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A PRIORI ESTIMATES FOR OPERATIONAL DIFFERENTIAL INCLUSIONS

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A priori estimates for operational differential inclusions

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FOREWORD

The author proves a set-valued Gronwall lemma and a relaxation theorem for the semilinear differential inclusion

$$x' \in Ax + F(t,x), \;\; x(0) = x_0$$

where A is the infinitesimal generator of a C_0-semigroup on a separable Banach space X and $F : [0,T] \times X \to X$ is a set-valued map. This result is important for investigation of many futures of semilinear inclusions, for instance, infinitesimal generators of reachable sets, variational inclusions, etc.

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Introduction

This paper is concerned with the multivalued operational equation (differential inclusion)

$$(1) \qquad x' \in Ax + F(t,x), \; x(0) = x_0$$

where A is the infinitesimal generator of a C_0-semigroup $\{G(t)\}_{t\geq 0}$ on a separable Banach space X and $F : [0,T] \times X \to X$ is a set-valued map. Such inclusion is a convenient tool to investigate for instance the semilinear control system

$$x' \in Ax + f(t,x,u), \; u \in U(t,x), \; x(0) = x_0$$

where $U : [0,T] \times X \to X$ is a set-valued map of controls (depending on the time and on the state). Setting $F(t,x) = f(t,x,U(t,x))$ we reduce the above control system to the inclusion (1).

Differential inclusion

$$(2) \qquad x'(t) \in F(t,x(t))$$

in finite dimensional context was extensively studied in the literature since 30$^{\underline{ies}}$. It was initiated by the Polish and French mathematicians Zaremba in [28], [29] and Marchaud [20]. They were mostly interested by existence results and also investigated some of their qualitative properties. While Zaremba studied the so-called paratingent solutions, Marchaud was mainly concerned with the contingent ones. Later on Ważewski [26] have shown that one may reduce his interest to more "classical", Caratheodory type solutions, i.e., absolutely continuous functions verifying (2) almost everywhere.

The interest to the differential inclusion (2) was renewed in earlier sixties, when mathematicians got attracted by a new domain: control theory. Filippov [10] and Ważewski [27] have shown that under very mild assumptions the control system

$$(3) \qquad x' = f(t,x,u(t)), \; u(t) \in U \text{ is measurable}$$

may be reduced to the differential inclusion (2). This tremendously simplified the study of the closure of trajectories to (3) and led to the celebrated Filippov-Ważewski relaxation theorem (see also [2, p.123]).

Control system (3) with state-independent control subset U can be considered as a family of differential equations: with every control $u(\cdot)$ (measurable selection of U) one can associate the ordinary differential equation

$$x'(t) = \varphi_u(t, x(t))$$

where $\varphi_u(t,x) = f(t,x,u(t))$. Differential inclusion also encompass much more sophisticated control systems:

1. closed loop control systems

$$x'(t) = f(t, x(t), u(t)), \ u(t) \in U(t, x(t)) \tag{4}$$

2. implicit control systems

$$f(t, x(t), x'(t), u(t)) = 0, \ u(t) \in U(t, x(t)) \tag{5}$$

3. systems with uncertainties

$$x'(t) \in f(t, x(t), u(t)) + \varepsilon(t,x)B, \ u(t) \in U(t, x(t)) \tag{6}$$

   where $\varepsilon(t,x)$ is a function incorporating the errors of the model.

Setting $F(t,x) = f(t,x,U(t,x))$ in the first case, $F(t,x) = \{v | 0 \in f(t,x,v,U(t,x))\}$ in the second one and $F(t,x) = f(t,x(t),U(t,x)) + \varepsilon(t,x)B$ in the third one, we replace the control systems (4)-(6) by the differential inclusion (2). To proceed further a differential calculus of set-valued maps adequate for control theory problems had to be developed. We refer to [2], [11], [4], [13]-[16] for many results on differential inclusions and their applications to control theory for finite dimensional control systems.

The last years there were many attempts to get similar results for infinite dimensional differential inclusions (see for instance [24], [6], [5] and the bibliographies contained therein). In monograph [24], it was shown that many results on (2) known in the finite dimensional context may be extended to compact valued maps in infinite dimensional Banach spaces. However the field of applications of results obtained so far is very restrictive. On one hand the compactness hypothesis is too strong, on the other, (3) keep us far from the distributed parameter systems. Let us mention also that state-constraint problems for (1) were studied by Shi Shuzhong in [22] and [23].

In this paper we study (1) and its mild trajectories, i.e., mild solutions of the Cauchy problem

$$\begin{cases} x' = Ax + f(t,x), \; f(t,x) \in F(t,x) \\ x(0) = \xi \end{cases}$$

It is well known that in general the Cauchy problem

$$\begin{cases} x' = Ax + f(t,x) \\ x(0) = x_0 \end{cases}$$

does not have classical solutions and that a way to overcome this difficulty is to look for continuous solutions to the integral equation

$$x(t) = G(t)x_0 + \int_0^t G(t-s)f(s,x(s))ds$$

This is why the concept of the mild solution is so convenient for solving (1).

We show here that many results which allow to apply differential inclusions to finite dimensional control systems are valid as well for (1). We start in Section 1 by a theorem analogous to the Filippov theorem [11], a kind of set-valued Gronwall's lemma (see also [2], [4]). This allows to prove in Section 2 a relaxation theorem for (1). Namely, that under some technical assumptions, the mild trajectories of (1) are dense in the mild trajectories of the convexified inclusion:

$$x' \in Ax + \overline{co}\, F(t,x), \; x(0) = x_0$$

In Section 3 we investigate infinitesimal generators of the reachable map associated to (1) and in Section 4, the variational inclusion for (1). We prove in Section 5 a necessary condition for optimality for an infinite dimensional optimal control problem, obtained thanks to the relaxation theorem.

Some theorems similar to the one presented in Section 5 may be found in [8] and [9] for the Hilbert space X and in [17] for the separable Banach space X with the norm Gâteaux differentiable away from zero. The main difficulty we overcome is the lack of such smoothness of the norm. In this way our result applies when X is for instance the space of essentially bounded maps or the space of continuous functions.

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1 Quasitrajectories and trajectories

Consider a separable Banach space X and a strongly continuous semigroup $G(t) \in \mathcal{L}(X, X)$, $t \geq 0$ of bounded linear operators from X to X having the infinitesimal generator A. Let $0 \leq t_0 < T$ be given and F be a set-valued map from $[t_0, T] \times X$ into closed nonempty subsets of X. We associate with it the differential inclusion

$$(7) \qquad x'(t) \in Ax(t) + F(t, x(t))$$

Denote by $\mathcal{C}(t_0, T; X)$ the Banach space of continuous functions from $[t_0, T]$ to X with the norm $\|x\|_{\mathcal{C}} = \sup_{t \in [t_0, T]} \|x(t)\|$ and by $\mathcal{L}^1(t_0, T; X)$ the Banach space of Bochner's integrable functions (see for instance [19, p.78]) from $[t_0, T]$ to X with the norm $\|x\|_1 = \int_{t_0}^{T} \|x(t)\| dt$. Set $\mathcal{L}^1(t_0, T) = \mathcal{L}^1(t_0, T; \mathbf{R}_+)$.

A continuous function $x \in \mathcal{C}(t_0, T; X)$ is called a *mild trajectory* of (7), if there exist $x_0 \in X$ and a Bochner integrable function $f \in \mathcal{L}^1(t_0, T; X)$ such that

$$(8) \qquad f(t) \in F(t, x(t)) \text{ a.e. in } [t_0, T]$$

$$(9) \qquad \forall\, t \in [t_0, T], \; x(t) = G(t - t_0)x_0 + \int_{t_0}^{t} G(t - s) f(s) ds$$

i.e. f is a Bochner integrable selection of the set-valued map $t \to F(t, x(t))$ and x is the mild solution of the initial value problem

$$(10) \qquad \begin{cases} x'(t) = Ax(t) + f(t), & t \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

We shall call $(x, f) \in \mathcal{C}(t_0, T; X) \times \mathcal{L}^1(t_0, T; X)$ a trajectory-selection pair of (7) if f verifies (8) and x is a mild solution of (10). This notion extends the definition of solution of differential inclusion for finite dimensional spaces because of the following

Remark — When X is a finite dimensional space and $G(\cdot) \equiv Id$, then every mild trajectory x is an absolutely continuous function satisfying

$$x'(t) \in F(t, x(t)) \text{ a.e. in } [t_0, T]$$

Indeed in this case the function x defined by (9) is absolutely continuous and $x'(t) = f(t)$ a.e. in $[t_0, T]$. □

We study here some qualitative properties of mild trajectories.

A set-valued map $\Phi : X \to 2^X$ is called L-Lipschitz on $K \subset X$ if for all $x \in K$, $\Phi(x) \neq \emptyset$ and

$$\forall\, x,\, y \in K, \;\; \Phi(x) \;\subset\; \Phi(y) \;+\; L\|x - y\|\, B$$

where B denotes the closed unit ball in X.

Remark — An equivalent definition may be given using the Hausdorff pseudometric. Namely let $d(\cdot,\cdot)$ denote the Hausdorff pseudometric on nonempty subsets of X. If for some $M > 0$ and all $x, y \in K$ we have $d(\Phi(x), \Phi(y)) \leq M\|x - y\|$ then for every $L > M$ and $x, y \in K$, $\Phi(x) \subset \Phi(y) + L\|x - y\|\, B$. □

Throughout the whole paper we shall consider the Lebesgue measure μ on $[t_0, T]$.

Theorem 1.1 ([3]) *Let $U : [t_0, T] \to X$ be a set-valued map with closed nonempty images. Then the following statements are equivalent*

i) — U is measurable in the sense that for every open set $\mathcal{O} \subset X$ the set $\{\, t \in [t_0, T] \mid U(t) \cap \mathcal{O} \neq \emptyset \,\}$ is measurable

ii) — There exist measurable selections $u_n(t) \in U(t)$ such that for every $t \in [t_0, T]$, $U(t) = \overline{\bigcup_{n \geq 1} u_n(t)}$.

Moreover if U is single-valued, then the above statements are equivalent to

iii) There exist measurable functions $u_n : [t_0, T] \to X$ assuming only finite number of values such that for almost every $t \in [t_0, T]$, $\lim_{n\to\infty} u_n(t) = U(t)$

iv) There exist a negligible set $\mathcal{N} \subset [t_0, T]$ and measurable functions $u_n : [t_0, T] \to X$ assuming only countable number of values such that u_n converge to U uniformly on $[t_0, T] \backslash \mathcal{N}$.

Consider the solution set of (7) from the point $x_0 \in X$ on $[t_0, T]$:

$$S_{[t_0,T]}(x_0) = \{x \mid x \text{ is a mild trajectory of (7) on } [t_0, T], x(t_0) = x_0\}$$

Let $y_0 \in X$, $g \in \mathcal{L}^1(t_0, T; X)$ and $y \in C(t_0, T; X)$ be a mild solution of the Cauchy problem

$$\begin{cases} y'(t) \;=\; Ay(t) \;+\; g(t) \\ y(t_0) \;=\; y_0 \end{cases}$$

The aim of this section is to estimate the distance from y to the set $S_{[t_0,T]}(x_0)$ under several assumptions on F:

$$\left\{\begin{array}{ll} H_1) & \forall\, x \in X \text{ the set-valued map } F(\cdot, x) \text{ is measurable} \\ H_2) & \exists\, \beta > 0,\ k \in \mathcal{L}^1(t_0,T) \text{ such that for almost all } t \in [t_0,T] \\ & \text{the map } F(t,\cdot) \text{ is } k(t) - \text{Lipschitzian on } y(t) + \beta B \\ H_3) & \text{The function } t \to dist(g(t), F(t,y(t))) \text{ belongs to } \mathcal{L}^1(t_0,T) \end{array}\right.$$

Remark — From Lemmas 1.4 and 1.5 proved below follows that under the assumptions $H_1)$ and $H_2)$ the function $t \to dist(g(t), F(t,y(t)))$ is always measurable. □

Theorem 1.2 *Let $\delta \geq 0$, $M = \sup_{t\in[0,T-t_0]} \|G(t)\|$. Assume that $H_1) - H_3)$ hold true and set $\gamma(t) = dist(g(t), F(t,y(t)))$, $m(t) = Mexp\left(M \int_{t_0}^t k(s)ds\right)$*

$$\eta(t) \;=\; m(t)\left(\delta \;+\; \int_{t_0}^t \gamma(s)ds\right)$$

If $\eta(T) < \beta$, then for all $x_0 \in X$ with $\|y_0 - x_0\| \leq \delta$ and all $\varepsilon > 0$, there exist $x \in S_{[t_0,T]}(x_0)$ and $f \in \mathcal{L}^1(t_0,T;X)$ satisfying (8), (9) such that for all $t \in [t_0,T]$

$$\|x(t) - y(t)\| \;\leq\; \eta(t) + \;\varepsilon(t - t_0)m(t)$$

and for almost every $t \in [t_0,T]$

$$\|f(t) - g(t)\| \leq k(t)(\eta(t) + \varepsilon(t-t_0)m(t)) + \gamma(t) + \varepsilon$$

Remark — When X is a finite dimensional space, the above estimation holds true with $\varepsilon = 0$. This follows from the celebrated Filippov theorem [11] (see also [2], [4]). □

The proof reminds in many aspects the one from [2] for the finite dimensional case. We need three following lemmas.

Lemma 1.3 *Let $U : [t_0,T] \to X$ be a measurable set-valued map with closed nonempty images and $g : [t_0,T] \to X$, $k : [t_0,T] \to \mathbf{R}_+$ be measurable single-valued maps. Assume that*

$$W(t) \;:=\; U(t) \,\cap\, (g(t) \;+\; k(t)B) \;\neq\; \emptyset \text{ a.e. in } [t_0,T]$$

where B denotes the closed unit ball in X. Then there exists a measurable function $u : [t_0,T] \to X$ such that $u(t) \in W(t)$ almost everywhere.

The proof follows from [3, pp.87, 88].

Lemma 1.4 *Let F and y be as in Theorem 1.2 and $x \in C(t_0, T; X)$ be such that $\|x - y\|_C \le \beta$. Then the map $t \to F(t, x(t))$ is measurable.*

Proof — By Theorem 1.1 *iv*) there exist a negligible set $\mathcal{N} \subset [t_0, T]$ and measurable functions $x_n : [t_0, T] \to X$, $k_n : [t_0, T] \to \mathbf{R}_+$, $n \ge 1$ assuming only a countable number of values and converging to x (respectively k) uniformly on $[t_0, T] \backslash \mathcal{N}$. It is not restrictive to assume that $k_n > k$ on $[t_0, T] \backslash \mathcal{N}$ and that for every $t \in [t_0, T] \backslash \mathcal{N}$, $F(t, \cdot)$ is $k(t)-$ Lipschitz on $y(t) + \beta B$. Set $q_n = \sup_{t \in [t_0, T] \backslash \mathcal{N}} \|x_n(t) - x(t)\|$. Let $\mathcal{O} \subset X$ be an open set. For all $n \ge 1$, $t \in [t_0, T] \backslash \mathcal{N}$ define the open sets

$$V_n(t) \ = \ \{x \in \mathcal{O} \mid dist\,(x,\ X \backslash \mathcal{O}) \ > \ k_n(t) q_n\}$$

Since k_n are measurable and assume only countable number of values, so does V_n. From H_1) we deduce that for all $n \ge 1$ the set-valued map $t \to F(t, x_n(t))$ is measurable. Hence, by the definition of V_n, the sets $\{t \in [t_0, T] \backslash \mathcal{N} \mid F(t, x_n(t)) \cap V_n(t) \neq \emptyset\}$ are measurable. This yields that the set

$$\bigcup_{N \ge 1} \bigcap_{n \ge N} \{t \in [t_0, T] \setminus \mathcal{N} \mid \ F(t, x_n(t)) \ \cap \ V_n(t) \ \neq \ \emptyset\}$$

is measurable. To end the proof it is enough to show that

$$(11) \quad \left\{ \begin{array}{ll} \bigcup_{N \ge 1} \bigcap_{n \ge N} & \{t \in [t_0, T] \setminus \mathcal{N} \mid \ F(t, x_n(t)) \ \cap \ V_n(t) \ \neq \ \emptyset\} \ = \\ & \{t \in [t_0, T] \setminus \mathcal{N} \mid \ F(t, x(t)) \cap \mathcal{O} \ \neq \ \emptyset\} \end{array} \right.$$

Fix $t \in [t_0, T] \backslash \mathcal{N}$, $v \in F(t, x(t)) \cap \mathcal{O}$ and set $\rho = dist(v, X \backslash \mathcal{O}) > 0$. By the Lipschitz continuity of $F(t, \cdot)$, for every $n \ge 1$ $v \in F(t, x_n(t)) + k(t)\|x_n(t) - x(t)\| B \subset F(t, x_n(t)) + k_n(t) q_n B$. Hence there exist $v_n \in F(t, x_n(t))$ satisfying $\|v_n - v\| \le k_n(t) q_n$. But $k_n(t) q_n \to 0+$ as $n \to \infty$ and therefore for all large n, $dist(v_n, X \backslash \mathcal{O}) \ge dist(v, X \backslash \mathcal{O}) - \|v_n - v\| \ge \rho - k_n(t) q_n > k_n(t) q_n$. Thus for all large n, $v_n \in V_n(t)$.

Conversely assume that $t \in [t_0, T] \backslash \mathcal{N}$ is such that for all large n the set $F(t, x_n(t)) \cap V_n(t) \neq \emptyset$. Pick $v_n \in F(t, x_n(t)) \cap V_n(t)$ and $w_n \in F(t, x(t))$ such that $\|v_n - w_n\| \le k(t)\|x_n(t) - x(t)\| \le k_n(t) q_n$. Thus $dist(w_n, X \backslash \mathcal{O}) \ge dist(v_n, X \backslash \mathcal{O}) - \|w_n - v_n\| > k_n(t) q_n - k_n(t) q_n = 0$ and therefore $w_n \in \mathcal{O}$. This proves (11) and ends the proof. □

Lemma 1.5 *Let $U : [t_0, T] \to X$ be a measurable set-valued map with closed nonempty images and $u : [t_0, T] \to X$ be a measurable function. Then the function $t \to dist(u(t), U(t))$ is measurable.*

Proof — By Theorem 1.1 *ii*) there exist measurable selections $u_n(t) \in U(t)$ such that for all $t \in [t_0, T]$, $\overline{\bigcup_{n\geq 1} u_n(t)} = U(t)$. Set

$$g_i(t) = dist\left(u(t), \bigcup_{n=1}^{n=i} u_n(t)\right)$$

Then $\lim_{i\to\infty} g_i(t) = dist(u(t), U(t))$. Thus the map $t \to dist(u(t), U(t))$ is the pointwise limit of measurable functions g_i and from [3, p.61] follows that g is measurable. □

Observe that the two last lemmas yield that the function γ defined in Theorem 1.2 is measurable.

Proof of Theorem 1.2 — It is not restrictive to assume that $t_0 = 0$. Let $\varepsilon > 0$ be so small that $\eta(T)+\varepsilon T m(T) \leq \beta$. Set $\chi(t) = M\left(\delta + \int_0^t \gamma(s)ds + \varepsilon t\right)$.

We claim that it is enough to construct sequences $x_n \in C(0,T;X)$, $f_n \in \mathcal{L}^1(0,T;X)$, $n = 0, 1, \ldots$ such that

$$\forall\, t \in [0,T], \quad x_n(t) = G(t)x_0 + \int_0^t G(t-s)f_n(s)ds \tag{12}$$

$$\forall\, t \in [0,T], \quad \|x_1(t) - y(t)\| \leq \chi(t) \tag{13}$$

$$f_0 = g, \quad \|f_1(t) - g(t)\| \leq \gamma(t) + \varepsilon \text{ a.e. in } [0,T] \tag{14}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \text{ for } n \geq 1 \text{ and } t \in [0,T] \tag{15}$$

$$\|f_{n+1}(t) - f_n(t)\| \leq k(t)\,\|x_n(t) - x_{n-1}(t)\| \text{ for } n \geq 1 \text{ a.e. in } [0,T] \tag{16}$$

Indeed observe that (12), (16) and (13) together imply that for almost every $t \in [t_0, T]$

$$\begin{gathered}
\|x_{n+1}(t) - x_n(t)\| \leq \int_0^t \|G(t-t_1)\|\,\|f_{n+1}(t_1) - f_n(t_1)\|\,dt_1 \leq \\
M\int_0^t k(t_1)\,\|x_n(t_1) - x_{n-1}(t_1)\|\,dt_1 \leq \\
M\int_0^t k(t_1)\int_0^{t_1} \|G(t_1-t_2)\|\,\|f_n(t_2) - f_{n-1}(t_2)\|dt_2 dt_1 \leq \\
M^2\int_0^t k(t_1)\int_0^{t_1} k(t_2)\,\|x_{n-1}(t_2) - x_{n-2}(t_2)\|\,dt_2 dt_1 \leq \ldots \leq \\
M^n\int_0^t k(t_1)\int_0^{t_1} k(t_2)\ldots\int_0^{t_{n-1}} k(t_n)\,\|x_1(t_n) - y(t_n)\|\,dt_n \ldots dt_1 \leq \\
\chi(t)M^n\int_0^t k(t_1)\int_0^{t_2} k(t_2)\ldots\int_0^{t_{n-1}} k(t_n)dt_n\ldots dt_1 = \chi(t)\left[M\int_0^t k(\tau)d\tau\right]^n/n!
\end{gathered}$$

Thus $\{x_n\}$ is a Cauchy sequence in the Banach space $C(0,T;X)$. Hence, by (16), for almost every $t \in [0,T]$ the sequence $\{f_n(t)\}$ is Cauchy in X.

Moreover from (13) and the last inequality we get

$$(17)\quad \begin{cases} \|x_n(t) - y(t)\| \le \|x_1(t) - y(t)\| + \sum_{i=1}^{n-1} \|x_{i+1}(t) - x_i(t)\| \le \\ \chi(t)\left\{1 + M\int_0^t k(\tau)d\tau + \left[M\int_0^t k(\tau)d\tau\right]^2/2! + \dots\right\} \le \\ \chi(t)exp\left\{M\int_0^t k(\tau)d\tau\right\} = \eta(t) + \varepsilon t m(t) \end{cases}$$

and, by the choice of ε,

$$(18)\qquad \forall\, n \ge 0,\ \|x_n - y\|_C \le \beta$$

Furthermore from (16), (14) and from the sequence of inequalities (17) follows that

$$(19)\quad \begin{cases} \|f_n(t) - g(t)\| \le \sum_{i=1}^{n-1} \|f_{i+1}(t) - f_i(t)\| + \|f_1(t) - g(t)\| \le \\ k(t)\ \sum_{i=1}^{n-2} \|x_i(t) - x_{i-1}(t)\| + \gamma(t) + \varepsilon \le \\ k(t)\ (\eta(t) + \varepsilon t m(t)) + \gamma(t) + \varepsilon \text{ a.e. in } [0,T] \end{cases}$$

Since the sequence $\{x_n\}$ is Cauchy we define $x \in \mathcal{C}(0,T;X)$ as the limit of x_n. By (19) the sequence $\{f_n\}$ is integrably bounded and we have already seen that for almost all $t \in [0,T]$, $\{f_n(t)\}$ is Cauchy. Thus we may define $f \in \mathcal{L}^1(0,T;X)$ by $f(t) = \lim_{n\to\infty} f_n(t)$.

From (18) and the assumption H_2) follows that for almost every $t \in [0,T]$ the set

$$Q(t) := \{\, (x,v) \mid v \in F(t,x);\ \|x - y(t)\| \le \beta \,\}$$

is closed. Moreover from (15) and (18) for all $n \ge 1$ and all $t \in [0,T]$, $(x_{n-1}(t), f_n(t)) \in Q(t)$. Taking the limit we obtain that (8) holds true a.e. in $[0,T]$. Furthermore taking the limit in (12) and using the Lebesgue dominated convergence theorem we get $x \in S_{[0,T]}(x_0)$. Passing to the limits in (17) and (19) yields the desired estimations on x and f ($t_0 = 0!$).

To construct x_n, f_n as above we proceed by the induction. From Lemma 1.4 the set-valued map $t \to F(t,y(t))$ is measurable and has closed images. Furthermore for almost every $t \in [t_0,T]$, $F(t,y(t)) \cap \{g(t) + (\gamma(t)+\varepsilon)B\} \ne \emptyset$. Hence by the Lemma 1.3 applied with $k(t) = \gamma(t) + \varepsilon$ the set-valued map

$$[0,T] \ni t \mapsto U_1(t) := F(t,y(t)) \cap \{g(t) + (\gamma(t)+\varepsilon)B\}$$

admits a measurable selection $f_1(t) \in U_1(t)$. Clearly f_1 satisfies (14). Define x_1 by (12). Then

$$\begin{aligned} \|x_1(t) - y(t)\| \le \|G(t)(x_0 - y_0)\| + \left\|\int_0^t G(t-s)(f_1(s) - g(s))ds\right\| \le \\ M\delta + M\int_0^t (\gamma(s) + \varepsilon)ds \le \eta(t) + M\varepsilon t \le \beta \end{aligned}$$

Assume that we already have constructed $x_n \in \mathcal{C}(0,T;X)$ and $f_n \in \mathcal{L}^1(0,T;X)$, $n = 0,...,N$ verifying (12) - (16). Define the set-valued map $[0,T] \ni t \to U_{N+1}(t)$ by

$$U_{N+1}(t) = F(t, x_N(t)) \cap \{f_N(t) + k(t)\, \|x_N(t) - x_{N-1}(t)\|\, B\}$$

By Lemma 1.4 the set-valued map $t \to F(t, x_N(t))$ is measurable. Moreover $t \to k(t)\, \|x_N(t) - x_{N-1}(t)\|$ is a measurable function. By the Lipschitz continuity of $F(t,\cdot)$ for almost every $t \in [0,T]$, $U_{N+1}(t) \neq \emptyset$. From Lemma 1.3 we deduce that there exists a measurable selection $f_{N+1}(t) \in F(t, x_N(t))$ satisfying $\|f_{N+1}(t) - f_N(t)\| \leq k(t)\|x_N(t) - x_{N-1}(t)\|$ on $[0,T]$. Define x_{N+1} by (12) with $n = N+1$. Then for almost all $t \in [0,T]$

$$\|f_{N+1}(t) - f_N(t)\| \;\leq\; k(t)\, \|x_N(t) - x_{N-1}(t)\|$$

Thus $\{f_n\}$, $\{x_n\}$ verify (12)-(16) with $n = N+1$. □

Consider the following norm on $\mathcal{C}(0,T;X) \times \mathcal{L}^1(0,T;X)$:

$$\forall\, (x,f) \in \mathcal{C}(0,T;X) \times \mathcal{L}^1(0,T;X), \;\; \|(x,f)\|_{\mathcal{C}\times\mathcal{L}} \;=\; \|x\|_{\mathcal{C}} \;+\; \|f\|_1$$

Corollary 1.6 (Lipschitz dependence on the initial condition) *Let (y,g) be a trajectory-selection pair of (7) on $[t_0,T]$ and assume that F, y satisfy $H_1) - H_3)$. Then there exists $L > 0$ such that for all η near $y(0)$ we have*

$$dist_{\mathcal{C}\times\mathcal{L}}\left((y,g),\ \{(x,f) \text{ is a trajectory-selection pair of (7) on } [t_0,T]\ \}\right) \leq\; L\,\|\eta - y(0)\|$$

Proof — Let m be defined as in the proof of Theorem 1.2. Fix $0 < \varepsilon \leq 1$ and set $L = m(T)(T - t_0 + \int_{t_0}^T k(s)ds + 1)$. By Theorem 1.2 we can find $\delta > 0$ such that for all $\eta \in B_\delta(y(t_0))$ there exists a trajectory selection pair $(x_\varepsilon, f_\varepsilon)$ of (7) satisfying $x_\varepsilon(t_0) = \eta$ and such that

$$\begin{aligned} \|x_\varepsilon - y\|_{\mathcal{C}} \;&\leq\; m(T)(\|\eta - y(t_0)\| \;+\; \varepsilon(T - t_0)) \\ &\leq\; L\|\eta - y(t_0)\| \;+\; L\varepsilon(T - t_0) \end{aligned}$$

and

$$\|f_\varepsilon - g\|_1 \;\leq\; L\|\eta - y(t_0)\| \;+\; \varepsilon(T - t_0)(L+1)$$

Since $\varepsilon > 0$ is arbitrary, the proof follows. □

We define next the reachable set of (7) from (t_0, x_0) at time $t_0 + h$:

$$R(t_0 + h, t_0)x_0 \;=\; \left\{\, x(t_0 + h) \mid x \;\in\; S_{[t_0,t_0+h]}(x_0) \,\right\}$$

Theorem 1.7 *Let $x_0 \in X$. Assume that $F : [t_0, T] \times X \to X$ is continuous, has closed nonempty images and for some $\delta > 0$, $K > 0$ and for every $t \in [t_0, T]$, $F(t, \cdot)$ is K-Lipschitz on $x_0 + \delta B$. Then for every $u \in F(t_0, x_0)$*

$$dist\,(G(h)x_0 \;+\; hu,\; R(t_0 + h, t_0)x_0) \;=\; o(h)$$

where $\lim_{h\to 0+}\; o(h)/h = 0$.

Proof — Fix $u \in F(t_0, x_0)$ and set $y(t_0 + h) = G(h)x_0 + \int_{t_0}^{t_0+h} G(t_0 + h - s)uds = G(h)x_0 + hu + o(h)$. Set $\varepsilon(h) = \sup_{s\in[0,h]} dist(u, F(t_0+s, y(t_0+s)))$. Then, by continuity of F, $\lim_{h\to 0+} \varepsilon(h) = 0$. This and Theorem 1.2 yield that for some $C > 0$, and all small $h > 0$, there exist $x_h \in S_{[t_0,t_0+h]}(x_0)$ such that $\|y(t_0 + h) - x_h(t_0 + h\| \le C\varepsilon(h)h$. Therefore $dist(G(h)x_0 + hu, R(t_0 + h, t_0)x_0\}) = o(h)$.

2 Relaxation of differential inclusions

In this section we compare trajectories of (7) and of the convexified (relaxed) differential inclusion:

$$x'(t) \;\in\; Ax(t) \;+\; \overline{co}\, F(t, x(t)) \tag{20}$$

Recall that a set-valued map $U : [t_0, T] \to X$ is called integrably bounded if there exists $m \in \mathcal{L}^1(t_0, T)$ such that for almost every $t \in [t_0, T]$, $U(t) \subset m(t)B$.

Theorem 2.1 *Let (y, g) be a trajectory-selection pair of the relaxed inclusion (20) on $[t_0, T]$. Assume that F and y satisfy all the assumptions of Theorem 1.2 and that the map $t \to F(t, y(t))$ is integrably bounded on $[t_0, T]$. Let $\eta(\cdot)$ be defined as in Theorem 1.2. If $\eta(T) < \beta$ then for every $\delta > 0$ there exists a mild trajectory x of (7) on $[t_0, T]$ satisfying $\|x - y\|_C \le \delta$.*

To prove the above we shall use

Theorem 2.2 ([18]) *Let $U : [t_0, T] \to X$ be a measurable, integrably bounded set-valued map with closed nonempty images. Then*

$$\overline{\int_{t_0}^{T} \overline{co}\, U(t)dt} \;=\; \overline{\int_{t_0}^{T} U(t)dt}$$

We also need the following two lemmas.

Lemma 2.3 *Consider a measurable, integrably bounded set-valued map $U : [t_0, T] \to X$ with closed nonempty images and let $[t_0, T] \ni s \to g(s) \in G(T-s)U(s)$ be a measurable selection. Then there exists a measurable selection $u(s) \in U(s)$ such that $g(s) = G(T-s)u(s)$ almost everywhere in $[t_0, T]$.*

Proof — Define the continuous function $f : [t_0, T] \times X \to X$ by

$$f(s, u) = G(T-s)u$$

Then $g(s) \in f(s, U(s))$. From [3, p.85] we deduce the existence of a measurable selection $u(\cdot)$ as in the claim of the lemma. □

Lemma 2.4 *Consider a measurable, integrably bounded set-valued map $U : [t_0, T] \to X$ with closed nonempty images. Then*

$$\overline{\int_{t_0}^{T} G(T-t)U(t)dt} = \overline{\int_{t_0}^{T} \overline{G(T-t)U(t)}dt}$$

Proof — Since U is integrably bounded, also the map $s \to G(T-s)U(s)$ is integrably bounded. By Theorem 1.1 *ii*) there exist measurable selections $u_n(t) \in U(t)$ such that for all $t \in [t_0, T]$, $U(t) = \overline{\bigcup_{n \geq 1} u_n(t)}$. Set $U_n(t) = \bigcup_{i=1}^{i=n} u_i(t)$ and observe that for every $t \in [t_0, T]$, $G(T-t)U_n(t)$ is closed. Let $w(t) \in \overline{G(T-t)U(t)}$ be an integrable selection and set $\varepsilon_n(t) = dist\ (w(t), G(T-t)U_n(t))$. Then the sequence $\{\varepsilon_n\}_{n \geq 1}$ is integrably bounded. From Lemma 1.2 there exist integrable selections $w_n(t) \in G(T-t)U_n(t)$ such that $\left\| \int_{t_0}^{T} w_n(t)dt - \int_{t_0}^{T} w(t)dt \right\| \leq \int_{t_0}^{T} \varepsilon_n(t)dt + \frac{1}{n}$. It remains to show that for every $t \in [t_0, T]$, $\lim_{n \to \infty} \varepsilon_n(t) = 0$. But this follows immediately from the choice of u_n. □

Proof of Theorem 2.1 — It is not restrictive to assume that $t_0 = 0$ and that $\int_0^T k(t)dt > 0$. Fix $\delta > 0$. We use the main idea from [4, p.117] for the finite dimensional case. Let $M,\ m$ be defined as in Theorem 1.2 for $t_0 = 0$ and $\psi \in \mathcal{L}^1(0, T)$ be such that for almost all $t \in [0, T]$, $F(t, y(t)) \subset \psi(t)B$. Let

$$0 < \alpha < \gamma < min \left\{ \frac{\beta}{2m(T)\int_0^T k(t)dt},\ \frac{\delta}{2m(T)(\int_0^T k(t)dt + T)},\ \frac{\beta}{2}, \frac{\delta}{2} \right\}$$

Let $n \geq 1$ be so large, that for any measurable $I \subset [0, T]$ of $\mu(I) \leq \frac{1}{n}$ we have

$$\int_I \psi(t)dt \leq \frac{\alpha}{2M}$$

Denote by I_j the interval $[\frac{j-1}{n}, \frac{j}{n}]$, $j = 1, ..., n$. From Lemma 1.4 the map $t \to F(t, y(t))$ is measurable. By Lemma 2.4 for every j

$$(21) \quad \begin{cases} \overline{\int_{I_j} G(t_j - \tau)\overline{co}\, F(\tau, y(\tau))d\tau} &= \overline{\int_{I_j} \overline{G(t_j - \tau)\overline{co}\, F(\tau, y(\tau))}d\tau} \\ \overline{\int_{I_j} \overline{G(t_j - \tau)\, F(\tau, y(\tau))}d\tau} &= \overline{\int_{I_j} G(t_j - \tau)\, F(\tau, y(\tau))d\tau} \end{cases}$$

Observe that $\overline{co}\, \overline{G(t_j - \tau)F(\tau, y(\tau))} = \overline{G(t_j - \tau)\overline{co}\, F(\tau, y(\tau))}$. Hence from (21) and Theorem 2.2 we deduce that

$$\overline{\int_{I_j} G(t_j - \tau)\overline{co}\, F(\tau, y(\tau))d\tau} = \overline{\int_{I_j} G(t_j - \tau)\, F(\tau, y(\tau))d\tau}$$

This and Lemma 2.3 imply that for every j there exists a measurable selection $f_j(t) \in F(t, y(t))$ such that

$$\left\| \int_{I_j} G(t_j - t) f_j(t)dt \; - \; \int_{I_j} G(t_j - t)g(t)dt \right\| \; \leq \; \frac{\gamma - \alpha}{Mn}$$

Let f be the function equal to f_j on I_j and set $x_0(t) = G(t)y(0) + \int_0^t G(t - s)f(\tau)d\tau$. Then for every $t \in [0, T]$ there exists j such that $t \in I_j$ and

$$\begin{aligned} &\|x_0(t) - y(t)\| \; = \; \left\| \int_0^t (G(t - \tau)f(\tau) \; - \; G(t - \tau)g(\tau))\, d\tau \right\| \; \leq \\ &\left\| \sum_{i=1}^{j-1} G(t - t_i) \int_{I_i} G(t_i - \tau)(f(\tau) - g(\tau))d\tau \right\| + \int_{I_j} \|G(t_j - \tau)(f(\tau) - g(\tau))\|\, d\tau \\ &\leq \; M \sum_{i=1}^{j-1} \left\| \int_{I_i} G(t_i - \tau)(f(\tau) - g(\tau))d\tau \right\| \; + \; M \int_{I_j} (\|f(\tau)\| + \|g(\tau)\|)\, d\tau \; \leq \\ &\gamma - \alpha \; + \; 2M \int_{I_j} \psi(\tau)d\tau \; < \; \gamma - \alpha + \alpha \; = \; \gamma \; < \; min\left\{\frac{\beta}{2}, \frac{\delta}{2}\right\} \end{aligned}$$

Observe that for all $t \in [0, T]$, $F(t, \cdot)$ is $k(t)-$Lipschitz on $x_0(t) + \frac{\beta}{2}B$. Hence

$$dist\; (f(t), F(t, x_0(t))) \; \leq \; k(t)\|x_0(t) - y(t)\| \; \leq \; k(t)\gamma$$

Furthermore, by the choice of γ, $Mexp\left(M \int_0^T k(s)ds\right) \int_0^T k(t)\gamma dt < \beta/2$. By Theorem 1.2 applied with $\varepsilon = \gamma$ there exists a trajectory x of (7) satisfying $x(0) = x_0(0) = y(0)$ and

$$\|x - x_0\|_C \leq Mexp\left(M \int_0^T k(\tau)d\tau\right)\left(\int_0^T k(t)\gamma dt + \gamma T\right) < \frac{\delta}{2}$$

In this way we obtain that $\|x - y\|_C \leq \|x - x_0\|_C + \|x_0 - y\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. The proof is complete. □

Theorem 2.5 (Relaxation theorem) *Let $F : [t_0, T] \times X \to X$ be a set-valued map with closed nonempty images. Assume that there exists $k \in \mathcal{L}^1(t_0, T)$ such that for almost every $t \in [t_0, T]$, $F(t, \cdot)$ is $k(t)$-Lipschitz and for all $x \in X$, $F(t, x) \subset k(t)B$. Then the mild trajectories of (7) are dense in the mild trajectories of the relaxed inclusion (20) in the metric of uniform convergence.*

Corollary 2.6 *Under all assumptions of Theorem 2.5 assume that X is reflexive and that at least one of the following three conditions is satisfied*

i) The semigroup $G(\cdot)$ is compact

ii) The semigroup $G(\cdot)$ is uniformly continuous

iii) There exists a compact $K \subset X$ such that for every $(t, x) \in [t_0, T] \times X$, $F(t, x) \subset K$.

Let $S^{co}_{[t_0,T]}(\xi)$ denote the set of mild trajectories of (20) on $[t_0, T]$ with $x(t_0) = \xi$. Then for every $\xi \in X$, the closure of $S_{[t_0,T]}(\xi)$ in the metric of uniform convergence is equal to $S^{co}_{[t_0,T]}(\xi)$.

The above corollary follows from Theorem 2.5 and

Theorem 2.7 *Assume that X is reflexive and let $F : [t_0, T] \times X \to X$ be a set-valued map with nonempty closed convex images. Assume that there exists $k \in \mathcal{L}^1(t_0, T)$ such that for almost every $t \in [t_0, T]$, $F(t, \cdot)$ is $k(t)$-Lipschitz and for all $x \in X$, $F(t, x) \subset k(t)B$. If at least one of the following three conditions is satisfied*

i) The semigroup $G(\cdot)$ is compact

ii) The semigroup $G(\cdot)$ is uniformly continuous

iii) There exists a compact $K \subset X$ such that for every $(t, x) \in [t_0, T] \times X$, $F(t, x) \subset K$.

Then for every $\xi \in X$ the set $S_{[t_0,T]}(\xi) \subset C(0, T; X)$ is sequentially compact.

Proof — Fix $\xi \in X$ and let (x_n, f_n) be trajectory-selection pairs of (7) with $x_n(t_0) = \xi$. Thus

$$(22) \qquad \forall\, t \in [t_0, T], \;\; x_n(t) \; = \; G(t - t_0)\xi \; + \; \int_{t_0}^{t} G(t - s) f_n(s) ds$$

Set $M = \sup_{s \in [0, T - t_0]} \|G(s)\|$. We prove first that the family $\{x_n\}_{n \geq 1}$ is equicontinuous. Indeed for every $n \geq 1$ and for all $t_0 \leq t \leq t' \leq T$

$$\|x_n(t') - x_n(t)\| \; \leq \; \left\| (G(t' - t) - Id) \int_{t_0}^{t} G(t - s) f_n(s) ds \right\| + M \int_{t}^{t'} \|f_n(s)\| ds$$

Since f_n are integrably bounded by k for every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 \leq t \ < t' \leq t + \delta \leq T$ yields $\int_t^{t'} \|f_n(s)\|ds \leq \varepsilon$. It remains to show that

$$(23) \lim_{t' \to t+} (G(t'-t) - Id) \int_{t_0}^{t} G(t-s) f_n(s)ds \ = \ 0 \text{ uniformly in } n \text{ and } t$$

We shall use the assumptions on the semigroup G. Assume first that $G(\cdot)$ is compact. Fix $t_1 \in]0, T - t_0]$ and let $Q \subset X$ be a compact convex set containing $G(t_1)B$. Then

$$\forall\, s \in [t_1, T - t_0], \ \ G(s)B = G(t_1)G(s - t_1)B \subset G(t_1)MB \subset MQ$$

Thus for all $t_1 > 0$ verifying $t_0 \leq t - t_1$

$$\int_{t_0}^{t-t_1} G(t-s)f_n(s)ds \in \int_{t_0}^{t-t_1} G(t-s)k(s)Bds \subset M \int_{t_0}^{t-t_1} k(s)Qds \subset M \int_{t_0}^{T} k(s)dsQ$$

The set $M \int_{t_0}^{T} k(s)dsQ$ being compact and the semigroup G being strongly continuous we deduce that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\, h \ \in \ [0, \delta], \ \ (G(h) - Id) \left[0, \ M \int_{t_0}^{T} k(s)ds\right] \ Q \ \ \subset \ \varepsilon B$$

Consequently for every $t \leq t' \leq t + \delta$ and all n, $(G(t' - t) - Id) \int_{t_0}^{t-t_1} G(t - s)f_n(s)ds \in \varepsilon B$. Since $\int_{t-t_1}^{t} G(t - s)f_n(s)ds \to 0$ when $t_1 \to 0+$ uniformly in t and n we proved (23).

If $G(\cdot)$ is uniformly continuous, then $\|G(h) - Id\| \to 0$ when $h \to 0+$ and, again we derive (23).

If the assumption *iii*) holds true, then there exists a compact $Q \subset X$ such that for every n, $f_n([t_0, T]) \subset Q$. Therefore $(G(t' - t) - Id) \int_{t_0}^{t} G(t - s)f_n(s)ds \subset \int_{t_0}^{t} G(t - s)(G(t' - t) - Id)Qds$. The set Q being compact, for every $\varepsilon > 0$ and all sufficiently small $h > 0$, $(G(h) - Id)Q \subset \varepsilon B$. This completes the proof of (23).

Hence the sequence $\{x_n\}$ is equicontinuous. Clearly it is also bounded. From the Ascoli-Arzela theorem, taking a subsequence and keeping the same notations we may assume that it converge uniformly to some $x \in \mathcal{C}(0, T; X)$. We prove next that $x \in \mathcal{S}_{[t_0,T]}(\xi)$.

The sequence $\{f_n\}_{n \geq 1}$ being integrably bounded and X being reflexive, by the Danford-Pettis theorem, we may assume that it converge weakly in $\mathcal{L}^1(0, T; X)$ to some $f \in \mathcal{L}^1(0, T; X)$. By the Mazur lemma, there exist

$\lambda_i^n \geq 0$, $i = n, ..., k(n)$ such that $\sum_{i=n}^{k(n)} \lambda_i^n = 1$ and the sequence $g_n := \sum_{i=n}^{k(n)} \lambda_i^n f_i$ converge to f in $\mathcal{L}^1(0,T;X)$. Then from [7, p.150] a subsequence g_{n_j} converge to f almost everywhere. Hence for every $t \in [t_0, T]$

$$\lim_{j\to\infty} \int_{t_0}^{t} G(t-s) g_{n_j}(s) ds = \int_{t_0}^{t} G(t-s) f(s) ds$$

and, since $\{x_n\}$ converge uniformly to x, using (22), we get

$$x(t) = G(t-t_0)\xi + \int_{t_0}^{t} G(t-s) f(s) ds$$

To end the proof it is enough to show that $f(s) \in F(s, x(s))$ almost everywhere in $[t_0, T]$.

Observe that for almost every $t \in [t_0, T]$, $g_{n_j}(t) \in \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} F(t, x_i(t)) \subset F(t, x(t)) + k(t) \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} \|x(t) - x_i(t)\| B$. Using that $\lim_{i\to\infty} x_i(t) = x(t)$ we deduce that for almost every $t \in [t_0, T]$, $f(t) \in F(t, x(t))$. The proof is complete.

3 Infinitesimal generator of reachable map

Consider a set-valued map $F : [t_0, T] \times X \to X$ with closed images, where $0 \leq t_0 \leq T$. For all $t \in [t_0, T[$ and $t \leq t' \leq T$, $\xi \in X$ set

$$R(t', t)\xi = \{ x(t') \mid x \in S_{[t,t']}(\xi) \} \quad (24)$$

This is the so-called reachable set of (7) from (t, ξ) at time t'. It was proved in [12], [16] that when the dimension of X is finite, $G \equiv Id$ and F is sufficiently regular, then the set $\overline{co}\, F(t, \xi)$ is the infinitesimal generator of the semigroup $R(\cdot, t)\xi$ in the sense that the difference quotients $\frac{R(t+h,t)\xi - \xi}{h}$ converge to $\overline{co}\, F(t, \xi)$. In this section we extend this result to the infinite dimensional case.

Theorem 3.1 *Under all assumptions of Theorem 1.7*

$$F(t_0, x_0) \subset \liminf_{h\to 0+} \frac{R(t_0+h, t_0)x_0 - G(h)x_0}{h}$$

Consequently, if $x_0 \in Dom\, A$, then we have

$$Ax_0 + F(t_0, x_0) \subset \liminf_{h\to 0+} \frac{R(t_0+h, t_0)x_0 - x_0}{h}$$

If moreover $F(t_0, x_0)$ is bounded, then $F(t_0, x_0)$ in the above formulas may be replaced by its closed convex hull $\overline{co}\, F(t_0, x_0)$.

Proof — The first claim follows from Theorem 1.7. If $F(t_0, x_0)$ is bounded then F is bounded on a neighborhood of (t_0, x_0). By the proof of Theorem 1.7 and Theorem 2.1 we may replace F by $\overline{co}F$. □

When F has compact images the following "upper" estimate holds true:

Theorem 3.2 *Let $x_0 \in X$. Assume that $F : [t_0, T] \times X \to X$ is bounded, upper semicontinuous at (t_0, x_0) and that either the semigroup $G(\cdot)$ is uniformly continuous or $F(t_0, x_0)$ is compact. Then*

$$\limsup_{h \to 0+} \frac{R(t_0 + h, t_0)(x_0) - G(h)x_0}{h} \subset \overline{co}\, F(t_0, x_0)$$

Consequently if $x_0 \in Dom\ A$ then

$$w \in \limsup_{h \to 0+} \frac{R(t_0 + h, t_0)(x_0) - x_0}{h} \iff w - Ax_0 \in \overline{co}\, F(t_0, x_0)$$

Proof — Let $M = \sup_{t \in [0, T - t_0]} \|G(t)\|$, $K > 0$ be such that for all (t, x), $F(t, x) \subset KB$. Fix $u \in \limsup_{h \to 0+} \frac{R(t_0+h, t_0)x_0 - G(h)x_0}{h}$ and let (x_h, f_h) be trajectory-selection pairs such that $x_h(t_0) = x_0$ and $(x_h(t_0 + h) - x(t_0))/h \to u$. Then $\|x_h(t_0 + s) - G(h)x_0\| \le MKs$ and, by the upper semicontinuity of F, there exist $\varepsilon(h) \to 0+$ such that for all $h > 0$ and all $s \in [0, h]$, $F(t_0 + s, x_h(t_0 + s)) \subset F(t_0, x_0) + \varepsilon(h)B$. Hence, by the assumptions on G,

$$\begin{cases} x_h(t_0 + h) \; = \; G(h)x_0 \; + \; \int_{t_0}^{t_0+h} G(t_0 + h - s)f_h(s)ds \; \in \\ G(h)x_0 \; + \; \int_{t_0}^{t_0+h} G(t_0 + h - s)F(t_0, x_0)ds \; + \; \varepsilon(h)hMB \subset \\ G(h)x_0 \; + \; \int_{t_0}^{t_0+h} F(t_0, x_0)dt \; + \; \hat{\varepsilon}(h)hB \end{cases}$$

where $\lim_{h \to 0+} \hat{\varepsilon}(h) = 0$. But $\int_{t_0}^{t_0+h} F(t_0, x_0)dt \subset h\ \overline{co}\ F(t_0, x_0)$ and our claim follows. □

Theorem 3.3 *Let $t_1 \in [t_0, T]$, $x_1 \in X$ and assume that for some $\rho > 0$, F is continuous on $[t_1 - \rho, t_1 + \rho] \cap [t_0, T] \times B_\rho(x_1)$ and has bounded nonempty images. Further assume that for some $L > 0$ and all $t \in [t_1 - \rho, t_1 + \rho] \cap [t_0, T]$ the set-valued map $F(t, \cdot)$ is $L-$Lipschitz on $B_\rho(x_1)$. Then for all (t, ξ) near (t_1, x_1) and all small $h > 0$*

$$(25) \quad R(t + h, t)\xi \; = \; G(h)\xi \; + \; \int_0^h G(h - s)\overline{co}\ F(t_1, x_1)ds \; + \; o(t, \xi, h)$$

where $\lim_{(t,\xi) \to (t_1, x_1), h \to 0+} o(t, \xi, h)/h = 0$. Consequently if $F(t_1, x_1)$ is compact or if the semigroup $\{G(t)\}_{t \ge 0}$ is uniformly continuous, then

$$R(t + h, t)\xi \; = \; G(h)\xi \; + \; h\, \overline{co}\ F(t_1, x_1) \; + \; o(t, \xi, h)$$

Remark — Equality (25) has to be understood in the following way

$$R(t+h,t)\xi \subset G(h)\xi + \int_0^h G(h-s)\overline{co}\, F(t_1,x_1)ds + \|o(t,\xi,h)\|\, B \text{ and}$$
$$G(h)\xi + \int_0^h G(h-s)\, \overline{co}\, F(t_1,x_1)ds \subset R(t+h,t)\xi + \|o(t,\xi,h)\|\, B$$

Proof — Set $M = \sup_{t\in[0,T-t_0]} \|G(t)\|$. Since F is continuous and has bounded images we may assume that for some $M_1 \geq 1$ such that for all $|t-t_1| \leq \rho$, $\xi \in x_1 + \rho B$ we have $F(t,\xi) \subset M_1 B$. Define

$$\mathcal{N} \;=\; \{\,(t,\xi) \mid\; |t-t_1| \leq \rho/2,\;\; \|\xi - x_1\| \leq \rho/2\,\}$$

and observe that for all $(t,\xi) \in \mathcal{N}$ and $t' \in [t_0,T]$ satisfying $\|t'-t\| \leq \rho/2$ and every trajectory-selection pair (x,f) of (7) with $x(t) = \xi$ defined on the time interval $[t,t']$ and verifying $x([t,t']) \subset x_1 + \rho B$, we have $\|x(t') - x_1\| \leq \|x_1 - \xi\| + \|\xi - G(t'-t)\xi\| + \|G(t'-t)\xi - x(t')\| \;\leq\; \|G(t'-t)\xi - \xi\| + \int_t^{t'} \|G(t'-s)\|\,\|f(s)\|ds + \rho/2 \leq \|G(t'-t)\xi - \xi\| + MM_1(t'-t) + \rho/2$. By Theorem 1.7 for all small $h > 0$ and all t near t_1 the set $S_{[t,t+h]}(\xi) \neq \emptyset$. Moreover for all $h \in [0, \rho/4MM_1]$, $(t,\xi) \in \mathcal{N}$, and for every $x \in S_{[t,t+h]}(\xi)$

(26) $$\forall\, s \in [t,t+h],\;\; \|x(s) - G(s-t)\xi\| \leq MM_1(s-t)$$

Since F is continuous at (t_1,x_1) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\, x \in x_1 + \delta B,\; t \in [t_1-\delta, t_1+\delta],\; h \in [0,\delta],\;\; F(t+h,x) \subset F(t_1,x_1) + \varepsilon B$$

Using (26) we obtain that for all $0 < h \leq \min\{\delta/MM_1, \rho/4MM_1\}$, $(t,\xi) \in \mathcal{N} \cap [t_1-\delta, t_1+\delta] \times B_\delta(x_1)$ and trajectory-selection pair (x,f) of (7) on $[t,t+h]$

$$\begin{gathered} x(t+h) \;\in\; G(h)\xi \;+\; \int_t^{t+h} G(t+h-s)F(s,x(s))ds \;\subset \\ G(h)\xi \;+\; \int_t^{t+h} G(t+h-s)(F(t_1,x_1) \;+\; \varepsilon B)ds \;\subset \\ G(h)\xi + \int_0^h G(h-s)\overline{co}\, F(t_1,x_1)ds + \varepsilon h M B \end{gathered}$$

This implies that

$$R(t+h,t)\xi \subset G(h)\xi + \int_0^h G(h-s)\overline{co}\, F(t_1,x_1)ds + \varepsilon h M B$$

for all sufficiently small $h > 0$ and all $(t,\xi) \in \mathcal{N} \cap [t_1-\delta, t_1+\delta] \times B_\delta(x_1)$. Since $\varepsilon > 0$ is arbitrary we proved that for all (t,ξ) near (t_1,x_1) and small $h > 0$

$$R(t+h,t)\xi \subset G(h)\xi + \int_0^h G(h-s)\overline{co}\, F(t_1,x_1)ds + \|o(t,\xi,h)\|\, B$$

To prove the opposite inclusion, observe that by Theorem 2.1, for all (t,ξ) near (t_1,x_1) and all small $h>0$ reachable sets $R(t+h,t)\xi$ of (7) are dense in the reachable sets $R^{co}(t+h,t)\xi$ of the convexified inclusion (20). Thus we may assume that F has convex images. Fix $u\in F(t_1,x_1)$ and set

$$\delta(t,s,\xi,u) \;:=\; dist\left(u, F(t+s, G(s)\xi+\int_0^s G(s-\tau)ud\tau)\right)$$

Then from the continuity of F, $\delta(t,s,\xi,u)\to 0$ when $t\to t_1,\ s\to 0+,\ \xi\to x_1$ uniformly in u. By Theorem 1.2 there exist $\delta_1>0, M_2>0$ which depend only on L and M_1 such that for all $h\in[0,\delta_1]$

$$dist\ \left(G(h)\xi+\int_0^h G(h-\tau)ud\tau,\ R(t+h,t)\xi\right) \le M_2\left(\int_0^h \delta(t,s,\xi,u)ds+h^2\right)$$

This proves our claim.

4 Variational inclusion

This section is devoted to an analog of the variational equation of ODE for differential inclusions. For this we need to extend the notion of derivative to set-valued maps.

Definition 4.1 *Let $\mathcal{F}$ be a set-valued map from a Banach space X to another Y and let $y\in\mathcal{F}(x)$. The* **derivative** *$d\mathcal{F}(x,y)$ is the set-valued map from X to Y defined by*

$$v\in d\mathcal{F}(x,y)(u) \iff \lim_{h\to 0+} d\left(v,\ \frac{\mathcal{F}(x+hu_h)-y}{h}\right)=0 \text{ for some } u_h\to u$$

When $\mathcal{F}$ is locally Lipschitz at x then the above definition may be rewritten as

$$v\in d\mathcal{F}(x,y)(u) \iff \lim_{h\to 0+} dist\left(v,\ \frac{\mathcal{F}(x+hu)-y}{h}\right)=0$$

We refer to [13], [12], [14] and [15] for the applications of set-valued derivatives in the finite dimensional context.

Below we denote by $dF(t,x,y)$ the derivative of the set-valued map $F(t,\cdot,\cdot)$, i.e. its partial derivative with respect to the state variable.

Let (y,g) be a trajectory-selection pair of the differential inclusion (7) defined on the time interval $[t_0,T]$. We "linearize" (7) along (y,g) replacing it by the "**variational inclusion**":

$$(27)\qquad \begin{cases} w'(t) & \in & Aw(t)\ +\ dF(t,y(t),g(t))(w(t)) \\ w(t_0) & = & u \end{cases}$$

where $u \in X$.

In the theorem stated below we consider the solution map $S_{C,\mathcal{L}}$ from X to the space $C(0,T;X) \times \mathcal{L}^1(0,T;X)$ defined by

$$S_{C,\mathcal{L}}(u) = \{\ (x,f) \text{ is a trajectory-selection pair of (7) on } [t_0,T]\ \}$$

Theorem 4.2 (Variational inclusion) *If F, y verify the assumptions $H_1)$ - $H_3)$, then for all $u \in X$, every trajectory-selection pair (w,π) of the linearized inclusion (27) on $[t_0,T]$ satisfies $(w,\pi) \in dS_{C,\mathcal{L}}(y(0),(y,g))(u)$. In the other words,*

$$\{(w,\pi) \text{ is a trajectory-selection pair of (27) on } [t_0,T]\ \} \subset dS_{C,\mathcal{L}}(y(0),(y,g))(u)$$

Proof — Let $(w,\pi) \in C(t_0,T;X) \times \mathcal{L}^1(0,T;X)$ be a trajectory-selection pair of (27). By the definition of derivative and local Lipschitz continuity of $F(t,\cdot)$, for almost all $t \in [t_0,T]$,

$$\lim_{h\to 0+} dist\left(\pi(t), \frac{F(t,y(t)+hw(t)) - g(t)}{h}\right) = 0 \tag{28}$$

Moreover, since $g(t) \in F(t,y(t))$ a.e. in $[t_0,T]$, by $H_3)$, for all sufficiently small $h > 0$ and for almost all $t \in [t_0,T]$

$$dist\ (g(t) + h\pi(t), F(t,y(t)+hw(t))) \leq h\,(\|\pi(t)\| + k(t)\,\|w(t)\|)$$

From Lemmas 1.4 and 1.5 the function

$$t \to dist\,(g(t) + h\pi(t), F(t,y(t)+hw(t)))$$

is measurable. This, (28) and the Lebesgue dominated convergence theorem yield

$$\int_{t_0}^{T} dist\,(g(t) + h\pi(t), F(t,y(t)+hw(t)))\,dt = o(h) \tag{29}$$

where $\lim_{h\to 0+} o(h)/h = 0$. By Theorem 1.2 applied with $\varepsilon = h^2$ and by (29) there exist $M_1 \geq 0$ and trajectory-selection pairs (y_h, g_h) of (7) satisfying

$$\|y_h - y - hw\|_C + \|g_h - g - h\pi\|_1 \leq M_1(o(h)+h^2);\ \ y_h(t_0) = y(t_0) + hw(t_0)$$

This implies that

$$\lim_{h\to 0+} \frac{y_h - y}{h} = w \text{ in } C(0,T;X);\ \lim_{h\to 0+} \frac{g_h - g}{h} = \pi \text{ in } \mathcal{L}^1(0,T;X)$$

Hence

$$\lim_{h\to 0+} dist_{\mathcal{C}\times\mathcal{L}}\left((w,\pi), \frac{S_{\mathcal{C},\mathcal{L}}(y(0)+hu)-(y,g)}{h}\right) = 0$$

Since u and (w,π) are arbitrary the proof is complete. □

A stronger result may be proved when we assume in addition that the map $t \to F(t,y(t))$ is integrably bounded.

Consider the "convex" linearization of (7) along (y,g):

$$(30) \qquad \begin{cases} w'(t) & \in & Aw(t) \; + \; d\,\overline{co}\,F(t,y(t),g(t))(w(t)) \\ w(t_0) & = & u \end{cases}$$

where $u \in X$. In the theorem stated below we consider the solution map $S_{\mathcal{C}}(\xi) = S_{[t_0,T]}(\xi)$ as the set-valued map from X to the space $\mathcal{C}(0,T;X)$.

Theorem 4.3 *Under all assumptions of Theorem 1.2 assume that $F(t,y(t))$ is integrably bounded. Then for all $u \in X$, every mild trajectory w to the linearized inclusion* (30) *defined on* $[t_0,T]$ *satisfies $w \in d\,S_{\mathcal{C}}(y(0),y)(u)$. In the other words,*

$$\{w(\cdot) \mid \; w \text{ is a trajectory of (30) on } [t_0,T]\,\} \;\subset\; dS_{\mathcal{C}}(y(0),y)(u)$$

Proof — From Theorem 2.1 we may replace F by $\overline{co}\,F$. Then the result follows from Theorem 4.2. □

The derivative of the set-valued map $\overline{co}\,F(t,x)$ has the following useful property:

If $F(t,\cdot)$ is locally Lipschitz on a neighborhood of x, then for every $y \in F(t,x)$

$$(31) \qquad dF(t,x,y) \; + \; T_{\overline{co}\,F(t,x)}(y) \;\subset\; d\,\overline{co}\,F(t,x,y)$$

where $T_{\overline{co}F(t,x)}(y)$ denote the tangent cone of convex analysis to $\overline{co}F(t,x)$ at y. This follows from a more general

Theorem 4.4 *Let $\mathcal{F}$ be a set-valued map from a Banach space X to another Y having convex images and assume that it is Lipschitz continuous at x. Then for every $y \in \mathcal{F}(x)$*

$$d\mathcal{F}(x,y)(0) \;=\; \overline{\bigcup_{\lambda\geq 0} \lambda(\mathcal{F}(x)-y)} \;\; \text{(tangent cone to } \mathcal{F}(x) \text{ at } y)$$

and

$$\forall\, u \in X \text{ with } d\mathcal{F}(x,y)(u) \neq \emptyset,\;\; d\mathcal{F}(x,y)(u) \;+\; d\mathcal{F}(x,y)(0) \;=\; d\mathcal{F}(x,y)(u)$$

Proof — The first statement follows immediately from Definition 4.1 and the Lipschitz continuity of $\mathcal{F}$. Fix $u \in X$ such that $d\mathcal{F}(x,y)(u) \neq \emptyset$ and any $v \in d\mathcal{F}(x,y)(u)$, $w \in d\mathcal{F}(x,y)(0)$. Let $v_h \to v$ be such that $y + hv_h \in \mathcal{F}(x+hu)$ and $w_h \to w$ be such that $y + \sqrt{h}w_h \in \mathcal{F}(x)$. Then, by the Lipschitz continuity of $\mathcal{F}$, for all small $h > 0$ and for some w'_h we have

$$y + \sqrt{h}w'_h \in \mathcal{F}(x+hu) \; ; \; \|w'_h - w_h\| \leq k\sqrt{h}\,\|u\|$$

where k denotes a Lipschitz constant of $\mathcal{F}$. Using that $\mathcal{F}$ has convex images we get $(1-\sqrt{h})(y+hv_h) + \sqrt{h}(y+\sqrt{h}w'_h) = y + h(v_h + w'_h) - \sqrt{h}\; hv_h = y + h(v+w) + o(h) \in \mathcal{F}(x+hu)$. Hence

$$\lim_{h\to 0+} \text{dist}\left(v+w, \frac{\mathcal{F}(x+hu)-y}{h}\right) = 0$$

Consequently $d\mathcal{F}(x,y)(u) + d\mathcal{F}(x,y)(0) \subset d\mathcal{F}(x,y)(u)$. On the other hand $0 \in d\mathcal{F}(x,y)(0)$ and therefore $d\mathcal{F}(x,y)(u) + d\mathcal{F}(x,y)(0) \supset d\mathcal{F}(x,y)(u)$. This ends the proof. □

5 Application: semilinear optimal control problem with end point constraints

Let Z be a complete separable metric space, X be a separable Banach space and $f : [0,T] \times X \times Z \to X$ be such that for all $(x,u) \in X \times Z$ the function $f(\cdot, x, u)$ is measurable, for every $t \in [0,T]$, $f(t,\cdot,\cdot)$ is continuous, for every $(t,u) \in [0,T] \times Z$ the function $f(t,\cdot,u)$ is differentiable.

Consider a measurable set-valued map $U : [0,T] \to Z$ with closed nonempty images. We assume that there exists $k \in \mathcal{L}^1(0,T)$ such that

a) For almost every $t \in [0,T]$ and for all $u \in U(t)$, $f(t,\cdot,u)$ is $k(t)$-Lipschitz, i.e.,

$$\forall\, x',\, x'' \in X,\ \forall\, u \in U(t),\ \|f(t,x',u) - f(t,x'',u)\| \leq k(t)\,\|x' - x''\|$$

b) For almost all $t \in [0,T]$ and for all $x \in X$ the set $f(t,x,U(t))$ is closed and is contained in $k(t)B$

Let $K \subset X$. Recall that the contingent cone and the Dubovitskij-Miljutine tangent cone to K at $x \in K$ are defined by

$$T_K(x) = \limsup_{h\to 0+} \frac{K-x}{h} = \{\, v \in X \mid \liminf_{h\to 0+} \frac{dist(x+hv, K)}{h} = 0 \,\}$$

and

$$D_K(x) = \{w \in X \mid \exists\, \varepsilon > 0 \text{ such that } \forall\, h \in [0,\varepsilon],\ x + hB_\varepsilon(w) \subset K\}$$

respectively.

Set $\mathcal{U}_T = \{\, u : [0,T] \to Z \mid u(t) \in U(t) \text{ is measurable}\}$.

Consider a differentiable function $\varphi : X \times X \to \mathbf{R}$, $T > 0$ and closed subsets $K_0,\ K_T \subset X$. We study the optimal control problem

$$\text{(32)} \qquad \text{minimize } \varphi(x(0), x(T))$$

over mild solutions of the semilinear control system

$$\text{(33)} \qquad \begin{cases} x'(t) &= Ax(t) + f(t, x(t), u(t)), \quad u \in \mathcal{U}_T \\ x(0) &\in K_0, \quad x(T) \in K_T \end{cases}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{G(t)\}_{t\geq 0}$ of continuous linear operators on X.

Our aim is to prove necessary conditions satisfied by the optimal solutions of problem (32), (33).

Let $(z, \overline{u})$ be a trajectory-control pair of (33), i.e., for every $t \in [0,T]$

$$z(t) = G(t)z(0) + \int_0^t G(t-s) f(s, z(s), \overline{u}(s))ds$$

We associate with it the linear equation

$$\text{(34)} \qquad Z'(t) = AZ(t) + \frac{\partial f}{\partial x}(t, z(t), \overline{u}(t))Z(t)$$

Denote by $S_{\overline{u}}(t;s)$ the solution operator of (34). That is the only strongly continuous solution of the operator equation

$$\forall\, p \in X,\ \ S_{\overline{u}}(t;s)p = S(t-s)p + \int_s^t S(t-\sigma)\frac{\partial f}{\partial x}(\sigma, z(\sigma), \overline{u}(\sigma))S_{\overline{u}}(\sigma;s)p\, d\sigma$$

where $0 \leq s \leq t \leq T$.

Theorem 5.1 *Let $(z,\overline{u})$ be an optimal trajectory-control pair of the problem (32)-(33) and let $Q \subset D_{K_T}(z(T))$ be a convex cone with nonempty interior and $P \subset T_{K_0}(z(0))$ be a convex cone. Then there exist $\lambda \geq 0$, $\xi_0 \in P^-$, $\xi_T \in Q^-$ not vanishing simultaneously such that the function*

$$\text{(35)} \qquad p(t) = S_{\overline{u}}(T;t)^\star \left(-\lambda\, \frac{\partial \varphi}{\partial x_2}(z(0), z(T)) - \xi_T\right)$$

satisfies the maximum principle

$$(36) < p(t),\ f(z(t),\overline{u}(t)) > = \max_{u\in U(t)} < p(t),\ f(z(t),u) > \quad \text{a.e. in } [0,T]$$

and the transversality condition

$$(37) \qquad (p(0),\ -p(T)) \ = \ \lambda\nabla\varphi(z(0),z(T)) \ + \ (\xi_0,\ \xi_T)$$

Proof — Define the set-valued map $F : [0,T] \times X \to X$ by

$$F(t,x) \ = \ f(t,x,U(t))$$

Fix $x \in X$. From Theorem 1.1 *ii*) there exist measurable selections $u_n(t) \in U(t)$ such that for every $t \in [0,T], U(t) = \overline{\bigcup_{n\geq 1} u_n(t)}$. Set $v_n(t) = f(t,x,u_n(t))$. Then $v_n(\cdot)$ is measurable and from continuity of $f(t,x,\cdot)$, $F(t,x) = \overline{\bigcup_{n\geq 1} v_n(t)}$. Thus from Theorem 1.1 we deduce that for every $x \in X$, $F(\cdot,x)$ is measurable and, by the assumptions on f, for almost every $t \in [t_0,T]$, $F(t,\cdot)$ is $k(t)-$Lipschitz and for every $x \in X$, $F(t,x) \subset k(t)B$. Consider the differential inclusion

$$(38) \qquad x'(t) \ \in \ Ax(t) \ + \ F(t,x(t))$$

We claim that solutions of differential inclusion (38) and of the control system (33) defined on the time interval $[0,T]$ do coincide. To prove that, it is enough to consider a trajectory-selection pair (x,g) of (38) defined on the time interval $[0,T]$ and to prove that there exists $u \in \mathcal{U}_T$ such that

$$(39) \qquad g(t) \ = \ f(t,x(t),u(t)) \ \text{a.e. in } [0,T]$$

Define the function $\psi(t,u) = f(t,x(t),u)$. Then ψ is measurable in t and coninuous in u. Moreover for almost every $t \in [0,T]$, $g(t) \in f(t,x(t),U(t)) = \psi(t,U(t))$. Hence from [3, p.85] we deduce the existence of $u \in \mathcal{U}_T$ verifying (39). Thus we may replace the control system (40) by the differential inclusion (38).

Consider the linear control system

$$(40) \qquad \begin{cases} w'(t) & = & Aw(t) \ + \ \frac{\partial f}{\partial x}(t,z(t),\overline{u}(t))w(t) + y(t) \\ y(t) & \in & T_{\overline{co}\ f(t,z(t),U(t))}(f(t,z(t),\overline{u}(t))) \\ y(0) & = & \xi \end{cases}$$

The reachable set $R^L(\xi)$ of (40) by the mild trajectories from ξ at time T is given by

$$R^L(\xi) = \left\{ S_{\overline{u}}(T;0)\xi + \int_0^T S_{\overline{u}}(T;s)y(s)ds |\ y(s) \in T_{\overline{co}\ f(s,z(s),U(s))}(f(s,z(s),\overline{u}(s))) \right\}$$

Case 1. Assume that $Int\ Q \cap R^L(P) = \emptyset$. Since Q has a nonempty interior, by the separation theorem, there exists a nonzero $\xi_T \in X^\star$ such that

$$\inf_{e \in R^L(T)} < \xi_T, e > \geq \sup_{e \in Q} < \xi_T, e >$$

Because Q is a cone we deduce that $\xi_T \in Q^-$. Moreover the last inequality yield that for every measurable selection $y(t) \in \overline{co}\, f(t, z(t), U(t))$ and every $p \in \overline{P}$ we have

$$(41) \qquad < \xi_T, S_{\overline{u}}(T;0)p + \int_0^T S_{\overline{u}}(T;t)\left(y(t) - f(t, z(t), \overline{u}(t))\right) dt > \geq 0$$

Setting $y(t) = f(t, z(t), \overline{u}(t))$ in the above we get $p(0) = S_{\overline{u}}(T;0)^\star(-\xi_T) \in P^-$. On the other hand applying (41) with $p = 0$ we get: for every measurable selection $y(t) \in \overline{co}\, f(t, z(t), U(t))$

$$\int_0^T \langle S_{\overline{u}}(T;t)^\star \xi_T, y(t)\rangle\, dt \ \geq \ \int_0^T \langle S_{\overline{u}}(T;t)^\star \xi_T, f(t, z(t), \overline{u}(t))\rangle\, dt$$

Hence $\sup_{u \in U(t)} < p(t), f(t, z(t), u) >=< p(t), f(t, z(t), \overline{u}(t)) >$ almost everywhere in $[0, T]$. Therefore the maximum principle (36) and the transversality condition (37) hold true with $\lambda = 0,\ \xi_0 = p(0)$.

Case 2. We assume here that $Int\ Q \cap R^L(P) \neq \emptyset$. Let $\overline{w}$ be a mild trajectory of (40) on $[0, T]$ satisfying

$$(42) \qquad \overline{w}(T) \ \in \ Int\, Q$$

From Theorem 4.3 we deduce that that every mild trajectory $w \in \mathcal{C}(0, T; X)$ of the "linear" differential inclusion

$$w'(t) \ \in \ Aw(t) \ + \ d\, \overline{co}\, F(t, z(t))$$

on $[0, T]$ verifies $w \in d\mathcal{S}_C(z(0), z)(w(0))$. From the definition of the derivative for almost every $t \in [0, T]$

$$\forall\ w \in X,\ \ \frac{\partial f}{\partial x}(t, z(t), \overline{u}(t))w \ \subset \ dF(t, z(t), f(t, z(t), \overline{u}(t)))w$$

Hence, using (31) we deduce that every solution $w \in \mathcal{C}(0, T; X)$ of the linear control system (40) verifies $w \ \in \ d\mathcal{S}_C(z(0), z)(w(0))$. We claim that for every trajectory w of (40) satisfying $w(T) \in \overline{Q}$ we have $\varphi'(z(0), z(T))(w(0), w(T)) \geq 0$. Indeed pick such w and assume first that $w(T) \in Int\ Q$. Let $h_i \to$

$0+$, $y_i \to w(0)$ be such that $z(0) + h_i y_i \in K_0$. Theorems 4.3 and 1.2 imply that for every $i \geq 1$ there exists $x_i \in S_{[0,T]}(z(0) + h_i y_i)$ such that

$$w_i := \frac{x_i - z}{h_i} \to w \text{ in } \mathcal{C}(0,T;X)$$

On the other hand, by definition of $D_{K_T}(z(T))$, there exists $\varepsilon > 0$ such that for all $h \in [0,\varepsilon]$, $z(T) + hB_\varepsilon(w(T)) \subset K_T$. Therefore for all sufficiently large i, $z(T) + h_i w_i(T) \in K_T$.

Since z is an optimal solution we get $\varphi(z(0) + h_i w_i(0), z(T) + h_i w_i(T)) \geq \varphi(z(0), z(T))$ and, consequently, $\varphi'(z(0), z(T))(w(0), w(T)) \geq 0$. To prove the same statement in the general case define $w_\lambda := \lambda\overline{w} + (1-\lambda)w$, where $0 < \lambda < 1$. Then w_λ is a trajectory of the linear system (40). Since $w(T) \in Q$, by (42), we also have $w_\lambda(T) \in Int\ Q$. Thus $\varphi'(z(0), z(T))(w_\lambda(0), w_\lambda(T)) \geq 0$. Taking the limit when $\lambda \to 0+$ we end the proof of our claim.

We proved that the following relation holds true:

$$(43) \qquad \begin{cases} -\nabla\varphi(z(0), z(T)) \in \left(\{(p, R^L(p)) \mid p \in X\} \cap P \times Q\right)^- \subset \\ \overline{\{(p, R^L(p)) \mid p \in X\}^- + P^- \times Q^-} \end{cases}$$

Since $\overline{w}(T) \in Int\ Q$ we get

$$\{(p, R^L(p)) \mid p \in X\} - \overline{P} \times \overline{Q} = \{(p, R^L(p)) \mid p \in X\} + (\overline{w}(0), \overline{w}(T)) - \overline{P} \times \overline{Q} \supset \{(p, R^L(p)) \mid p \in X\} + (\overline{w}(0), X) = X \times X$$

and from a well known result of convex analysis we deduce that

$$\{(p, R^L(p)) \mid p \in X\}^- + P^- \times Q^- \quad \text{is closed}$$

Therefore, by (43), there exist $(\alpha_0, p(T)) \in \{(p, R^L(p)) \mid p \in X\}^-$ and $(\xi_0, \xi_T) \in P^- \times Q^-$ such that $(-\alpha_0, -p(T)) = \nabla\varphi(z(0), z(T)) + (\xi_0, \xi_T)$. Thus $p(T) \in R^L(0)^-$. Define p by (35). Exactly as in the Case 1 we deduce that p verifies the maximum priciple (36). Furthermore for every $x \in X$, $< (\alpha_0, p(T)), (x, S_{\overline{u}}(T;0)x) >=< \alpha_0 - p(0), x > \leq 0$. Consequently $\alpha_0 = p(0)$. Which completes the proof. □

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