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SOME FURTHER OPTIMALITY CONDITIONS FOR THE C_M-EMBEDDED PROBLEM

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FOREWORD

Normally, executing an algorithm for solving a C_M-embedded problem is stopped at a point where some necessary conditions are satisfied. However, for the C_M-embedded problem both necessary and sufficient conditions may be found. This paper is contributed to explore new optimality conditions some of which are both necessary and suffient conditions. They could be used to verify if a solution generated by an algorithm is at least a locally optimal solution to the C_M-embedded problem, and used to construct ascent algorithms for this problem with non-convex regions in practical calculations.

Alexander Kurzhanski
Chairman
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ABSTRACT

In this paper some new optimality conditions for the C_M-embedded problem with the Euclidean norm are presented. Some of them are both necessary and sufficient for certain non-convex regions. The results associated with optimality conditions given here could be used to construct ascent algorithms and in practical calculations.

Keywords. Design centering, lineality cone, quasi-differentiable functions, C_M-embedded problem.

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INTRODUCTION

We consider the C_M-embedded problem as follows:

$$\max_{x \in S} \min_{\xi \in \Omega} \min_{y \in D_\xi} \|x-y\|^2 \ , \tag{1.1}$$

where $intS \neq \varphi$ and

$$S = \{x \in \mathbf{R}^n | f_\xi(x) \leq f_\xi^o, \xi \in \Omega\}$$

being bounded and simply connected and satisfying $\nabla f_i(y) \neq 0$, $y \in bdS$, $i \in \Omega$, Ω is a finite index set, and

$$D_\xi = \{y \in \mathbf{R}^n \mid f_\xi(y) \geq f_\xi^o \ , f_\eta(y) \leq f_\eta^o \ , \eta \in \Omega \backslash \{\xi\}\} \ , \xi \in \Omega \ .$$

There are different ways to describe this problem, for instance,

$$\max_{r,x} \ r$$

$$s.t. \quad f_i(x+wr) \leq f_i^o \ , \ i \in \Omega \ , \ w \in B(0,1) \ ,$$

[1]-[5], [10]. The function

$$\varphi(x) = \min_{\xi \in \Omega} \min_{y \in D_\xi} \|x-y\|^2$$

is Lipschitzian and quasidifferentiable in the sense of [7]. Some optimality conditions concerned have been proposed, [1], [2], [5] - [10]. In the case where nonconvex regions are determined by convex and complementry convex constraints, say, the generalized Fritz-John necessary conditions,

$$0 \in \lambda_o e_{n+1} + \sum_{i \in J_g} \lambda_i \partial \gamma_i(c^*, r^*) + \sum_{j \in J_h} \lambda_j \ \partial \eta_j(c^*, r^*) \tag{1.2}$$

$$\lambda_o,\ \lambda_i \geq 0,\ \sum_o^m \lambda_i \neq 0,\ m = J_g + J_h,$$

$$\lambda_i(\gamma_i - g_i^o) = 0,\ \ i \in J_g,$$

$$\lambda_j(\eta_j - h_j^o) = 0,\ \ j \in J_h,$$

where (c^*, r^*) is an optimal solution to the problem

$$\max_{c,r}\ r$$

$$s.t.\ \gamma_i(c,r) \leq g_i^o,\ i \in J_g\ ,$$

$$\eta_j(c,r) \leq h_j^o,\ j \in J_h$$

where

$$\gamma(c,r) = \max_{w \in B(0,1)} g(c+rw)\ ,\ \ \eta(c,r) = \max_{w \in B(0,1)} h(c+rw)\ ,$$

and a sufficient condition have been proposed, due to [1]. Under the same constraint condition a necessary condition different from (1.2) could be deducted from [13], [14] and $\varphi(\cdot)^{1/2}$ is a d.c. function, [14]. In [9] some concrete necessary conditions and expansible directions were also described. In [10] a sufficient condition for general nonconvex regions was given. As for the case where all of the constraints which determine a feasible region are quasiconvex, or quasiconcave (complementary convex), the necessary and sufficient conditions could be found in some literatures, e.g., [6], [7]. The necessary conditions for this problem, e.g., (1.2), that have been proposed are not very efficient to be used in the practical computation. The sufficient conditions that were presented and proved in [1], [8] and [10], are more efficient to be used in the practical computation and in constructing numerical methods. But it seems that for the concrete problem, the C_M-embedded problem, it is possible to find both necessary and sufficient conditions in order to expect to provide new bases on which more efficient search techniques could be constructed.

This paper is contributed to explore new optimality conditions which are both necessary and sufficient conditions. This is the basic purpose of this paper.

It should be mentioned that some notations given in [10], and also used here, are listed below:

- $\psi(x,\xi,y) = \|x-y\|^2\ ,\ \ \forall\ y \in D_\xi\ ,\ \xi \in \Omega,\ x \in S\ .$

- $Y = \times_{\Omega} D_\xi \subset \mathbf{R}^{n\times m}$.
- $Y(x,\xi) = \{y\in\mathbf{R}^n \mid \psi(x,\xi,y) = \min_{y\in D_\xi} \psi(x,\xi,y)\}$, $\xi\in\Omega$.
- $\Omega(x) = \{\xi\in\Omega \mid \min_{y\in D_\xi} \psi(x,\xi,y) = \min_{\xi\in\Omega} \min_{y\in D_\xi} \psi(x,\xi,y)\}$, $x\in S$.
- $Y(x) = \bigcup_{\xi\in\Omega(x)} Y(x,\xi)$.
- $\partial\varphi(x)$ is the quasidifferential of φ at x in the sense of [7].
- $\mathbf{C}A$ is the conical hull of the set A.
- $L_{\mathbf{C}A}$ is the lineality space in the cone $\mathbf{C}A$.
- $G(x;Q) = \{x-y \mid y\in Q\subset \mathbf{R}^n\} = x-Q$. Normally, we suppose $Q\subset Y(x)$.
- $N^+(x;Q) = \{h\in\mathbf{R}^n \mid <u,h> \geq 0,\ u\in G(x;Q)\}$, or if $Q = Y(x)$, then

  $N^+(x;Y(x)) = \{h\in\mathbf{R}^n \mid <u,h> \geq 0,\ u\in\partial\varphi(x)\}$.

1. SOME FURTHER OPTIMALITY CONDITIONS

Some notations will be introduced below in order to further develop optimality conditions to the problem (1.1). The notation

$$A\ \underline{SP(y)}\ B$$

means that sets A and B can be separated by some hyperplane $H(y)$ at the point y. In our present discussion, the hyperplane $H(y)$ is just the tangent hyperplane $Tf_\xi(y)$ at the point $y\in bd\ S$, i.e., satisfying

$$\| y-x\|^2 = \varphi(x)\ , \tag{2.1}$$

$$f_\xi(y) = f_\xi^o\ ,$$

$$f_\eta(y) \leq f_\eta^o\ ,\ \eta\in\Omega\setminus\{\xi\}$$

for some $\xi\in\Omega(x)$. $U_{\epsilon(y)}(y)$ denotes a neighborhood at y, mostly an Euclidean ball with radium $\epsilon(y)$. Define

$$\check{P}(x) := \{y\in Y(x) \mid \exists\epsilon(y)>0 : B(x\ \varphi(x)^{1/2})\ \underline{SP(y)}\ U_{\epsilon(y)}(y)\cap bd\ S\}\ .$$

A point x is said to be a $\vee$-point if $x\in int\ S$ and $\check{P}(x) = Y(x)$. A point $y\in\check{P}(x)$ is said to be an open-type point associated with x. An open-type point is said to be strict if there exists $\epsilon(y)>0$ such that

$$U_{\epsilon(y)}(y) \bigcap bd\ S \bigcap H(y) = \{y\} . \tag{2.2}$$

A point x is called a strict $\vee$-pointif every point $y \in \overset{\vee}{P}(x)$ is strict.

Similarly, we can define the opposite type point. The symbol

$$A\ \underline{SS(y)}\ B$$

means that sets A and B are in the same closed half-space determined by a supporting hyperplane $H(y)$ of A or B at $y \in A \bigcap B$. Define

$$\hat{P}(x) := \{y \in Y(x) \mid \exists \epsilon(y) > 0 : B(x\ \varphi(x)^{1/2})\ \underline{SS(y)}\ U_{\epsilon(y)}\ (y) \bigcap bd\ S \text{ and}$$

$$B(x\ \varphi(x)^{1/2}) \bigcap bd\ S \bigcap U_{\epsilon(y)}\ (y) = \{y\}\} .$$

As mentioned above, the supporting hyperplane $H(y)$ is just the tangent hyperplane $Tf_\xi(y)$ at $y \in bd\ S$ satisfying the condition (2.1). A point x is said to be a $\wedge$-point if $x \in int\ S$ and $\hat{P}(x) = Y(x)$. Each point $y \in \hat{P}\ (x)$ is called a closed-type point associated with x. A point $y \in \hat{P}(x)$ is called strict if it satisfies the condition (2.2). A point x is called a strict $\wedge$-point if every point $y \in \hat{P}(y)$ is strict. A point $y \in \hat{P}(x)$ is called trivial if there exists $\epsilon(y) > 0$ such that

$$U_{\epsilon(y)}\ (y) \bigcap bd\ S \subset Tf_\xi(y)$$

for some $\xi \in \Omega(x)$. In the definition of $\hat{P}(x)$, we excluded the case in which for some point $y \in Y(x)$ and, for all $\epsilon > 0$, one has

$$U_\epsilon(y) \bigcap bd\ S \bigcap B(x\ \varphi(x)^{1/2}) \backslash \{y\} \neq \emptyset \text{ and} \tag{2.3}$$

$$B(x\ \varphi(x)^{1/2})\ \underline{SS(y)}\ U_\epsilon(y) \bigcap bd\ S .$$

The set consisting of the points, in $Y(x)$, satisfying (2.3) is denoted by $\bar{P}(x)$. Similarly, we can define strictness for the points in $\bar{P}(x)$ and $x \in int\ S$.

The other situation is defined as follows. Define

$$\tilde{P}(x) := \{y \in Y(x) \mid \forall\ \epsilon > 0 : U_\epsilon(y) \bigcap bd\ S \bigcap H(y; \nabla f_\xi(y), >) \neq \emptyset \text{ and}$$

$$U_\epsilon(y) \bigcap bd\ S \bigcap H(y; \nabla f_\xi(y), <) \neq \emptyset \text{ for some } \xi \in \Omega(x) \text{ satisfying (2.1)}\}$$

where $H(y; \nabla f_\xi(y), >)$ denotes an open half-space, i.e.,

$$H(y; \nabla f_\xi(y), >) = \{y\prime \in \mathbf{R}^n \mid < y\prime - y, \nabla f_\xi(y) > \ > 0\}$$

and $H(y; \nabla f_\xi(y), <)$ denotes the other open half-space , i.e.,

$$H(y;\nabla f_\xi(y),<) = \{y\prime\in\mathrm{R}^n \mid <y\prime-y,\nabla f_\xi(y)> \; < 0\} \; .$$

A point $y\in\tilde{P}(x)$ is said to be a general type point associated with x.

Theorem 2.1. Suppose that x is a strict $\vee$-point. If the point $x\in$ *int* S is a locally optimal solution to the problem (1.1), then $N^+(x;Y(x))$ is only the singleton containing the zero-vector, i.e.,

$$dim\ N^+(x;Y(x)) = 0 \; . \tag{2.4}$$

Proof. For the sake of contradiction, suppose that $dim\ N^+(x;Y(x))\neq 0$. From the opposite assumption there exists such an $h\in\mathrm{R}^n$ that $h\in N^+(x;Y(x))$ and $h\neq 0$. Since x is a locally optimal solution, there exists a $\delta_0>0$ such that

$$\varphi(x+\lambda h)-\varphi(x)\leq 0 \tag{2.5}$$

for all $\lambda\in(0,\delta_o)$. Now we prove that there exists a $\delta>0$ such that $\varphi(x+\lambda h)-\varphi(x)>0$ for all $\lambda\in(0,\delta)$, that is, for all $\lambda\in(0,\delta)$, one has

$$d^2(x+\lambda h, bd\,S)>\varphi(x) \; . \tag{2.6}$$

Suppose that (2.6) is not true. Then there exist sequences $\{\delta_i\}_1^\infty \downarrow 0$ and $\{\lambda_i\}_1^\infty$ such that $0<\lambda_i\leq\delta_i$ and

$$d^2(x+\lambda_i h, bd\ S)\leq\varphi(x), \forall\ i \; . \tag{2.7}$$

A sequence of points $\{z_i\}\subset$ $bd\ S$ can be found such that, for any i,

$$d^2(x+\lambda_i h, z_i) = d^2(x+\lambda_i h,\ bd\ S)\leq\varphi(x) \; . \tag{2.8}$$

Since Ω is finite and $Y(x)$ is compact, there exist $\bar{\xi}\in\Omega$ and a subsequence $\{i_k\}_1^\infty\subset\{i\}_1^\infty$ such that the corresponding subsequence $\{z_{i_k}\}$ converges to a point $\bar{y}$ of the boundary of S and the whole subsequence $\{z_{i_k}\}$ is on the $(bd\ S)_{\bar{\xi}}$ where $(bd\ S)_{\bar{\xi}}$ denotes $S\bigcap D_{\bar{\xi}}$. Since $(bd\ S)_{\bar{\xi}}$ is compact, $\bar{y}\in(bd\ S)_{\bar{\xi}}$. In consequence of the continuity of the function $d(.,.)$ and (2.8), one has

$$d^2(x+\lambda_{i_k}\, h, z_{i_k}) = d^2(x+\lambda_{i_k} h,(bd\ S)_{\bar{\xi}}) \underset{as\ k\to\infty}{\longrightarrow} d^2(x,\bar{y}) \tag{2.9}$$

$$= d^2(x\ ,\ (bd\ S)_{\bar{\xi}}) = d^2(x\ ,\ bd\ S) = \varphi(x)$$

that is, $\bar{y}\in Y(x)$. In view of $h\in N^+(x;Y(x))$, we have

$$\lambda_{i_k}\, h\in N^+(x;Y(x)) \text{ or } <x-\bar{y},\lambda_{i_k}\, h> \;\geq 0, \ \forall\ i_k \; .$$

We showed that $\{x+\lambda_{i_k} h\}_1^\infty$ is included in the closed half-space $H^+(x;x-\bar{y})$, i.e.,

$$\{x+\lambda_{i_k} h\}_1^\infty \subset \{w \mid <x-\bar{y},w-x> \geq 0\} .$$

However the hyperplane $\{w \mid <x-\bar{y},w-x> = 0\}$ is parallel to the tangent hyperplane $Tf_\xi(\bar{y})$. Since x is strict, it is impossible for (2.8) to hold. This contradiction shows that (2.6) is true. But (2.6) contradicts (2.5). This contradiction shows that $N^+(x;Y(x)) = \{0\}$, i.e., *dim* $N^+(x;Y(x)) = 0$. The proof of this theorem is completed. □

Remark 2.2. If we only assume that x is a v-point, then (2.4) is still a necessary condition for a strictly locally optimal solution to the problem (1.1).

From [10, Theorem 2.4] and Theorem 2.9 we are able to get a necessary and sufficient condition to a locally optimal solution of the problem (1.1) in the case where the center is a v-point.

Optimality Condition 2.3

Suppose that x is a v-point and strict. x is a locally optimal solution to the problem (1.1) if and only if

$$dim\ N^+(x;Y(x)) = 0 \tag{2.10}$$

or

$$dim\ L_{C\partial\varphi(x)} = n . \tag{2.11}$$

Remark 2.4. Suppose that $Y(x) = \hat{P}(x) \bigcup \check{P}(x)$ and furthermore that

$$L_{CG(x;Y(x))} \bigcap G(x;\hat{P}(x)) = \{\emptyset\} \tag{2.12}$$

Then x is not a locally optimal solution, and (2.12) holds if and only if

$$N^+(x;\check{P}(x)) \bigcap int\ N^+(x;\hat{P}(x) \neq \emptyset . \tag{2.13}$$

Lemma 2.5. If x is a locally optimal solution to the problem (1.1), then *dim* $L_{C\partial\varphi(x)} \neq 0$, i.e., *dim* $N^+(x;Y(x)) \neq n$.

Proof. Suppose that *dim* $L_{CG(x;Y(x))} = 0$. From Theorem 2.1, there exists a non-zero direction $h \in int\ N^+(x;Y(x))$. Here, for all $u \in G(x;Y(x))$, $<u,h> > 0$. It follows immediately that

$$<u,h> > 0 \quad \forall\ u \in \partial\ \varphi(x) .$$

Because $\partial\ \varphi(x)$ is compact and $\partial\varphi(x) = co\ \{2\ u \mid u \in G(x;Y(x))\}$, one has

$$\varphi\prime(x;h) = \min_{u \in \partial\ \varphi(x)} <u,h> > 0\ .$$

Thus $B(x,\varphi(x)^{1/2})$ is able to expand continuously along the direction h, and for all $\epsilon > 0$, there exists $z \in U_\epsilon(x)$ such that $\varphi(z) > \varphi(x)$. This contradicts the fact that x is a locally optimal solution. □

Theorem 2.6. If the condition

$$dim\ L_{\mathbf{C}G(x,\hat{P}(x))} \neq 0 \tag{2.14}$$

holds, then x is a locally optimal solution to the problem (1.1).

Proof. Let

$$dim\ L_{\mathbf{C}G(x;\hat{P}(x))} = k > 0\ .$$

For all $y \in \hat{P}(x)$, there exists $\delta(y) > 0$ and $\epsilon(y)$ satisfying $0 < \epsilon(y) < \delta(y)$ such that

$$B(x,\varphi(x)^{1/2})\ \underline{SS(y)}\ U_{\delta(y)}\ (y) \cap\ bd\ S$$

and, for any $z \in [U_{\epsilon(y)}(x) \cap \{z \mid <x-y,z-x> = 0\}]$, one has

$$d(z,U_{\delta(y)}(y) \cap bd\ S) \leq \varphi(x)^{1/2}\ . \tag{2.15}$$

In fact, let $u := Proj\ z/Tf(y)$. w denotes such a point that $w \in U_{\delta(y)}(y) \cap\ bd\ S$ and

$$d(z,w) = d(z,U_{\delta(y)}(y) \cap\ bd\ S)\ .$$

Since, for $\epsilon(y)$ small enough, $\| u-z \| \geq \| \bar{u}-z \|$ where $f(\bar{u}) = f^o$, $\bar{u} = z+\alpha(u-z)$ and $0<\alpha\leq 1$, we have $\| u-z \| \geq \| w-z \|$, i.e., (2.15). So, for all $z \in U_{\epsilon(y)}(y) \cap \{z \mid <x-y,z-x> \leq 0\}$, (2.15) is true. Let $\bar{y}$ be such a point that

$$x-\bar{y} \in (x-\hat{P}(x)) \cap L_{\mathbf{C}G(x;\hat{P}(x))}\ .$$

Then, for any $z \in U_{\epsilon(\bar{y})}(x) \cap \{z \mid <x-\bar{y},z-x> \leq 0\}$, the relation

$$d(z,U_{\delta(\bar{y})}(\bar{y}) \cap\ bd\ S) \leq \varphi(x)^{1/2} \tag{2.16}$$

holds.

Since $dim\ L_{\mathbf{C}G(x;\hat{P}(x))} \neq 0$, there exists a finite set $A \subset [G(x;\hat{P}(x)) \setminus\ \{x-\bar{y}\}] \cap L_{\mathbf{C}G(x;\hat{P}(x))}$ such that $\bar{y}-x \in \mathbf{C}A$, i.e., there exists a set of nonnegative scalars $\{\lambda_1,\dots,\lambda_l\}$ satisfying $\sum\lambda_i>0$ and

$$\bar{y}-x = \sum \lambda_i \bar{u}_i, \quad \bar{u}_i \in A \ . \tag{2.17}$$

For any $v \in \{z \mid \langle x-\bar{y}, z-x\rangle > 0\}$, there exists at least one $y(v) \in X-A$ such that

$$v \in \{z \mid \langle x-y(v), z-x\rangle < 0\} = H(x; x-y(v), <) \ . \tag{2.18}$$

In fact, from (2.17), one has the following relation

$$\langle x-\bar{y}, v-x\rangle > 0 \Longleftrightarrow \langle \bar{y}-x, v-x\rangle = \sum_{\bar{u}_i \in A} \lambda_i \langle \bar{u}_i, v-x\rangle < 0 \ .$$

Therefore there exists at least one $\bar{u}_i \in A$ such that

$$\langle \bar{u}_i, v-x\rangle < 0$$

This implies that there exists $y(v) \in x-A$ such that (2.18) holds. Take a neighborhood

$$U_{\bar{\epsilon}}(x) \subset \Big[\bigcap_{y \in x-A} (U_{\epsilon(y)}(x))\Big] \bigcap (U_{\epsilon(\bar{y})}(x)) \ ,$$

where $\bar{\epsilon}$ satisfies $0 < \bar{\epsilon} < \min\limits_{y \in (x-A) \bigcup \{\bar{y}\}} \epsilon(y)$ and $\epsilon(y)$ satisfies (2.16). Of course, the following relation is true

$$U_{\bar{\epsilon}}(x) \subset \bigcup_{y \in x-A} (U_{\epsilon(y)}(x)) \bigcup (U_{\epsilon(\bar{y})}(x)) \ .$$

It follows from this and (2.18) that there exists a neighborhood of x, $U_{\bar{\epsilon}}(x)$, such that $\forall\, v \in U_{\bar{\epsilon}}(x)$ one has $\varphi(x)^{1/2} \geq \varphi(v)^{1/2}$. So x is a locally optimal solution. □

Optimality Condition 2.7

Suppose that x is a $\wedge$-point. x is a locally optimal solution to the problem (1.1) if and only if

$$dim\ L_{CG(x; Y(x))} \neq 0 \ . \ \square$$

It is not easy to treat the situation in which $G(x;\hat{P}(x)) \bigcap L_{CG(x;Y(x))} \neq \{\emptyset\}$ and $CG(x;\hat{P}(x))$ is a pointed cone, but (2.12) is not valid. The main trouble is that in this case it is possible that although the main body is not able to expand in a straight line but it is able to expand in a curvilinear path. This situation will be discussed briefly. To this end we start with the following optimality condition (from [9]).

Optimality Condition 2.8

A point x is a locally optimal solution if and only if there exists $\epsilon>0$ such that

$$\bigcap_{y_\xi \in Y(x)} [D_\xi + \|x-y_\xi\| B(0,1)]^c \bigcap (U_\xi(x)) = \emptyset \ . \ \square \tag{2.19}$$

Suppose $y\in Y(x)$ and $\varsigma' \in bd\ [D_y + \|x-y\| B(0,1)] \bigcap S \bigcap U_{\delta(y)}(y)$, where $U_{\delta(y)}(y)$ is some neighborhood of y and D_ξ is the same as D_y. ς' can be expressed as

$$\varsigma' = y' - \|x-y\| \nabla f(y') / \|\nabla f(y')\| \tag{2.20}$$

where $y' \in \{y' | f(y') = f^o\}$. For convenience subscripts of f and y are omitted for temporality. To begin with we approximate $\nabla f(y')/\|\nabla f(y')\|$. Since

$$\nabla f(y') = \nabla f(y) + H_y(y'-y) + \|y'-y\| \bar{o}(y, y'-y) \ . \tag{2.21}$$

where $\bar{o}(y' \,,\, y' - y) \in \mathbf{R}^n \longrightarrow 0$, as $y' \longrightarrow y$.

Calculate the unit vector in (2.21)

$$\begin{aligned} \nabla f(y')/\|\nabla f(y')\| &= \nabla f(y')[1 - <\nabla f(y), H_y(y'-y)>/\|\nabla f(y)\|^2 \\ &+ \|y'-y\| \bar{o}(y;y'-y)]/\|\nabla f(y)\| \\ &= \frac{\nabla f(y)}{\|\nabla f(y)\|}[1 - <\nabla f(y), H_y(y'-y)>/\|f(y)\|^2] \\ &+ \frac{H_y(y'-y)}{\|\nabla f(y)\|} + \|y'-y\| \bar{o}(y, y'-y) \ . \end{aligned} \tag{2.22}$$

Substituting (2.22) into (2.20) we obtain the expression of ς'

$$\begin{aligned} \varsigma' &= y' - \frac{\|x-y\| \nabla f(y)}{\|\nabla f(y)\|} - \frac{\|x-y\| \nabla f(y) <\nabla f(y), H_y(y'-y)>}{\|\nabla f(y)\|^3} \\ &+ \|x-y\| \frac{H_y(y'-y)}{\|\nabla f(y)\|} + \|y' - y\| \bar{o}(y, y'-y) \\ &= y' - (y-x) + q + \|y'-y\| \bar{o}(y', y'-y) \ . \end{aligned}$$

From this one has

$$\varsigma' - x = y' - y \ + q + \|y'-y\| o(y, y'-y) \ .$$

Let $\varsigma - x := y' - y$, then

$$\varsigma \in f(\varsigma - x + y) = f_y^0 = f(y) \ ,$$

and the following equations are satisfied

$$\varsigma\prime - x = \varsigma - x + q + \|\varsigma - x\| \bar{o}(y, \varsigma - x)$$

In some neighborhood of x we make the approximation

$$\{\varsigma \,|\, f(\varsigma - x + y) = f(y)\} \bigcap U_\epsilon(x) = bd[D_y + \|x - y\| B(0,1)] \bigcap U_\epsilon(x)$$

Consequently, the set

$$\{\varsigma \,|\, 2\nabla f(y)^T(\varsigma - x) + (\varsigma - x)^T H_y(\varsigma - x) = 0\} \tag{2.23}$$

can be regarded as an approximation to $bd\ [D_y + \|x - y\| B(0,1)] \bigcap S$ at x.

Finally combining (2.19) and (2.23), we obtain an approximate condition

$$\bigcap_{y_\xi \in Y(x)} \{z \,|\, 2\nabla f_\xi(y_\xi)(z - x) + (z - x)^T H_{y_\xi}(z - x) \geq 0\}^c \bigcap U_\epsilon(x) = \emptyset$$

i.e.

$$\bigcap_{y_\xi \in Y(x)} \{z \,|\, 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) < 0\} \bigcap U_\epsilon(x) = \emptyset \ . \tag{2.24}$$

Because $\varsigma - x = y\prime - y$, for some $\epsilon > 0$ small enough, one has

$$bd\ [D_\xi + \|x - y_\xi\| B(0,1)] \bigcap\ U_\epsilon(x)$$
$$\subset \{z \,|\, 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) \leq 0\} \bigcap U_\epsilon(x) \ .$$

Therefore

$$\bigcap_{y_\xi \in Y(x)} [D_\xi + \|x - y_\xi\| B(0,1)]^c \bigcap U_\xi(x) \tag{2.25}$$

$$\subset \bar{S} := \bigcap_{y_\xi \in Y(x)} \{z \,|\, 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) < 0\} \bigcap U_\epsilon(x) \ .$$

It follows from (2.25) that if x is not a locally optimal solution to the problem (1.1) then the left hand side of (2.24) is nonempty. So if (2.24) holds then x must be an optimal solution (at least locally).

The problem now is how to verify if (2.24) holds when $dim\ N^+(x; Y(x)) \neq 0$ and n. Take appropriate $\epsilon > 0$, a point $z_{N^+} \in N^+(x; Y(x)) \bigcap U_\epsilon(x)$ and find a $\bar{z}$ such that it satisfies

$$d(z_{N^+}, \bar{z}) = d(z_{N^+}, \bar{S}) \ . \tag{2.26}$$

In order to avoid infinite programming brought by (2.25) and (2.26) the following sub-problem can be adopted, for some $\xi \in \Omega(x)$

$$\min \|z - z_{N^+}\| \tag{2.27}$$

$$s.t.\ f_\xi(z - x + y_\xi) = f_\xi^o$$

$$f_\xi(z - x + y_\eta) \le f_\eta^o \quad \eta \in \Omega(x) \setminus \{\xi\}$$

with initial point $x^{(o)} = x$. A local solution $z(x;\xi)$ for the problem (2.27) is near the boundary $bd[D_\xi + \|x - y_\xi\| B(0,1)]$. According to the information resulting from the point one can choose a method suitable for further checking if (2.24) holds. If needed, the parameter ϵ used for controlling a neighborhood of x can be reduced successively.

When the region determined by convex constraints and complementary convex constraints and a norm associated with predefined unit convex body (in the case where it has a continuously differentiable surface corresponding to the unit convex body), the VD2 algorithm [1] described a method convergent to a locally optimal solution to the c_M-embedded problem (the DC problem). In this case the VD2 algorithm can be executed continuously without checking optimality conditions until meeting stopping criteria prescribed.

2. OTHER RESULTS

In this section some results related to checking optimality described in the last section will be presented. With this end in view we first give the following lemma.

Lemma 3.1. Suppose that $y \in Q \subset \mathbf{R}^n$ and $x \notin clQ$. Then $x - y \in L_{CG(x;Q)}$ if and only if both of $Proj\ (x-y)/N^+(x;Q) = 0$ and $Proj\ (y-x)/N^+(x;Q) = 0$ hold.

Proof. Since $Proj\ (x-y)/N^+(x;Q) = 0$, the hyperplane (or subspace) $\{u \mid \langle u, x - y \rangle = 0\}$ separates $x - y$ and $N^+(x;Q)$ such that

$$\|x - y\|^2 > 0 \text{ and } \langle x - y, w \rangle \le 0 ,\ \forall\, w \in N^+(x;Q) . \tag{3.1}$$

Likewise, for $y - x$ it is the same as above, i.e.,

$$\|y - x\|^2 > 0 \text{ and } \langle y - x, w \rangle \le 0 ,\ \forall\, w \in N^+(x;Q) . \tag{3.2}$$

From (3.1) snd (3.2), we have

$$\langle x - y, w \rangle = 0 ,\ \forall\, w \in N^+(x;Q) .$$

Since $dim\ N^+(x;Q) + dim\ L_{CG(x;Q)} = n$ and $L^\perp_{CG(x;Q)} = LN^+(x;Q)$, one has

$x-y\perp L^{\perp}_{CG(x;Q)}$, i.e., $x-y\in L_{CG(x;Q)}$. This is the proof of sufficiency. Conversely, it is easy to be proved. □

We now study with the aid of quadratic approximation how to analyze the local behavior of boundary at a point. According to this, an algorithm can be constructed, which can be used to identify the local behavior of boundary at a point.

Given a point $y\in bd\ S$ and its corresponding constraint function f, and suppose further that

$$y'_T\in Tf(y) := \{y'_T \mid <y'_T-y,\nabla f(y)>=0\}$$

$$y'\in f(y') = f(y) \ ,$$

i.e.

$$y'\in\nabla f(y)^T(y'-y)+1/2(y'-y)^T H_y(y'-y)+\|y'-y\|\, o(y,y'-y) = 0 \tag{3.3}$$

$$y'_q\in q(y'_q;y) = f(y) \ ,$$

i.e.

$$y'_q\in\nabla f(y)^T(y'_q-y) + 1/2(y'_q-y)^T H_y(y'_q-y) = 0 \tag{3.4}$$

where $q(y;.)$ is the approximation of order two for the Taylor expansion of f at y. For any $y'_T\in Tf(y)$ in a neighborhood of y, we can find two corresponding points on $f(y')=f(y)$ and $q(y;y'_q)=f(y)$ respectively

$$y' = y'_T - \beta(y'_T)\nabla f(y) \tag{3.5}$$

and

$$y'_q = y'_T - \alpha(y'_T)\nabla f(y) \ . \tag{3.6}$$

$\alpha(y'_T)$ and $\beta(y'_T)$ can be regarded as two mappings

$$\alpha:\mathbf{R}^{n-1}\to\mathbf{R} \ , \ \beta:\mathbf{R}^{n-1}\to\mathbf{R} \ .$$

Substituting (3.5) into (3.3), one has

$$2\nabla f(y)^T[y'_T-y-\beta(y'_T)\nabla f(y)] + (y'_T-y-\beta(y'_T)\nabla f(y))^T H_y(y'_T-y-\beta(y'_T)\nabla f(y))$$
$$+ \|y'-y\|\, o(y,y'-y) = 0 \ .$$

From this we have

$$\beta(y_T')(\nabla f(y)^T H_y \nabla f(y)) - 2\beta(y_T')(\|\nabla f(y)\|^2+(y_T'-y)^T H_y \nabla f(y)) \tag{3.7}$$
$$+ (y_T'-y)^T H_y (y_T'-y)+o(\|y'-y\|) = 0 .$$

Suppose that $\nabla f(y)$ is not self-conjugate to Hessian of f at y. Solving (3.7), one has

$$\beta(y_T') = [(\|\nabla f(y)\|^2+(y_T'-y)^T H_y \nabla f(y))\pm((\|\nabla f(y)\|^2 \tag{3.8}$$
$$+ (y_T'-y)^T H_y \nabla f(y))^2 - (\nabla f(y)^T H_y \nabla f(y))(y_T'-y)^T H_y (y_T'-y)$$
$$+ o(\|y'-y\|^2))]/\nabla f(y)^T H_y \nabla f(y) .$$

For the sake of simplicity, let

$$a := (y_T'-y)/\|\nabla f(y)\| , \quad b := \nabla f(y)/\|\nabla f(y)\| , \tag{3.9}$$
$$c := b^T H_y b , \quad d := a^T H_y a .$$

Made use of (3.9), (3.8) can be simplified as follows

$$\beta(y_T') = [1+a^T H_y b\pm((1+a^T H_y b)^2-cd+o(\|y'-y\|^2))^{1/2}]/c . \tag{3.10}$$

Making Taylor approximation of second order, one has

$$\beta(y_T') = \frac{1}{c}(1+a^T H_y b) \tag{3.11}$$
$$- \left[1+a^T H_y b+\frac{1}{2}\left(a^T H_y b b^T H_y a-cd\right)-\frac{1}{8}\left(2a^T H_y b+\dots\right)^2+o(\|y'-y\|^2)\right]/c .$$

Because $(y_T'-y)/(y'-y)\to 1$ as $y_T'\to y$, the "-" is taken in (3.10). Therefore,

$$\beta(y_T') = \frac{1}{2}d+o(\|y_T'-y\|^2) \tag{3.12}$$
$$= \frac{1}{2}(y_T'-y)H_y(y_T'-y)/\|\nabla f(y)\|^2+o_\beta(\|y_T'-y\|^2) .$$

For (3.4) and (3.6), we have the similar expansion

$$\alpha(y_T') = \frac{1}{2}(y_T'-y)H_y(y_T'-y)/\|\nabla f(y)\|^2) . \tag{3.13}$$

Define

$$\Delta(y_T') := \frac{1}{2}(y_T'-y)H_y(y_T'-y)/\|\nabla f(y)\|^2 \tag{3.14}$$
$$y_T'\in Tf(y)\cap U_\epsilon(y) , \text{ for some } \epsilon>0 .$$

The foregoing development is summarized in the following lemma.

Lemma 3.2. Suppose $y \in Y(x)$. If $y \in \check{P}(x)(\hat{P}(x)$ *or* $\bar{P}(x))$, then there exists a $U_\epsilon(y)$ such that $\Delta(y_T') \leq 0$ (≥ 0) for all $y_T' \in Tf(y) \cap U_\epsilon(y)$. Conversely, if there exists a $U_\epsilon(y)$ such that, for all $y_T' \in Tf(y) \cap U_\epsilon(y)$, $\Delta(y_T') > 0$ (<0), then $y \in \hat{P}(x)$ or $\bar{P}(x)$ $(\check{P}(x))$. □

According to our hypothesis that for any boundary point $y \in bd\ S$, $\nabla f(y) \neq 0$, for any $y \in bd\ S$ there exists a matrix

$$C(y) = [u_1, \ldots, u_{n-1}] ,$$

that is, the matrix consisting of vectors in a basis of subspace $Tf(y) - y$, such that, for each $y_T' \in Tf(y)$, there is the unique expression

$$y_T' - y = \|\nabla f(y)\| Cw , \quad w \in Tf(y) - y = \mathbf{R}^{n-1} . \tag{3.15}$$

Substituting (3.15) into (3.14), we get

$$\Delta = w^T C^T H_y C w . \tag{3.16}$$

From Lemma 3.2 and (3.16), it is easy to prove the following theorem.

Theorem 3.3. Suppose $y \in Y(x)$. If $y \in \check{P}(x)$ $(\hat{P}(x)$ or $\bar{P}(x))$, then $C^T H_y C \leq 0$ (≥ 0). Conversely, if $C^T H_y C < 0$ (>0), then $y \in \check{P}(x)$ $(\hat{P}(x)$ *or* $\bar{P}(x))$. If $C^T H_y C$ is indefinite, then $y \in \tilde{P}(x)$. □

Note that although from $C^T H_y C \geq 0$, we are not able to conjecture if y must belong to $\hat{P}(x)$. For convenience, y is put into $\hat{P}(x)$ because of $\beta(y_T') = o(\|y_T' - y\|^2)$ for every $y_T' \in Tf(y) \cap U_\epsilon(y)$ when ϵ is small enough.

Now we will discuss an important problem, but a very difficult one as well. How to find $Y(x)$? Clearly, it is the problem we are interested in. Suppose that the set $\Omega(x)$ has been found. For every $\xi \in \Omega(x)$, consider the subproblem

$$\min \|y - x\|^2 \tag{3.17}$$

$$s.t.\ f_\xi(y) = f_\xi^o ,$$

$$f_\eta(y) \leq f_\eta^o , \quad \eta \in \Omega(x) \backslash \{\xi\} .$$

The problem in question can be described as finding

$$Y(x) = \bigcup_{\xi \in \Omega(x)} Y(x, \xi)$$

$$= \bigcup_{\xi\in\Omega(x)} \{\bar{y}\in\mathbf{R}^n \,|\, \bar{y} \text{ is a solution to the problem(3.17)}\} \ .$$

The part of inequality constraints in (3.17) can be taken no account of, or straightforwardly, they can be dropped when finding $Y(x,\xi)$, i.e., the subproblem

$$\min \| y-x \|^2 \qquad (3.18)$$

$$s.t. \ f_\xi(y) = f_\xi^o$$

will be considered. It is evident that (3.17) and (3.18) are equivalent. We assume that $Y(x,f)$ is finite. Then

$$Y(x) = \hat{P}(x) \bigcup \check{P}(x) \bigcup \tilde{P}(x) \ .$$

For convenience, the subscript in (3.18) is omitted and $Y(x,f)$ is used instead of $Y(x,\xi)$ temporarily. According to the definitions of $\hat{P}(x)$, $\check{P}(x)$ and $\tilde{P}(x)$ for each $\bar{y}\in Y(x,f)$, there exists a $\delta_0'(\bar{y})>0$ such that

$$bd\ S \bigcap (U_{\delta_o'(\bar{y})}(\bar{y})) \bigcap bd\ B(x,\varphi(x)^{1/2}) = \{\bar{y}\} \ .$$

In other words, there exists a $\delta_o(\bar{y})>0$ such that

$$Y(x,f) \bigcap H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) \ ; \ \nabla f(\bar{y})) = \{\bar{y}\} \ ,$$

where

$$H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) \ ; \ \nabla f(\bar{y})) = \{z\in\mathbf{R}^n \,|\, \langle z-\bar{y}+\delta_0(\bar{y})\nabla f(\bar{y}) \ , \ \nabla f(\bar{y})\rangle \geq 0\} \ .$$

There exists a set $\{\delta_0(\bar{y}) \,|\, \bar{y}\in Y(x,f)\}$ such that

$$\bigcap_{\bar{y}\in Y(x,f)} H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) \ ; \ -\nabla f(\bar{y})) \bigcap B(x,\varphi(x)^{1/2}) \bigcap Y(x,f) = \emptyset \ .$$

Define

$$X := \bigcap_{\bar{y}\in Y(x,f)} H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) \ ; \ -\nabla f(\bar{y}) \bigcap B(x,\varphi(x)^{1/2}) \ . \qquad (3.19)$$

It is closed, convex and bounded. Clearly

$$d(X,f(y) = f^o)>0 \ .$$

It can be proved that, if $Y(x,f)$ is finite, then there exists a polyhedron P_l such that

$$X\subset int P_l \ \text{ and } \ f(y) = f^o \subset P_l^c \ ,$$

i.e., $bd\ P_l$ can separate X and $f(y) = f^0$ strictly.

Given $Q \neq \emptyset$ and $Q \subset Y(x,f)$, consider the following problem

$$\min \|y-x\|^2$$

$$s.t.\ f(y) = f^0 ,$$

$$y \in \bigcap_{\bar{y} \in Q} H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) ; -\nabla f(\bar{y})) \qquad (SC) \tag{3.20}$$

$$(y \in M \text{ if } needed\ )$$

where M is a bounded domain containing $B(x,\varphi(x)^{1/2})$.

Suppose that $Y(x,f)$ is finite and $Y(x,f) \backslash Q \neq \emptyset$. The set determined by the constraints of (3.20) is nonempty, i.e.,

$$\{y \mid f(y) = f^0 \bigcap (\bigcap_{\bar{y} \in Q} H^+(\bar{y}-\delta_0(\bar{y})\nabla f(\bar{y}) ; -\nabla f(\bar{y})))\} \neq \emptyset . \tag{3.21}$$

After solved (3.20) each time two cases may happen. One is that $y^* \in bd\ B(x,\varphi(x)^{1/2})$, the other is that $y^* \in bd\ B(x,\varphi(x)^{1/2})$, i.e., $d(y^*,x) > \varphi(x)^{1/2}$ where y^* is a local solution to (3.20), obtained after some algorithms were executed for solving (3.20), e.g., [12].

Case I. In this case, a new touching point has been obtained. The set Q can be enlarged after $Q = Q \bigcup \{u^*\}$. If there is a need, a new hyperplane

$$Tf(y^*) - \delta_0(y^*)\nabla f(y^*)$$

can be introduced and

$$H^+(y^* - \delta_0(y^*)\ \nabla f(y^*) ;\ -\nabla f(y^*))$$

can be added to the original constraint set of the problem (3.20). Resolve the problem again.

Case II. $y^* \in bd\ B(x,\varphi(x)^{1/2})$. A hyperplane

$$\Pi(y^*) = \{z \mid <z-y^*, x-y^*> - \delta_0(y^*) \|x-y^*\|^2 = 0\}$$

can be constructed where $\delta_0(y^*)$ satisfies

$$\delta(y^*) \|x-y^*\| \geq (1/2)[\|x-y^*\| - \varphi(x)^{1/2}] \tag{3.22}$$

and

$$B(x,\varphi(x)^{1/2}) \subset H^+(y^* + \delta_0(y^*)(x-y^*) ;\ x-y^*) .$$

Then the right-hand side of the above inclusion relation is added to the set of original constraints of the problem (3.20). Resolve the problem (3.20) with the enlarged constraint set and an appropriately initial point $y^*(0)$ such that

$$\|x-y^*_{(0)}\| \le (1/2)[\|x-y^*\|+\varphi(x)^{1/2}] \ . \tag{3.23}$$

From this, if (3.20) is resolved infinitely, then a sequence of solutions can be formed, denoted by $\{y_k^*\}_1^\infty$. In terms of (3.23), one has

$$0 \le \|y_k^*-x\|-\varphi(x)^{1/2} \le 1/2[\|x-y_{k-1}^*\|-\varphi(x)^{1/2}]$$
$$\le (1/2)^{k-1}[\|x-y_1^*\|-\varphi(x)^{1/2}] \ .$$

So $\{y_k^*\} \to y^{**} \in B(x,\varphi(x)^{1/2})$. Summarizinng the description above, we get the following lemma.

Lemma 3.5. If $Y(x,f)$ is finite and $Y(x,f)\backslash Q \neq \emptyset$ and $Q \subset Y(x,f)$, then, by solving (3.20) repeatedly, a new touching point can be found provided that initial points in an infinitely iterative process satisfy (3.23). □

There are two points we have to mention before a strategy is proposed. The first is that, if $\bar{y} \in Q$ and for some $\delta>0$, the intersection of the hyperplane $Tf(\bar{y})-\delta\nabla f(\bar{y})$ and the hypersurface is empty, i.e.

$$(Tf(\bar{y})-\delta\nabla f(\bar{y})) \cap (f(y) = f^0) = \emptyset \ .$$

then the hypersurface is included in the halfspace $H^+(\bar{y}-\delta\nabla f(\bar{y});\nabla f(\bar{y}))$, because of $f \in C^2$. The second point concerns a basis in $(n-1)$-dimensional space. Suppose that the columns of $C(\bar{y}) = [u_1,\dots,u_{n-1}]$ form a basis in $Tf(\bar{y})-\bar{y}$. For any $\delta>0$, $\{u_1,\dots,u_{n-1}\}$ $C(\bar{y})+\bar{y}-\delta\nabla f(\bar{y})$ is a basis on the hyperplane (manifold) $Tf(\bar{y})-\delta\nabla f(\bar{y})$. In addition to these, an engineering infinite ∞_E is defined by a positive number large enough, r_∞, i.e., $\infty_E := r_\infty$ controlled by M in (3.20).

Finally it is necessary to mention the case in which $Y(x,f)$ is an infinite set. But it is not our purpose to elaborate on the details. We only make a short discussion below.

It is clear that

$$Y(x,f) \subset A := [\bigcup_{y \in Y(x,f)} (U_{\epsilon(y)}(y)] \cap bd\ B(x,\varphi(x)^{1/2}) \ , \ \ \epsilon(y)>0$$

and

$$N^+(x;Y(x,f)) \supset N^+(x;A) \ .$$

Since $bd\ B(x,\varphi(x)^{1/2})$ and $Y(x,f)$ are compact, there exists a finite family of neighborhoods

$$\{U_{\epsilon(y)}(y)\,|\,y\in Q\subset Y(x,f)\ ,\ \epsilon(y)>0\}\ .$$

covering $Y(x,f)$, where Q is a finite subset of $Y(x,f)$. Let $\epsilon = \max\{\epsilon(y)\,|\,y\in Q\}$. Then

$$Y(x,f)\subset \bigcup_{y\in Q}(U_\epsilon(y))\ ,\quad Q\subset Y(x,f)\ .$$

In general, we are not able to infer from $Q\subset Y(x,f)$ that

$$N^+(x;Y(x,f))\supset N^+(x;\bigcup_{y\in Q} U_\epsilon(y))$$

for some $\epsilon>0$. A finite subset of $Y(x,f)$, Q, is called a finite $N^+-\epsilon$ approximation to $Y(x,f)$ if

$$N^+(x;Y(x,f))\supset N^+(x;\bigcup_{y\in Q} U_\epsilon(y))\ . \tag{3.25}$$

Theorem 3.8. Suppose that $Q_i\subset Y(x,f)$, $i = 0,1,\ldots,\{Q_i\}_0^\infty\uparrow$ in the sense of the inclusion relation, $\{\epsilon_i\}_0^\infty\downarrow 0$ and $Y(x,f)$ is infinite. If for any $i = 0,1,\ldots$, the following condition is satisfied

$$Y(x,f)\subset \bigcup_{y\in Q_i}(U_{\epsilon_i}(y))\ , \tag{3.26}$$

then

$$\lim_{i\to\infty} N^+(x;Q_i) = N^+(x;Y(x,f))\ . \tag{3.27}$$

Proof. From (3.26), one has

$$Y(x,f)\subset\bigcap_{i=0}^{\infty}\ \bigcup_{y\in Q_i}(U_{\epsilon_i}(y))\ . \tag{3.28}$$

Now we prove the inclusion relation opposite to (3.28). For any fixed z such that

$$z\in\bigcap_{i=0}^{\infty}\ \bigcup_{y\in Q_i}(U_{\epsilon_i}(y))$$

and any i, there exist $y_i\in Q_i$ and α_i such that $0\le\alpha_i\le\epsilon_i$ and

$$z = y_i+\alpha_i\omega_i\ ,$$

where $\|\omega_i\| \leq 1$. From this we have

$$\|z - y_i\| < \epsilon_i \ .$$

Since $\epsilon_i \to 0$ and $Y(x,f)$ is compact, $\{y_i\}_0^\infty \to z$ and $z \in Y(x,f)$. Hence

$$Y(x,f) = \bigcap_{i=0}^{\infty} \bigcup_{y \in Q_i} (U_{\epsilon_i}(y)) \ . \tag{3.29}$$

It can be seen that $\lim_{i \to \infty} Q_i$ is dense in $Y(x,f)$ because of $\{Q_i\}\uparrow$ and (3.29). Thus

$$N^+(x; \lim_{i \to \infty} Q_i) = N^+(x; Y(x,f)) \tag{3.30}$$

because of $cl \lim_{i-.\infty} Q_i = Y(x,f)$. Note that $\{N^+(x;Q_i)\}\downarrow$ as $\{Q_i\}\uparrow$. It is easy to prove that

$$N^+(x; \lim_{i \to \infty} Q_i) = N^+(x; \bigcup_{i=0}^{\infty} Q_i) = \bigcap_{i=0}^{\infty} N^+(x;Q_i) = \lim_{i \to \infty} N^+(x;Q_i) \ . \tag{3.31}$$

In view of (3.30) and (3.31), one has (3.27). The proof is completed. □.

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