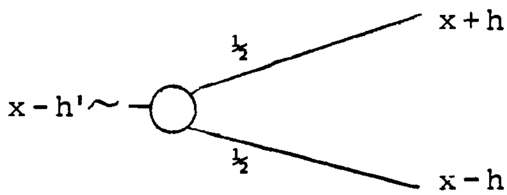
(a) Preferences will be called p. absolute risk constant provided that, for any amount $h > 0$ and any probability $0 < p < 1$, the indifference




(8)

is satisfied uniformly in x with $x + h$ and $x - h$ in C .

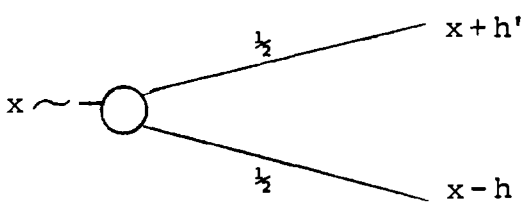
(b) Preferences will be called c. absolute risk constant provided that, for any amounts $h > 0$ and h', the indifference




(9)

is satisfied uniformly in x with $x + h$ and $x - h$ in C .

(c) Preferences will be called g. absolute risk constant provided that, for any amounts $h > 0$ and $h' > 0$, the indifference




(10)

is satisfied uniformly in x with $x + h'$ and $x - h$ in C .

In these definitions, the terms p., c., and g. are abbreviations for probability equivalent, certainty equivalent, and gain equivalent. These "equivalents" refer to the probability $p$, certainty amount $h'$, or gain $h'$ that is to be assessed for a fixed amount $h$. A fourth definition could be included, that of <u>ℓ. absolute risk constant</u> preferences in which the roles of $h$ and $h'$ in (10) are reversed: for a fixed amount $h' > 0$, the decision maker is to assess a "loss equivalent" $h$ so that the indifference (10) is satisfied. For discussions of similar lottery comparisons, see, e.g., Farquhar (1984), Harvey (1981), and Wehrung et al. (1984).

The verification of absolute risk constancy for any one of the classes of lotteries in (8), (9), (10) suffices to imply that the utility function $u$ belongs to the parametric family of linear-exponential functions, that is, that $u$ is one of the functions,

$$u(x) = \begin{cases} e^{rx}, & r > 0 \\ x, & r = 0 \\ -e^{rx}, & r < 0 \end{cases} \tag{11}$$

up to a positive linear transformation.

<u>Theorem 5</u>. For an expected-utility model, the following are equivalent:

(a) The preference relation $\succsim$ is p. absolute risk constant.

(b) The preference relation $\succsim$ is c. absolute risk constant and continuous.

(c) The preference relation $\succsim$ is g. absolute risk constant and continuous.

(d) The utility function $u$ for $\succsim$ belongs to the family of linear-exponential functions (11).

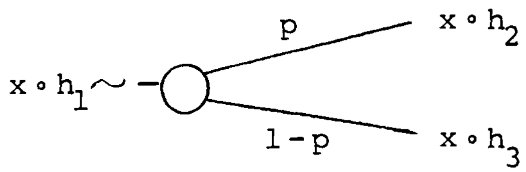
Theorem 5 differs from similar previous results in that the hypotheses (a) - (c) are weaker. Unlike results in Arrow (1971), Harvey (1981), and Pratt (1964), it is not assumed that u is twice continuously differentiable with u' positive; unlike the result in Pfanzagl (1959, Theorem 5), smaller classes of lotteries are considered and in (a) it is not assumed that u is continuous. (This non-assumption of continuity is used in Harvey, 1986b.) Moreover, the proofs in Theorem 5 are quite different - and perhaps more direct - than the proofs of similar previous results.

Parts (b) and (c) raise the question of whether the continuity assumption used there is needed or can be omitted.

<u>Proposition 1</u>. For any interval C, there exists a function u defined on C that is increasing but discontinuous on C, and such that the preference relation $\succsim$ corresponding to u is c. absolute risk constant and g. absolute risk constant.

In the Appendix, such counterexample functions are constructed by means of a Cantor-type subdivision of the interval (0 , 1).

A family of conditions of risk constancy can be defined in terms of the group operations $x \circ x'$ discussed in Section 2. Preferences will be called <u>$\circ$-risk constant</u> provided that for any amounts $h_1, h_2, h_3$ and any probability $0 < p < 1$, the indifference

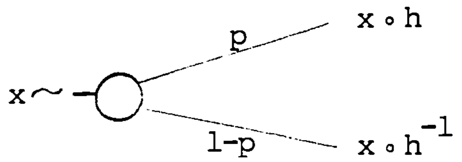



(12)

is satisfied uniformly in x. As in Definition 3, the class of lotteries to be considered can be reduced.

<u>Definition 4</u>. Consider a lottery space $(L, C, \succsim)$ and a group operation $\circ$ defined on $C$.

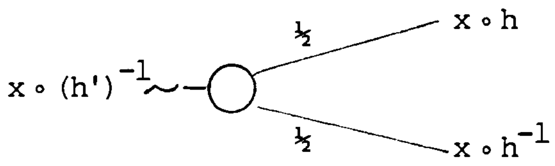
(a) Preferences will be called <u>p. $\circ$-risk constant</u> provided that, for any amount $h > e$ and any probability $0 < p < 1$, the indifference




(13)

is satisfied uniformly in $x$ with $x \circ h$ and $x \circ h^{-1}$ in $C$.

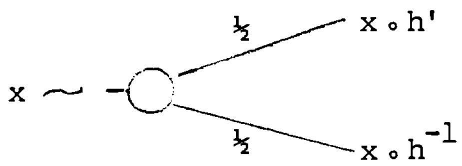
(b) Preferences will be called <u>c. $\circ$-risk constant</u> provided that, for any amounts $h > e$ and $h'$, the indifference




(14)

is satisfied uniformly in $x$ with $x \circ h$ and $x \circ h^{-1}$ in $C$.

(c) Preferences will be called <u>g. $\circ$-risk constant</u> provided that, for any amounts $h > e$ and $h' > e$, the indifference




(15)

is satisfied uniformly in $x$ with $x \circ h'$ and $x \circ h^{-1}$ in $C$.

Any one of the above conditions implies that the utility function $u$ belongs to the following parametric family of functions,

$$u(x) = \begin{cases} e^{rg(x)} & , r > 0 \\ g(x) & , r = 0 \\ -e^{rg(x)} & , r < 0 \end{cases} \tag{16}$$

where $g(x)$ is a scaling function for the group operation $\circ$.

Theorem 6. For an expected-utility model, the following are equivalent:

(a) The preference relation $\succsim$ is p. $\circ$-risk constant.

(b) The preference relation $\succsim$ is c. $\circ$-risk constant and continuous.

(c) The preference relation $\succsim$ is g. $\circ$-risk constant and continuous.

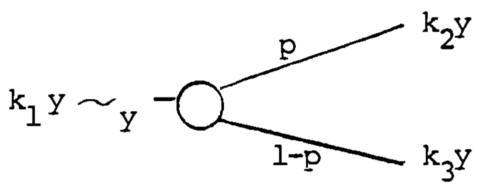
(d) The utility function u for $\succsim$ belongs to the parametric family of functions (16).

As in Theorem 5, the continuity assumption in parts (b) and (c) cannot be omitted since, for a function u as in Proposition 1 that is defined on $g(C)$, the function $u(g(x))$, x in C, is a discontinuous utility function for a preference relation $\succsim$ that is c. $\circ$-risk constant and g. $\circ$-risk constant.

## 6. Relative Risk Constancy

This section discusses those conditions of $\circ$-risk constancy which are concerned with relative changes in consequences. The distinction discussed in Section 3 between a decision maker's net gains x and his or her final asset positions $y = a + x$ is also important here.

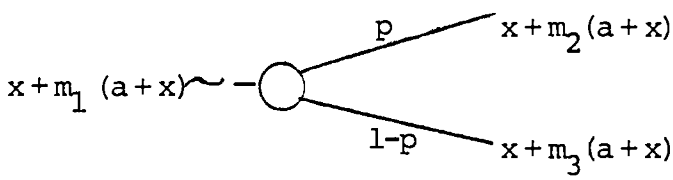
For final asset positions y, consider the condition that, for any multiplier changes $k_1, k_2, k_3 > 0$ and any probability $0 < p < 1$, the indifference




(17)

is satisfied uniformly in y. This condition is a version of the condition of constant proportional risk aversion defined in Pratt (1964).

For net gains x, the corresponding condition is that, for any proportions $m_1, m_2, m_3 > -1$ and any probability $0 < p < 1$, the indifference



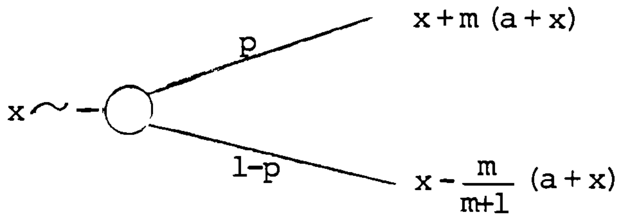

(17)'

is satisfied uniformly in $x$. This condition is a version of the condition of a <u>linear</u> <u>risk</u> <u>attitude</u> defined in Harvey (1981).

The concern here is to weaken the condition (17)' in the same manner that the condition of absolute risk constancy was weakened in Definition 3. The following three conditions are restatements of the $\circ$-risk constant conditions in Definition 4 where changes are viewed as percent changes and thus the group operation $\circ$ is shift multiplication with respect to an initial asset position $a$.

<u>Definition 5</u>. Consider a lottery space $(L, C, \succsim)$ and an initial asset position $a$:

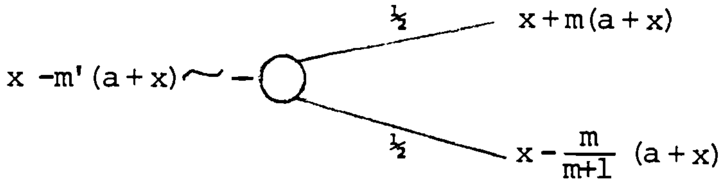
(a) Preferences will be called <u>p. relative risk constant</u> provided that for any proportion $m > 0$ and any probability $0 < p < 1$, the indifference




(18)

is satisfied uniformly in $x$ with $x+m(a+x)$ and $x - \frac{m}{m+1}(a+x)$ in $C$.

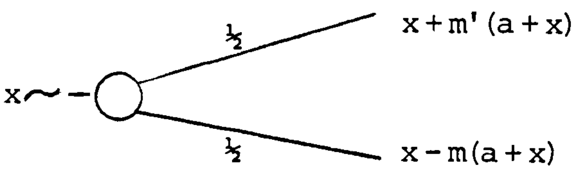
(b) Preferences will be called <u>c. relative risk constant</u> provided that, for any proportions $m > 0$ and $m' > -1$, the indifference




(19)

is satisfied uniformly in $x$ with $x+m(a+x)$ and $x - \frac{m}{m+1}(a+x)$ in $C$.

(c) Preferences will be called <u>g. relative risk constant</u> provided that, for any proportions $0 < m < 1$ and $m' > 0$, the indifference




(20)

is satisfied uniformly in $x$ with $x+m'(a+x)$ and $x-m(a+x)$ in $C$.

As observed in Theorem 4, the function $g(x) = \log(a+x)$ is a scaling function for the group operation of shift multiplication with respect to an initial asset position $a$. Thus, the parametric family of functions (16) is here,

$$u(x) = \begin{cases} (a+x)^r & , r > 0 \\ \log(a+x) & , r = 0 \\ -(a+x)^r & , r < 0 . \end{cases} \tag{21}$$

Theorem 6 implies as the following corollary that the verification of relative risk constancy for any one of the classes of lotteries in (18),(19),(20) suffices to imply that the utility function $u$ belongs to the parametric family of functions (21).

Theorem 7. For an expected-utility model, the following are equivalent:

(a) The preference relation $\succsim$ is p. relative risk constant.

(b) The preference relation $\succsim$ is c. relative risk constant and continuous.

(c) The preference relation $\succsim$ is g. relative risk constant and continuous.

(d) The utility function $u$ for $\succsim$ belongs to the parametric family of functions (21).

Theorem 7 differs from similar previous results in that the hypotheses (a)-(c) are weaker. Unlike results in Harvey (1981) and Pratt (1964), it is not assumed that $u$ is twice continuously differentiable with $u'$ positive. That the continuity assumption in parts (b), (c) cannot be omitted follows from the remark on parts (b), (c) in Theorem 6.

Proposition 2. Suppose that preferences are risk averse and relative risk constant. Then:

(a) Preferences are represented by a utility function (21) with $r < 1$.

(b) (Pratt (1964)) There is decreasing risk aversion for all $x$ in $C$.

(c) The risk attitude tends to risk neutrality as $x$ tends to $+\infty$; that is, for any fixed $h$ the difference between $x$ and the certainty equivalent of an even-chance lottery $\ell_{x+h,\, x-h}$ tends to 0 as $x$ tends to $+\infty$.

## 7. Assessment Methods

In order to verify one of the above conditions of risk constancy, it suffices to consider only one of the types of indifference comparisons (8)-(10) or (18)-(20). For the certainty equivalence and gain equivalence cases, (9)-(10) and (19)-(20), one of the continuity conditions (E1)-(E4) must also be verified. For this step, attention should be focused on amounts $x$ such as $x = 0$, the "status quo," that may have a special significance for the decision maker. To evaluate continuity at $x = 0$, for example, choose a fixed loss or gain $h \neq 0$ and determine the certainty equivalents of various lotteries $p\ell_h + (1-p)\ell_0$ as the probability $p$ tends to zero. Harvey (1986b) discusses a model in which the continuity conditions are not satisfied at $x = 0$.

If a decision maker's preferences are absolute risk constant or are relative risk constant with respect to a known amount $a$, then a utility function (11) or (21) is determined up to a single parameter $r$. Thus, the Methods 1 and 2 in Section 4 can use either of these conditions in the same manner as the condition of relative risk neutrality with respect to an unknown amount $a$.

If a decision maker's preferences are relative risk constant with respect to an unknown amount $a$, then the utility function belongs to the two-parameter family of functions (21) with parameters $r$ and $a$. The following assessment methods that are analogous to Methods 1 and 2 use these utility functions.

Method 1'. To verify the condition of relative risk constancy with respect to an unknown amount $a$, assess indifference comparisons (18), (19), or (20) for two different ranges of consequences and calculate the resulting values of $r$ and $a$ (thereby obtaining the utility function). Then, assess indifference comparisons for other ranges of consequences, and ask whether these assessments are in accord with the calculated utility function.

A utility function (21) represents an attitude of decreasing risk aversion when the parameter $r$ has a value $r < 1$. The following method is based on the idea that these utility functions form a sufficiently rich family of functions to model the preference issue of decreasing risk aversion.

Method 2'. Determine that the issue of risk is important and that it will be helpful to consider the degree of risk aversion for two different, e.g., opposite, ranges of consequences. For each of these ranges, select an even-chance lottery in that range that is relatively convenient to consider. For any possible certainty equivalents of these lotteries, calculate first the corresponding values of the parameters $r$ and $a$, and then the resulting preferences between the decision maker's alternative actions. Report this information by showing for which combinations of the two certainty equivalents each of the alternative actions is most preferred.

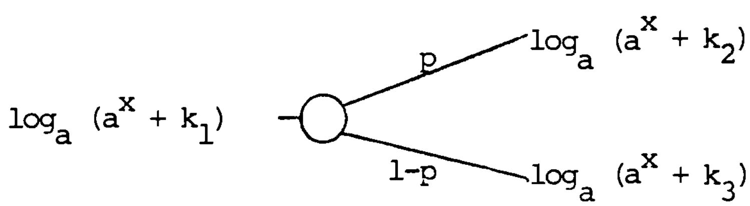
## 8. Other Types of Special Conditions

This section discusses three directions in which the results in the preceding sections can be extended.

9.1. There are conditions of $\circ$-risk constancy other than those of absolute risk constancy and relative risk constancy. Two such conditions are as follows.

Double-exponential utility functions. Consider an exponential scaling function, $g(x) = c^{-1} e^{cx}$, where the constant $c$ is positive or negative. The corresponding operation is $x \circ y = c^{-1} \log (e^{cx} + e^{cy})$, which is a semi-group operation on $(-\infty, \infty)$. It is related to addition much as addition is related to multiplication; for example, addition is distributive over $\circ$, i.e., $(x+z) \circ (y+z) = (x \circ y) + z$.

The condition (12) of $\circ$-risk constancy for this operation requires that, for any amounts $k_1, k_2, k_3 > 0$ and any probability $0 < p < 1$, the indifference



(22)

is satisfied uniformly in $x$. Here, the base $a$ equals $e^c$.

The family (15) of utility functions implied by (22) is:

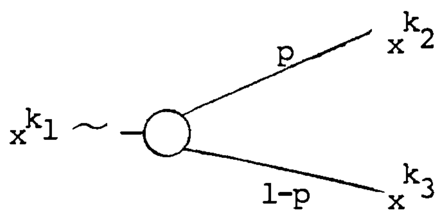
$$u(x) = \begin{cases} \exp\left(\frac{r}{c}e^{cx}\right), & r > 0 \\ \frac{1}{c}e^{cx}, & r = 0 \\ -\exp\left(\frac{r}{c}e^{cx}\right), & r < 0 \end{cases} \qquad (23)$$

For $r \neq 0$, these are the double-exponential utility functions introduced in Bell (1986).

As shown in that paper, a double-exponential utility function represents an attitude of decreasing risk aversion for any parameter values $c < 0$ and $r < 0$. For such values of $c$ and $r$, preferences also have the property that the risk attitude tends to risk constancy with parameter $c$ as $x$ tends to $+\infty$.

Double-logarithmic utility functions. In symmetry to the above move from absolute risk constancy in an exponential direction, it is possible to move from relative risk constancy in a logarithmic direction. Consider a scaling function of the form $g(x) = \log(a_1 + \log(a_2 + x))$, and assume for simplicity that $a_1, a_2 = 0$ and $C = (1, \infty)$. The corresponding operation is $x \circ y = x^{\log y} = y^{\log x}$.

The condition (12) of $\circ$-risk constancy for this operation requires that, for any amounts $k_1, k_2, k_3 > 0$ and any probability $0 < p < 1$, the indifference




(24)

is satisfied uniformly in $x$.

The family (15) of utility functions impled by (24) is:

$$u(x) = \begin{cases} (\log x)^r & , r > 0 \\ \log \log x & , r = 0 \\ -(\log x)^r & , r < 0 \end{cases} \tag{25}$$

These functions will be called double-logarithmic utility functions.

A double-logarithmic utility function represents an attitude of decreasing risk aversion for all $x > 1$ when $r \leq 1$ and for $x$ sufficiently large when $r > 1$. For any $r$, the risk attitude tends to risk neutrality as $x$ tends to $+\infty$.

9.2 There are a variety of conditions on risk attitude that are closely related to that of absolute risk constancy. Farquhar and Nakamura (1985), for example, introduce a number of such conditions; e.g., their augmented constant exchange property implies that $u(x)$ belongs to a parametric family that contains the linear and exponential functions and also contains four other types of functions, one of which is the sumex fucntions. Thus, this condition is a weakening of absolute risk constancy. By means of the scaling function $g(x) = \log(a + x)$, it would be possible to define a "relative exchange property" and to show that this condition implies a utility function of the form $u(g(x))$ where $u$ belongs to the above family of functions.

Harvey (1986b) introduces two conditions that restrict preferences among lotteries when a pair of lotteries stochastically dominates another pair of lotteries. One condition is equivalent to that of absolute risk consistency and the other condition is equivalent to relative risk constancy.

Epstein (1985) and Machina (1982) are concerned with strengthening the condition of decreasing absolute risk aversion (DARA) in an expected-utility model and, more generally, in a model having a Frechet differentiable preference function. Several of their conditions are a strengthening of absolute risk constancy in the sense that they imply risk neutrality in an expected-utility model. See, for example, the conditions C.1 and C.2 in Machina (1982) and the conditions R-DARA together with R.1, R.2, and R.3 in Epstein (1985). It might be of interest to examine the implications of conditions in the present paper, e.g., relative risk constancy, in a variety of non-expected-utility models.

9.3 For the preference conditions that are discussed in this paper, the single variable $x$ is regarded as a measure of monetary changes. However, these preference conditions may also be appropriate in decision problems in which the consequences are described by a single non-monetary variable $x$. In a medical decision problem, for example, $x$ might denote extra days of life in normal health.

For decision problems in which each consequence $c$ is described by a number of variables, that is, $c = (x_1, \ldots, x_n)$, conditions of preferential independence and expected utility imply that preferences are represented by a utility function of the form,

$$u(c) = f\,(\,v_1(x_1) + \ldots + v_n(x_n)\,)\,, \tag{26}$$

where $v = v_1(x_1) + \ldots + v_n(x_n)$ is an additive value function. If $x$ denotes one of the variables $x_1, \ldots, x_n$, then preference conditions regarding $x$ are conditions on conditional risk attitude, while if $x$ denotes the additive value function $v$, then preference conditions regarding $x$ are conditions on multivariable risk attitude. All these remarks are well known.

When $x$ measures the amounts of an additive value function $v$, it is often useful to restate a preference condition in terms of the consequences $(x_1, \ldots, x_n)$ rather than in terms of the amounts $v$. The following list provides terminology and references for the restatements of several preference conditions:

(a) Absolute risk neutrality: additive utility independence (Fishburn, 1965, 1970) and multivariate risk neutrality (Richard, 1975).

(b) Relative risk neutrality: proportional multiperiod risk neutrality (Harvey, 1986a).

(c) Absolute risk constancy: mutual utility independence (Keeney, 1968, 1974, Meyer, 1970, 1972, and Meyer and Pratt (in Keeney and Raiffa, 1976, p. 330)), and weak additivity (Pollak, 1967).

(d) Relative risk constancy: proportional utility dependence (Harvey, 1984), coinciding standard and equal utility (Harvey, 1985), and timing independence (Harvey, 1986a).

The results in this paper imply that assumptions of differentiability of the function f in (26) are not needed as part of the conditions on multivariable risk attitude cited above. Thus, the same removal of inessential assumptions is possible for the modeling of multivariable preferences as for the modeling of single-variable preferences.

## Appendix: Proofs of Results

Proof of Theorem 1. Herstein and Milnor (1953) have shown the hard part of this result, namely that conditions (A), (C), (D) imply the existence of a function u defined on C that represents the preference relation $\succsim$ as in (1). Then, (1) and condition (B) immediately imply that u is increasing. The converse implications are straightforward to verify.

Proof of Theorem 2. The interval C is the union of any pair of sets in (E1). If (E1) is satisfied, then since C is connected any such pair of sets must have a non-empty intersection. Thus, (E2) is satisfied. Conversely, (E2) implies that for $x_0 \sim \ell$, $\{x \text{ in } C : x \succsim \ell\} = [x_0, \infty) \cap C$ and $\{x \text{ in } C : x \precsim \ell\} = (-\infty, x_0] \cap C$. Thus, (E1) is satisfied. The equivalence of (E3) and (E4) can be shown in a similar manner.

Clearly, (E1) implies (E3), and (E2) implies (E4). To show that (E4) implies the continuity of the utility function u, suppose that u is not continuous. Then, since u is increasing, it has a jump discontinuity at some point $x_0$ in C. Thus, there exist $y \leq x_0$ and $y' \geq x_0$ in C such that $\frac{1}{2}u(y) + \frac{1}{2}u(y')$ is not equal to $u(x)$ for any x in C, and so (E4) is false. Finally, the continuity of u implies that the sets in (E1) are closed in C.

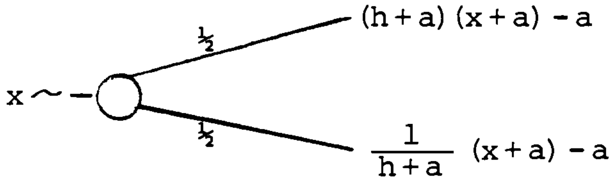
Consider a function f as described and a continuous utility function u. Let $z = u(y)$ and $z' = u(y')$. Then, for any points z, z' in the interval u(C), $f \circ u^{-1}(\frac{1}{2}z + \frac{1}{2}z') = \frac{1}{2} f \circ u^{-1}(z) + \frac{1}{2} f \circ u^{-1}(z')$. Since $f \circ u^{-1}$ is increasing, it follows that $f \circ u^{-1}(z) = az + b$, z in u(C), for some constants $a > 0$, b, and so $f(y) = a\,u(y) + b$, y in C.

Proof of Theorem 3. Suppose that $\succsim$ is $\circ$-risk neutral with respect to a group operation $\circ$ . For any $y$ and $y'$ in $C$ , the "$\circ$-midpoint" $\bar{y} = g^{-1}(\frac{1}{2}g(y) + \frac{1}{2}g(y'))$ of $y$ and $y'$ is between $y$ and $y'$ and hence is in $C$. Moreover, $y = \bar{y} \circ h$ and $y' = \bar{y} \circ h^{-1}$ where $h = g^{-1}(g(y) - g(\bar{y}))$ .

According to (4), it follows that for any $y$ and $y'$ in $C$ , the lottery $\ell_{y,y'}$ has the certainty equivalent $\bar{y} \sim \ell_{y,y'}$ . Thus, condition (E4) is satisfied, and so any increasing function $f$ is a utility function for $\succsim$ provided that (2) is satisfied. But, for any $x,y,y'$ in $C$ , $x \sim \ell_{y,y'}$ implies that $x = \bar{y}$ , and so $g(x) = g(\bar{y}) = \frac{1}{2}g(y) + \frac{1}{2}g(y')$. Therefore, a scaling function $g$ for $\circ$ is a utility function for $\succsim$ .

Conversely, suppose that a scaling function $g$ for a group operation $\circ$ is a utility function for the preference relation $\succsim$ . Since $g(x) = \frac{1}{2}g(x \circ h) + \frac{1}{2}g(x \circ h^{-1})$ for any $x, x \circ h, x \circ h^{-1}$ in $C$ , it follows that the condition (4) of $\circ$-risk neutrality is satisfied.

Proof of Theorem 4. The condition (6) can be rewritten as




where $h = m + 1 - a$ is in $I_a$ when $m > 0$ . Thus, (6) states that $\succsim$ is $\circ$-risk neutral for the shift multiplication $\circ$ on $I_a$. The function $g(x) = \log(x + a)$ is a scaling function for $\circ$ since $\log(x \circ x' + a) = \log(x + a) + \log(x' + a)$ for any $x, x'$ in $I_a$. Thus, by Theorem 3, the preference relation $\succsim$ is relative risk neutral if and only if $g(x) = \log(x + a)$ is a utility function for $\succsim$ .

Proof of Theorem 5. We first show that p. absolute risk constancy implies a linear-exponential utility function (11). Select two amounts $x_1$ and $x_{-1}$ in C with $x_1 > x_{-1}$, and define

$$x_t = \tfrac{1}{2}(1+t)x_1 + \tfrac{1}{2}(1-t)x_{-1} \qquad \text{(A1)}$$

for all real $t$. This definition is consistent with the notation $x_1, x_{-1}$. It specifies a linear, increasing correspondence between the variables $x$ and $t$. As a second functional dependence, define $q = 1 - p$ for any probability $p$.

Since preferences satisfy the conditions of expected utility, there exists a unique probability $0 < p < 1$ such that the indifference (8) is satisfied with $x = x_0$, $x + h = x_1$, and $x - h = x_{-1}$. The argument will be divided into three cases, depending on whether $p = \tfrac{1}{2}$, $p < \tfrac{1}{2}$, or $p > \tfrac{1}{2}$.

Suppose that $p = \tfrac{1}{2}$. Normalize the utility function $u$ for the lottery space $(L, C, \succsim)$ so that $u(x_1) = 1$ and $u(x_{-1}) = -1$. Then, by the indifference (8), $u(x_0) = \tfrac{1}{2}u(x_1) + \tfrac{1}{2}u(x_{-1}) = 0$. Therefore, $u(x_t) = t$ for $t = 1, 0, -1$.

Since $\succsim$ is p. absolute risk constant, there exists a single probability $\hat{p}$ such that

$$\begin{aligned} u(x_{\frac{1}{2}}) &= \hat{p}\,u(x_1) + \hat{q}\,u(x_0) \\ u(x_{-\frac{1}{2}}) &= \hat{p}\,u(x_0) + \hat{q}\,u(x_{-1}) \qquad \text{(A2)} \\ u(x_0) &= \hat{p}\,u(x_{\frac{1}{2}}) + \hat{q}\,u(x_{-\frac{1}{2}}) \end{aligned}$$

Therefore, $u(x_{\frac{1}{2}}) = \hat{p}$, $u(x_{-\frac{1}{2}}) = -\hat{q}$, and $0 = \hat{p}\,u(x_{\frac{1}{2}}) + \hat{q}\,u(x_{-\frac{1}{2}})$. It follows that $\hat{p} = \tfrac{1}{2}$. Thus, $u(x_{\frac{1}{2}}) = \tfrac{1}{2}$ and $u(x_{-\frac{1}{2}}) = -\tfrac{1}{2}$. Arguing by mathematical induction it follows that $u(x_t) = t$ for any dyadic number $t$ in the interval $[-1, 1]$. Since $u$ is increasing, it follows that $u(x_t) = t$ for any real number $t$ in $[-1, 1]$. According to (A1), however, $t = (x_t - x_0)/(x_1 - x_0)$ for any real number $x_t$. Therefore, $u(x) = (x - x_0)/(x_1 - x_0) = ax + b$ where $a = (x_1 - x_0)^{-1} > 0$ and $b = -x_0/(x_1 - x_0)$. Here, $x$ is any real number in the interval $[x_{-1}, x_1]$.

It remains to consider any $x_t$ in $C$ such that $x_t > x_1$ or $x_t < x_{-1}$. Suppose that $x_t > x_1$. Then, there exists a single probability $\hat{p}$ such that

$$\begin{aligned} u(x_1) &= \hat{p}\,u(x_t) + \hat{q}\,u(x_0) \\ u(x_0) &= \hat{p}\,u(x_{t-1}) + \hat{q}\,u(x_{-1}) . \end{aligned} \tag{A3}$$

Thus, $1 = \hat{p}\,u(x_t)$ and $0 = \hat{p}\,u(x_{t-1}) - \hat{q}$. It follows that $u(x_t) = u(x_{t-1}) + 1$. Arguing by iteration, it then follows that $u(x_t) = t$. For any $x_t < x_{-1}$ in $C$, it can be shown by a similar argument that $u(x_t) = t$. Thus, in conclusion, $u(x) = (x - x_0)/(x_1 - x_0) = ax + b$ with $a > 0$ for all $x$ in $C$.

Next, suppose that $p < \frac{1}{2}$. Normalize the utility function $u$ so that $u(x_1) = q/p$ and $u(x_{-1}) = p/q$. Then, by the indifference (8), $u(x_0) = p\,u(x_1) + q\,u(x_{-1}) = 1$. If $s > 0$ is defined so that $e^s = q/p > 1$, then $u(x_t) = e^{st}$ for $t = 1, 0, -1$.

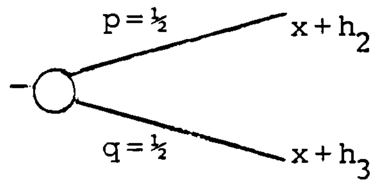
Since $\succsim$ is p. absolute risk constant, there exists a single probability $\hat{p}$ such that (A2) is satisfied. Therefore, $u(x_{\frac{1}{2}}) = \hat{p}(q/p) + \hat{q}$, $u(x_{-\frac{1}{2}}) = \hat{p} + \hat{q}(p/q)$, and $1 = \hat{p}\,u(x_{\frac{1}{2}}) + \hat{q}\,u(x_{-\frac{1}{2}})$. It follows that $\hat{p}(q/p)^{\frac{1}{2}} + \hat{q}(p/q)^{\frac{1}{2}} = 1$, and hence $u(x_{\frac{1}{2}}) = (q/p)^{\frac{1}{2}} = e^{s(\frac{1}{2})}$, $u(x_{-\frac{1}{2}}) = (p/q)^{\frac{1}{2}} = e^{s(-\frac{1}{2})}$. Arguing by mathematical induction, it follows that $u(x_t) = e^{st}$ for any dyadic number $t$ in $[-1, 1]$.

Since $u$ is increasing, it follows that $u(x_t) = e^{st}$ for any real number $t$ in $[-1, 1]$. However, $t = (x_t - x_0)/(x_1 - x_0)$, and thus $u(x) = e^{r(x - x_0)} = a\,e^{rx}$ where $r = s/(x_1 - x_0) > 0$ and $a = e^{-rx_0} > 0$. Here, $x$ is any real number in $[x_{-1}, x_1]$.

It remains to consider any $x_t$ in $C$ such that $x_t > x_1$ or $x_t < x_{-1}$. Suppose that $x_t > x_1$. Then, there exists a single probability $\hat{p}$ such that (A3) is satisfied. Thus, $e^s = \hat{p}\,u(x_t) + \hat{q}$ and $1 = \hat{p}\,u(x_{t-1}) + \hat{q}\,e^{-s}$. It follows that $u(x_t) = e^s\,u(x_{t-1})$. Arguing by iteration, it then follows that $u(x_t) = e^{st}$. For any $x_t < x_{-1}$ in $C$, it can be shown by a similar argument that $u(x_t) = e^{st}$. Thus, in conclusion, $u(x) = a\,e^{rx}$ with $a > 0$, $r > 0$ for all $x$ in $C$.

Now, suppose that $p > \frac{1}{2}$. In this case, normalize the utility function $u$ so that $u(x_1) = -q/p$ and $u(x_{-1}) = -p/q$. Then, by an argument similar to that in the case $p < \frac{1}{2}$, it can be shown that $u(x) = -a\,e^{rx}$ with $a > 0$, $r < 0$ for all $x$ in $C$.

We next show that each of c. absolute risk constancy and g. absolute risk constancy implies a linear-exponential utility function. Observe that by means of relabeling the consequences in (9) and (10) each of these conditions of uniform indifference implies that for any amounts $h_1, h_2, h_3$ the indifference

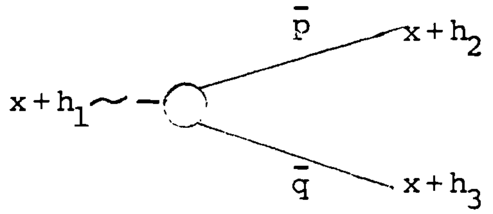
$x + h_1 \sim \ell_p =$ 

(A4)

is satisfied uniformly in $x$ with $x + h_2$ and $x + h_3$ in $C$.

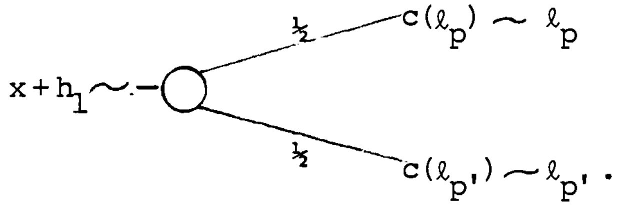
The condition (A4) is therefore satisfied uniformly in $x$ when $p = \frac{1}{2}, 1$, or $0$. Assume that (A4) is satisfied uniformly in $x$ for two probabilities $p$ and $p'$. Then, (A4) is also satisfied uniformly in $x$ for the probability $p = \frac{1}{2}p + \frac{1}{2}p'$. A crucial part of the following proof is the assumption in parts (b) and (c) of Theorem 5 that any lottery $\ell$ has a certainty equivalent $c(\ell) \sim \ell$.

Suppose that for some $x+h_2$ and $x+h_3$ in C,

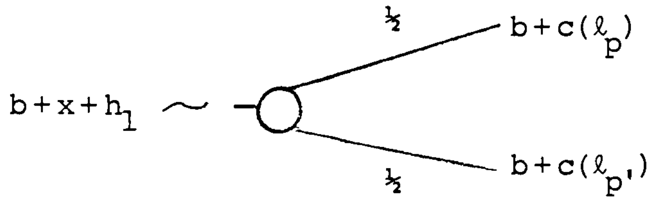


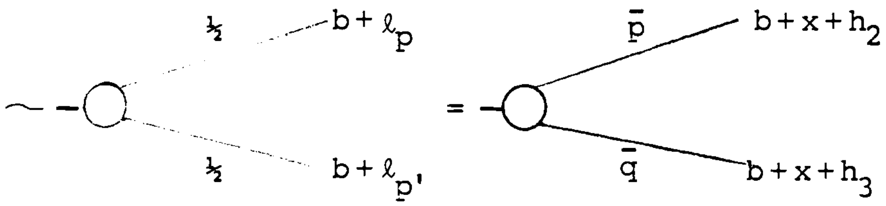

(A5)

Then,




Consider any other amount $x' = b + x$ with $x' + h_2$ and $x' + h_3$ in C. By assumption, $c(\ell_p) \sim \ell_p$ implies $b + c(\ell_p) \sim b + \ell_p$, and $c(\ell_{p'}) \sim \ell_{p'}$ implies $b + c(\ell_{p'}) \sim b + \ell_{p'}$. Thus, (A5) implies that




Thus, (A4) is satisfied uniformly in x for the probability $\bar{p}$.

Arguing by mathematical induction, it follows that (A4) is satisfied uniformly in $x$ for any dyadic probability $p$. For any real probability $p$, consider a sequence of dyadic probabilities $p(n)$, $n = 1, 2, \ldots,$ such that $\lim_n p(n) = p$. For given amounts $x + h_2$ and $x + h_3$ in $C$, let $x + h_1$ and $x + h_1(n)$, $n = 1, 2, \ldots,$ denote the certainty equivalents of $\ell_p$ and $\ell_{p(n)}$, $n = 1, 2, \ldots$. Then, $x + h_1 = u^{-1}(p\,u(x + h_2) + q\,u(x + h_3))$ and $x + h_1(n) = u^{-1}(p(n)\,u(x + h_2) + q(n)\,u(x + h_3))$, $n = 1, 2, \ldots,$ and thus $h_1 = \lim_n h_1(n)$ as a result of the continuity of $u^{-1}$. Consider any amount $x' = b + x$ with $x' + h_2$ and $x' + h_3$ in $C$. Then, $b + x + h_1(n)$ is the certainty equivalent of $b + \ell_{p(n)}$ for $n = 1, 2, \ldots$. Therefore, $b + x + h_1$ is the certainty equivalent of $b + \ell_p$ as a result again of the continuity of $u^{-1}$.

The above argument establishes that, for a continuous expected-utility model, each of c. absolute risk constancy and g. absolute risk constancy implies p. absolute risk constancy. Thus, by the first part of this proof, each of these conditions implies a linear-exponential utility function.

It remains to show that if there is a linear-exponential utility function, then the three conditions of risk constancy are satisfied. The verification is straightforward, and hence can be omitted.

Proof of Proposition 1. We will construct a function $u(x)$ that is defined and increasing on the interval $[0, 1)$ such that, for the corresponding preference relation $\succsim$, any lottery $\ell_{x,x'}$ with $x, x'$ in $[0, 1)$ does not have a certainty equivalent. Thus, the preference relation $\succsim$ is discontinuous and vacuously satisfies the conditions of c. absolute risk constancy and g. absolute risk constancy. A similar result can be obtained for any interval $C$ by choosing a continuous, increasing function $f$ with domain $C$ and range a subinterval of $[0, 1)$, and considering $u(f(x))$ as a utility function defined on $C$.

Any real number $x$ in $[0,1)$ can be represented as a sum

$$x = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \ldots + \frac{a_n}{2^n} + \ldots \tag{A6}$$

where $a_n = 0$ or $1$ for any $n = 1, 2, \ldots$ and for any $N$ there exists an $n \geq N$ such that $a_n = 0$. The representation (A6) defines a one-to-one correspondence between $[0,1)$ and the set of sequences $\{a_n\}$ as described. For any $x$ in $[0,1)$, define

$$u(x) = \frac{2a_1}{3^1} + \frac{2a_2}{3^2} + \ldots + \frac{2a_n}{3^n} + \ldots$$

where $\{a_n\}$ is the unique sequence corresponding to $x$.

Then, $u(x)$ is an increasing function. For if $x > x'$ are two amounts in $[0,1)$ and $\{a_n\}$, $\{a'_n\}$ are the corresponding sequences, then there exists an $N$ such that $a_N > a'_N$ but $a_n = a'_n$ for all $n < N$. Hence,

$$u(x) \geq \frac{2a_1}{3^1} + \ldots + \frac{2a_{N-1}}{3^{N-1}} + \frac{2}{3^N} > u(x') .$$

Moreover, there is not a number $\hat{x}$ between $x$ and $x'$ such that $u(\hat{x}) = \frac{1}{2}u(x) + \frac{1}{2}u(x')$. For if $\bar{u} = \frac{1}{2}u(x) + \frac{1}{2}u(x')$, then

$$\bar{u} = \frac{2\bar{a}_1}{3^1} + \frac{2\bar{a}_2}{3^2} + \ldots + \frac{2\bar{a}_n}{3^n} + \ldots$$

where $\bar{a}_n = \frac{1}{2}(a_n + a'_n)$, $n = 1, 2, \ldots$ Let $N$ denote the least $n$ such that $a_n \neq a'_n$. Hence, $\bar{a}_N = \frac{1}{2}$, and thus for $S = 2a_1/3^1 + \ldots + 2a_{N-1}/3^{N-1}$ it follows that $S + 3^{-N} \leq \bar{u} < S + 3^{-N} + 3^{-N}$. Let $\{\hat{a}_n\}$ denote the sequence corresponding to a number $\hat{x}$ in $[0,1)$. If $\hat{a}_M \neq a_M$ for a least integer $M<N$, then $u(\hat{x}) < \bar{u}$ if $\hat{a}_M < a_M$ and $u(\hat{x}) > \bar{u}$ if $\hat{a}_M > a_M$. If $\hat{a}_n = a_n$ for all $n<N$, then $u(\hat{x}) < S+0+3^{-N}$ if $\hat{a}_N = 0$ and $u(\hat{x}) \geq S + 2\cdot 3^{-N} + 0$ if $\hat{a}_N = 1$. Thus in every case, $u(\hat{x}) \neq \bar{u}$.

Proof of Theorem 6. Let $C_g$ denote the range of the scaling function $t=g(x)$ restricted to the domain $C$. Let $L_g$ denote the set of lotteries having consequences in $C_g$. Then, the given expected-utility model $(L, C, \succsim, u)$ corresponds to an expected-utility model $(L_g, C_g, \succsim_g, u_g)$ induced by the scaling function $g$. Here, $\langle t_i, p_i \rangle \succsim_g \langle t'_i, p'_i \rangle$ in $L_g$ if and only if $\langle g^{-1}(t_i), p_i \rangle \succsim \langle g^{-1}(t'_i), p'_i \rangle$ in $L$, and $u_g(t) = u(g^{-1}(t))$.

The preference relation $\succsim$ satisfies one of the conditions (a)-(c) in Theorem 6 if and only if the preference relation $\succsim_g$ satisfies the corresponding condition (a)-(c) in Theorem 5. Moreover, the utility function $u$ belongs to the parametric family (16) if and only if the utility function $u_g$ belongs to the linear-exponential family (11). Thus, Theorem 5 implies Theorem 6 by means of the correspondence between $(L, C, \succsim, u)$ and $(L_g, C_g, \succsim_g,$ and $u_g)$.

Proof of Theorem 7. As shown in Section 4, $x \circ h = (h+a)(a+x) - a$ for the group operation of shift multiplication. Thus, for any $k=a+h>0$, the $\circ$-risk constant conditions (13)-(15) become the relative risk constant conditions (18)-(20) when $\circ$ is shift multiplication. Moreover, the family of functions (16) becomes the family of functions (21). Thus, Theorem 7 is a corollary of Theorem 6.

Proof of Proposition 2. Part (a) is a direct corollary of Theorem 7 since a utility function (21) is strictly concave if and only if $r<1$. Part (b) is due to Pratt (1964) in that such a utility function is shown there to represent decreasing risk aversion.

To show part (c), consider a utility function $u(x)$ of the form (21) with $r < 1$. For a fixed amount $x$, let $x - f(h)$ denote the certainty equivalent of a lottery $\ell_{x+h,\, x-h}$. Risk aversion implies that $f(h) \geq 0$.

In general, $f(h) = x - u^{-1}\left(\tfrac{1}{2}u(x+h) + \tfrac{1}{2}u(x-h)\right) = \int_0^h f'(t)\,dt$ where

$$f'(t) = -\frac{\tfrac{1}{2}u'(x+t) - \tfrac{1}{2}u'(x-t)}{u'\left(u^{-1}\left(\tfrac{1}{2}u(x+t) + \tfrac{1}{2}u(x-t)\right)\right)} \leq \frac{\tfrac{1}{2}u'(x-t) - \tfrac{1}{2}u'(x+t)}{u'(x)}$$

Hence,

$$0 \leq f(h) \leq \frac{2u(x) - u(x+h) - u(x-h)}{2u'(x)} = -\frac{h^2}{2}\;\frac{u''(x+\theta h)}{u'(x)}$$

for some $-1 < \theta < 1$. However, for a utility function (21) the ratio $-u''(x+\theta h)/u'(x)$ is equal to $(1-r)\left((x+\theta h)/x\right)^{r-1}/(x+\theta h)$. This expression tends to 0 as $x$ tends to $+\infty$, and hence $f(h)$ tends to 0 as $x$ tends to $+\infty$.

REFERENCES


Aczel, J., Lectures on Functional Equations and their Applications, Academic Press, New York, 1966.

Arrow, K. J., Essays in the Theory of Risk-Bearing, Markham Publishing Co., Chicago, 1971.

Bell, D. E., "Double-Exponential Utility Functions." Mathematics of Operations Research, 11 (May 1986), 351-361.

Darboux, G., "Sur la Composition des Forces en Statique," Bull. des Sciences Math., 9 (1875), 281-288.

Debreu, G., "Topological Methods in Cardinal Utility Theorey," in K. J. Arrow, S. Karlin, and P. Suppes (eds.), Mathematical Methods in the Social Sciences, Stanford University Press, Stanford, Calif. (1960), 16-26.

deFinetti, B., "Sul Concetto di Media," Giornale dell' Instituto Italiano degli Attuari, 2 (1931), 369-396.

deGroot, M. H., Optimal Statistical Decisions, McGraw Hill, New York, 1970.

Dybvig, P. H. and S. A. Lippman, "An Alternative Characterization of Decreasing Absolute Risk Aversion," Econometrica, 51 (1983), 223-224.

Epstein, L. G., "Decreasing Risk Aversion in Mean-Variance Analysis," Econometrica, 53 (July 1985), 945-961.

Farquhar, P. H., "Utility Assessment Methods," Management Science, 30 (November 1984), 1283-1300.

Farquhar, P. H. and Y. Nakamura, "Constant Exchange Risk Properties," To appear in Operations Research (Working Paper, 1986) .

Fishburn, P. C., "Independence in Utility Theory with Whole Product Sets," Operations Research, 13 (1965), 28-45.

Fishburn, P. C., Utility Theory for Decision Making, Wiley, New York, 1970

Fishburn, P. C., "Unbounded Expected Utility," Ann. Statist., 3 (1975), 884-896.

Fishburn, P. C., The Foundations of Expected Utility, Reidel Publishing Co., Dordrecht, Holland, 1982.

Grayson, C. J., "Decisions under Uncertainty: Drilling Decisions by Oil and Gas Operators," Working paper, Graduate School of Business, Harvard University, Boston, Mass., 1960.

Hamel, G., "Eine Basis aller Zahlen und die Unstatigen Losungen der Funktionalgleichung $f(x+y) = f(x) + f(y)$," Math. Annalen, 60 (1905), 459-462.

Hammond, J. S., "Simplifying the Choice between Uncertain Prospects," Management Science, 20 (1974), 1047-1072.

Hardy, G. H., J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, U. K, 1934.

Harvey, C. M., "Conditions on Risk Attitude for a Single Attribute," Management Science, 27 (February 1981), 190-203. (Erratum in Vol. 30, No. 12, p. 1548.).

Harvey, C. M., "Cost-Benefit Study of a Proposed Daylight Running Lights Safety Programme," J. of the Operational Research Society, 34 (1983), 37-43.

Harvey, C. M., "Assessment of Risk Attitudes by a Utility Dependence Condition," Operations Research, 32 (1984), 457-464.

Harvey, C. M., "Preference Functions for Catastrophe and Risk Inequity," Large Scale Systems, 8 (1985), 131-146.

Harvey, C. M., "Utility Functions for Infinite-Period Planning," Working paper, Dickinson College, 1986a .

Harvey, C. M., "A Prescriptive Model for Averse-Prone Risk Attitudes," Working paper, IIASA, A-2361 Laxenburg, Austria, 1986b .

Herstein, I. N. and J. Milnor, "An Axiomatic Approach to Measurable Utility," Econometrica, 21 (1953), 291-297.

Jensen, N. E., "An Introduction to Bernoullian Utility Theory. I: Utility Functions," Swedish Journal of Economics, 69 (1967), 163-183.

Keeney, R. L., "Quasi-Separable Utility Functions," Naval Research Logistics Quarterly, 15 (1968), 551-565.

Keeney, R. L., "Multiplicative Utility Functions," Operations Research, 22 (1974), 22-34.

Keeney, R. L., "Decision Analysis: An Overview," Operations Research, 30 (1982), 803-838.

Keeney, R. L. and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs, Wiley, New York, 1976.

Kihlstrom, R. E., D. Romer, and S. Williams, "Risk Aversion with Random Initial Wealth," Econometrica, 49 (1981), 911-920.

Ledyard, J. C., "A Pseudo-Metric Space of Probability Measures and the Existence of Measurable Utility," Ann. Math. Statist. 42 (1971), 794-798.

Luce, R. D. and H. Raiffa, Games and Decisions, Wiley, New York, 1957.

Machina, M. J., "A Stronger Characterization of Declining Risk Aversion, Econometrica, 50 (July 1982), 1069-1079.

Magee, J. F., "Decision Trees for Decision Making," Harvard Business Review (1964), 126-135.

Marschak, J., "Rational Behavior, Uncertain Prospects, and Measurable Utility," Econometrica, 18 (1950), 111-141.

Meyer, R. F., "On the Relationship among the Utility of Assets, the Utility of Consumption, and Investment Strategy in an Uncertain, but Time Invariant World," in OR 69: Proceedings of the Fifth International Conference on Operational Research, J. Lawrence, ed. Tavistock Publications, London, 1970.

Meyer, R. F., "Some Notes on Discrete Multivariate Utility," Working paper, Graduate School of Business, Harvard University, 1972.

Nielsen, L. T., "Unbounded Expected Utility and Continuity," Mathematical Social Sciences, 8 (1984), 201-216.

Pfanzagl, J., "A General Theory of Measurement Applications to Utility," Naval Research Logistics Quarterly, 6 (1959), 283-294.

Pollak, R. A., "Additive Von Neumann-Morgenstern Utility Functions," Econometrica, 35 (1967), 485-494.

Pratt, J. W., "Risk Aversion in the Small and in the Large," Econometrica, 32 (January-April 1964), 122-136.

Raiffa, H., Decision Analysis: Introductory Lectures on Choices Under Uncertainty, Addison-Wesley, Reading, Mass., 1968.

Richard, S. F., "Multivariate Risk Aversion, Utility Independence and Separable Utility Functions," Management Science, 22 (1975), 12-21.

Roberts, F. S., Measurement Theory with Applications to Decisionmaking, Utility and the Social Sciences, Addison-Wesley, Reading Mass., 1979.

Ross, S. A., "Some Stronger Measures of Risk Aversion in the Small and the Large with Applications," Econometrica, 49 (May 1981), 621-638.

Rothblum, U. G., "Multivariate Constant Risk Posture," J. of Economic Theory, 10 (1975), 309-332.

Rubenstein, M., "The Strong Case for the Generalized Logarithmic Utility Model as the Premier Model of Financial Markets," in H. Levy and M. Sarnot (eds.), Financial Decision Making under Uncertainty, Academic Press, New York, 1977.

Savage, L. J., The Foundations of Statistics, Wiley, New York, 1954.

Toulet, C., "An Axiomatic Model of Unbounded Utility Functions," Mathematics of Operations Research, 11 (February 1986), 81-94.

von Neumann, J. and O. Morgenstern, Theory of Games and Economic Behavior, 2nd ed., Princeton University Press, Princeton, N.J., 1947.

Wehrung, D. A., K. R. MacCrimmon, and K. M. Brothers, "Utility Assessment: Domains, Stability, and Equivalence Procedures," INFOR, 22 (1984), 94-115.