WORKING PAPER

An ϵ-Approximation Scheme for Minimum Variance Resource Allocation Problems

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Foreword

The minimum variance resource allocation problem asks to allocate a given amount of discrete resource to a given set of activities so that the variance of the profits among activities is minimized. The author presents a fully polynomial time approximation scheme for this problem.

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1. Introduction

The problem of allocating a limited resource to relevant activities in a fair manner on the basis of a certain general objective function has recently been considered by Katoh, Ibaraki and Mine [13]. Fujishige, Katoh and Ichimori [5] extended this result to the one with submodular constraints. The problem considered by [13] is written as follows.

$$\text{FAIR: minimize } g(\max_{1\le j\le n} f_j(x_j),\ \min_{1\le j\le n} f_j(x_j)) \tag{1.1}$$

$$\text{subject to } \sum_{j=1}^{n} x_j = N \quad , \tag{1.2}$$

$$x_j \in \{0,1,2,\ldots,u_j\},\ j = 1,\ldots,n \quad , \tag{1.3}$$

where g is a function from R^2 to R such that $g(u,v)$ is monotone nondecreasing in u and monotone nonincreasing in v, and $f_j, j = 1,2,\ldots,n$, are nondecreasing functions from $[0, u_j]$ to R. $f_j(x_j)$ denotes the profit resulting from allocating x_j amount of resource to activity j. N and u_j, $j = 1,\ldots,n$, are positive integers satisfying

$$N < \sum_{j=1}^{n} u_j \quad , \tag{1.4}$$

$$u_j \le N\ ,\ j = 1,\ldots,n \quad . \tag{1.5}$$

If (1.4) is not satisfied, the problem is infeasible or has a trivial solution. If (1.5) is not satisfied for some j, replacing it by $u_j \le N$ does not change the feasible set. Therefore assumptions of (1.4) and (1.5) do not lose the generality.

This problem arises whenever the distribution of a given amount of integer resource to a given set of activities is required so that the profit differences among activities are minimized. The fairness of the allocation is measured by the function g in problem FAIR. Zeitlin [18] and Burt and Harris [1] considered the special case of FAIR such as $g(u,v) = u - v$, and gave a finite algorithm. [13] and [5] gave polynomial time algorithms for the general case.

The fairness of the allocation can be measured alternatively by the variance among the profits resulting from the allocation. Letting $x = (x_1,x_2,\ldots,x_n)$ be a feasible allocation, the variance among profits is defined by

$$var(x) \equiv \frac{1}{n}\sum_{j=1}^{n}(f_j(x_j) - \frac{1}{n}\sum_{j=1}^{n}f_j(x_j))^2 \quad . \tag{1.6}$$

The minimum variance resource allocation problem is then described as follows

$$P : \text{minimize } var(x) \tag{1.7}$$

$$\text{subject to the constraints of (1.2) and (1.3)} \quad .$$

We assume that all $f_j, j = 1,\ldots,n$, are nondecreasing, or all $f_j, j = 1,\ldots,n$, are nonincreasing. Notice that all $f_j, j = 1,\ldots,n$, can be assumed to be nonnegative valued without loss of generality. Let us consider the case in which all f_j are nondecreasing (the case in which all f_j are nonincreasing can be similarly treated). Let

$$a \equiv \min_{1\le j\le n} f_j(0) \quad ,$$

and define for each j with $1 \le j \le n$

$$g_j(x_j) \equiv f_j(x_j) - a \quad , \qquad x_j \in [0,u_j] \quad .$$

Let P' denote problem P with all f_j replaced by g_j. It is easy to see from (1.6) that a solution is optimal to P' if and only if it is optimal to P, and that the objective value of P for a solution x is equivalent to that of P' for x. This proves the above claim.

We first give a parametric characterization stating that an optimal solution of the following parametric problem $P(\lambda)$ defined below provides an optimal solution of P, if an appropriate number λ is chosen.

$$P(\lambda) : z(\lambda) \equiv \text{minimize } \sum_{j=1}^{n}(\{f_j(x_j)\}^2 - \lambda f_j(x_j)) \quad . \tag{1.8}$$

Thus, solving P is reduced to find a $\lambda = \lambda^*$ with which an optimal solution to $P(\lambda^*)$ is also optimal to P. Such characterizations can be obtained in the same manner as was done by Katoh [11] (Sniedovich [16, 17] and Katoh and Ibaraki [12] treat the more general cases). [14] also gave the similar result for variance constrained markov decision process.

This characterization, however, does not tell how to find such λ^*. The straightforward approach for finding λ^* is to compute optimal solutions of $P(\lambda)$ over the entire range of λ. Based on this idea, we shall present a pseudopolynomial algorithm for P (see [7] for the definition of a "pseudopolynomial algorithm"). We assume throughout this paper that the evaluation of $f_j(x_j)$ for each integer x_j can be done in constant time.

The number of optimal solutions of $P(\lambda)$ generated over the entire range of λ is not polynomially bounded in most cases (see Chapter 10 of Ibaraki and Katoh [10]). In addition, solving $P(\lambda)$ for a given λ cannot be done in polynomial time in general unless $\{f_j(x_j)\}^2 - \lambda f_j(x_j)$ is convex. Notice that $\{f_j(x_j)\}^2 - \lambda f_j(x_j)$ is not convex in general even if $f_j(x_j)$ is convex. Therefore it seems to be difficult to develop polynomial time algorithms, and we then focus on approximation schemes in this paper. A solution is said to be an *ϵ-approximate* solution if its relative error is bounded above by ϵ. An *approximation scheme* is an algorithm containing $\epsilon > 0$ as a parameter such that, for any given ϵ, it can provide an ϵ-approximate solution. If it runs in time polynomial in the input size of each problem instance, and $1/\epsilon$, the scheme is called a *fully polynomial time approximation scheme* (FPAS) [7,15].

We shall show that, if $P(\lambda)$ for each nonnegative λ can be solved in polynomial time, we can develop an FPAS for P. The idea is to solve $P(\lambda)$ only for a polynomially bounded number of λ's, which are systematically generated so that the relative error of the achieved objective value is within ϵ. We shall then show that if all $f_j(x_j), j = 1,\ldots,n$, are convex, we can develop an FPAS for P.

We should mention here relationships between this paper and related papers [11, 12]. Recently, Katoh [11] studied the minimum variance combinatorial problems and gave an FPAS under the assumption that the corresponding minimum sum problem can be solved in polynomial time. [11] is based on the parametric characterization which is the same as this paper and the scaling technique. Notice that the scaling technique cannot be applied to our problem since f_j are nonlinear in general. An FPAS for the problems similar to P of (1.7) has been proposed by Katoh and Ibaraki [12]. Though the techniques employed therein are similar to those developed here, our problem P does not belong to the class of problems for which they developed an FPAS (especially the condition (A5) given in Section 5 of [12] does not hold for P).

This paper is organized as follows. Section 2 gives the relationship between P and $P(\lambda)$, and shows that P can be solved in pseudopolynomial time. Section 3 gives an outline of an FPAS for P, assuming that $P(\lambda)$ for any nonnegative λ can be solved in polynomial time. Section 4 describes the FPAS for P. Section 5 shows that if all $f_j(x_j), j = 1,\ldots,n$, are convex, the procedure of Section 4 with slight modifications becomes an FPAS.

2. Relationship between P and P(λ)

Katoh and Ibaraki [12] and Sniedovich [16, 17] considered the following problem Q.

$$Q: \underset{x \in X}{\text{minimize}}\ h(q_1(x),\ q_2(x)) \quad , \tag{2.1}$$

where x denotes an n-dimensional decision vector and X denotes a feasible region. $q_i, i = 1,2$, are real-valued functions and $h(u_1,u_2)$ is quasiconcave over an appropriate region and differentiable in u_i, $i = 1,2$. They proved the following lemma.

Lemma 2.1 [12, 16, 17] Let x^* be optimal to Q and let $u^*_i = q_i(x^*)$, $i = 1,2$. Define λ^* by

$$\lambda^* = \left\{ \frac{\partial h(u^*_1,u^*_2)}{\partial u_2} \right\} \Big/ \left\{ \frac{\partial h(u^*_1,u^*_2)}{\partial u_1} \right\} \quad . \tag{2.2}$$

Then an optimal solution of the following problem $Q(\lambda)$ with $\lambda = \lambda^*$ is optimal to Q.

$$Q(\lambda): \underset{x \in X}{\text{minimize}}\ q_1(x) + \lambda q_2(x) \quad . \qquad \square$$

The following lemma is obtained by specializing Lemma 2.1 to problem P. Let x^* and $x(\lambda)$ be optimal to P and $P(\lambda)$ respectively.

Theorem 2.1 Let λ^* be defined by

$$\lambda^* = 2\sum_{j=1}^{n} f_j(x^*_j)/n \quad . \tag{2.3}$$

Then $x(\lambda^*)$ is optimal to P.

Proof. First note that for any n-dimensional vector $x = (x_1,x_2,\ldots,x_n)$,

$$var(x) = \frac{1}{n}\sum_{j=1}^{n}(f_j(x_j) - \frac{1}{n}\sum_{j=1}^{n} f_j(x_j))^2$$

$$= \frac{1}{n}\sum_{j=1}^{n}\{f_j(x_j)\}^2 - \frac{1}{n^2}(\sum_{j=1}^{n} f_j(x_j))^2 \quad . \tag{2.4}$$

Let X be the set of all n-dimensional vectors satisfying (1.2) and (1.3), and let

$$q_1(x) \equiv \sum_{j=1}^{n}\{f_j(x_j)\}^2 \,, \quad q_2(x) \equiv \sum_{j=1}^{n} f_j(x_j) \tag{2.5}$$

and

$$h(u_1,u_2) \equiv \frac{1}{n}(u_1 - \frac{1}{n}(u_2)^2) \quad .$$

Then it is easy to see that for any $x \in X$

$$var(x) = \frac{1}{n}[q_1(x) - \frac{1}{n}\{q_2(x)\}^2] \quad .$$

Therefore P can be rewritten into

$$\underset{x \in X}{\text{minimize}}\, \frac{1}{n}[q_1(x) - \frac{1}{n}\{q_2(x)\}^2] \quad .$$

Since $h(u_1,u_2)$ is clearly quasiconcave, it turns out that P is a special case of Q. As a result, by $\partial h(u_1,u_2)/\partial u_1 = 1/n$ and $\partial h(u_1,u_2)/\partial u_2 = -2u_2/n^2$, the theorem follows from Lemma 2.1. □

Notice that λ^* is nonnegative since all f_j are assumed to be nonnegative valued. Although this theorem states that $P(\lambda)$ for an appropriate λ can solve P, such λ is not known unless P is solved. A straightforward approach to resolve this dilemma is to solve $P(\lambda)$ for all λ; the one with the minimum $var(x)$ is an optimal solution. This idea leads to a pseudopolynomial algorithm for P. For this, we shall give basic properties.

It is well known in the theory for parametric programming (see for example [2, 6, 8, 9]) that $z(\lambda)$ (the optimal objective value of $P(\lambda)$) is a piecewise linear concave function as illustrated in Fig. 1, with a finite number of joint points $\lambda_{(1)},\lambda_{(2)},\ldots,\lambda_{(M)}$ with $0 < \lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(M)}$. Here M denotes the number of total joint points, and let $\lambda_{(0)} = 0$ and $\lambda_{(M+1)} = \infty$ by convention. In what follows, for two real numbers a,b with $a \leq b$, (a,b) and $[a,b]$ stand for the open interval $\{x | a < x < b\}$ and the closed interval $\{x | a \leq x \leq b\}$ respectively. The following two lemmas are also known in the parametric combinatorial programming. Let X be the one as defined in the proof of Theorem 2.1.

Lemma 2.2 [8, 9] For any $\lambda' \in (\lambda_{(k-1)},\lambda_{(k)})$, $k = 1,\ldots,M+1, x(\lambda')$ is optimal to $P(\lambda)$ for all $\lambda \in [\lambda_{(k-1)},\lambda_{(k)}]$.

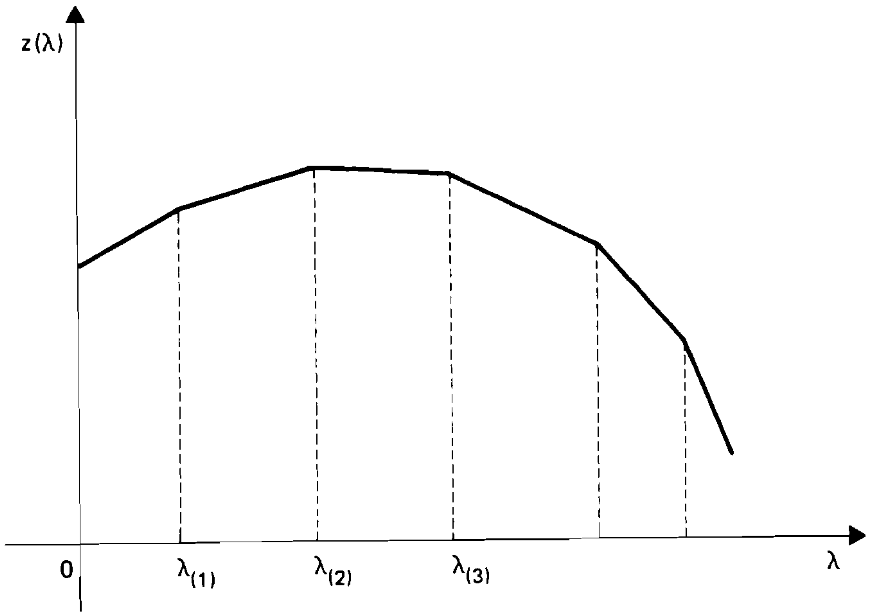

Figure 1 Illustration of $z(\lambda)$.

Let for $k = 1,\ldots,M+1$

$$X^*_k \equiv \{x \in X | x \text{ is optimal to } P(\lambda) \text{ for all } \lambda \in [\lambda_{(k-1)}, \lambda_{(k)}]\}$$

Lemma 2.3 [8, 9] (i) For any two $x,x' \in X^*_k$ with $1 \leq k \leq M+1$,

$$\sum_{j=1}^{n} \{f_j(x_j)\}^2 = \sum_{j=1}^{n} \{f_j(x_j')\}^2 \text{ and } \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n} f_j(x_j')$$

hold.

(ii) For any $x \in X^*_{k-1}$ and any $x' \in X^*_k$ with $2 \leq k \leq M+1$,

$$\sum_{j=1}^{n} f_j(x_j) < \sum_{j=1}^{n} f_j(x_j')$$

holds.

Lemmas 2.2 and 2.3(i) imply that in order to determine $z(\lambda)$ for all $\lambda \geq 0$, it is sufficient to compute $x(\lambda')$ for an arbitrary $\lambda' \in (\lambda_{(k-1)}, \lambda_{(k)})$ for each $k = 1,2,\ldots,M+1$. We shall use the notation x^k to stand for any $x \in X^*_k$.

Eisner and Severence [3] proposed an algorithm that determines $z(\lambda)$ for all $\lambda \geq 0$ and $x^k, k = 1,\ldots,M+1$, for a large class of combinatorial parametric problems including $P(\lambda)$ as a special case. They showed the following result.

Lemma 2.4 [3] Let $\tau(n,N)$ denote the time required to solve $P(\lambda)$ for any fixed $\lambda \geq 0$. Then $z(\lambda)$ for all $\lambda \geq 0$ and $x^k, k = 1,\dots,M+1$, can be determined in $0(M \cdot \tau(n,N))$ time.

Lemma 2.5 (Chapter 10 of [10])

$$M \leq 2n\sqrt{n}N \quad .$$

Since $P(\lambda)$ for a fixed λ can be viewed as the resource allocation problem with a separable objective function, it can be solved in $0(nN^2)$ time by applying the dynamic programming technique (see Chapter 3 of [10] for the details). Thus, by Lemmas 2.4 and 2.5, we have the following theorem.

Theorem 2.2 Problem P can be solved in $0(n^2\sqrt{n}N^3)$ time.

Notice that this running time is not polynomial in the input size but pseudopolynomial.

3. The Outline of an FPAS for P

We assume in this section that $P(\lambda)$ for any given $\lambda \geq 0$ can be solved in polynomial time. Based on this assumption, we shall develop an FPAS for P. Consider the following two problems MINIMAX and MAXIMIN associated with the original problem P. Let X be as defined in Section 2.

$$\text{MINIMAX:} \quad \underset{x \in X}{\text{minimize}} \max_{1 \leq j \leq n} f_j(x_j) \quad , \tag{3.1}$$

$$\text{MAXIMIN:} \quad \underset{x \in X}{\text{maximize}} \min_{1 \leq j \leq n} f_j(x_j) \quad . \tag{3.2}$$

Let $v_{MINIMAX}$ and $v_{MAXIMIN}$ denote the optimal objective values of MINIMAX and MAXIMIN respectively. Since all $f_j, j = 1,\dots,n$, are assumed to be nondecreasing or nondecreasing, problems MINIMAX and MAXIMIN can be reduced to problems of minimizing certain separable convex functions over X (see Chapter 5 of [10] for the reduction), and hence these problems can be solved in polynomial time. If we apply the Frederickson and Johnson algorithm [4] to solve MINIMAX and MAXIMIN, we have the following lemma.

Lemma 3.1 (Chapter 5 of [10]) $v_{MINIMAX}$ and $v_{MAXIMIN}$ can be computed in $0(\max\{n, n\log(N/n)\})$ time.

Lemma 3.2

$$v_{MAXIMIN} \leq v_{MINIMAX} \quad . \tag{3.3}$$

Now let us consider problem FAIR with $g(u,v) = u - v$. Let $d(x)$ denote the objective value of this problem for an $x \in X$, and let x° denote its optimal solution. Though [5] and [13] treated only the nondecreasing case of f_j, the nonincreasing case can be treated in the same manner, since replacing all f_j by $-f_j$ does not change the problem. Therefore, we have the following lemma.

Lemma 3.3 [5,13] x° can be computed in $0(\max\{n \log n, n\, log(N/n)\})$ time.

Lemma 3.4 For any $x \in X$, we have

$$\frac{2(n-1)}{n^3} \cdot \{d(x)\}^2 \leq var(x) \leq \frac{n-1}{2n} \cdot \{d(x)\}^2 \tag{3.4}$$

Proof. Assume without loss of generality that $f_1(x_1) \leq f_2(x_2) \leq \cdots \leq f_n(x_n)$ holds. First notice that

$$var(x) = \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (f_j(x_j) - f_i(x_i))^2 \tag{3.5}$$

holds. By $f_j(x_j) - f_i(x_i) \leq f_n(x_n) - f_1(x_1)$ for i,j with $1 \leq i < j \leq n$, the second inequality of (3.4) immediately follows. By the well known inequality $q \sum_{k=1}^{q} a_k^2 \geq (\sum_{k=1}^{q} a_k)^2$ for nonnegative numbers $a_1, a_2, \ldots, a_q$,

$$\frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (f_j(x_j) - f_i(x_i))^2 \geq \frac{2}{n^3(n-1)} \left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (f_j(x_j) - f_i(x_i)) \right]^2 \tag{3.6}$$

holds. Since

$$\begin{aligned}\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (f_j(x_j) - f_i(x_i)) &= (n-1)(f_n(x_n) - f_1(x_1)) + (n-3)(f_{n-1}(x_{n-1}) - f_2(x_2)) \\ &\quad + \cdots\cdots \\ &\geq (n-1)(f_n(x_n) - f_1(x_1)) \\ &= (n-1)d(x) \quad ,\end{aligned}$$

the first inequality of (3.4) follows from (3.5) and (3.6). □

Lemma 3.5

$$\frac{2(n-1)}{n^3}\cdot\{d((x^\circ)\}^2 \le var(x^*) \le \frac{n-1}{2n}\cdot\{d(x^\circ)\}^2 \quad . \tag{3.7}$$

Proof. Since $d(x^\circ) \le d(x^*)$ holds by the optimality of x° to FAIR with $g(u,v) = u - v$, the first inequality of (3.7) follows from the first inequality of (3.4). Since $var(x^*) \le var(x^\circ)$ holds by the optimality of x^* to P, the second inequality of (3.7) follows from the second inequality of (3.4). □

Lemma 3.6 For any optimal solution x^* of P, we have

$$\max_{1\le j\le n} f_j(x^*_j) \le v_{MAXIMIN} + \frac{n}{2}\cdot d(x^\circ) \quad , \tag{3.8}$$

$$\min_{1\le j\le n} f_j(x^*_j) \ge v_{MINIMAX} - \frac{n}{2}\cdot d(x^\circ) \quad . \tag{3.9}$$

Proof. Let

$$v^* = \max_{1\le j\le n} f_j(x^*_j) \text{ and } v_* = \min_{1\le j\le n} f_j(x^*_j) \quad . \tag{3.10}$$

By the minimality of $v_{MINIMAX}$ and the maximality of $v_{MAXIMIN}$,

$$v^* \ge v_{MINIMAX} \tag{3.11}$$

and

$$v_* \le v_{MAXIMIN} \tag{3.12}$$

follow. If (3.8) does not hold,

$$d(x^*) = v^* - v_* > \frac{n}{2}\cdot d(x^\circ) \tag{3.13}$$

follows from (3.8) and (3.12). By the first inequality of (3.4),

$$\frac{2(n-1)}{n^3}\cdot\{d(x^*)\}^2 \le var(x^*) \tag{3.14}$$

holds. Then it follows that

$$\begin{aligned} var(x^*) &\le \frac{n-1}{2n}\cdot\{d(x^\circ)\}^2 \quad \text{(by the second inequality of (3.7))} \\ &< \frac{n-1}{2n}\cdot\frac{4}{n^2}\cdot\{d(x^*)\}^2 \quad \text{(by (3.13))} \\ &= \frac{2(n-1)}{n^3}\cdot\{d(x^*)\}^2 \le var(x^*) \quad . \quad \text{(by (3.14))} \end{aligned}$$

This is a contradiction. Hence (3.8) is derived. (3.9) can be similarly proved. □

Lemma 3.7 For λ^* defined in (2.3),

$$\max\{2v_{MINIMAX} - n \cdot d(x^\circ), 0\} \le \lambda^* \le 2v_{MAXIMIN} + n \cdot d(x^\circ) \tag{3.15}$$

holds.

Proof. Immediate from (2.3), (3.8) and (3.9). □

Now we shall describe the outline of FPAS for P. First note that if $d(x^\circ) = 0$, it is obvious that $var(x^\circ) = 0$ and thus x° is optimal to P. By Lemma 3.3, P can be solved in polynomial time if $d(x^\circ) = 0$. Therefore assume $d(x^\circ) > 0$ in the following discussion.

Define

$$\delta \equiv \sqrt{\frac{8(n-1)\epsilon}{n^3}} \cdot d(x^\circ) \tag{3.16}$$

$$K \equiv \lceil (2v_{MAXIMIN} + n \cdot d(x^\circ) - \lambda_0)/\delta \rceil \quad , \tag{3.17}$$

$$\lambda_0 \equiv \max\{2v_{MINIMAX} - n \cdot d(x^\circ), 0\} \quad , \tag{3.18}$$

$$\lambda_K \equiv 2v_{MAXIMIN} + n \cdot d(x^\circ) \quad , \tag{3.19}$$

$$\lambda_k \equiv \lambda_0 + \frac{k(\lambda_K - \lambda_0)}{K} \quad , \quad k = 1, \ldots, K-1 \quad , \tag{3.20}$$

where $\lceil a \rceil$ denotes the smallest integer not less than a. Then solve $P(\lambda)$ for $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_K$. Among $K + 1$ solutions obtained, the one with minimum $var(x(\lambda_k))$ is output as an ϵ-approximate solution of P. This is proved as follows.

Lemma 3.8 Let $\lambda_0, \lambda_1, \ldots, \lambda_K$ be as defined above, and let λ_{k^*} satisfy

$$var(x(\lambda_{k^*})) = \min_{0 \le k \le K} var(x(\lambda_k)) \quad . \tag{3.21}$$

Then $x(\lambda_{k^*})$ is an ϵ-approximate solution of P.

Proof. By Lemma 3.7 and (3.16)-(3.20), there exists l with $0 \le l \le K$ such that

$$0 \le \lambda_l - \lambda^* \le \delta \tag{3.22}$$

holds. Since $var(x(\lambda_l)) \ge var(x(\lambda_{k^*}))$ holds by (3.21), it is sufficient to show that $x(\lambda_l)$ is an ϵ-approximate solution. Define δ' by

$$\delta' \equiv \lambda_l - \lambda^* (\le \delta) \quad . \tag{3.23}$$

For the sake of simplicity, let

$$\tilde{z}_1 = \sum_{j=1}^{n} \{f_j(x_j(\lambda_l))\}^2 \quad , \quad \tilde{z}_2 = \sum_{j=1}^{n} f_j(x_j(\lambda_l)) \quad ,$$

$$z^*_1 = \sum_{j=1}^{n} \{f_j(x^*_j)\}^2 \quad , \quad z^*_2 = \sum_{j=1}^{n} f_j(x^*_j) \quad .$$

Since $x(\lambda_l)$ is optimal to $P(\lambda_l)$, we have

$$z(x(\lambda_l)) = \tilde{z}_1 - \lambda_l \tilde{z}_2 \leq (z(x^*) =)z^*_1 - \lambda_l z^*_2 \quad . \tag{3.24}$$

It then follows that

$$var(x(\lambda_l)) = \frac{1}{n} \cdot \tilde{z}_1 - \frac{1}{n^2} \tilde{z}_2 \quad \text{(by (2.4))}$$

$$\leq \frac{1}{n} z^*_1 - \frac{\lambda^* + \delta'}{n} \cdot z^*_2 + \frac{\lambda^* + \delta'}{n} \cdot \tilde{z}_2 - \frac{1}{n^2} \tilde{z}_2 \quad \text{(by (3.23) and (3.24))}$$

$$= \frac{1}{n} \cdot z^*_1 - \frac{\lambda^* + \delta'}{n} \cdot z^*_2$$

$$- \frac{1}{n^2} (\tilde{z}_2 - \frac{n}{2} (\lambda^* + \delta'))^2 + \frac{1}{4} (\lambda^* + \delta')^2$$

$$\leq \frac{1}{n} \cdot z^*_1 - \frac{\lambda^* + \delta'}{n} \cdot z^*_2 + \frac{1}{4} (\lambda^* + \delta')^2$$

$$\leq \frac{1}{n} \cdot z^*_1 - \frac{2}{n^2} \cdot (z^*_2)^2 - \frac{\delta'}{n} \cdot z^*_2$$

$$+ \frac{1}{n^2} \cdot (z^*_2)^2 + \frac{\delta'}{n} \cdot z^*_2 + \frac{1}{4} (\delta')^2 \quad \text{(by substituting } \lambda^* = \frac{2 z^*_2}{n} \text{ from (2.3))}$$

$$= \frac{1}{n} \cdot z^*_1 - \frac{1}{n^2} \cdot (z^*_2)^2 + \frac{1}{4} (\delta')^2$$

$$= var(x^*) + \frac{1}{4} (\delta')^2 \quad \text{(by (2.4))}$$

$$\leq var(x^*) + \frac{1}{4} \delta^2 \quad . \quad \text{(by (3.23))} \tag{3.25}$$

Therefore

$$\frac{var(x(\lambda_l)) - var(x^*)}{var(x^*)} \leq \frac{\delta^2}{4 \cdot var(x^*)} \quad \text{(by (3.25))}$$

$$\leq \frac{n^3 \cdot \delta^2}{8(n-1) \cdot \{d((x^\circ)\}^2} \quad \text{(by the first inequality of (3.7))}$$

$$= \epsilon \ . \qquad \text{(by (3.16))}$$

This implies that $x(\lambda_l)$ is an ϵ-approximate solution. □

4. Description of FPAS for P

Based on the results given in the previous section, we shall describe an FPAS for P.

Procedure APPROX

Input: The minimum variance resource allocation problem P with n,N,f_j and $u_j,\ j = 1,2,...,n$.

Output: An ϵ-approximate solution of P.

Step 1: Solve MINIMAX and MAXIMIN with n,N,f_j and $u_j, j = 1,2,...,n$, and let $v_{MINIMAX}$ and $v_{MAXIMIN}$ be their optimum values, respectively. Solve FAIR with $g(u,v) = u - v, n, N$ and f_j and $u_j, j = 1,2,...,n$, and let x° and $d(x^\circ)$ be its optimal solution and optimum value, respectively.

Step 2: If $d(x^\circ) = 0$, then output x° as an optimal solution of P and halt. Else go to Step 3.

Step 3: Compute $\delta, \lambda_0, \lambda_1, ..., \lambda_K$ and K by (3.16)-(3.20).

Step 4: For each $k = 0,1,...,K$, compute $x(\lambda_k)$.

Step 5: Compute $x(\lambda_{k^*})$ determined by

$$var(x(\lambda_{k^*})) = \min_{0 \leq k \leq K} var(x(\lambda_k)) \quad .$$

and output $x(\lambda_{k^*})$ as an ϵ-approximate solution of P. Halt. □

Theorem 4.1 Procedure APPROX correctly computes an ϵ-approximate solution of P in

$$0(\tau(n,N)n^2/\sqrt{\epsilon} + \max\{n \log n, n \log (N/n)\}) \tag{4.1}$$

time, where $\tau(n,N)$ is the time required to compute an optimal solution $x(\lambda)$ of $P(\lambda)$.

Proof. The correctness follows from Lemma 3.8. The running time is analyzed as follows. Step 1 requires $0(max\{n \log n, n \log(N/n)\})$ time from Lemmas 3.1 and 3.3. Step 2 requires $0(n)$ time to output an n-dimensional vector x°.

Since

$$2v_{MAXIMIN} + n \cdot d(x^\circ) - \lambda_0$$

$$\leq 2v_{MAXIMIN} - 2v_{MINIMAX} + 2n \cdot d(x^\circ) \quad \text{(by (3.17))}$$

$$\leq 2n \cdot d(x^\circ), \quad \text{(by Lemma 3.2)}$$

$$K \leq 2n^2\sqrt{n}/\sqrt{8(n-1)\epsilon} = 0(n^2/\sqrt{\epsilon}) \tag{4.2}$$

follows. Thus , K is determined in $0(\log n - \log \epsilon)$ time by applying the binary search. By (4.2), $0(\tau(n,N) \cdot n^2/\sqrt{\epsilon})$ time is required in Step 4. Step 5 requires $0(n)$ time to output $x(\lambda_{k^*})$. The total time required by APPROX is therefore given by (4.1). □

Corollary 4.1 If $\tau(n,N)$ is polynomial in the input size of a problem instance $P(\lambda)$, procedure APPROX is an FPAS. □

5. The Case Where All f_j are Convex

We shall discuss the case in which all $f_j, j = 1,\ldots,n$, are convex. It should be mentioned that $\{f_j(x_j)\}^2 - \lambda f_j(x_j)$ may not be convex for some positive λ. Therefore, $P(\lambda)$ cannot, in general, be solved in polynomial time. Recall that all f_j are nondecreasing or all f_j are nonincreasing. First consider the case in which all f_j are nondecreasing. Let

$$\alpha \equiv \max_{1 \leq j \leq n} f_j(u_j) \quad , \tag{5.1}$$

and let for each j with $1 \leq j \leq n$

$$g_j(x_j) \equiv \alpha - f_j(x_j), \qquad x_j \in [0,u_j] \quad . \tag{5.2}$$

Notice that g_j is nonincreasing and nonnegative valued. Then apply procedure APPROX with all f_j replaced by g_j. We shall claim that this gives an ϵ-approximate solution of P and that its running time is polynomial in input size and $1/\epsilon$. Let P' denote P with all f_j replaced by g_j. It is easy to see from (1.6) that a solution is optimal to P if and only if it is optimal to P' and that the objective value of P for a solution x is equivalent to that of P' for x. This proves the first claim.

To prove the second claim, note that $g_j(x_j)$ is concave and nonnegative valued over $[0,u_j]$, and that $-g_j(x_j)$ is convex. With this observation it is easy to show that $\{g_j(x_j)\}^2 - \lambda g_j(x_j)$ is convex. By the convexity of $-g_j(x_j)$ and the nonnegativity of λ, it is sufficient to show that $\{g_j(x_j)\}^2$ is convex. For any y and y' with $0 \le y < y' \le u_j$, we have

$$\{g_j(y)\}^2 + \{g_j(y')\}^2 - 2\{g_j(\frac{y+y'}{2})\}^2$$

$$= (f_j(y) - \alpha)^2 + (f_j(y') - \alpha)^2 - 2(f_j(\frac{y+y'}{2}) - \alpha)^2$$

$$= (f_j(y') - \alpha - f_j(\frac{y+y'}{2}) + \alpha)(f_j(y') - \alpha + f_j(\frac{y+y'}{2}) - \alpha)$$

$$- (f_j(\frac{y+y'}{2}) - \alpha - f_j(y) + \alpha)(f_j(\frac{y+y'}{2}) - \alpha + f_j(y) - \alpha)$$

$$= (f_j(y) + f_j(y') - 2f_j(\frac{y+y'}{2}))(f_j(\frac{y+y'}{2}) - \alpha + f_j(y) - \alpha)$$

$$+ (f_j(y') - f_j(\frac{y+y'}{2}))(f_j(y') - f_j(y)) \quad . \tag{5.3}$$

By the convexity of f_j,$f_j(y) + f_j(y') - 2f_j(\frac{y+y'}{2}) \ge 0$ holds. By definition of α and the nondecreasingness of f_j, $f_j(\frac{y+y'}{2}) - \alpha + f_j(y) - \alpha \ge 0$ holds. Thus, the first term of (5.3) is nonnegative. Since f_j is nondecreasing, $f_j(y') - f_j(\frac{y+y'}{2}) \ge 0$ and $f_j(y') - f_j(y) \ge 0$ follow from $y' > y$. Hence, the second term of (5.3) is also nonnegative. This shows the convexity of $\{g_j(x_j)\}^2$. Thus, the second claim is proved.

The case in which all f_j,$j = 1,...,n$, are convex and nonincreasing can be similarly treated after replacing $f_j(x_j)$ by $h_j(x_j)$ defined as follows.

$$h_j(x_j) \equiv \beta - f_j(x_j), \quad x_j \in [0,u_j] \quad , \tag{5.4}$$

where

$$\beta \equiv \max_{1 \le j \le n} f_j(0) \tag{5.5}$$

An FPAS for the case where all f_j are convex is described as follows.

Procedure APPROXCONV

Input: The minimum variance resource allocation problem P with n,N,f_j and

$u_j, j = 1,2,\ldots,n$, where all f_j are convex.

Output: An ϵ-approximate solution of P.

Step 1: If all f_j are nondecreasing (resp. nondecreasing), replace $f_j(x_j)$ by $g_j(x_j)$ of (5.3) (resp. $h_j(x_j)$ of (5.4)), and call APPROX. Output x returned by APPROX as an ϵ-approximate solution of P. □

Theorem 5.1 Procedure APPROXCONV correctly computes an ϵ-approximate solution of P with convex $f_j, j = 1,\ldots,n$, in

$$0(\max\{n, n\log(N/n)\} \cdot n^2/\sqrt{\epsilon} + \max\{n\log n, n\log(N/n)\}) \qquad (5.6)$$

time.

Proof. The correctness is immediate from the discussion given prior to the description of APPROXCONV. Since $\{g_j(x_j)\}^2 - \lambda g_j(x_j)$ (resp. $\{h_j(x_j)\}^2 - \lambda h_j(x_j)$) is convex as shown above, $P(\lambda)$ with all f_j replaced by g_j (resp. h_j) can be solved in $0(\max\{n, n\log(N/n)\})$ time by applying the Frederickson and Johnson algorithm [4]. This and Theorem 4.1 prove that the running time of APPROXCONV is given by (5.6). □

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