WORKING PAPER

VICTORY AND DEFEAT IN DIFFERENTIAL GAMES

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September 1988
WP-88-079

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Victory and Defeat in Differential Games

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FOREWORD

The author constructs the set-valued **feedback** map wich allow players in a differential game the possiblity of winning, separately or colletively, or the certainty of winning or loosing and characterizes the indicator functions of their graphs as solutions to (contingent) partial differential equations. Decisions are defined to be the derivatives of the controls of players, and decision rules for each of these set-valued feedback maps allowing the players to abide by them as time elapses are provided.

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1 Description of the Game

Let our two players Xavier and Yves act on the evolution of the state $z(t) \in \mathbf{R}^n$ of the differential game governed by the differential equation

$$(1) \qquad z'(t) = h(z(t, u(t), v(t)))$$

by choosing Xavier's controls

$$(2) \qquad \forall\, t \geq 0,\ u(t) \in U(z(t))$$

and by choosing Yves's controls

$$(3) \qquad \forall\, t \geq 0,\ v(t) \in V(t)$$

Here, h, describing the dynamics of the game, maps continuously $\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ into $\mathbf{R}^n$, and $U : \mathbf{R}^n \rightsquigarrow \mathbf{R}^p$ and $V : \mathbf{R}^n \rightsquigarrow \mathbf{R}^q$ are closed[1] set-valued maps describing the state-dependent constraints bearing on the players.

We shall assume that the open-loop controls $u(\cdot)$ and $v(\cdot)$ are absolutely continuous and obey a growth condition of the type[2]

$$(4) \qquad \begin{cases} i) & \|u'(t)\| \leq \rho(\|u(t)\| + 1) \\ ii) & \|v'(t)\| \leq \sigma(\|v(t)\| + 1) \end{cases}$$

We shall refer to them as "smooth open-loop controls", the non negative parameters[3] ρ and σ being fixed once and for all. The domain K of the game is the subset of

$$(5) \qquad \begin{cases} (z, u, v) \in \mathbf{R}^n \times \mathbf{R}^p \in \mathbf{R}^q \text{ such that} \\ u \in U(z) \ \&\ v \in V(z) \end{cases}$$

Roughly speaking, Xavier may win as long as its opponent allows him to choose at each instant $t \geq 0$ controls $u(t)$ in the subset $U(z(t))$, and must loose if for any choice of open-loop controls, there exists a time $T > 0$ such that $u(T) \notin U(z(T))$.

[1]This means that the graph of the set-valued map is closed. Upper semicontinuous set-valued maps with compact values are closed, and thus, closedness can be regarded as a weak continuity requirement.

[2]one can replace $\rho(\|u\| + 1)$ by any continuous function $\phi(u)$ with linear growth.

[3]or any other linear growth condition $\phi(\cdot)$ or $\psi(\cdot)$ which makes sense in the framework of a game under investigation.

Definition 1.1 *Let (u_0, v_0, z_0) be an initial situation such that initial controls $u_0 \in U(z_0)$ and $v_0 \in V(z_0)$ of the two players are consistent with the initial state z_0.*

We shall say that

— *Xavier* **must win** *if and only if for all smooth open-loop controls $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0, there exists a solution $z(\cdot)$ to (1) starting at z_0 such that (2) is satisfied.*

— *Xavier* **may win** *if and only if there exist smooth open-loop controls $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (1) starting at z_0 such that (2) is satisfied.*

— *Xavier* **must loose** *if and only if for all smooth open-loop control $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and solution $z(\cdot)$ to (1) starting at z_0, there exists a time $T > 0$ such that*

$$u(T) \notin U(z(T))$$

— *The initial situation is* **playable** *if and only if there exist open-loop controls $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (1) starting at z_0 satisfying* **both** *relations (2)and (3).*

Naturally, if both Xavier and Yves must win, then both relations (2)and (3) are satisfied. This is not necessarily the case when both Xavier and Yves may win, and this is the reason why we are led to introduce the concept of playability.

2 The Main Theorems

Theorem 2.1 *Let us assume that h is continuous with linear growth and that the graphs of U and V are closed. Let the growth rates ρ and σ be fixed.*

There exist five (possibly empty) closed set-valued feedback maps from $\mathbf{R}^n$ to $\mathbf{R}^p \times \mathbf{R}^q$ having the following properties:

— *$R_U \subset U$ is such that whenever $(u_0, v_0) \in R_U(z_0)$, Xavier* **may win** *and that whenever $(u_0, v_0) \notin R_U(z_0)$, Xavier* **must loose**

— *If h is lipschitzean, $S_U \subset R_U$ is the largest closed set-valued map such that whenever $(u_0, v_0) \in S_U(z_0)$, Xavier must win.*

— $S_V \subset R_V \subset V$, *which have analogous properties.*

— $R_{UV} \subset R_U \cap R_V$ *is the largest closed set-valued map such that any initial situation satisfying* $(u_0, v_0) \in R_{UV}(z_0)$ *is playable.*

Knowing these five set-valued feedback maps, we can split the domain K of initial situations into ten areas which describe the behavior of the differential game from the position of the initial situation.

$(z_0, u_0, v_0) \in$	Graph(S_U)	Graph(R_U)	$K\backslash$Graph(R_U)
Graph(S_V)	Xavier must win Yves must win	Xavier may win Yves must win	Xavier must loose Yves must win
Graph(R_V)	Xavier must win Yves may win	? ? ? ? PLAYABILITY ? ? ? ?	Xavier must loose Yves may win
$K\backslash$Graph(R_V)	Xavier must win Yves must loose	Xavier may win Yves must loose	Xavier must loose Yves must loose

The 10 areas of the domain of the differential game

In particular, the complement of the graph of R_{UV} in the intersection of the graphs of R_U and R_V is the instability region, where either Xavier or Yves may win, but not both together.

The problem is to characterize these five set-valued maps, the existence of which is now guaranteed, by solving the "contingent extension" of the partial differential equation[4]

$$(6) \qquad \frac{\partial \Phi}{\partial z} \cdot h(z,u,v) - \rho(\|u\|+1)\left\|\frac{\partial \Phi}{\partial u}\right\| - \sigma(\|v\|+1)\left\|\frac{\partial \Phi}{\partial v}\right\| = 0$$

[4] If Φ is a solution to this partial differential equation, one can check that for any initial situation $(z_0, u_0, v_0) \in \mathrm{Dom}(\Phi)$, there exists a smooth solution $(z(\cdot), u(\cdot), v(\cdot))$ such that

$$t \to \Phi(z(t), u(t), v(t)) \text{ is non increasing}$$

This property remains true for the solutions to the contingent partial differential equation (9).

which can be written in the following way:

$$\frac{\partial \Phi}{\partial z} \cdot h(z,u,v) + \inf_{\|u'\| \le \rho(\|u\|+1)} \frac{\partial \Phi}{\partial u} \cdot u' + \inf_{\|v'\| \le \sigma(\|v\|+1)} \frac{\partial \Phi}{\partial v} \cdot v' = 0$$

We shall also introduce the partial differential equation[5]

$$(7) \qquad \frac{\partial \Phi}{\partial z} \cdot h(z,u,v) + \rho(\|u\|+1) \left\| \frac{\partial \Phi}{\partial u} \right\| + \sigma(\|v\|+1) \left\| \frac{\partial \Phi}{\partial v} \right\| = 0$$

which can be written in the following way:

$$\frac{\partial \Phi}{\partial z} \cdot h(z,u,v) + \sup_{\|u'\| \le \rho(\|u\|+1)} \frac{\partial \Phi}{\partial u} \cdot u' + \sup_{\|v'\| \le \sigma(\|v\|+1)} \frac{\partial \Phi}{\partial v} \cdot v' = 0$$

The link between the feedback maps and the solutions to the solutions to these partial differential equations is provided by the indicators of the graphs: we associate with the set-valued maps S_U, R_U and R_{UV} the functions Φ_U, Ψ_U and Ψ from $\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ to $\mathbf{R}_+ \cup \{+\infty\}$ defined by

$$(8) \qquad \begin{cases} i) & \Phi_U(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in S_U(z) \\ +\infty & \text{if } (u,v) \notin S_U(z) \end{cases} \\ ii) & \Psi_U(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in R_U(z) \\ +\infty & \text{if } (u,v) \notin R_U(z) \end{cases} \\ iii) & \Psi(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in R_{UV}(z) \\ +\infty & \text{if } (u,v) \notin R_{UV}(z) \end{cases} \end{cases}$$

and the functions Ψ_V and Φ_V associated to the set-valued map R_V and S_V in an analogous way.

These functions being only lower semicontinuous, but not differentiable, cannot be solutions to either partial differential equations (6) and (7). But we can define the **contingent epiderivatives** of any function $\Phi : \mathbf{R}^n \times$

[5] One can check that if f is lipschitzean and Φ is a solution to this partial differential equation, for any initial situation $(z_0, u_0, v_0) \in \mathrm{Dom}(\Phi)$, any smooth solution $(z(\cdot), u(\cdot), v(\cdot))$ satisfies

$$t \to \Phi(z(t), u(t), v(t)) \text{ is non increasing}$$

This property remains true for the solutions to the contingent partial differential equation (10).

$\mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R} \cup \{+\infty\}$ and replace the partial differential equations (6)and (7) by the contingent partial differential equations

$$(9) \qquad \inf_{\substack{\|u'\| \leq \rho(\|u\|+1) \\ \|v'\| \leq \sigma(\|v\|+1)}} D_{\uparrow}\Phi(z,u,v)(h(z,u,v),u',v')$$

and

$$(10) \qquad \sup_{\substack{\|u'\| \leq \rho(\|u\|+1) \\ \|v'\| \leq \sigma(\|v\|+1)}} D_{\uparrow}\Phi(z,u,v)(h(z,u,v),u',v')$$

respectively.

Let Ω_U and Ω_V be the indicators of the graphs of the set-valued maps U and V defined by

$$(11) \qquad \begin{cases} i) \quad \Omega_U(z,u,v) := \begin{cases} 0 & \text{if } u \in U(z) \\ +\infty & \text{if } u \notin U(z) \end{cases} \\ ii) \quad \Omega_V(z,u,v) := \begin{cases} 0 & \text{if } v \in V(z) \\ +\infty & \text{if } v \notin V(z) \end{cases} \end{cases}$$

Theorem 2.2 *We posit the assumptions of Theorem 2.1. Then*

— Ψ_U *is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to* Ω_U

— Ψ_V *is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to* Ω_V

— Ψ *is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to* $\max(\Omega_U, \Omega_V)$

— *If h is lipschitzean,* Φ_U *is the smallest lower semicontinuous solution to the contingent partial differential equation (10) larger than or equal to* Ω_U

— *If h is lipschitzean,* Φ_V *is the smallest lower semicontinuous solution to the contingent partial differential equation (10) larger than or equal to* Ω_V

If any of the above solutions is the constant $+\infty$, the corresponding feedback map is empty.

Proof of Theorem 2.1 — Let us denote by B the unit ball and introduce the set-valued map F defined by

$$H(z,u,v) := \{h(z,u,v)\} \times \rho(\|u\|+1)B \times \sigma(\|v\|+1)B$$

The evolution of the differential game described by the equations (1) and (4) is governed by the differential inclusion

$$(z'(t), u'(t), v'(t)) \in H(z(t), u(t), v(t))$$

— Since the graph of U is closed, we know that there exists a largest closed viability domain contained in $\text{Graph}(U) \times \mathbf{R}^q$, which is the set of initial situations (z_0, u_0, v_0) such that there exists a solution $(z(\cdot), u(\cdot), v(\cdot))$ to this differential inclusion remaining in this closed set. This is the graph of R_U. Indeed, if $(u_0, v_0) \in R_U(z_0)$, there exists a solution to the differential inclusion remaining in the graph of U, i.e., Xavier may win. If not, all solutions starting at (z_0, u_0, v_0) must leave this domain in finite time.

The set-valued feedback map is defined in an analogous way.

— For the same reasons, the graph of the set-valued feedback map R_{UV} is the largest closed viability domain of the set K of initial situations.

— When h is lipschitzean, so is F. Then the solution-map $S(z_0, u_0, v_0)$ is also lipschitzean thanks to Filippov's Theorem[6], so that the subset of initial situations such that all the functions of $S(z_0, u_0, v_0)$ remain in a closed subset is also closed. This is the largest closed invariant domain by F of this closed subset. Then the largest closed invariant domain contained in $\text{Graph}(U) \times \mathbf{R}^q$ is the graph of the set-valued feedback map S_U. □

Proof of Theorem 2.2 — We recall that thanks to Haddad's viability Theorem, a subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is a viability domain of F if and only if

$$\forall\, (z,u,v) \in L,\ T_L(z,u,v) \cap H(z,u,v) \neq \emptyset$$

Let Ψ_L denote the indicator of L. We know that the epigraph of the contingent epiderivative $D_\uparrow \Psi_L(z,u,v)$ of Ψ_L is the contingent cone to the epigraph of Ψ_L at $((z,u,v),0)$. Since the latter subset is equal to $L \times \mathbf{R}_+$, its contingent cone is equal to $T_L(z,u,v) \times \mathbf{R}_+$, and coincides with the epigraph of the indicator of $T_L(z,u,v)$. Hence *the indicator of the contingent cone*

[6] See [3, p.120]

$T_L(z,u,v)$ *is the contingent epiderivative* $D_\uparrow\Psi_L(z,u,v)$ *of the indicator* Ψ_L *of* L *at* (z,u,v).

Therefore, the above tangential condition can be reformulated in the following way:

$$\forall\ (z,u,v) \in L,\ \exists\ w \in H(z,u,v) \text{ such that } D_\uparrow\Psi_L(z,u,v)(w) = \Psi_{T_L(z,u,v)}(w) = 0$$

Since the epiderivative is lower semicontinuous and the images of F are compact, this is equivalent to say that

$$\forall\ (z,u,v) \in L,\ \inf_{w \in H(z,u,v)} D_\uparrow\Psi_L(z,u,v)(w) = 0$$

By the very definition of the set-valued map F, we have proved that L is a closed viability domain if and only if its indicator function Ψ_L is a solution to the contingent partial differential equation (9).

— Hence to say that the graph of R_U is the largest closed viability domain contained in the graph of U amounts to saying that its indicator Ψ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to the indicator Ω_U of $\text{Graph}(U) \times \mathbf{R}^q$. The same reasoning shows that indicator Ψ_V of R_V is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to Ω_V and that the indicator Ψ of the graph of R_{UV} is the smallest lower semicontinuous solution to the contingent partial differential equation (9) larger than or equal to the indicator of K, which is equal to $\max(\Omega_U, \Omega_V)$.

— We know that the a closed subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is "invariant" by a lipschitzean set-valued map F if and only if

$$\forall\ (z,u,v) \in L,\ T_L(z,u,v)\ \subset\ H(z,u,v)$$

This condition can be reformulated in terms of contingent epiderivative of the indicator function Ψ_L of L by saying that

$$\forall\ (z,u,v) \in L,\ \sup_{w \in H(z,u,v)} D_\uparrow\Psi_L(z,u,v)(w) = 0$$

Hence to say that the graph of S_U is the largest closed invariance domain contained in the graph of U amounts to saying that its indicator Φ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (10) larger than or equal to the indicator Ω_U of $\text{Graph}(U) \times \mathbf{R}^q$.
□

3 Closed-Loop Decision Rules

When the initial situation (z_0, u_0, v_0) belongs to one of the following subsets:

(12) $\quad \mathrm{Graph}(S_U) \cap \mathrm{Graph}(S_V) \;\text{ or }\; K \backslash (\mathrm{Graph}(R_U) \cup \mathrm{Graph}(R_V))$

then the players has nothing to worry about because both of them must either win or loose whatever the choice of their control.

In the other areas, at least one of the players may win, but for achieving victory, he has to find open-loop or closed-loop controls which remain in the appropriate set-valued feedback map.

Let us denote by R one of the feedback maps R_U, R_V, R_{UV} and assume that the initial situation belongs to the graph of the set-valued feedback map R (when it is not empty). The theorem states only that there exists at least a solution $(z(\cdot), u(\cdot), v(\cdot))$ to the differential game such that

$$\forall\, t \geq 0,\; (u(t), v(t)) \;\in\; R(z(t))$$

To implement these strategy, players have **to make decisions, i.e., to choose velocities of controls** in an adequate way:

We observe that playable solutions

Proposition 3.1 *The solutions to the game satisfying*

$$\forall\, t \geq 0,\; (u(t), v(t)) \;\in\; R(z(t))$$

are the solutions to the system of differential inclusions

(13) $$\begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & (u'(t), v'(t)) \in G_R(z(t), u(t), v(t)) \end{cases}$$

*where we have denoted by G_R the R-***decision map** *defined by*

(14) $$G_R(z, u, v) \;:=\; DR_R(z, u, v)(h(z, u, v))$$

For simplicity, we shall set $G := G_R$ whenever there is no ambiguity.

Proof— Indeed, since the absolutely continuous function $(z(\cdot), u(\cdot), v(\cdot))$ takes its values into Graph(R), then its derivative $(z'(\cdot), u'(\cdot), v'(\cdot))$ belongs almost everywhere to the contingent cone

$$T_{\mathrm{Graph}(R)}(z(t), u(t), v(t)) \;:=\; \mathrm{Graph}(DR(z(t), u(t), v(t)))$$

We then replace $z'(t)$ by $h(z(t), u(t), v(t))$.

The converse holds true because equation (13) makes sense only if $(z(t), u(t), v(t))$ belongs to the graph of R. □

The question arises whether we can construct selection procedures of the decision components of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 3.2 () *We shall say that a selection $(\tilde{c}, \tilde{d})$ of the contingent derivative of the smooth regulation map R in the direction h defined by*

$$(15)\quad \forall\ (z, u, v) \ \in\ \mathrm{Graph}(R),\ \ \tilde{c}(z, u, v) \ \in\ DR(z, u, v)(h(z, u, v))$$

is a **closed-loop decision rule.**

The system of differential equations

$$(16)\qquad \begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & u'(t) = c(z(t), u(t), v(t)) \\ iii) & v'(t) = d(z(t), u(t), v(t)) \end{cases}$$

is called the associated **closed-loop decision game.**

Therefore, closed-loop decision rules being given for each player, the closed-loop decision system is just a system of ordinary differential equations.

It has solutions whenever the maps c and d are continuous (and if such is the case, they will be continuously differentiable).

But they also may exist when c or d or both are no longer continuous. This is the case when the decision map is lower semicontinuous thanks to Michael's Theorem:

Theorem 3.3 *Let us assume that the decision map $G := G_R$ is lower semicontinuous with non empty closed convex values on the graph of R. Then there exist continuous decision rules c and d, so that the decision system 16 has a solution whenever the initial situation $(u_0, v_0) \in R(z_0)$*

But we can obtain explicit decision rules which are not necessarily continuous, but for which the decision system 16 has a still solution.

It is useful for that purpose to introduce the following definition:

Definition 3.4 (Selection Procedure) *A* selection procedure of the regulation map $G : \mathbf{R}^n \rightsquigarrow \mathbf{R}^p \times \mathbf{R}^q$ *is a set-valued map* $S_G : \mathbf{R}^n \rightsquigarrow \mathbf{R}^p \times \mathbf{R}^q$

$$(17) \quad \begin{cases} i) & \forall z \in K, \;\; S(G(z)) := S_G(z) \cap G(z) \neq \emptyset \\ ii) & \text{the graph of } S_G \text{ is closed} \end{cases}$$

and the set-valued map $S(G) : z \rightsquigarrow S(G(z))$ *is called the* selection *of* G.

It is said convex-valued *or simply,* convex *if its values are convex and* strict *if moreover*

$$(18) \quad \forall z \in \mathrm{Dom}(G), \;\; S_G(z) \cap G(z) = \; \{\tilde{d}(z)), \tilde{c}(z)\}$$

is a singleton.

Hence, we obtain also the following existence theorem for closed-loop decision rules obtained through sharp convex selection procedures.

Theorem 3.5 *Let* S_G *be a convex selection of the set-valued map* G. *Then, for any initial state* $(z_0, u_0, v_0) \in \mathrm{graph}(R)$, *there exists a starting at* (z_0, u_0, v_0) *to the associated system of differential inclusions*

$$(19) \quad \begin{cases} i) & z'(t) & = & h(z(t), u(t), v(t)) \\ ii) & (u'(t), v'(t)) & \in & S(DR(z(t), u(t), v(t))h(z(t), u(t), v(t))) \\ & & := & G(z(t), u(t), v(t)) \cap S_G(z(t), u(t), v(t)) \end{cases}$$

In particular, if we assume further that the selection procedure S_G *is sharp, then the single-valued map*

$$(\tilde{c}(z,u,v), \tilde{d}(z,u,v)) := S(G)(z,u,v)$$

is closed-loop decision rule, for which decision system 16 has a solution for any initial state $(z_0, u_0, v_0) \in \mathrm{graph}(R)$.

Proof — We shall replace the system of differential inclusions (13) by the system of differential inclusions

$$(20) \quad \begin{cases} i) & z'(t) & = & h(z(t), u(t), v(t)) \\ ii) & (u'(t), v'(t)) & \in & S_G(z(t), u(t), v(t)) \end{cases}$$

Since the convex selection procedure S_G has a closed graph and convex values, the right-hand side is upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. It remains to check that Graph$R$ is still a viability domain for this new system of differential inclusions. Indeed, by construction, we know that there exists an element w in the intersection of $G(z,u,v)$ and $S_G(z,u,v)$. This means that the pair $(h(z,u,v),w)$ belongs to $h(z,u,v) \times S_G(z,u,v)$ and that it also belongs to

$$\text{Graph}(G) \; := \; T_{\text{Graph}R}(z,u)$$

Therefore, we can apply Haddad's Viability Theorem. For any initial situation (z_0,u_0,v_0), there exists a solution $(z(\cdot),u(\cdot),v(\cdot))$ to the new system of differential inclusions (20) which is viable in Graph(R). Consequently, for almost all $t > 0$, the pair $(z'(t),u'(t),v'(t))$ belongs to the contingent cone to the graph of R at $(z(t),u(t),v(t))$, which is the graph of the contingent derivative $DR(z(t),u(t),v(t))$. In other words,

$$\text{for almost all } \; t > 0, \;\; (u'(t),v'(t)) \; \in \; G(z(t),u(t),v(t))$$

We thus deduce that for almost all $t > 0$, $(u'(t),v'(t))$ belongs to the selection $S(G)(z(t),u(t),v(t))$ of the set-valued map $G(z(t),u(t),v(t))$. Hence, we have found a solution to the system of differential inclusions (19). □

We can now multiply the possible corollaries, since we have given several instances of selection procedures of set-valued maps.

Example— COOPERATIVE BEHAVIOR

Let $\sigma : \text{Graph}(G) \mapsto \mathbf{G}$ be continuous.

Corollary 3.6 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R). *Let σ be continuous on* Graph(G) *and convex with respect to the pair (u,v). Then, for all initial situation $(u_0,v_0) \in R(z_0)$, there exist a solution starting at (z_0,u_0,v_0) and to the differential game (1)-(4) which are regulated by:*

$$(21) \qquad \begin{cases} \text{for almost all } \geq 0, \;\; (u'(t),v'(t)) \in G(z(t),u(t),v(t)) \;\; \text{and} \\ \sigma(z(t),u(t),v(t),u'(t),v'(t)) \\ = \; \inf_{u',v' \in G(z(t),u(t),v(t))} \sigma(z(t),u(t),v(t),u',v') \end{cases}$$

In particular, the game can be played by the **heavy decision** *of minimal norm:*

$$\begin{cases} (c^\circ(z,u,v), d^\circ(z,u,v)) \in G(z,u,v) \\ \|c^\circ(z,u,v)\|^2 + \|d^\circ(z,u,v)\|^2) = \min_{(u',v')\in G(z,u,v)}(\|u'\|^2 + \|v'\|^2) \end{cases}$$

Proof — We introduce the set-valued map S_G defined by:

$$S_G(z) := \{(c,d) \in Y \mid \sigma(z,u,v,c,d) \le \inf_{(u',v')\in G(z,u,v)} \sigma(z,u,v,u',v')\}$$

It is a convex *selection procedure* of G. Indeed, since G is lower semicontinuous, the function

$$(z,u,v,c,d) \mapsto \sigma(z,u,v,c,d) + \sup_{(u',v')\in G(z,u,v)} (-\sigma(z,u,v,u',v'))$$

is lower semicontinuous thanks to the Maximum Theorem. Then the graph of S_G is closed because

$$\begin{array}{l} \mathrm{Graph}(S_G) = \\ \{(z,u,v) \mid \sigma(z,u,v,c,d) + \sup_{(u',v')\in G(z,u,v)}(-\sigma(z,u,v,u',v')) \le 0\} \end{array}$$

The images are obviously convex. Consequently, the graph of G being also closed, so is the selection $S(G)$ equal to:

$$S(G)(z,u,v) = \{(c,d) \in G(z,u,v) \mid \sigma(z,u,v,c,d) \le \inf_{(u',v')\in G(z,u,v)} \sigma(z,u,v,u',v'))\}$$

We then apply Theorem 3.5. We observe that when we take

$$\sigma(z,u,v,c,d) := \|c\|^2 + \|d\|^2$$

the selection procedure is strict and yields the decisions of minimal norm. □

Example— NONCOOPERATIVE BEHAVIOR

We can also choose controls in the regulation sets $G(z,u,v)$ in a non cooperative way, as saddle points of a function $a(z,u,v,\cdot,\cdot)$.

Corollary 3.7 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R) *and that* $a : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}$ *satisfies*

$$(22) \qquad \begin{cases} i) & a \text{ is continuous} \\ ii) & \forall(z,u,v,d),\ c \mapsto a(z,u,v,c,d) \text{ is convex} \\ iii) & \forall(z,u,v,c),\ d \mapsto a(z,u,v,c,d) \text{ is concave} \end{cases}$$

Then, for all initial situation $(u_0, v_0) \in R(z_0)$, there exist a solution starting at (z_0, u_0, v_0) and to the differential game (1)-(4) which are regulated by:

$$\text{for almost all } t \geq 0, \quad \begin{cases} i) & (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \\ ii) & \forall\, (u', v') \in G(z(t), u(t), v(t)), \\ & a(z(t), u(t), v(t), u'(t), v') \\ & \leq a(z(t), u(t), v(t), u'(t), v'(t)) \\ & \leq a(z(t), u(t), v(t), u', v'(t)) \end{cases}$$

Proof — We prove that the set-valued map S_G associating to any $(z, u, v) \in \text{Graph}(R)$ the subset

$$\begin{array}{l} S_G(z, u, v) \ := \ \{ \ (c, d) \ \text{such that} \\ \forall (u', v') \in G(z, u, v), \ \ a(z, u, v, c, v') \leq a(z, u, v, u', d) \ \} \end{array}$$

is a convex selection procedure of G. The associated selection map $S(G(\cdot))$ associates with any (z, u, v) the subset

$$\begin{array}{l} S(G(z, u, v)) \ := \ \{ \ (c, d) \in G(z, u, v) \ \text{such that} \\ \forall (u', v') \in G(z, u, v), \ \ a(z, u, v, c, v') \leq a(z, u, v, u', d) \ \} \end{array}$$

of saddle-points of $a(z, u, v, \cdot, \cdot)$ in $G(z, u, v)$. Von Neumann' Minimax Theorem states that the subsets $S(G(z, u, v))$ of saddle-points are not empty since $G(z, u, v)$ are convex and compact. The graph of S_G is closed thanks to the assumptions and the Maximum Theorem because it is equal to the lower section of a lower semicontinuous function:

$$\text{Graph}(S_G) = \{(z, u, v, c, d) \mid \sup_{(u', v') \in G(z, u, v)} (a(z, u, v, c, v') - a(z, u, v, u', d)) \leq 0\}$$

We then apply Theorem 3.5. □

Remark — Whenever the subset $R_{UV}(z(t)) \backslash R_V(z(t))$ is not empty, Xavier may be tempted to choose a control $u(t)$ such that

$$(u(t), v(t)) \ \in \ R_{UV}(z(t)) \backslash R_V(z(t))$$

because in this case, Xavier may win and Yves is sure to loose eventually. Naturally, Yves will use the opposite behavior.

Hence we can attach to the game two functions

$$(23) \quad \begin{cases} i) & a_U(z,u,v) := d((u,v), R_{UV}(z) \backslash R_V(z)) \\ ii) & b_V(z,u,v) := d((u,v), R_{UV}(z) \backslash R_U(z)) \end{cases}$$

and look for closed-loop controls $(\hat{u}(z), \hat{v}(z))$ which are Nash equilibria of this game:

$$(24) \quad \begin{cases} i) & (\hat{u}(z, \hat{v}(z))) \in R_{UV}(z) \\ ii) & \forall (u,v) \in R_{UV}(z), \ a_U(z, \hat{u}(z), \hat{v}(z)) \leq a_U(z, u, \hat{v}(z)) \\ & \& b_V(z, \hat{u}(z), \hat{v}(z)) \leq b_V(z, \hat{u}(z), v) \ \square \end{cases}$$

Unfortunately, the selection procedure which could yield such behavior are not convex. The answer to this question remains unknown for the time.

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