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ADAPTIVE VARIABLE METRIC ALGORITHMS FOR GENERALIZED EQUATIONS

S.P. Uryas'ev

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International Institute
for Applied Systems Analysis

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FOREWORD

Many of the problems in mathematical economics and game theory may be reduced to the investigation of a generalized equation with a multivalued right-hand side.

This paper deals with methods for solving generalized equations. The author has developed a new approach to the construction of variable metric algorithms for these equations. The convergence of the suggested algorithm is proved for X^*-antimonotone multivalued maps.

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ADAPTIVE VARIABLE METRIC ALGORITHMS FOR GENERALIZED EQUATIONS

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1. INTRODUCTION

In this paper we study the algorithms to solve generalized equation

$$0 \in G(x) \quad , \tag{1}$$

where $G: R^n \to 2^{R^n}$ is some multivalued map on R^n. Often such equations appear when we formulate necessary and sufficient conditions for different game theoretic and mathematical economics problems (saddle points, Nash equilibrium etc.).

The problem (1) is a partial case of variational inequality

$$<x^* - x,\, g(x^*)> \geq 0 \tag{2}$$

for all $x \in X \subseteq R^n$ and some $g(x^*) \in G(x^*)$. If $X = R^n$ then problems (1) and (2) are equivalent, i.e. if x^* is a solution of generalized equation (1) then x^* is a solution of variational equality (2) and vice versa.

To solve variational inequality (2) the following projection algorithm can be used

$$x^{s+1} = \Pi_X(x^s + \rho_s g(x^s)),\ s = 0, 1, \cdots \tag{3}$$

$$\rho_s > 0,\ g(x^s) \in G(x^s) \quad ,$$

$\Pi_X(\cdot)$ denotes the projection of x on the set X with respect to Euclidean norm $\|\cdot\|$.

Many authors (see, for example papers [1]–[7]) studied this algorithm for a antimonotone or strictly antimonotone map $G(x)$.

We call a multivalued operator $G: R^n \to 2^{R^n}$ *(strictly) antimonotine* on $X \subset R^n$ if

$$<g(x) - g(y), y - x> \geq 0$$
$$(>)$$

for all $y, x \in X$, $g(x) \in G(x)$, $g(y) \in G(y)$.

Denote X^* the solution set of problem (2). The convergence of algorithm (3) for strictly X^*-antimonotone multivalued map $G(x)$ was proved in paper [8]. We call a multivalued map $G: R^n \to 2^{R^n}$ *(strictly)* X^*-antimonotone on $X \subset R^n$ if

$$<x^* - x, g(x)> \geq 0$$
$$(>)$$

for all $x^* \in X^* \subset X$, $x \in X \backslash X^*$, $g(x) \in G(x)$. It is easy to see that if X^* is the solution set of variational inequality (2) then (strict) antimonotonicity of the map $G(x)$ implies the (strict) X^*-antimonotonicity of $G(x)$. Indeed

$$<g(x) - g(x^*), x^* - x> \geq 0 \ ,$$
$$(>)$$

consequently

$$0 \leq <g(x^*), x^* - x> \leq <x^* - x, g(x)>$$
$$(<)$$

for all $x^* \in X^*$, $x \in X/X^*$, $g(x) \in G(x)$. The X^*-antimonotonicity is considerably weaker assumption then antimonotonicity. For example the map $G(x) = 1 - \cos x$ is not antimonotone on $X = \{x \in R : x \geq 0\}$, but $G(x)$ is X^*-antimonotone with $X^* = \{0\}$ on X. It is also important that map $G(x)$ can be strictly X^*-antimonotone, while $G(x)$ is only antimonotone map. For example if X is a compact convex subset of R^n and $f: X \to R$ is a concave function then the subdifferential $\partial f(x)$ of a function $f(x)$ is antimonotone map [9] and strictly X^*-antimonotone (X^* in this case is the set of minimum points of the function f). It can be easily seen for the function $f(x) = -|x|$, $x \in R$. Algorithm (3) is simple, but usually it has a low practical rate of convergence. If the inner product

$$<x^* - x^s, g(x^s)> \qquad (x^* \in X^*)$$

is close to zero then algorithm (3) practically does not reduce the distance between X^* and x^s. Indeed, let $X = R^n$, if

$$|<g(x^s), x^s - x^*>| \le \frac{1}{2}\rho_s\|g(x^s)\|^2 ,$$

then

$$\|x^{s+1} - x^*\|^2 = \|x^s + \rho_s g(x^s) - x^*\|^2 =$$

$$= \|x^s - x^*\|^2 + 2\rho_s<g(x^s), x^s - x^*> + \rho_s^2\|g(x^s)\|^2 \ge$$

$$\ge \|x^s - x^*\|^2 .$$

To overcome such difficulties we use the following variable metric algorithm

$$x^{s+1} = \Pi_X(x^s + \rho_s H^s g(x^s)),\ s = 0, 1,... \qquad (4)$$

$$\rho_s \ge 0,\ H^s \in R^{n\times n},\ g(x^s) \in G(x^s) .$$

Algorithm (4) with matrices $H^s = G^{-1}$, $s = 0, 1,...$ where G is some symmetric positive definite matrix was studied by M. Sibony [1], J.-S. Pang and D. Chan [6], S. Dafermos [10]. For this case next approximation x^{s+1} can be obtained as the solution of the problem

$$\text{minimize} - <x - x^s, g(x^s)> + \frac{1}{2\rho_s}<x - x^s, G(x - x^s)>$$

subject to $x \in X$.

In this case the projection in algorithm (4) is with respect to the norm $\|x\| = <x, Gx>^{1/2}$. D.P. Bertsekas and E.M. Gafni [11] investigated a more complicated algorithm

$$x^{s+1} = \Pi_x(x^s + \rho_s G^{-1} A_s^T g(A_s x^s)) ,$$

where A_s is some $n \times n$ matrix, but they did not give a rule for changing of the matrix A_s.

2. ESSENCE OF THE APPROACH

Let us consider the algorithm (4) for the case $X = R^n$

$$x^{s+1} = x^s + \rho_s H^s g(x^s),\ s = 0, 1, \cdots \tag{5}$$

$$\rho_s \geq 0,\ H^s \in R^{n \times n},\ g(x^s) \in G(x^s) .$$

To constract the matrix H^s we use the following idea. At the s^{th} iteration the natural criterion defining the best choice of the matrix H^s is via the multivalued map $\psi_s : R^{n \times n} \to 2^{R^n}$

$$\varphi_s(H) = G(x^s + \rho_s H g(x^s))$$

the best matrix H^s is a solution of the generalized equation

$$0 \in \varphi_s(H) . \tag{6}$$

It is easy to see that problem (6) is a reformulation of the source problem (1), since if H^* is a solution of problem (6), then the point $x^s + \rho_s H^* g(x^s)$ is a solution of the problem (1). More than that, problem (6) is more complicated than (1) because the dimension of problem (6) is in n times higher the dimension of (1). However, at the s^{th} iteration of algorithm (5) we do not need the optimal matrix, it is enough to correct (update) the matrix H^s. To make corrections of the matrix we reformulate problem (6). Define $\Psi_s : R^{n\times n} \to 2^{R^{n\times n}}$

$$\Psi_s(H) = \{gg^T(x^s) : g \in \psi_s(H)\} \ ,$$

here and below sign T means transposition of a vector or a matrix. It is easy to verify that equation (6) is equivalent to the equation

$$0 \in \Psi_s(H) \tag{7}$$

if $g(x^s) \neq 0$. Indeed, if for some H^*

$$0 \in \psi_s(H^*)$$

then

$$0 \cdot g^T(x^s) = 0 \in \Phi_s(H^*)$$

Vice versa, if

$$0 \in \Psi_s(H^*)$$

then for some $g \in \psi_s(H^*)$

$$(gg_1(x^s), \ldots, gg_n(x^s)) = 0,\ g(x^s) = (g_1(x^s), \ldots, g_n(x^s))$$

and $g = 0$, because $g(x^s) \neq 0$. We prove that if multivalued map $G(x)$ is strictly X^*-antimonotone, then multivalued map $\Psi_s(H)$ is also strictly H^*_{opt}-antimonotone, where

$$H^*_{\text{opt}} = \{H \in R^{n\times n} : x_s + \rho_s H g(x^s) \in X^*\} \ .$$

Let $H^* \in H^*_{\text{opt}}$, $H \in R^{n\times n}$, $x^s + \rho_s H^* g(x^s) = x^* \in X^*$ and

$$\psi = g(x^s + \rho_s H g(x^s)) g^T(x^s) \in \Psi_s(H) \quad ,$$

then we have

$$<H^* - H, \psi> = <H^* - H, g(x^s + \rho_s H g(x^s)) g^T(x^s)> =$$

$$= \frac{1}{\rho_s} <\rho_s H^* g(x^s) - \rho_s H g(x^s), g(x^s + \rho_s H g(x^s))> =$$

$$= \frac{1}{\rho_s} <x^s + \rho_s H^* g(x^s) - (x^s + \rho_s H g(x^s)), g(x^s + \rho_s H g(x^s))> =$$

$$= \frac{1}{\rho_s} <x^* - (x^s + \rho_s H g(x^s)), g(x^s + \rho_s H g(x^s))> > 0 \quad .$$

Thus we can apply algorithm (3) in space $R^{n \times n}$ to correct the matrix in algorithm (5). If we already have some matrix H_0^s at the iterations s, then the next approximation is equal

$$H_1^s = H_0^s + \lambda_0^s g_0^s g^T(x^s), \lambda_0^s > 0 \quad ,$$

where g_0^s is some element of the set $G(x^s + \rho_s H_0^s g(x^s))$. It is possible either to take $H^s = H_1^s$, or to continue the iterations of method (3) with respect to the matrix

$$H_{i+1}^s = H_i^s + \lambda_i g_i^s g^T(x^s), i = 1, 2, \cdots \tag{8}$$

where

$$\lambda_i > 0, g_i^s \in G(x^s + \rho_s H_i^s g(x^s))$$

and

$$g_i^s g^T(x^s) \in \Psi_s(H_i^s) \quad .$$

For some $i(s) \geq 1$ assume $H^s = H_{i(s)}^s$. At the next iteration $H_0^{s+1} = H^s$. The number $i(s)$ can be taken independently upon s, for example $i(s) = 1$ for all s.

Note that matrix updating requires additional calculations of the multivalued map $G(x)$ elements. This can be avoided by taking $g^{s+1} = g_0^s$, $i(s) = 1$ and using the matrix H_1^s at $(s+1)^{\text{th}}$ iteration. Therefore it is possible also to use the following formula for matrix updating

$$H^{s+1} = H^s + \lambda_s g(x^{s+1}) g^T(x^s) \ , \tag{9}$$

$$\lambda^s > 0, \ s = 0, 1, \cdots$$

In formula (9) additional calculations fo the multivalued map $G(x)$ elements are not required.

3. CONVERGENCE

Let us consider algorithm (5), (8). We suppose that at the s^{th} iteration of the main algorithm for the updating of the matrix H_0^s formula (8) is used $i(s)$ times. At the iteration $s+1$ we take $H_0^{s+1} = H_{i(s)}^s$. Denote by g^s some vector from the set $G(x^s + \rho_s H_i^s g^s)$. It is convenient to normalize the test vector g_i^s, therefore denote by

$$\xi_i^s = \begin{cases} 0 \ , & \text{if } \ g_i^s = 0 \ , \\ g_i^s / \|g_i^s\|^{-1} \ , & \text{otherwise} \end{cases}$$

Let $\{\rho_s\}$, $\{\epsilon_s\}$ be some sequences of positive numbers and for each $s = 0, 1, \cdots$ let the sequence $\{\lambda_{si}\}$, $i = 0, 1, \cdots$ of positive values be given. We write the algorithm in more detail.

ALGORITHM

Step I Initialization

$s = 0,\ i = -1,\ x^0 = x_{\text{init}},\ g^0 \in G(x^0)$,

$\xi^0 = g^0/\|g^0\|,\ H_0^{-1} = I$

Step II

1 $H_0^s = H_{i+1}^{s-1},\ i = 0$

2 $x_i^s = x^s + \rho_s H_i^s \xi^s$

3 compute $g_i^s \in G(x_i^s)$, if $g_i^s = 0$ then STOP, otherwise $\xi_i^s = g_i^s \|g_i^s\|^{-1}$

4 $H_{i+1}^s = H_i^s + \lambda_{si} \xi_i^s \xi^{s^T}$

5 if $\rho_s \sum_{l=0}^{i} \lambda_{sl} \geq \epsilon_s$, then $i(s) = i$ and go to step III

6 $i = i + 1$, return to point 2.

Step III $x^{s+1} = x_i^s,\ \xi^{s+1} = \xi_i^s$.

Step IV $s = s + 1$ and return to step II.

We now formulate a theorem about the convergence of the algorithm. Here we consider that the set X^* consists only of one point x^*.

THEOREM 1 *Let:*

1 *there exist constant $\alpha > 0$ such that*

$<g(x),\ x^* - x> \ \geq \alpha \|g(x)\|\ \|x^* - x\|$

for all $x \in R^n$, $g(x) \in G(x)$;

2 *$\{\epsilon_s\}$ be a sequence of positive numbers satisfying*

$\epsilon_s \to \infty$ for $s \to \infty$;

3 $\{\rho_s\}$ *be a sequence of positive numbers, such that*

$$\rho_s \| H_0^s \| \leq h = \text{const for } s = 0, 1, \cdots , \tag{11}$$

and

$$0 \leq \rho_s \leq \bar{\rho} \quad \text{for } s = 0, 1, \cdots ; \tag{12}$$

4 $\{\lambda_{si}\} s = 0, 1, \cdots ; i = 0, 1, \cdots$ *be a given sequence of positive numbers satisfying*

$$\lambda_{si} > 0 \quad \text{for } s = 0, 1, \cdots ; i = 0, 1, \cdots , \tag{13}$$

$$\sum_{i=0}^{\infty} \lambda_{si} = \infty \quad \text{for } s = 0, 1, \cdots , \tag{14}$$

$$\sum_{i=0}^{\infty} \lambda_{si}^2 \leq \Lambda = \text{const} \quad \text{for } s = 0, 1, \cdots \tag{15}$$

Then

$$\| x^* - x^s \| \to 0 \quad \text{for } s \to \infty. \tag{16}$$

PROOF We start the proof from the following lemma.

LEMMA 1 *If the sequence $\{x^s\}$ does not converge to the point x^* then there exists a subsequence $\{x^{s_k}\}$ such that*

$$\| x^{s_k + 1} - x^* \| \geq \delta > 0 \tag{17}$$

for some $\delta > 0$ and

$$\| x^{s_k + 1} - x^* \| \geq \frac{1}{2} \| x^{s_k} - x^* \| \tag{18}$$

for sufficiently large k.

PROOF At first we consider the case when there exists some S such that for $s > S$

$$\|x^{s+1} - x^*\| \le \|x^s - x^*\| \ .$$

If $\{x^s\}$ does not converge to x^* then there exists a limit

$$\lim_{s \to \infty} \|x^s - x^*\| = \delta > 0 \ .$$

and

$$\|x^{s+1} - x^*\| \ge \frac{1}{2}\|x^s - x^*\|$$

for sufficiently large s. Next let us consider the case when the sequence $\{\|x^s - x^*\|\}$ is not monotone. Let the subsequence $\{x^{m_l}\}$ be such that for all $l = 0, 1, \cdots$

$$\|x^{s_l+1} - x^*\| \ge \|x^{s_l} - x^*\|$$

Since the sequence $\{x^s\}$ does not converge to zero, then the subsequence $\{x^{s_l}\}$ also does not converge to zero and a subsequence $\{x^{s_{l_m}}\}$ can be selected from the subsequence $\{x^{s_l}\}$ such that

$$\|x^{s_l+1} - x^*\| \ge \delta > 0$$

for some $\delta > 0$. The lemma is proved.

Using the formulae of point 2 and 4 of step II and condition 1 of the theorem we get

$$\|x^* - x_{i+1}^s\|^2 = \|x^* - x^s - \rho_s H_{i+1}^s \xi^s\|^2 =$$

$$= \|x^* - x^s - \rho_s (H_i^s + \lambda_{si} \xi_i^s \xi^{s^T}) \xi^s\|^2 =$$

$$= \|x^* - x^s - \rho_s H_i^s \xi^s - \rho_s \lambda_{si} \|\xi^s\|^2 \xi_i^s\|^2 \le$$

$$\le \|x^* - x_i^s - \rho_s \lambda_{si} \xi_i^s\|^2 = \|x^* - x_i^s\|^2 -$$

$$- 2\rho_s \lambda_{si} <\xi_i^s,\ x^* - x_i^s> + \rho_s^2 \lambda_{si}^2 =$$

$$= \|x^* - x_m^s\|^2 - 2\rho_s \sum_{l=m}^{i} \lambda_{sl} <\xi_l^s,\ x^* - x_l^s> + \rho_s^2 \sum_{l=m}^{i} \lambda_{sl}^2 \leq$$

$$\leq \|x^* - x_m^s\|^2 - 2\rho_s \sum_{l=m}^{i} \lambda_{sl} \alpha \|x^* - x_l^s\| + \rho_s^2 \sum_{l=m}^{i} \lambda_{sl}^2 \qquad (19)$$

for $0 \leq m \leq i(s) - 1$. Applying to the last inequality we have

$$\|x^* - x^{s_k+1}\|^2 = \|x^* - x_{i(s_k)}^{s_k}\|^2 \leq$$

$$\leq \|x^* - x_{i(s)-1}^{s_k}\|^2 - 2\rho_{s_k} \lambda_{s_k, i(s_k)-1} \alpha \|x^* - x_{i(s_k)-1}^{s_k}\| +$$

$$+ \rho_{s_k}^2 \lambda_{s_k, i(s_k)-1} \leq \|x^* - x_{i(s)-1}^{s_k}\|^2 + \rho_{s_k, i(s_k)-1}^2 \ . \qquad (20)$$

Consequently, taking into account (12), (15) and (17) we get

$$\|x^* - x_{i(s)-1}^{s_k}\|^2 \geq \|x^* - x^{s_k+1}\|^2 - \rho_{s_k}^2 \lambda_{s_k, i(s_k)-1}^2 \geq$$

$$\geq \delta^2 - \rho_{s_k}^2 \lambda_{s_k, i(s_k)-1}^2 \geq \delta^2/2$$

for sufficiently large k. Substituting the last inequality into (20) we have for sufficiently large k

$$\|x^* - x_{i(s)-1}^{s_k}\|^2 \geq \|x^* - x_{i(s)}^{s_k}\|^2 + \rho_{s_k} \lambda_{s_k, (s_k)-1} \alpha \|x^* - x_{i(s_k)-1}^{s_k}\| -$$

$$- \rho_{s_k}^2 \lambda_{s_k, i(s_k)-1}^2 \geq \|x^* - x_{i(s)}^{s_k}\|^2 + \rho_{s_k} \lambda_{s_k, i(s_k)-1} \alpha \delta^2/2 -$$

$$- \rho_{s_k}^2 \lambda_{s_k, i(s_k)-1}^2 \geq \|x^* - x_{i(s)}^{s_k}\|^2 + \rho_{s_k} \lambda_{s_k, i(s_k)-1} (\alpha \delta^2/2 - \rho_{s_k} \lambda_{s_k, i(s_k)-1} \geq$$

$$\geq \|x^* - x_{i(s)}^{s_k}\|^2 = \|x^* - x^{s_k+1}\|^2 \geq \delta^2$$

Analogously from (19) and (20) we receive

$$||x^* - x^{s_k}_{i(s)-2}||^2 \geq ||x^* - x^{s_k}_{i(s)-1}||^2 \geq \delta^2 \tag{21}$$

and so on. Thus

$$||x^* - x^{s_k}_m|| \geq ||x^* - x^{s_k+1}|| > \delta \quad \text{for } 0 \leq m \leq i(s_k) \tag{22}$$

From (17) and (22) we obtain

$$0 < \delta \leq ||x^* - x^{s_k+1}|| = ||x^* - x^{s_k}_{i(s_k)}|| \leq ||x^* - x^{s_k}_0|| =$$

$$= ||x^* - x^{s_k} - \rho_{s_k} H_0^{s_k} \xi^{s_k}|| \leq ||x^* - x^{s_k}|| + \rho_{s_k} ||H_0^{s_k}|| \, ||\xi^{s_k}|| \leq$$

$$\leq ||x^* - x^{s_k}|| + \rho_{s_k} ||H_0^{s_k}|| \ . \tag{23}$$

Inequalities (18), (19), (22) and (23) imply for sufficiently large k

$$0 \leq ||x^* - x^{s_k}_0||^2 - 2\rho_{s_k} \sum_{l=0}^{i(s_k)} \lambda_{s_k l} \alpha ||x^* - x^s_l|| + \rho^2_{s_k} \sum_{l=0}^{i(s_k)} \lambda^2_{s_k l} \leq$$

$$\leq ||x^* - x^{s_k}||^2 + 2\rho_{s_k} ||H_0^{s_k}|| \, ||x^* - x^{s_k}|| + \rho^2_{s_k} ||H_{s_k}||^2 - 2\rho_{s_k} \sum_{l=0}^{i(s_k)} \lambda_{s_k l} \alpha ||x^* -$$

$$- x^{s_k+1}|| + \rho^2_{s_k} \sum_{l=0}^{i(k)} \lambda^2_{s_k l} \leq ||x^* - x^{s_k}||^2 + 2\rho_{s_k} ||H_0^{s_k}|| \, ||x^* - x^{s_k+1}|| -$$

$$- 2\rho_{s_k} \sum_{l=0}^{i(s_k)} \lambda_{s_k l} \alpha ||x^* - x^{s_k+1}|| + \rho^2_{s_k} ||H_{s_k}||^2 + \rho^2_{s_k} \sum_{l=0}^{i(s_k)} \lambda^2_{s_k l} \leq$$

$$\leq ||x^* - x^{s_k}||^2 + 2||x^* - x^{s_k+1}|| \left[\rho_{s_k} ||H_0^{s_k}|| - \alpha \rho_{s_k} \sum_{l=0}^{i(s_k)} \lambda_{s_k l} \right] +$$

$$+ \rho^2_{s_k} ||H_{s_k}||^2 + \rho^2_{s_k} \sum_{l=0}^{i(s_k)} \lambda^2_{s_k l} \ . \tag{24}$$

Taking into account point 5 of step II and conditions (11), (12), (16) and (17) we have from the previous inequality

$$0 \leq \|x^* - x^{s_k}\|^2 + 2\delta(h - \alpha\epsilon_{s_k}) + h^2 + \bar{\rho}^2\Lambda \ .$$

But this inequality contradicts condition 2 of the theorem. The contradiction proves the theorem.

4. HOW THE X^*-ANTIMONOTONICITY CAN BE CHECKED?

Below we discuss some important problem examples. Let function $\Psi : X \times X \to R$, $X \subseteq R^n$ be differentiable in some generalized sense with respect to the second group of variables and $G(x) = \partial_y \Psi(x, y)|_{y=x}$ be a differential with respect to y on the diagonal. A lot of game theoretic and mathematical economic problems (see, for example papers [12] and [13]) can be reduced to the variational inequality

$$<g(x^*),\, x - x^*> \geq 0 \tag{25}$$

for some $g(x^*) \in G(x^*)$, $x^* \in X$ and for all $x \in X$.

EXAMPLE *A Nash equilibria for n-person games* Let x be a convex closed subset of the product $R^{m1} \times \cdots \times R^{m_n}$ of the Euclidean spaces R^{m_i}, $i = 1, \ldots, n$. A point $x_i \in R^{m_i}$ is a strategy of i-th player $i = 1, \ldots, n$ and $\varphi_i(x) = \varphi_i(x_1, \ldots, x_n)$ is his payoff function. The vector $(x_1, \ldots, x_{i-1},\ y_i,\ x_{i+1}, \ldots, x_i)$ is denoted by (y_i/x). The point $x^* = (x_1^*, \ldots, x^*) \in X$ is referred to as the *Nash equilibrium* of n-person game if for $i = 1, \ldots, n$

$$\varphi_i(x^*) = \max_{y_i} \{\varphi_i(y_i/x^*) : (y_i/x^*) \in X\} \ .$$

Let us introduce the function $\Psi(x, y)$:

$$\Psi(x, y) = \sum_{i=1}^{n} (\varphi_i(y_i/x) - \varphi_i(x)), \; y = (y_1, \ldots, y_n) \; .$$

It is obvious that $\Psi(x, x) = 0$ for $x \in X$. We suppose that the functions $\varphi_i(x)$, $i = 1, \ldots, n$ are continuous on X. The point $x^* \in X$ is defined as the ***normalized equilibrium*** point if

$$\max_{y \in X} \Psi(x^*, y) = 0 \; . \tag{26}$$

LEMMA 2 (See, for example [14]) *The normalized equilibrium point is the equilibrium point, the reverse is true if* $X = X_1 \times \cdots \times X_n$, $X_i \subset R^{m_i}$.

Variational inequality (2) is a necessary optimality condition for the problem (26), for this reason the problem of finding Nash equilibrium is reduced to the problem (26).

We consider the case with a weakly convex-concave function $\Psi(x, y)$. The function $\Psi(x, y)$ is *weakly convex on* X with respect to the first argument i.e.

$$\alpha_1 \Psi(x, z) + \alpha_2 \Psi(y, z) \geq \Psi(\alpha_1 x + \alpha_2 y, z) + \alpha_1 \alpha_2 r_z(x, y)$$

for all $x, y, z \in X$; $\alpha_1 + \alpha_2 = 1$; $\alpha_1 \alpha_2 \geq 0$ and

$$\frac{r_z(x, y)}{||x - y||} \to 0 \quad \text{if } ||x - y|| \to 0 \quad \text{for all } z \in X \; .$$

We suppose that the function $\Psi(x, y)$ is *weakly concave* with respect to the second argument on X i.e.

$$\alpha_1 \Psi(z, x) + \alpha_2 \Psi(z, y) \leq \Psi(z, \alpha_1 x + \alpha_2 y) + \alpha_1 \alpha_2 \mu_z(x, y)$$

for all $x, y, z \in X$; $\alpha_1 + \alpha_2 = 1$; $\alpha_1, \alpha_2 \geq 0$ and also

$$\frac{\mu_z(x, y)}{||x - y||} \to 0 \quad \text{if } ||x - y|| \to 0 \quad \text{for all } z \in X \ .$$

Denote $G(x) = \partial_y \Psi(x, y)|_{y=x}$ the *differential* of the function $\Psi(x, z)$ with respect to the second argument at a point (x, x) i.e. (see [15] and [13]) $G(x)$ is a set of vectors g such that

$$\Psi(x, y) - \Psi(x, x) \leq <g, y - x> + \mu_z(x, y)$$

THEOREM 2 [13] *Let X be an open convex subset of R^n, a function $\Psi : X \times X \to R$ be weakly convex-concave, the remainder $r_z(x, y)$ be continuous with respect to z, the function Ψ satisfies equation $\Psi(x, x) = 0$ for all $x \in X$. Then*

$$<g(x) - g(y), y - x> \geq r_y(x, y) - \mu_z(y, x) \tag{27}$$

for all $x, y \in X$; $g(x) \in G(x)$, $g(y) \in G(y)$.

Let us consider now the case $X = R^n$. In this case if x^* is a solution of problem (25) then

$$0 \in G(x^*) \ .$$

Inequality (27) implies

$$<g(x), x^* - x> \geq r_{x^*}(x, x^*) - \mu_z(x^*, x)$$

Consequently the inequality

$$r_{x^*}(x, x^*) - \mu_z(x^*, x) \geq \alpha ||g(x)|| \, ||x^* - x||$$

is a necessary condition for condition 1 of theorem 1.

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