WORKING PAPER

SMALLEST LYAPUNOV FUNCTIONS OF DIFFERENTIAL INCLUSIONS

Jean-Pierre Aubin

September 1988
WP-88-081

IIASA
International Institute for Applied Systems Analysis

SMALLEST LYAPUNOV FUNCTIONS OF DIFFERENTIAL INCLUSIONS

Jean-Pierre Aubin

September 1988
WP-88-081

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

FOREWORD

This paper provides a first answer to the question: does there exist a **smallest** Lyapunov function of a differential inclusion **larger than a given function**. For that purpose, they have to be looked for in the class of lower semicontinuous functions, and thus, the concept of derivative has to be replaced by the one of contingent epiderivative to characterize lower semicontinuous Lyapunov functions. The existence of a largest closed viability (and/or invariance) domain of a differential inclusion contained in a given closed subset is then proved and used to infer the existence of such a Lyapunov function.

Alexander B. Kurzhanski
Chairman
System and Decision Science Program

Smallest Lyapunov Functions of Differential Inclusions

Jean-Pierre Aubin

CEREMADE, Université de Paris-Dauphine
&
IIASA, International Institute for Applied Systems Analysis

Contents

Smallest Lyapunov Functions of Differential Inclusions

Jean-Pierre Aubin

Introduction

Introduction

We provide a first answer to the question: given a differential inclusion, does there exist a **smallest** nonnegative extended lower semicontinuous (i.e., take their values in $\mathbf{R}_+ \cup \{+\infty\}$) Lyapunov function **larger than a given** lower semicontinuous function? Since lower semicontinuous functions are involved in the statement of this problem are not necessarily differentiable, we have to weaken the usual definition of a derivative and replace it by the one of epicontingent derivative. This allows to characterize lower semicontinuous Lyapunov functions of a differential inclusion. With this definition at hand, we shall answer this question.

The tool for achieving this objective is the existence of largest closed viability (and/or invariance) domains of a differential inclusion contained in a given closed subset. Hence, we shall provide in the appendix the proof of their existence as well as the division of the boundary of a closed subset in areas from where some or all solutions to the differential inclusion remain or leave this closed subset.

Contents

1 Lyapunov Functions

We consider a differential inclusion

(1) $$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t))$$

and time-dependent functions $w(\cdot)$ defined as solutions to a differential equation

(2) $$w'(t) = -\phi(w(t)), \;\; w(0) \;=\; V(x(0))$$

where $\phi : \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth. This function ϕ is used as a parameter in what follows. (The main instance of such a function ϕ is the affine function $\phi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a})$.

Our problem is to characterize either functions enjoying the ϕ-Lyapunov property, i.e., nonnegative extended functions $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ satisfying

(3) $$\forall\, t \geq 0, \;\; V(x(t)) \;\leq\; w(t), \;\; w(0) = V(x(0))$$

along at least a solution to the differential inclusion (1) or the ϕ- universal Lyapunov property, for which property (3) is satisfied along **all** solutions to (1).

We recall that the contingent epiderivative $D_\uparrow V(x)(v)$ of V at x in the direction v is defined by

$$D_\uparrow V(x)(v) \;:=\; \liminf_{h \to 0+, u \to v} \frac{V(x+hu) - V(x)}{h}$$

because the epigraph of the function $v \to D_\uparrow V(x)(v)$ is the contingent cone to the epigraph of V at $(x, V(x))$ (see [1], [3, Chapter VII] for further information).

We say that V is contingently epidifferentiable if for all $x \in \text{Dom}(V)$,

$$\forall\, v \in X, \; D_\uparrow V(x)(v) > -\infty \;\;\&\;\; D_\uparrow V(x)(v) < \infty \text{ for at least a } v \in X$$

Definition 1.1 *We shall say that a nonnegative contingently epidifferentiable extended function V is a* **Lyapunov function** *of F associated with* *a function* $\phi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$ *if and only if* V *is a solution to the contingent Hamilton-Jacobi inequalities*

$$(4) \qquad \forall\, x \in \mathrm{Dom}(V), \quad \inf_{v\in F(x)} D_\uparrow V(x)(v) + \phi(V(x)) \;\leq\; 0$$

and a **universal Lyapunov function** *of* F *associated with a function* ϕ *if and only if* V *is a solution to the upper contingent Hamilton-Jacobi inequalities*

$$(5) \qquad \forall\, x \in \mathrm{Dom}(V), \quad \sup_{v\in F(x)} D_\uparrow V(x)(v) + \phi(V(x)) \;\leq\; 0$$

(We refer to [5,6,7] and the references of these papers for a thorough study of contingent Hamilton-Jacobi equations arising from optimal control and comparison with viscosity solutions.)

Theorem 1.2 *Let* V *be a nonnegative contingently epidifferentiable lower semicontinuous extended function and* $F : X \rightsquigarrow X$ *be a nontrivial set-valued map.*

— *Let us assume that* F *is upper semicontinuous with compact convex images and linear growth. Then* V *is a Lyapunov function of* F *associated with* $\phi(\cdot)$ *if and only if for any initial state* $x_0 \in \mathrm{Dom}(V)$, **there exist solutions** $x(\cdot)$ *to differential inclusion (1) and* $w(\cdot)$ *to differential equation (2) satisfying property (3).*

— *If* F *is lipschitzean on the interior of its domain with compact values, then* V *is a universal Lyapunov function associated with* ϕ *if and only if for any initial state* $x_0 \in \mathrm{Dom}(V)$, **all solutions** $x(\cdot)$ *to differential inclusion (1) and* $w(\cdot)$ *to differential equation (2) do satisfy property (3).*

Proof — We consider the system of differential inclusions

$$(6) \qquad \begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & w'(t) = -\phi(w(t)) \end{cases}$$

— We provide a simpler proof than the ones of a stronger result (see [1], [2, Theorem 6.3.1] and [4]) by observing that the epigraph $\mathcal{E}pV$ of V, (which is closed) is a viability domain (see Definition 3.3 of the appendix) of the set-valued map $(x, w) \rightsquigarrow F(x) \times -\phi(w)$ if and only if V is a Lyapunov function.

Indeed, if $v \in F(x)$ is such that $(v, -\phi(V(x)))$ belongs to the contingent cone $T_{\mathcal{E}p(V)}(x, V(x))$ to the epigraph of V at $(x, V(x))$, which is equal to the epigraph $\mathcal{E}pD_{\uparrow}V(x)$ of the contingent epiderivative, we deduce that $D_{\uparrow}V(x)(v) + \phi(V(x)) \leq 0$.

Conversely, since $F(x)$ is compact and $v \mapsto D_{\uparrow}V(x)(v)$ is lower semicontinuous, there exists $v \in F(x)$ such that the pair $(v, -\phi(V(x)))$ belongs to $T_{\mathcal{E}p(V)}(x, V(x))$. Hence $(v, -\phi(V(x)))$ belongs to the intersection of $F(x) \times -\phi(V(x))$ and the contingent cone $T_{\mathcal{E}p(V)}(x, V(x))$. When $w > V(x)$, we deduce also that the pair $(v, -\phi(w))$, which belongs to $\mathrm{Dom}(D_{\uparrow}V(x)) \times \mathbf{R}$, is contained in the intersection of $F(x) \times -\phi(w)$ and the contingent cone $T_{\mathcal{E}p(V)}(x, w)$ because if $w > V(x)$,

$$\mathrm{Dom}(D_{\uparrow}V(x)) \times \mathbf{R}_{+} \subset T_{\mathcal{E}p(V)}(x, w) \subset T_{\mathrm{Dom}(V)} \times \mathbf{R}$$

Then the epigraph of V enjoys the viability property: there exists a solution $(x(\cdot), w(\cdot))$ to the system of differential inclusions (6) which is viable in $\mathcal{E}p(V)$, i.e., which satisfies property (3).

— In the same way, one can check that the closed subset $\mathcal{E}pV$ is an invariant domain Definition 3.3) of the set-valued map $(x, w) \rightsquigarrow F(x) \times -\phi(w)$ if and only if V is a universal Lyapunov function. By the Invariance Theorem (see [2, Theorem 4.6.2], which can be applied because F is lipschitzean, we deduce that V is a universal Lyapunov function if and only if $\mathcal{E}pV$ is invariant by $(x, w) \rightsquigarrow F(x) \times -\phi(w)$, i.e., if and only if property (3) holds true for all solutions to the system (6). □

For $\phi \equiv 0$, we obtain the following consequence:

Corollary 1.3 *Let V be an nonnegative contingently epidifferentiable extended function and $F : X \rightsquigarrow X$ be a nontrivial set-valued map.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth. Then V is a Lyapunov function of F in the sense that*

$$\inf_{v \in F(x)} D_{\uparrow}V(x)(v) \leq 0$$

if and only if for any initial state $x_0 \in \mathrm{Dom}(V)$, V decreases along a solution $x(\cdot)$ to differential inclusion (1).

— *If F is lipschitzean on the interior of its domain with compact* *values, then V is a universal Lyapunov function in the sense that*

$$\sup_{v \in F(x)} D_{\uparrow}V(x)(v) \leq 0$$

if and only if for any initial state $x_0 \in \mathrm{Dom}(V)$, V decreases along all solutions $x(\cdot)$ to differential inclusion (1).

We can reformulate the viability and invariance theorems in the following way:

Corollary 1.4 *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth.*

A closed subset K enjoys the viability property if and only if its indicator Ψ_K is a solution to the contingent equation

$$\inf_{v \in F(x)} D_{\uparrow}\Psi_K(x)(v) = 0$$

— *If F is lipschitzean on the interior of its domain with compact values, then K is invariant by F if and only if its indicator Ψ_K is a solution to the contingent equation*

$$\sup_{v \in F(x)} D_{\uparrow}\Psi_K(x)(v) = 0$$

We introduce now attractors:

Definition 1.5 *We shall say that a closed subset K is an "attractor" of order $a \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, there exists a solution $x(\cdot)$ to the differential inclusion (1) such that*

$$(7) \qquad \forall\, t \geq 0, \;\; d_K(x(t)) \leq d_K(x_0)e^{-at}$$

It is said to be an "universal attractor" of order $a \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, all solutions $x(\cdot)$ to the differential inclusion (1) satisfy the above property.

We can recognize attractors sets K by checking whether the distance function to K is a Lyapunov function:

Corollary 1.6 *Assume that F is a nontrivial upper semicontinuous set-valued map with nonempty compact convex images and with linear growth.*

Then a closed subset $K \subset \text{Dom}(F)$ is an attractor if and only if the function $d_K(\cdot)$ is a solution to the contingent inequalities:

$$\forall x \in \text{Dom}(F), \inf_{v \in F(x)} D_\uparrow d_K(x)(v) + a d_K(x) \leq 0$$

If F is lipschitzean with compact images, then K is a universal attractor if and only if

$$\forall x \in \text{Dom}(F), \sup_{v \in F(x)} D_\uparrow d_K(x)(v) + a d_K(x) \leq 0$$

For $a = 0$, a sufficient condition for K to be an attractor of order 0 is then to satisfy

$$\forall x \in \text{Dom}(F), \exists\, y \in \pi_K(x) \mid F(x) \cap T_K(y) \neq \emptyset$$

because we know that

$$D_\uparrow d_K(x)(v) \leq d(v, T_K(\pi_K(x)))$$

This a particular case of the situation where the function V is defined through a nonnegative function $U : X \times Y \to \mathbf{R}_+ \cup \{+\infty\}$ in the following way:

$$V(x) := \inf_{y \in Y} U(x, y)$$

When we assume that the infimum is achieved at a point y_x, formula

$$D_\uparrow V(x) \leq \inf_{v \in Y} D_\uparrow U(x, y_x)(u, v)$$

holds true (we take $U(x,y) := \|x-y\| + \Psi_K(y)$, whose contingent epiderivative is equal to $\|u - v\| + \Psi_{T_K(y)}(v)$).

We deduce it from the fact that the epigraph of V is contained in the closure of the projection of the epigraph of U onto $X \times \mathbf{R}$ and that the closures of the image of the contingent cone by a linear operator is contained in the contingent cones of the image:

$$\begin{cases} \pi \mathcal{E}p D_\uparrow(U)(x, y_x) \subset (\overline{\pi T_{\mathcal{E}p(U)}(x, y_x, U(x, y_x))}) \\ \subset T_{\pi(\mathcal{E}pU)}(x, V(x)) = T_{\mathcal{E}p(V)}(x, V(x)) \\ = \mathcal{E}p(D_\uparrow(V)(x)) \end{cases}$$

which can be easily translated into this inequality.

Hence, under the assumptions of Theorem 1.2, we infer that assumption

$$(8) \qquad \forall\, x, \inf_{u\in F(x), v\in Y} D_\uparrow U(x,y_x) + \phi(U(x,y_x)) \;\leq\; 0$$

implies there exists a solution (that all solutions) $x(\cdot)$ satisfy

$$\forall\, t \geq 0, \; \inf_{v\in Y} U(x(t),y) \;\leq\; w(t)$$

We can derive from this inequality and the calculus of contingent epiderivatives many consequences.

Example W-Monotone Set-Valued Maps

Let $W : X \to \mathbf{R}_+ \cup \{+\infty\}$ be a nonnegative extended function. We say that a set-valued map F is W-monotone (with respect to ϕ) if

$$(9)\, \forall\, x,y, \; \forall\, u \in F(x), v \in F(y), \; D_\uparrow W(x-y)(v-u) + \phi(W(x-y)) \leq 0$$

We obtain for instance the following consequence:

Corollary 1.7 *Let W be an nonnegative contingently epidifferentiable extended lower semicontinuous function and $F : X \rightsquigarrow X$ be a nontrivial upper semicontinuous set-valued map with compact convex images and linear growth which is W-monotone with respect to some ϕ. Let $\bar{x}$ be some equilibrium of F. Then, for any initial state x_0, there exist solutions $x(\cdot)$ and $w(\cdot)$ satisfying*

$$\forall\, t \geq 0, \; W(x(t) - \bar{x}) \;\leq\; w(t)$$

In particular, for $W(z) := \frac{1}{2}\|z\|^2$, we find the usual concepts of monotonicity (with respect to ϕ):

$$\forall\, x,y, \;\; \forall\, u \in F(x), v \in F(y), \; < u-v, x-y > \;\geq\; \phi(\frac{1}{2}\|x-y\|^2) \;\; \square$$

Remark — Given an extended nonnegative function V, we can associate with it affine functions $w \to aw - b$ for which V is a solution to the contingent Hamilton-Jacobi inequalities (4).

For that purpose, we consider the convex function b defined by

$$b(a) \;:=\; \sup_{x\in \mathrm{Dom}(F)} \;(\inf_{v\in F(x)} D_\uparrow V(x)(v) + aV(x))$$

Then it is clear that V is a solution to the contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(F),\ \inf_{v\in F(x)} D_{\uparrow}V(x)(v) + aV(x) - b(a) \le 0$$

Therefore, we deduce that there exists a solution to the differential inclusion satisfying

$$\forall\, t \ge 0,\ V(x(t)) \ \le\ (V(x_0) - \frac{b(a)}{a})e^{-at} + \frac{b(a)}{a}$$

A reasonable choice of a is the largest of the minimizers of $a \in]0,\infty[\rightarrow \max(0, b(a)/a)$, for which $V(x(t))$ decreases as fast as possible to the smallest level set $V^{-1}(]-\infty,\frac{b}{a}])$ of V. □

Remark — By using the necessary condition of the Viability Theorem, we obtain the following result. First, we denote by $D_{\downarrow}V(x)(v)$ the contingent hypoderivative of V, whose hypograph is the contingent cone to the hypograph of V at $(x, V(x))$, and defined by

$$D_{\downarrow}V(x)(v) \ :=\ \limsup_{h\to 0+,u\to v} \frac{V(x+hu)-V(x)}{h}$$

Theorem 1.8 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth and a continuous function $\phi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$.*

Let V be an nonnegative contingently epidifferentiable extended function. If, for some $x_0 \in \mathrm{Dom}(F)$, we have

$$\sup_{v\in F(x_0)} D_{\downarrow}V(x_0)(v) + \phi(V(x_0)) \ <\ 0$$

then, for any solution $x(\cdot)$ to the differential inclusion starting at x_0 and any solution $w(\cdot)$ to the differential equation starting at $V(x_0)$, there exists $T > 0$ such that

$$\forall\, t \in]0,T],\ \ V(x(t)) < w(t)$$

Proof — Assume the contrary: there exists a solution $x(\cdot)$ to (1) starting at x_0 and a solution $w(\cdot)$ to (3) starting at $V(x_0)$ satisfying

$$\forall\, T > 0,\ \exists\, t \in]0,T] \ \mid\ W(x(t)) \ \ge\ w(t)$$

It is easy to deduce that there exists $v \in F(x_0)$ such that

$$(v, -\phi(V(x_0))) \in T_{\mathcal{H}yp(V)}(x_0) := \mathcal{H}yp(D_\downarrow V(x_0))$$

by the very definition of the contingent hypoderivative. Hence

$$-\phi(V(x_0)) \;\leq\; D_\downarrow V(x_0)(v) \;\leq\; \sup_{v \in F(x_0)} D_\downarrow V(x_0)(v)$$

which contradicts our assumption. □

2 Smallest Lyapunov Functions

The functions ϕ and $U : X \to \mathbf{R}_+ \cup \{+\infty\}$ being given, we shall construct the smallest lower semicontinuous Lyapunov function of a set-valued map F associated to ϕ larger than or equal to U, i.e., the smallest nonnegative lower semicontinuous solution U_ϕ to the contingent Hamilton-Jacobi inequalities (4) larger than or equal to U.

Theorem 2.1 *Let us consider a nontrivial set-valued map $F : X \rightsquigarrow X$, a continuous function $\phi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth and a proper nonnegative extended function U.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth. Then there exists a smallest nonnegative lower semicontinuous solution $U_\phi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the contingent Hamilton-Jacobi inequalities (4) larger than or equal to U (which can be the constant $+\infty$), which then enjoys the property:*

$$\begin{array}{l} \forall\, x \in \mathrm{Dom}(U_\phi), \quad \textbf{there exists solutions} \text{ to (1) and (2) satisfying} \\ \qquad \forall\, t \geq 0, \;\; U(x(t)) \;\leq\; U_\phi(x(t)) \;\leq\; w(t) \end{array}$$

— *If F is lipschitzean on the interior of its domain with compact values and ϕ is lipschitzean, then there exists a smallest nonnegative lower semicontinuous solution $\tilde{U}_\phi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the upper contingent Hamilton-Jacobi inequalities (4) larger than or equal to U (which can be the constant $+\infty$), which then enjoys the property:*

$$\begin{array}{l} \forall\, x \in \mathrm{Dom}(U_\phi), \quad \textbf{all solutions} \text{ to (1) and (2) satisfy} \\ \qquad \forall\, t \geq 0, \;\; U(x(t)) \;\leq\; U_\phi(x(t)) \;\leq\; w(t) \end{array}$$

In particular, for $\phi(w) := aw$, we deduce that

$$\forall\, x \in \text{Dom}(U_a),\ U(x(t)) \leq U_a(x_0)e^{-at} \text{ and thus, converges to } 0$$

Proof

— By Theorem 3.4 of the appendix, we know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(U)$ (the viability kernel of the epigraph of U) of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$. If it is empty, it is the epigraph of the constant function equal to $+\infty$.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function U_ϕ defined by

$$U_\phi(x) \ := \ \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, the epigraph of any solution U to the contingent inequalities (4) being a closed viability domain of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$, is contained in the epigraph of U_ϕ, so that U_ϕ is the smallest of the lower semicontinuous solutions to (4).

— For that purpose, assume for a while that the following claim is true:

if $\mathcal{M} \subset Dom(F) \times \mathbf{R}_+$ is a closed viability domain of the set-valued map $(x,w) \rightsquigarrow F(x) \times -\phi(w)$, then so is the subset

$$\mathcal{M} \ + \ \{0\} \times \mathbf{R}_+$$

If this is the case, $\mathcal{K}$ is contained in the closed viability domain $\mathcal{K} \ + \ \{0\} \times \mathbf{R}_+$, so that, being the largest one, is equal to it. Let us prove this claim.

First, $\mathcal{M} + \{0\} \times \mathbf{R}_+$ is closed. Indeed, let a sequence (x_n, λ_n) of this subset converges to some (x,λ).Then there exists a sequence of elements $(x_n, \mu_n) \in \mathcal{M}$ with $0 \leq \mu_n \leq \lambda_n$. A subsequence (again denoted) μ_n does converge to some $\mu \in [0,\lambda]$ because the sequence remains in a compact interval of $\mathbf{R}_+$. Therefore (x,μ) belongs to $\mathcal{M}$ (which is closed) and (x,λ) belongs to $\mathcal{M} + \{0\} \times \mathbf{R}_+$.

Second, $\mathcal{M} + \{0\} \times \mathbf{R}_+$ is a viability domain. Let (x,w) belong to $\mathcal{M} + \{0\} \times \mathbf{R}_+$. Hence $w \geq U_M(x)$ defined by

$$U_M(x) \ := \ \inf_{(x,\lambda)\in\mathcal{M}} \lambda$$

We set $d := -\phi(U_M(x))$. By assumption, there exists $v \in F(x)$ such that (v, d) belongs to the contingent cone to $\mathcal{M}$ at the point $(x, U_M(x)) \in \mathcal{M}$. We shall check that the pair $(v, -\phi(w))$ does belong to the contingent cone to $\mathcal{M} + \{0\} \times \mathbf{R}_+$ at (x, w). Indeed, there exist sequences $h_n > 0$ converging to 0, v_n converging to v and d_n converging to d such that

$$\forall\, n \geq 0, \quad (x + h_n v_n, U_M(x) + h_n d_n) \;\in\; \mathcal{M}$$

This proves the claim when $w = U_M(x)$. If not, $\varepsilon := w - U_M(x)$ is strictly positive, so that, for h_n sufficiently small,

$$\begin{array}{l} (x + h_n v_n, w - h_n \phi(w)) = \\ (x + h_n v_n, U_M(x) + h_n d_n) + (0, \varepsilon + h_n(\phi(w) - d_n) \;\in\; \mathcal{M} \;+\; \{0\} \times \mathbf{R}_+ \end{array}$$

because d_n converges to d and $\varepsilon + h_n(\phi(w) - d_n)$ is nonnegative for small enough h_n.

— When F and ϕ are lipschitzean, Theorem 3.7 of the appendix implies that there exists a largest closed invariance domain $\widetilde{\mathcal{K}}$ contained in the epigraph of U (the invariant kernel). We prove that it is the epigraph of smallest lower semicontinuous solution

$$\widetilde{U}_\phi \;=\; \inf_{(x,\lambda) \in \widetilde{\mathcal{K}}} \lambda$$

to the upper contingent Hamilton-Jacobi inequalities (2) we are looking for.

For that purpose, we check in an analogous way that the claim

if $\mathcal{M} \subset Dom(F) \times \mathbf{R}_+$ is a closed invariance domain of the set-valued map $(x, w) \rightsquigarrow F(x) \times -\phi(w)$, then so is the subset

$$\mathcal{M} \;+\; \{0\} \times \mathbf{R}_+$$

is true. We conclude in the same way. □

Corollary 2.2 *We posit the assumptions of Theorem 2.1.*

a/ Let us assume that F is upper semicontinuous with compact convex images and linear growth.

— *The indicator* $\Psi_{\mathrm{Viab}(K)}$ *of the viability kernel* $\mathrm{Viab}(K)$ *of a closed subset* K *(i.e., the largest closed viability domain of* F *contained in* K*) is the smallest nonnegative lower semicontinuous solution to*

$$(10) \qquad \forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\leq\; 0$$

larger than or equal to Ψ_K.

— *For all* $a \geq 0$*, there exists a smallest lower semicontinuous function* $d_{M_a} : X \to \mathbf{R} \cup \{+\infty\}$ *larger than or equal to* d_M *such that*

$$\forall\, x_0 \in \mathrm{Dom}(d_{M_a}), \text{ there exists a solution } x(\cdot) \text{ to (1) such that } d_M(x(t)) \leq d_{M_a}(x_0)e^{-at}$$

b/ Assume that F *is lipschitzean on the interior of its domain with compact values.*

— *The indicator* $\Psi_{\mathrm{Inv}(K)}$ *of the invariant kernel* $\mathrm{Inv}(K)$ *of a closed subset* K *(i.e., the largest closed invariance domain of* F *contained in* K*) is the smallest nonnegative lower semicontinuous solution to*

$$(11) \qquad \forall\, x \in \mathrm{Dom}(V), \quad \sup_{v \in F(x)} D_{\uparrow}V(x)(v) \;\leq\; 0$$

larger than or equal to Ψ_K.

— *For all* $a \geq 0$*, there exists a smallest lower semicontinuous function* $\widetilde{d_{M_a}} : X \to \mathbf{R} \cup \{+\infty\}$ *larger than or equal to* d_M *such that*

$$\forall\, x_0 \in \mathrm{Dom}(d_{M_a}), \text{ any solution } x(\cdot) \text{ to (1) satisfies } d_M(x(t)) \leq \widetilde{d_{M_a}}(x_0)e^{-at}$$

We can regard the subsets $\mathrm{Dom}(d_{M_a})$ and $\mathrm{Dom}(\widetilde{d_{M_a}})$ as the basins of exponential attraction and of universal exponential attraction of M.

Proof

— Let us check that the smallest lower semicontinuous solution U_0 larger than or equal to $U \equiv 0$ is equal to the indicator of $\mathrm{Viab}(K)$. Since it is clear that it is a solution to the above contingent inequalities (10), then

$$\forall\, x \in \mathrm{Viab}(K), \quad U_0(x) \;\leq \Psi_{\mathrm{Viab}(K)}(x)$$

Let x_0 belong to the domain of U_0. Then there exists a solution $x(\cdot)$ to the system of differential inclusions (6) starting at $(x_0, U_0(x_0))$ satisfying $U_0(x(t)) \leq U_0(x_0)$ since $w(t) \equiv U_0(x_0)$. Therefore x_0 belongs to the largest closed viability domain Viab(K). Hence $U_0(x_0) \leq \Psi_{\text{Viab}(K)}(x_0) = 0$.

— Let us check now that the smallest lower semicontinuous solution $\widetilde{U}_0$ larger than or equal to $U \equiv 0$ is equal to the indicator of Inv(K). Since it is clear that it is a solution to contingent inequalities (11), then

$$\forall\, x \in \text{Inv}(K), \;\; \widetilde{U}_0(x) \;\; \leq \Psi_{\text{Inv}(K)}(x)$$

Let x_0 belong to the domain of $\widetilde{U}_0$. Then all solutions $x(\cdot)$ to the system of differential inclusions (6) starting at $(x_0, \widetilde{U}_0(x_0))$ satisfy $U_0(x(t)) \leq \widetilde{U}_0(x_0)$, so that x_0 belongs to the largest closed invariance domain Inv(K). Hence $\widetilde{U}_0(x_0) \leq \Psi_{\text{Inv}(K)}(x_0) = 0$. □

Remark — If $0 \leq \phi \leq \psi$, then

$$U_\psi \;\geq\; U_\phi \;\geq\; U_0 \;\geq\; U \tag{12}$$

Therefore, if the extended function U_ψ is proper, (i.e., different from the constant function $+\infty$), we obtain the inclusions

$$\begin{cases} U_\phi^{-1}(0) & \subset & \text{Dom}(U_\phi) & \subset & \text{Dom}(U) \\ \cup & & \cup & & \\ U_\psi^{-1}(0) & \subset & \text{Dom}(U_\psi) & \subset & \text{Dom}(U) \end{cases} \tag{13}$$ □

Proposition 2.3 *We posit the assumptions of Theorem 2.1. Assume furthermore that ϕ vanishes at 0. Then if U vanishes on an equilibrium $\bar{x}$ of F, so does the function U_ϕ associated with ϕ.*

Let L be the set-valued map associating to any solution $x(\cdot)$ to the differential inclusion (1) its limit set and $\mathcal{S}$ be the solution map. If ϕ is asymptotically stable, then for any $x_0 \in \text{Dom}(U_\phi)$, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that $L(x(\cdot)) \subset U^{-1}(0) \cap F^{-1}(0)$.

Proof

— If $\bar{x}$ is an equilibrium of F such that $U(\bar{x}) = 0$, then $(\bar{x}, 0)$ is an equilibrium of $(x, w) \to F(x) \times -\phi(w)$ restricted to the epigraph of U

(because $\phi(0) = 0$), so that the singleton $(\bar{x}, 0)$, which is a viability domain, is contained in the epigraph of U_ϕ. Hence $0 \leq U(\bar{x}) \leq U_\phi(\bar{x}) \leq 0$.

— If ϕ is asymptotically stable, then the solutions $w(\cdot)$ to the differential equation

$$(14) \qquad w'(t) = -\phi(w(t))$$

do converge to 0 when $t \to +\infty$. Let x_0 belong to the domain of U_ϕ and $x(\cdot)$ be a solution satisfying

$$U(x(t)) \leq U_\phi(x(t)) \leq w(t)$$

Hence any cluster point ξ of $L(x(\cdot))$, which is the limit of a subsequence $x(t_n)$, belongs to $U_\phi^{-1}(0)$, because the limit $(\xi, 0)$ of the sequence of elements $(x(t_n), w(t_n))$ of the epigraph of U_ϕ belongs to it, for it is closed. Hence $0 \leq U(\xi) \leq U_\phi(\xi) \leq 0$. □

Remark — Since the epigraph of U_ϕ is the viability kernel of the epigraph of U, we deduce that for any initial situation (x_0, w_0) such that $w_0 < U_\phi(x_0)$, for any solution $(x(\cdot), w(\cdot))$ to the system (6) starting at (x_0, w_0), then

$$\exists\, T > 0 \mid w(T) < U(x(T))$$

This happens whenever the initial state x_0 does not belong to the domain of U_ϕ.

If $U_\phi \equiv +\infty$, then the above property holds true for any solution to the differential inclusion (2.1). □

3 Appendix: the Anatomy of a Closed Subset

Let us consider the differential inclusion (1)

Definition 3.1 (Viability and invariance properties) *Let K be a subset of Ω. We shall say that K enjoys the local* viability property *(for the set-valued map F) if for any initial state x_0 of K,* there exist *$T > 0$ and a* viable *solution on $[0, T]$ to the differential inclusion (1) starting at x_0. It enjoys the global viability property (or, simply, the viability property) if we can always take $T = \infty$*

The subset K is said to be invariant by F *if for any initial state x_0 of K,* all *solutions to the differential inclusion (1) are* viable .

Remark — We should emphasize again that the concept of invariance *depends upon the behavior of F on the domain Ω outside K.* □

There are two ways to extend the concept of viability domain K to set-valued maps. The first one is to require that for any state x, there exists **at least** a velocity $v \in F(x)$ which is **contingent** to K at x. The second demands that **all** velocities $v \in F(x)$ are **contingent** to K at x.

We would naturally like to characterize the viability property by the first condition and the invariance property by the second. This is more or less the situation that we shall meet.

Definition 3.2 (Viability and Invariance Domains) *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \mathrm{Dom}(F)$ is a* viability domain *of F if and only if*

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset \tag{15}$$

and that it is an invariance domain *if and only if*

$$\forall x \in K, \quad F(x) \subset T_K(x) \tag{16}$$

The main Viability and Invariance Theorems (equivalent to the statements of Theorems 1.4) state that under the assumptions described in this theorem, a closed domain enjoys the viability (invariance) property if and only if it is a viability (invariance) domain.

Let K be a closed subset of the domain of F. We shall prove the existence of the largest closed viability and invariance domains contained in K.

Definition 3.3 (Viability and Invariance Kernels) *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$, We shall say that the largest closed viability domain contained in K (which may be empty) is the* **viability kernel of** *K and denote it by $\mathrm{Viab}_F(K)$ or, simply, $\mathrm{Viab}(K)$. The largest closed invariance domain contained in K, which we denote by $\mathrm{Inv}_F(K)$ or $\mathrm{Inv}(K)$, is called the* **invariance kernel** *of K.*

We begin by proving that such a viability kernel does exist and characterize it.

Theorem 3.4 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth. Then the viability kernel does exist and is the subset of initial points such that at least a solution starting from them is viable in K.*

Proof — Let us denote by $\mathcal{K} \subset \mathcal{C}(0,T;X)$ the closed subset of functions viable in K and set

$$\text{Viab}(K) := \{x \in K \mid \mathcal{S}(x) \cap \mathcal{K} \neq \emptyset\}$$

— It is closed: indeed, let us consider a sequence $x_n \in \text{Viab}(K)$ converges to x, and thus, remains in a compact subset of the finite dimensional vector-space X. Let us choose a sequence of solutions $x_n(\cdot) \in \mathcal{S}(x_n) \cap \mathcal{K}$.

Since the graph of the restriction of $\mathcal{S}$ to any compact subset of the finite dimensional vector-space X is compact by the Convergence Theorem (see [3, Theorem 2.2.1.]), we infer that $(x_n, x_n(\cdot))$ remains in the compact set Graph($\mathcal{S}$). A subsequence converges to some $(x, x(\cdot))$ of the graph of $\mathcal{S}$, so that $x(\cdot)$ belongs to both $\mathcal{S}(x)$ and $\mathcal{K}$, which is closed. Therefore, the limit x belongs to Viab(K).

— The subset Viab(K) is also a viability domain. Indeed, for any element $x_0 \in \text{Viab}(K)$, there exists a viable solution $x(\cdot)$ to the differential inclusion starting from x_0. For all $t > 0$, the function $y(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is also a viable solution to the differential inclusion, starting at $x(t)$. Hence $x(t) \in \text{Viab}(K)$, so that Viab(K) enjoys the viability property, and thus, is a viability domain thanks to Viability Theorem (see [8], [2, Proposition 4.2.1]).

— Let us assume that $L \subset K$ is a closed viability domain of F. Viability Theorem (see [2, Theorem 4.2.1]) implies that for all $x_0 \in L$, there exists a solution $x(\cdot)$ to the differential inclusion (1) starting from x_0 which is viable in L, and thus, in K. □

In particular, the above proof implies the existence of a viability kernel of the domain of F.

Corollary 3.5 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth. Then the domain of the solution map $\mathcal{S}$ is the viability kernel of the domain of F.*

The viability kernels may inherit properties of both F and K. For instance, if the graph of F and the subset K are convex, so is the viability kernel of K. If F is a closed convex process (i.e., its graph is a closed convex cone) and if K is a closed convex cone, the viability kernel is a closed convex cone.

It may be useful to state the following consequence:

Corollary 3.6 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and linear growth. Then if the initial state x_0 does not belong to the viability kernel of a closed subset K, all solutions $x(\cdot) \in \mathcal{S}(x_0)$ must eventually leave K in the sense that for some $T > 0$, $x(T) \notin K$.*

We prove now the existence of an invariance kernel:

Theorem 3.7 *Let us assume that F is lipschitzean on the interior of its domain and has compact values. For any closed subset $K \subset \mathrm{Dom}(F)$, there exists an invariance kernel of K. It is the subset of initial points such that all solutions starting from them are viable in K.*

Proof — Let us denote by $\mathcal{K} \subset \mathcal{C}(0,T;X)$ the subset of continuous functions $x(\cdot)$ which are viable in K and by Inv(K) the subset of initial state $x \in K$ such that $\mathcal{S}(x) \subset \mathcal{K}$.

Filippov's Theorem (see [2, Corollary 2.4.1, p.121]) states that for all $T > 0$, the solution map $\mathcal{S}$ is lipschitzean from the interior of the domain of F to $\mathcal{C}(0,T;X)$ or even, to $W^{1,1}(0,T;X)$. In particular, it is lower semicontinuous, and thus, lower semicontinuous from the interior of the domain of F to $\mathcal{C}(0,T;X)$ supplied with the topology of pointwise convergence. Since $\mathcal{K}$ is closed, we deduce that Inv(K) is also a closed subset of K, possibly empty.

It contains obviously any closed invariance domain of F contained in K.

It remains to check that it is also invariant by F. For that purpose, let us take $x \in$ Inv(F) and show that any solution $x(\cdot) \in \mathcal{S}(x)$ is viable on Inv(K), by checking that for any $T > 0$, $x(T) \in$ Inv(K). Let $y(\cdot)$ belongs to $\mathcal{S}(x(T))$. Hence the function $z(\cdot)$ defined by

$$z(t) := \begin{cases} x(t) & \text{if } t \in [0,T] \\ y(t-T) & \text{if } t \in [T,\infty[\end{cases}$$

is a solution to the differential inclusion (1) starting at x at time 0, and thus, is viable in K by the very definition of Inv(K). Hence for all $t \geq 0$, $y(t) = z(t+T)$ belongs to K, so that we have proved that $S(x(T)) \subset K$, i.e., $x(T) \in \text{Inv(K)}$. □

Remark — We used only the lower semicontinuity of the solution map S to prove that Inv(K) is closed. Hence, any criterion implying it will implies that the largest invariance set contained in a closed subset K is closed. There is no known such criterion besides the lipschitzianity of S, since, even in the case of ordinary differential equation with continuous right-hand side (and no uniqueness), the solution map may not be lower semicontinuous. □

It is clear that

$$\text{Inv}(K_1 \cap K_2) = \text{Inv}(K_1) \cap \text{Inv}(K_2) \tag{17}$$

and more generally, that the invariance kernel of any intersection of closed subsets K_i ($i \in I$) is the intersection of the invariance kernels of the K_i.

It may be useful to state the following consequence:

Corollary 3.8 *Let us assume that F is lipschitzean on the interior of its domain and has compact values. Then if the initial state x_0 does not belong to the invariance kernel of a closed subset K, there exists a solution $x(\cdot) \in S(x_0)$ such that for some $T > 0$, $x(T) \notin K$.*

Let us consider now any closed subset of a viability domain. The introduction of the Dubovitsky-Miliutin cone defined by

Definition 3.9 *The "Dubovitsky-Miliutin tangent cone" $D_K(x)$ to K is defined by:*

$$\begin{cases} v \in D_K(x) \text{ if and only if} \\ \exists\ \varepsilon > 0,\ \exists\ \alpha > 0 \text{ such that } x +]0,\alpha](v + \varepsilon B) \subset K \end{cases} \tag{18}$$

is justified by the following

Lemma 3.10 *The complement of the contingent cone $T_K(x)$ to K at $x \in \partial K$ is the "Dubovitsky-Miliutin cone" $D_{\widehat{K}}(x)$ to the closure $\widehat{K}$ of the complement of K.*

These definitions and the proof of the Viability Theorem imply the following useful result:

Proposition 3.11 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images. Assume that the interior of K is not empty. Let x_0 belong to the boundary ∂K of the closed subset K. Then each condition implies the next one:*

$$\left\{\begin{array}{ll} i) & F(x_0) \subset D_K(x_0) \\ ii) & \text{for any solution starting from } x_0, \exists\, T > 0 \mid \forall\, t \in]0,T], x(t) \in \text{Int}(K) \\ iii) & \text{for any solution starting from } x_0, \text{ there exists } T > 0 \text{ such that} \\ & x(T) \in \text{Int}(K) \\ iv) & \exists \text{ a sequence } x_n \in \partial K \text{ converging to } x_0 \\ & \text{such that } F(x_n) \subset D_K(x_n) \end{array}\right.$$

All these statements are equivalent if we assume that the set-valued map R defined by

$$x \in \partial K \rightsquigarrow R(x) := F(x) \cap T_{\widehat{K}}(x)$$

is lower semicontinuous on ∂K at x_0 (see [3, Theorem 1.2.3.] for a criterion of lower semicontinuity for such maps).

Proof — The statement of this proposition can be reformulated in this way: each condition implies the next one

$$\left\{\begin{array}{ll} i) & \exists\, r > 0 \text{ such that for all } x \in \partial K \cap (x_0 + rB), \text{ we have} \\ & F(x) \cap T_{\widehat{K}}(x) \neq \emptyset \\ ii) & \exists\, T > 0 \text{ and a viable solution starting at } x_0 \text{ on } [0,T] \\ iii) & \exists \text{ a solution starting at } x_0 \text{ such that } \forall T > 0,\ \exists t \in]0,T] \mid x(t) \in \widehat{K} \\ iv) & F(x_0) \cap T_{\widehat{K}}(x_0) \neq \emptyset \end{array}\right.$$

The first implication follows from the proof of the sufficient condition of the Viability Theorem applied to the closure $\widehat{K}$ of the complement of K, the second implication is obvious and the third one ensues from the proof of the necessary condition of the Viability Theorem still applied to $\widehat{K}$.

Condition (3)i) follows from (3)iv) whenever

$$x \in \partial K \rightsquigarrow R(x) := F(x) \cap T_{\widehat{K}}(x) \text{ is lower semicontinuous at } x_0 \in \partial K$$

Indeed, by the very definition of lower semicontinuity of R, if $v_0 \in R(x_0)$, there exists a neighborhood $\partial K \cap (x_0 + rB)$ such that $(v_0 + B) \cap R(x) \neq \emptyset$ on this neighborhood. Hence the pointwise viability property implies the local one, and thus, the existence of at least a local viable solution starting from 0. □

As a consequence, we obtain the

Theorem 3.12 (Strict Invariance Theorem) *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and assume that the interior of K is not empty. If*

$$\forall x \in \partial K, \;\; F(x) \subset D_K(x) \tag{19}$$

then, for any initial state x_0 in the boundary ∂K of K, any solutions to the differential inclusion (1) starting from x_0 remains in the interior of K on some interval $]0, T]$.

We then can divide the boundary of ∂K into five areas:

$$\begin{cases} \overline{K_e} & := \{ x \in \partial K \mid F(x) \cap T_K(x) \neq \emptyset \} \\ \overline{K_s} & := \{ x \in \partial K \mid F(x) \cap T_{\widehat{K}}(x) \neq \emptyset \} \\ K_e & := K \backslash \overline{K_s} = \{ x \in \partial K \mid F(x) \subset D_K(x) \} \subset \overline{K_e} \\ K_s & := K \backslash \overline{K_e} = \{ x \in \partial K \mid F(x) \subset D_{\widehat{K}}(x) \} \subset \overline{K_s} \\ K_b & := \{ x \in \partial K \mid F(x) \cap T_{\partial K}(x) \neq \emptyset \} \subset \overline{K_e} \cap \overline{K_s} \end{cases}$$

Proposition 3.13 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and a closed subset K of its domain with a nonempty interior.*

— *Whenever $x \in K_e$, all solutions starting at x must enter the interior of K on some open time interval $]0, T[$, and whenever $x \in K_s$, all solutions starting at x must leave the subset K on some $]0, T[$.*

— *If $\partial K \cap (x + rB) \subset \overline{K_e}$ for some $r > 0$, then at least one solution starting at x is viable in K on some $[0, T]$ and the analogous statement holds true for $\overline{K_s}$.*

— *If $\partial K \cap (x + rB) \subset K_b$ for some $r > 0$, then at least one solution starting at x remains in the boundary ∂K on some $[0, T]$.*

In summary, the boundary of K can be partitioned into the four K_e, K_s, $\overline{K_e} \cap \overline{K_s}$ and K_b. From K_e, all solutions must enter K, from K_s, all solutions must leave K, from K_b, a solution can remain in the boundary if $F(\cdot) \cap T_{\partial K}(\cdot)$ is lower semicontinuous, from $(\overline{K_e} \cap \overline{K_s}) \backslash K_b$, a solution can remain in K or in $\widehat{K}$ according that either $F(\cdot) \cap T_K(\cdot)$ or $F(\cdot) \cap T_{\widehat{K}}(\cdot)$ is lower semicontinuous on the boundary of K.

References

[1] AUBIN J.-P. (1981) *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions.* Advances in Mathematics, Supplementary studies, Ed. Nachbin L., 160-232

[2] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIALINCLUSIONS. Springer-Verlag (Grundlehren der Math. Wissenschaften, Vol.264)

[3] AUBIN J.-P. & EKELAND I. (1984) APPLIED NONLINEAR ANALYSIS. Wiley-Interscience

[4] DEIMLING K. (1988) *Multivalued Differential Equations on Closed Sets.* Differential and Integral Equations, 1, 23-30

[5] FRANKOWSKA H. (1987) *L'équation d'Hamilton-Jacobi contingente.* Comptes Rendus de l'Académie des Sciences, PARIS,

[6] FRANKOWSKA H. *Optimal trajectories associated to a solution of contingent Hamilton-Jacobi Equation* Appl. Math. Opt.

[7] FRANKOWSKA H. (to appear) *Hamilton-Jacobi Equations:viscosity solutions and generalized gradients.* J. Math.Anal. Appli.

[8] HADDAD G. (1981) *Monotone trajectories of differential inclusions with memory.* Israel J. Maths, 39, 38-100