THE MORPHOLOGY OF GEOGRAPHICAL CHANGE

W.R. Tobler

September 1975 WP-75-112

The Morphology of Geographical Change

W.R. Tobler

Geography is the study of the processes which result in changes of phenomena on the surface of the earth, and thus it is useful to be able to describe change in an analytical manner. Here a single geographical variable is assumed to vary with time and this change is modelled in a number of simple ways. The objective is to see if this excursion is of use in the geographical study of growth poles. The topic is not one with which I am excessively familiar.

The most elementary case is to consider, in a rather mechanical fashion, a scalar variable and its changes through time. This variable may appear in three guises, which however may be used fairly interchangeably:

(I) as a function, $G(x,y)$
(II) as a finite set of real numbers, G_{ij}
(III) as a finite set of real numbers, G_k .

In (I), x,y are considered as latitude and longitude coordinates (although no explicit recognition will be given to the roundness of the earth). G represents the value of the phenomenon of concern at the location x,y and exists at every location in the domain of interest. For example, G might represent the number of people residing within a circle of one kilometer radius centered at x,y; or it might represent the average annual income of these people; or the percent of them which are unemployed, etc.

In (II), i and j are indices describing geographical locations when the data are assembled on a chessboard-like lattice. An interpretation is that G_{ij} is a sample obtained from $G(x,y)$ by multiplication with the brush function, the two-dimensional equivalent to Dirac's comb function. An alternate interpretation is provided by grid coordinate data collection schemes as recently promulgated in Sweden and elsewhere.

The interpretation for (III) is similar, but the subscript now refers to a bounded two-dimensional region of irregular shape; a country or a census tract for example. A conversion might be that

$$G(x,y) = G_k \quad , \quad \text{if and only if } x,y \text{ is in region } k.$$

Figure 1 shows such a function.

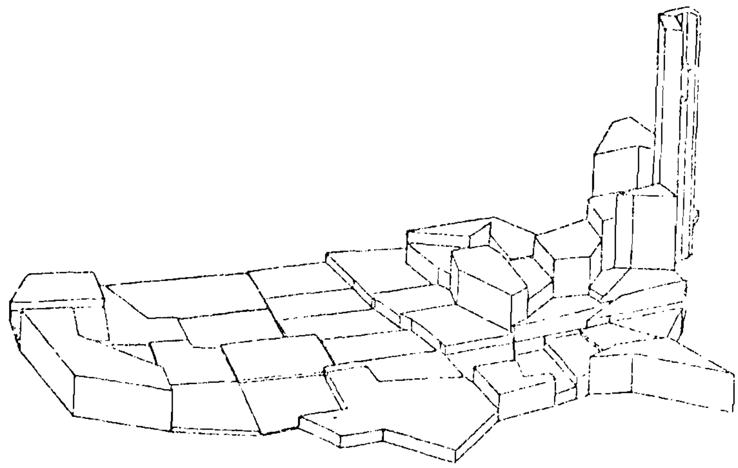


Figure 1. A piecewise continuous function $z = f(x,y)$; 1970 population density by states.

The single symbol G, with or without subscripts, will refer to the state of the phenomenon at all locations at a single time period. The state of the phenomenon at a different time will be distinguished by an asterisk; G^* for a later time period, G^{**} for two periods later, G^{-*} for an earlier time period, etc. Time will here be considered either continuous or in equal discrete steps.

Geographical Change Models

Regional policy is, in one elementary interpretation, concerned with the morphology of the function G(x,y); whether its variance is too large or too small; whether it has too many peaks or too few; whether $\int\int G(x,y)\,dxdy$ is too small or too large; and so on to include all of the static, descriptive, questions which one might ask of a social topography, as well as the normative desirata which one might envision. Of course, the concerns of regional policy are at least vector valued and not scalars, and should be more related to people than places. In any event, the dynamics can be viewed as an empirical question in which one asks how the morphology is changing. Let

$$\frac{dG}{dt}(x,y) \approx \frac{G_k^* - G_k}{\Delta t}$$

be the temporal rate of change of the phenomenon of interest. A number of specific models come to mind, e.g.,

$$\frac{dG}{dt}(x,y) = R(x,y) \quad ,$$

where R(x,y) is a two-dimensional continuous random variable. The change at each location is independent of that at all other locations, and is random. Presumably one needs to reject this model with one's data before going further, but I do not ever recall seeing such a test in the literature.

The next simplest model would appear to be that of constant regional growth:

$$\frac{dG(x,y)}{dt} = \text{constant} \quad .$$

Every place grows (or shrinks) by the same amount. If the measure G is in dollars this might reflect a static situation with inflation. Observationally, of course, the more usual case is that in which the rate of change depends on where one is located. We can now speculate on the form of this function. One, still simple, model is that of proportional growth:

$$\frac{dG}{dt}(x,y) = kG(x,y) \quad ,$$

where k is the same for all locations. Combining the three foregoing models would yield

$$\frac{dG(x,y)}{dt} = c + kG(x,y) + R(x,y) \quad .$$

This looks like a simple linear regression. A more complicated version, which however does not seem well specified, would be to allow spatial variations in both constants; that is, $c = C(x,y)$, $k = K(x,y)$.

Continuing,

$$\frac{dG}{dt}(x,y) = k(1 - \alpha)\; G(x,y)$$

describes logistic growth. The alpha can be interpreted as a target, and this might be interesting from a control theoretic viewpoint. Alpha could be a constant or could vary spatially, $\alpha = A(x,y)$.

Alternate hypotheses might include

$$\frac{dG}{dt}(x,y) = k\left[\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2\right]^{\frac{1}{2}}$$

in which the rate of change is proportional to the spatial gradient, or

$$\frac{dG}{dt}(x,y) = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right)$$

in which the temporal rate of change is proportional to the spatial rate of change of the gradient, or

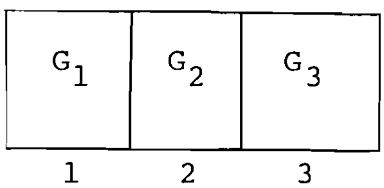
$$\frac{d^2G}{dt^2}(x,y) = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right)$$

where the temporal acceleration of growth is proportional to the spatial curvature of the phenomenon. These last two linear partial differential equations could be considered special cases of

$$\sum_{u=1}^{n} \ell_u \frac{d^uG}{dt^u}(x,y) = \sum_{\nu=1}^{m} k_\nu\left(\frac{\partial^\nu G}{\partial x^\nu} + \frac{\partial^\nu G}{\partial y^\nu}\right)$$

with $\ell_u = L_u(x,y)$ and $k_\nu = K_\nu(x,y)$ to make for further generality.

The motivation for such models can be made clearer. Consider a system of three adjacent spatial cells with populations G_1, G_2, and G_3:



Suppose further that the population of any cell is reduced by out-migration to neighbors and is increased by in-migration from neighbors. Let the *proportion* migrating from cell 1 to cell 2 be W_{21}, with comparable interpretations for W_{23}, W_{32}, W_{12}. If there are no other ways of changing the system, then at some later time the population G_2^* of cell 2 will be

$$G_2^* = G_2 - W_{21}G_2 - W_{23}G_2 + W_{32}G_3 + W_{12}G_1 \quad ,$$

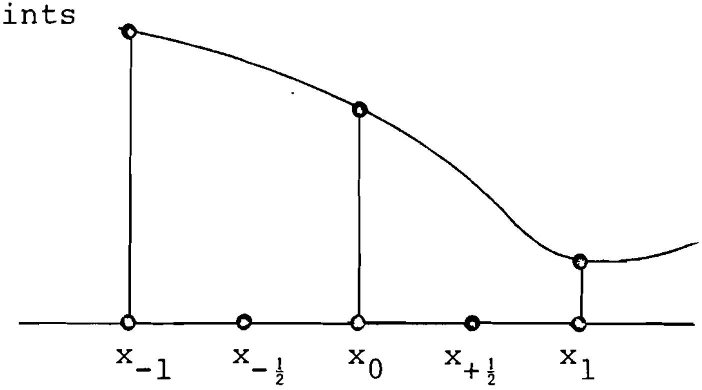
in which everything is accounted for. Now assume (but only for expository convenience) spatial isotropy, namely, $W_{21} = W_{23} = W_{32} = W_{12} = W$. As a consequence

$$G_2^* = G_2 + W(G_1 - 2G_2 + G_3) \quad .$$

The change in population due to migration is then $G_2^* - G_2 = W(G_1 - 2G_2 + G_3)$, but this has occurred over some time period, say Δt. Consequently,

$$\frac{dG}{dt} \simeq \frac{G_2^* - G_2}{\Delta t} = W(G_1 - 2G_2 + G_3) \quad .$$

Now consider the derivatives of a function f(x) at a finite set of points

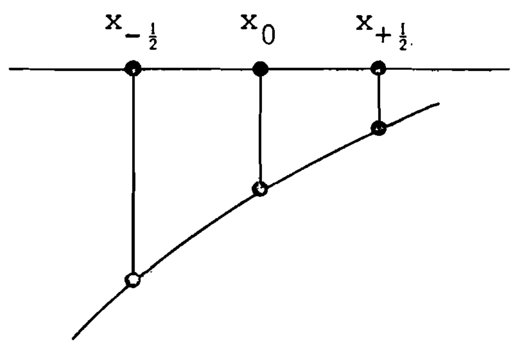


Define $x_{-\frac{1}{2}}$ to be $(f(x_{-1}) + f(x_0))/2$ and $x_{+\frac{1}{2}}$ to be $(f(x_0) + f(x_1))/2$, then the derivative at these points is approximated by

$$\frac{df}{\partial x_{-\frac{1}{2}}} \simeq \frac{f(x_{-1}) - f(x_0)}{\Delta x}$$

$$\frac{df}{\partial x_{+\frac{1}{2}}} \approx \frac{f(x_0) - f(x_{+1})}{\Delta x}$$

where $x_0 - x_{-1} = x_{+1} - x_0 = \Delta x$. Thus we can consider the new function $f'(x) = \frac{df}{dx}$ at the points $x_{-\frac{1}{2}}$ and $x_{+\frac{1}{2}}$:



Now the value $\frac{f'(x_{-\frac{1}{2}}) - f'(x_{+\frac{1}{2}})}{\Delta x}$ is an approximation to the derivative of f' at x_0, or $\frac{d^2 f}{\partial x^2}$. By substitution using the above values one finds

$$\frac{d^2 f}{dx^2} \approx f(x_{-1}) - 2f(x_0) + f(x_{+1}) \quad .$$

Changing the labels to go back to the G's, we have

$$\frac{d^2 G}{dx^2} \approx G_1 - 2G_2 + G_3 \quad .$$

So that, approximately

$$\frac{dG}{dt} = W \frac{d^2 G}{dx^2} \quad .$$

By analogous arguments in an orthogonal direction, a second spatial dimension can be added to obtain

$$\frac{dG}{dt} = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right)$$

which is one of the equations exhibited earlier, and it no longer seems so mysterious and implausible. A comparable, but of course different, discussion could be provided for some of the other equations. The assumptions were, one recalls, that changes in the system occur only through migration, and spatial isotropy. The first situation is easily accounted for by adding terms for new entrants and departures, viz.

$$\frac{dG}{dt}(x,y) = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \Big(\beta(x,y) - \delta(x,y)\Big)\ G(x,y) \quad .$$

The second assumption is most easily removed by reverting to the equation

$$G_2^* - G_2 = W_{12}G_1 - (W_{21} + W_{23})\ G_2 + W_{32}G_3$$

which has now to be completed by adding the births and deaths

$$G_2^* - G_2 = W_{12}G_1 - (W_{21} + W_{23})\ G_2 + W_{32}G_3 + (\beta_2 - \delta_2)\ G_2 \quad .$$

Adding more spatial cells to this three cell scheme is clearly not difficult in an accounting sense. More interesting perhaps is to interpret $k = K(x,y)$ in

$$\frac{dG}{dt} = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + (\beta + \delta)\ G \quad .$$

This must surely mean that the earlier constant W varies from place to place. If one were to write out the entire system of equations, one accounting equation for each place,

$$\begin{bmatrix} G_1^* - G_1 \\ G_2^* - G_2 \\ G_3^* - G_3 \end{bmatrix} = \begin{bmatrix} -(W_{12} + W_{13}) & W_{21} & W_{31} \\ W_{12} & -(W_{21} + W_{23}) & W_{32} \\ W_{12} & W_{23} & -(W_{31} + W_{32}) \end{bmatrix} \cdot \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} + \begin{bmatrix} \beta_1 & - & \delta_1 \\ \beta_2 & & \delta_2 \\ \beta_3 & & \delta_3 \end{bmatrix}^t \cdot \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

then the isotropic assumption would yield a constant W matrix. An intermediate form is obtained if the matrix is a band-matrix with entries some distance from the diagonal equal to zero. This can be used to imply that there is no migration between distant cells. A convenient isotropic assumption is then that

$$G_{ij}^* = \sum_{p=-n}^{p=+n} \sum_{q=-n}^{q=+n} W_{pq} G_{i+p,j+q} + (\beta_{ij} - \delta_{ij})\ G_{ij}$$

where the summation is over the n^2 cells which make up the influence field around each cell.

A geographical growth pole is supposed to be a place from which growth effects spread. Re-examine the diffusion equation

$$\frac{dG}{dt} = k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + (\beta - \delta)\ G \quad .$$

The first term on the right describes the spread effect, migration to neighboring places, and the second term describes the "natural increase" and decrease. Suppose we remove the spread effects, subtracting from both sides

$$\frac{dG}{dt} - k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) = (\beta - \delta)\ G$$

and now rewrite this in finite difference form as

$$G^*_{ij} - \sum_p \sum_q W_{pq} G_{i+p,j+q} = (\beta_{ij} - \delta_{ij})\ G$$

(wherein $W_{ij} = 1 - W_{21} - W_{23}$ of the earlier equations). That is, if we separate out the spread effects, we can estimate the *spatial growth multiplier* in the "natural increase" term. It, of course, can also be less than one, an inhibiting effect may exist. There seems to be hope that this can be done empirically using data from two time periods. If one writes out the wave equation

$$\frac{d^2G}{dt^2} - k\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + (\beta - \delta)\ G$$

in finite difference form then the second derivative term on the left hand side means that we need data from an addition time period; i.e., the model is one with temporal lags, just as the presence of spatial derivatives calls our attention to the spatial lags.

An alternate type of model would be the purely spatial function

$$G^*(x,y) - G(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x^i y^j$$

using an algebraic polynomial, or some other two-dimensional form (spherical harmonics, for example). Here the low-order terms (linear, quadratic) would indicate a spatial trend in the growth. A geographical trend, from East to West say, might then indicate the effects of an external influence. Other than such effects the model is more descriptive than interesting.

Casetti and Semple have proposed an alternate model as follows

$$G_k^* - G_k = \sum_{i=0}^{n} a_i \exp(-\beta_i r_{ik})$$

where $r_{ik} = \left[(x_i - x_k)^2 - (y_i - y_k)^2\right]^{\frac{1}{2}}$ is the separation of the location i from the location k. Basically, the model says that there are n + 1 points around which growth declines. The magnitude a_i and the rate of decline β_i are different for each of the several centers. The rate of decline can be made to vary in different directions around each of the centers by putting a separate Fourier series in the exponents, as has been done by Medvedkov. One of the difficulties of this model is its non-linearity, although an iterative computer program is available.

Suppose that a government decides to allocate funds according to F(x,y). Then one might postulate that

$$\frac{dG}{dt} = kF(x,y) \quad .$$

Suppose that the allocation is proportional to the population P(x,y), then

$$\frac{dG}{dt} = kP(x,y)$$

and if the population is the phenomenon of concern

$$\frac{dP}{dt} = kP(x,y) \quad ,$$

one of the models encountered earlier. We must always subtract the source of the funds too, the taxes T(x,y), and

$$\int\int F(x,y)\ dxdy \equiv \int\int T(x,y)\ dxdy \quad .$$

Assuming negligible costs for the bureaucracy. Thus

$$\frac{dG}{dt} = k\left[F(x,y) - T(x,y)\right]$$

would be one model. What is now needed is a more process-oriented morphology.