WORKING PAPER

HAZY DIFFERENTIAL INCLUSIONS

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FOREWORD

This paper is devoted to differential inclusions the right-hand sides of which are **hazy** subsets, which are fuzzy subsets whose membership functions are cost functions taking their values in $[0, \infty]$ instead of $[0, 1]$. By doing so, the concept of uncertainty involved in differential inclusions becomes more precise, by allowing the velocities not only to depend in a multivalued way upon the state of the system, but also in a fuzzy way. The viability theorems are adapted to hazy differential inclusions and to sets of state constraints which are either usual or hazy. The existence of a largest closed hazy viability domain contained in a given closed hazy subset is also provided.

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Introduction

Instead of characterizing a given subset $K \subset X$ by its characteristic function χ_K taking its values in $\{0,1\}$, we represent it by its "indicator" ψ_K, which is the non negative extended function defined by

$$\psi_K(x) \quad := \quad \begin{cases} 0 & \text{if} \quad x \in K \\ +\infty & \text{if} \quad x \notin K \end{cases}$$

Hence, instead of introducing **fuzzy sets**[1] which are (membership) functions χ taking their values in the closed interval $[0,1]$, we shall use **hazy sets** which are extended nonnegative functions $V : X \to \mathbf{R}_+ \cup \{+\infty\}$. An element belongs to the hazy subset V if and only if $V(x) < \infty$. This slight modification in the definition of fuzzy sets allows us to represent convex (respectively, closed) hazy subsets by convex (respectively, lower semicontinuous) functions.

A hazy set-valued map from X to itself is then defined by its graph, which is a hazy subset of $X \times X$ described by a membership function $U : X \times X \to \mathbf{R}_+ \cup \{+\infty\}$. Since a usual differential inclusion $x' \in F(x)$ can be written in the form

$$\text{for almost all } t \geq 0, \;\; (x(t), x'(t)) \in \text{Graph}(F)$$

hence a hazy differential inclusion can be written

$$\text{for almost all } t \geq 0, \;\; U(x(t), x'(t)) < \infty$$

since this says that $(x(t), x'(t))$ belongs to the hazy subset whose membership function is U, which is the graph of the hazy set-valued map.

We shall characterize first the usual closed subsets K which enjoy the viability property for hazy differential inclusion: for any initial state $x_0 \in$

[1] We refer to [7] for a presentation of fuzzy sets and the bibliography of this book.

K, there exists a solution $x(\cdot)$ to the hazy differential inclusion starting at x_0 which is viable in K, in the sense that for all $t \geq 0$, $x(t)$ belongs to K (see [3, Chapters 4,5 & 6] for a presentation of viability theory).

The next natural step is to use hazy subsets for representing state constraints which are not of the form: either live or die. The idea is then to replace the closed subset K by a closed hazy subset V and to replace the viability property by the hazy viability property: for all initial state $x_0 \in \mathrm{Dom}(V)$, there exists a solution to the hazy differential inclusion which is hazily viable in the sense that

$$\forall\, t \geq 0,\ \ V(x(t)) \ \leq \ w(t),\ \ w(0) = V(x_0) \tag{1}$$

where $w(\cdot)$ is a function (such as $w(t) := V(x_0)e^{-at}$) which describes a given upper estimate of the viability cost, so to speak. It will be convenient to provide these functions $w(\cdot)$ as solutions to usual differential equations

$$w'(t) = -\phi(w(t)),\ \ w(0) \ = \ V(x_0)$$

Finally, we shall prove the existence of a largest closed hazy viability domain of a hazy differential inclusion contained in a given hazy closed subset V (which can be empty; in this case, we shall prove that all solutions must eventually leave the hazy subset V). This may be as useful in further applications as the existence of a largest closed viability domain of a usual differential inclusion, which we use by the way for deriving the hazy case.

1 Hazy sets and set-valued maps

We recall that any subset $K \subset X$ can be characterized by its "indicator" ψ_K, which is the non negative extended function defined by:

$$\psi_K(x) \ := \ \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases} \tag{2}$$

It can be regarded as a "cost function" or a "penalty function", assigning to any element $x \in X$ an infinite cost when x is outside K, and no cost at all when x belongs to K.

We also recall that K is closed (respectively convex) if and only if its indicator is lower semicontinuous (respectively convex).

We are led to regard any non negative extended function U from X to $\mathbf{R}_+ \cup \{+\infty\}$ as another implementation of the idea underlying "fuzzy sets", in which indicators replace characteristic functions. Instead of using membership functions taking values in the interval $[0,1]$, we shall deal with membership functions taking their values anywhere between 0 and $+\infty$.

Definition 1.1 *We shall regard an extended nonnegative function $U : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ as a* **hazy set**. *Its domain is the domain of U, i.e., the set of elements x such that $U(x)$ is finite, and the core of U is the set of elements x such that $U(x) = 0$. The complement of the hazy set U is the complement of its domain and the complement of its core is called the hazy boundary.*

We shall say that the hazy set U is **closed** *(respectively* **convex***) if the extended function U is lower semicontinuous (respectively convex).*

Hence the membership function of the empty set is the constant function equal to $+\infty$.

Definition 1.2 *We shall say that a set-valued map $\mathbf{U} : X \rightsquigarrow Y$ associating to any $x \in X$ a hazy subset $U(x)$ of Y is a* **hazy set-valued map**. *Its* **graph** *is the hazy subset of $X \times Y$ associated to the extended nonnegative function $(x,y) \mapsto U(x,y) := U(x)(y)$.*

A hazy set-valued map $\mathbf{U}$ is said to be **closed** *if and only if its graph is closed, i.e., if its membership function is lower semicontinuous. Its values are closed (respectively convex) if and only if the hazy subset $U(x)$ are closed (respectively convex). It has linear growth if and only if, for some constant $c > 0$,*

$$U(x,v) < +\infty \implies \|v\| \leq c(\|x\| + 1)$$

2 Hazy Differential Inclusions

By using indicators, we can reformulate the differential inclusion

$$\text{(3)} \qquad \text{for almost all } t, \; x'(t) \in F(x(t))$$

as

$$\text{for almost all } t, \; \psi_{F(x(t))}(x'(t)) < +\infty)$$

Then we are led to define "hazy dynamics" of a system by a hazy set-valued map **U** associating to any $x \in X$ a hazy set $U(x)$ of velocities $\{v \mid U(x,v) < +\infty\}$. In this case, we can write the associated **hazy differential inclusion** in the form

$$\text{(4)} \qquad \text{for almost all } t \geq 0, \;\; U(x(t), x'(t)) < +\infty$$

or, equivalently, in the form

$$\text{for almost all } t \geq 0, \;\; (x(t), x'(t)) \in \text{Graph}(\mathbf{U})$$

which is a hazy subset instead of a usual subset.

We begin by characterizing usual subsets K enjoying the viability property for hazy differential inclusion: for any initial state $x_0 \in K$, there exists a solution $x(\cdot)$ to the hazy differential inclusion (4) which is viable in K.

For usual differential inclusion $x' \in F(x)$, the Viability Theorem (see [10], [3, Theorem 4.2.1]) states that under adequate assumptions, a closed subset K enjoys the viability property if and only if it is a viability domain of F, i.e., a subset satisfying

$$\forall\, x \in K, \quad F(x) \cap T_K(x) \neq \infty$$

where $T_K(x)$ is the Bouligand contingent cone (introduced in the 30's), defined in the following way:

$$T_K(x) \;:=\; \{\, v \in X \mid \liminf_{h \to 0+} d_K(x + hv)/h = 0 \,\}$$

This is always a closed cone, equal to the whole space X when x belongs to the interior of K, equal to the usual tangent space of differential geometry when K is a smooth manifold and to the tangent cone of convex analysis when K is convex. It is a very convenient way to implement the concept of tangency for arbitrary subsets, the price to pay being that the collection of contingent vectors is only a closed cone instead of a vector space.

Definition 2.1 *We shall say that a subset $K \subset$ Dom(**U**) is a viability domain of the hazy set-valued map **U** if and only if*

$$\text{(5)} \qquad \forall\, x \in K, \;\; \exists\, v \in T_K(x) \;\text{ such that }\; U(x,v) \;<\; +\infty$$

We begin by proving an extension to the Viability Theorem to hazy differential inclusions.

Theorem 2.2 (Hazy Viability Theorem) *Let us consider a nontrivial hazy set-valued map* $\mathbf{U}$ *from a finite dimensional vector-space* X *to itself. Let us assume that it is upper semicontinuous with closed convex images and has linear growth. Any closed subset* $K \subset \mathrm{Dom}(\mathbf{U})$ *enjoying the viability property with respect of* U *is a viability domain and the converse holds true if*

$$(6) \qquad \lambda \ := \ \sup_{x\in K} \inf_{v\in T_K(x)} U(x,v) < +\infty$$

Proof — Let us introduce the set-valued map $F : K \rightsquigarrow X$ defined by

$$(7) \qquad F(x) \ := \ \{ \ v \in X \ | \ U(x,v) \leq \lambda \ \}$$

The subset K enjoys the viability property (is a viability domain) for the hazy differential inclusion (4) if and only if it does so for this set-valued map F. The set-valued map satisfies the assumptions of the Viability Theorem (see [3, Theorem 4.2.1]), because the graph of F is closed, its images are convex and its growth is linear. Then we infer that K enjoys the viability property if and only if it is a viability domain of F, and thus, of $\mathbf{U}$. □

When the hazy set-valued map $\mathbf{U}$ is continuous, we can select a viable solution to the hazy differential inclusion (4) which is **sharpest**, in the sense that the cost of its velocity's membership is minimal:

$$(8) \qquad \text{for almost all } t, \ \ U(x(t),x'(t)) = \inf_{v\in T_K(x(t))} U(x(t),v)$$

We say that a closed subset K is "sleek" the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous. Closed convex subsets and smooth manifolds are sleek.

Theorem 2.3 *We posit the assumptions of Theorem 2.2. We assume moreover that the restriction of the membership function* U *to its domain (the graph of* $\mathbf{U}$*) is continuous and that the viability domain* K *is sleek. Then there exists a sharpest viable solution to the differential inclusion (4) (i.e., which satisfies condition (8)).*

Proof — We introduce the function λ defined by

$$(9) \qquad \lambda(x) \; := \; \inf_{v \in T_K(x)} U(x,v)$$

Since the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous by assumption, the Maximum Theorem implies that the function λ is upper semicontinuous, because we have assumed that U is upper semicontinuous.

We then introduce the set-valued map G defined by

$$(10) \qquad G(x) \; := \; \{ \, v \in X \; | \; U(x,v) \leq \lambda(x) \, \}$$

Then G has a closed graph, and the other assumptions of the Viability Theorem (see [3, Theorem 4.2.1]) are satisfied. There exist a viable solution to differential inclusion $x'(t) \in G(x(t))$, which is a sharpest viable solution to hazy differential inclusion (4.) □

3 Hazy Viability Domains

Is it possible to speak of hazy subsets having the viability property?

A way to capture this idea is to introduce a continuous function ϕ : $\mathbf{R}_+ \rightarrow \mathbf{R}$ with linear growth (which is used as a parameter in what follows) and the associated differential equation

$$(11) \qquad w'(t) = -\phi(w(t)), \;\; w(0) \; = \; V(x_0)$$

whose solutions $w(\cdot)$ set an upper bound to the membership of a hazy subset when time elapses. (The main instance of such a function ϕ is the affine function $\phi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a})$.

We shall say that a hazy set $V \subset \mathrm{Dom}(\mathbf{U})$ enjoys the "hazy viability property" (with respect to ϕ) if and only if for all initial state $x_0 \in \mathrm{Dom}(V)$, there exist solutions to the hazy differential inclusion (4) and to the differential equation (11) which are hazily viable in the sense that

$$(12) \qquad \forall\, t \geq 0, \;\; V(x(t)) \; \leq \; w(t), \;\; w(0) = V(x_0)$$

In order to extend the concept of contingent cone to a hazy subset, we need to adapt the concept of directional derivative to membership functions, which are only lower semicontinuous. Among the many possibilities, we choose the contingent epiderivative $D_\uparrow V(x)(v)$ of V at x in the direction v, which is defined by

$$D_\uparrow V(x)(v) \ := \ \liminf_{h \to 0+, u \to v} \frac{V(x+hu) - V(x)}{h}$$

because the epigraph of the function $v \to D_\uparrow V(x)(v)$ is the contingent cone to the epigraph of V at $(x, V(x))$ (see [1], [3, Chapter VII] for further information).

We say that V is contingently epidifferentiable if for all $x \in \mathrm{Dom}(V)$,

$$\forall\, v \in X,\ D_\uparrow V(x)(v) > -\infty \ \ \& \ \ D_\uparrow V(x)(v) < \infty \text{ for at least a } v \in X$$

We introduce now the "contingent set" $T_V^\phi(x)$ (also denoted $T_V(x)$), the closed subset defined by:

$$T_V^\phi(x) \ := \ \{\, v \in X \mid D_\uparrow V(x)(v) + \phi(V(x)) \leq 0 \,\} \tag{13}$$

Definition 3.1 (Hazy Viability Domain) *Let the continuous function ϕ with linear growth be given. We shall say that a hazy subset V is a hazy viability domain of a hazy set-valued map* $\mathbf{U}$ *(with respect to ϕ) if and only if*

$$\forall\, x \in \mathrm{Dom}(V),\ \ \exists\, v \in T_V^\phi(x) \ \ \textit{such that} \ \ U(x,v) < +\infty \tag{14}$$

Theorem 2.2 can be extended to hazy viability domains:

Theorem 3.2 *The hazy set-valued map* $\mathbf{U}$ *satisfies the assumptions of Theorem 2.2. We assume that $V \subset \mathrm{Dom}(\mathbf{U})$ is a closed hazy subset which is contingently epidifferentiable. If a closed hazy subset V enjoys the viability property, then it is a closed hazy viability domain of* $\mathbf{U}$ *and the converse holds true if*

$$\lambda \ := \ \sup_{x \in K} \ \inf_{v \in T_V^\phi(x)} U(x,v) < +\infty$$

Proof — Let us consider the set-valued map F defined by (7) and associate with it the system of differential inclusions

$$\begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & w'(t) = -\phi(w(t)) \end{cases} \tag{15}$$

We first observe that the epigraph $\mathcal{E}pV$ of V (which is closed) is a viability domain of the set-valued map $(x, w) \rightsquigarrow F(x) \times -\phi(w)$ if and only if V is a hazy viability domain.

Indeed, if $v \in F(x)$ is such that $(v, -\phi(V(x)))$ belongs to the contingent cone $T_{\mathcal{E}p(V)}(x, V(x))$ to the epigraph of V at $(x, V(x))$, which is equal to the epigraph $\mathcal{E}pD_\uparrow V(x)$ of the contingent epiderivative, we deduce that $D_\uparrow V(x)(v) + \phi(V(x)) \leq 0$.

Conversely, since $F(x)$ is compact and $v \mapsto D_\uparrow V(x)(v)$ is lower semicontinuous, there exists $v \in F(x)$ such that the pair $(v, -\phi(V(x)))$ belongs to $T_{\mathcal{E}p(V)}(x, V(x))$. Hence $(v, -\phi(V(x)))$ belongs to the intersection of $F(x) \times -\phi(V(x))$ and the contingent cone $T_{\mathcal{E}p(V)}(x, V(x))$. When $w > V(x)$, we deduce also that the pair $(v, -\phi(w))$, which belongs to $\mathrm{Dom}(D_\uparrow V(x)) \times \mathbf{R}$, is contained in the intersection of $F(x) \times -\phi(w)$ and the contingent cone $T_{\mathcal{E}p(V)}(x, w)$ because if $w > V(x)$,

$$\mathrm{Dom}(D_\uparrow V(x)) \times \mathbf{R}_+ \subset T_{\mathcal{E}p(V)}(x, w) \subset T_{\mathrm{Dom}(V)} \times \mathbf{R}$$

Then the epigraph of V enjoys the viability property: there exists a solution $(x(\cdot), w(\cdot))$ to the system of differential inclusions (15) which is viable in $\mathcal{E}p(V)$, i.e., which is hazily viable. □

For $\phi \equiv 0$, we obtain the following consequence:

Corollary 3.3 *We posit the assumptions of Theorem 3.2. Then a closed hazy subset V is a hazy viability domain (with respect to $\phi \equiv 0$) if and only if for all initial state $x_0 \in \mathrm{Dom}(V)$, the membership function V decreases along a solution $x(\cdot)$ to the hazy differential inclusion (4).*

Remark — Given an closed hazy subset V, we can associate with it affine functions $w \to aw - b$ for which V is a hazy viability domain.

For that purpose, we consider the convex function b defined by

$$b(a) := \sup_{x \in \mathrm{Dom}(\mathbf{U})} \left(\inf_{\{v | U(x,v) \leq \lambda(x)\}} D_\uparrow V(x)(v) + aV(x) \right)$$

Then it is clear that V is a hazy viability domain:

$$\forall\, x \in \mathrm{Dom}(F), \ \inf_{v \in F(x)} D_\uparrow V(x)(v) + aV(x) - b(a) \leq 0$$

Therefore, we deduce that there exists a solution to the hazy differential inclusion satisfying

$$\forall\, t \geq 0,\ V(x(t)) \ \leq\ (V(x_0) - \frac{b(a)}{a})e^{-at} + \frac{b(a)}{a}$$

A reasonable choice of a is the largest of the minimizers of $a\ \in]0,\infty[\to \max(0, b(a)/a)$, for which $V(x(t))$ decreases as fast as possible to the smallest level set $V^{-1}(]-\infty, \frac{b}{a}])$ of V. □

We proceed by extending Theorem 2.3 on selection of hazy viable solutions to hazy differential inclusion which are sharpest, in the sense that

$$\text{(16)} \qquad \text{for almost all } t,\ \ U(x(t), x'(t)) = \inf_{v\in T_V^{\phi}(x(t))} U(x(t), v)$$

Theorem 3.4 *We posit the assumptions of Theorem 2.2. We assume moreover that the restriction of the membership function U to its domain (the graph of* **U***) is continuous and that the hazy viability domain V satisfies*

$$\text{(17)} \qquad x \rightsquigarrow T_V^{\phi}(x) \ \text{ is lower semicontinuous}$$

Then there exists a sharpest viable solution to the differential inclusion (4) (which satisfies condition (16)).

Proof — The proof is the same than the one of Theorem 2.3, where the function λ is now defined by

$$\lambda(x) \ := \ \inf_{v\in T_V^{\phi}(x)} U(x,v) \ \square$$

We then need a sufficient condition for the set-valued map $x \rightsquigarrow T_V^{\phi}(x)$ to be lower semicontinuous:

Lemma 3.5 *Let us assume that the epigraph of V is sleek, (i.e., that the set-valued map $x \rightsquigarrow \mathcal{E}p(D_\uparrow V(x))$ is lower semicontinuous) and that the restriction of V to its domain is continuous. If for any x, there exists $\bar{v}$ such that*

$$D_\uparrow V(x)(\bar{v}) + \phi(V(x)) \ < \ 0$$

then $x \rightsquigarrow T_V^{\phi}(x)$ is lower semicontinuous at x.

Proof — We set $g(x) := -\phi(V(x))$, which is continuous by assumption. Let v belong to $T_V^{\phi}(x)$ be chosen and a sequence $x_n \in \mathrm{Dom}(D_\uparrow(V))$ converge to x. Since the set-valued map $\mathcal{E}p(D_\uparrow V(\cdot))$ is lower semicontinuous, and since $(v, g(x))$ belongs to $\mathcal{E}p(D_\uparrow V(x))$, there exist a subsequence (again denoted x_n), a sequence v_n converging to v and a sequence $\varepsilon_n \geq 0$ converging to 0 such that

$$(v_n, g(x_n) + \varepsilon_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

Since by assumption the pair $(\bar{v}, g(x) - a_0)$ belongs also to $\mathcal{E}p(D_\uparrow V(x))$, where $a_0 := g(x) - D_\uparrow V(x)(\bar{v}) > 0$, we deduce that there exist sequences $\bar{v}_n$ converging to $\bar{v}$ and $a_n > 0$ converging to a_0 such that

$$(\bar{v}_n, g(x_n) - a_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

We introduce now $\theta_n := \frac{\varepsilon_n}{2(\varepsilon_n + a_n)} \in [0,1]$ converging to 0 and $u_n := (1-\theta_n)v_n + \theta_n \bar{v}_n$ converging to v. The lower semicontinuity of the contingent cone to the epigraph of V, which is the epigraph of $D_\uparrow V(\cdot)$, implies that these cones are convex. Hence

$$(u_n, g(x_n) - \varepsilon_n/2) = (1-\theta_n)(v_n, g(x_n) + \varepsilon_n) + \theta_n(\bar{v}_n, g(x_n) - a_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

which can be written

$$D_\uparrow V(x_n)(u_n) \leq g(x_n) - \varepsilon_n/2 < g(x_n)$$

Hence u_n belongs to $T_V^{\phi}(x_n)$ and converges to v. □

4 Largest Closed Hazy Viability Domains

Let us consider now any closed hazy subset of the domain of $\mathbf{U}$, which is not necessarily a hazy viability domain. The functions ϕ being given, we shall construct the largest closed hazy viability domain V_ϕ contained in V.

Theorem 4.1 *The hazy set-valued map satisfies the assumptions of Theorem 2.2. We assume that $V \subset \mathrm{Dom}(\mathbf{U})$ is a closed hazy subset which is contingently epidifferentiable.*

Then for any $\lambda > 0$, there exists a largest closed hazy viability domain V_ϕ contained in V, which enjoys furthermore the property:

$$\text{for almost all } t \geq 0, \;\; U(x(t), x'(t)) \leq \lambda$$

Proof — We know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(V)$ of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$. If it is empty, it is the epigraph of the constant function equal to $+\infty$, and in this case, the largest closed hazy viability domain is empty.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function V_ϕ defined by

$$V_\phi(x) \quad := \quad \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, the epigraph of any membership function of a hazy viability domain W being a closed viability domain of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$, is contained in the epigraph of V_ϕ, so that V_ϕ is the largest closed viability domain contained in V.

For that purpose, assume for a while that the following claim is true:

if $\mathcal{M} \subset \mathrm{Dom}(F) \times \mathbf{R}_+$ is a closed viability domain of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$, then so is the subset

$$\mathcal{M} \; + \; \{0\} \times \mathbf{R}_+$$

If this is the case, $\mathcal{K}$ is contained in the closed viability domain $\mathcal{K} \; + \; \{0\} \times \mathbf{R}_+$, so that, being the largest one, is equal to it. Let us prove this claim.

First, $\mathcal{M} + \{0\} \times \mathbf{R}_+$ is closed. Indeed, let a sequence (x_n, λ_n) of this subset converges to some (x, λ).Then there exists a sequence of elements $(x_n, \mu_n) \in \mathcal{M}$ with $0 \le \mu_n \le \lambda_n$. A subsequence (again denoted) μ_n does converge to some $\mu \in [0, \lambda]$ because the sequence remains in a compact interval of $\mathbf{R}_+$. Therefore (x, μ) belongs to $\mathcal{M}$ (which is closed) and (x, λ) belongs to $\mathcal{M} + \{0\} \times \mathbf{R}_+$.

Second, $\mathcal{M} + \{0\} \times \mathbf{R}_+$ is a viability domain. Let (x, w) belong to $\mathcal{M} + \{0\} \times \mathbf{R}_+$. Hence $w \ge V_M(x)$ defined by

$$V_M(x) \quad := \quad \inf_{(x,\lambda)\in\mathcal{M}} \lambda$$

We set $d := -\phi(V_M(x))$. By assumption, there exists $v \in F(x)$ such that (v, d) belongs to the contingent cone to $\mathcal{M}$ at the point $(x, V_M(x)) \in \mathcal{M}$. We shall check that the pair $(v, -\phi(w))$ does belong to the contingent cone to

$\mathcal{M} + \{0\} \times \mathbf{R}_+$ at (x, w). Indeed, there exist sequences $h_n > 0$ converging to 0, v_n converging to v and d_n converging to d such that

$$\forall\, n \geq 0, \quad (x + h_n v_n, V_M(x) + h_n d_n) \;\in\; \mathcal{M}$$

This proves the claim when $w = V_M(x)$. If not, $\varepsilon := w - V_M(x)$ is strictly positive, so that, for h_n sufficiently small,

$$\begin{array}{l}(x + h_n v_n, w - h_n \phi(w)) = \\ (x + h_n v_n, V_M(x) + h_n d_n) + (0, \varepsilon + h_n(\phi(w) - d_n) \;\in\; \mathcal{M} \;+\; \{0\} \times \mathbf{R}_+\end{array}$$

because d_n converges to d and $\varepsilon + h_n(\phi(w) - d_n)$ is nonnegative for small enough h_n. □

Remark — If the initial state $x_0 \in \mathrm{Dom}(V)$ does not belong to the domain of V_ϕ, then for any solution $(x(\cdot), w(\cdot))$ to the system (15) starting at (x_0, w_0), the state leaves eventually the hazy subset V:

$$\exists\, T > 0 \mid w(T) \;<\; V(x(T))$$

This happens for all initial state $x_0 \in \mathrm{Dom}(V)$ whenever the largest closed hazy viability domain contained in V is empty, i.e., when $V_\phi \equiv +\infty$. □

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