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Contingent Cones to Reachable Sets of Control Systems

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Foreword

The author studies High order necessary conditions for optimality for an optimal control problem via properties of contingent cones to reachable sets along the optimal trajectory. It is shown that the adjoint vector of Pontriagin's maximum principle is normal to the set of variations of reachable sets. Results are applied to study optimal control problems for dynamical systems described by:

1) Closed loop control systems
2) Nonlinear implicit systems
3) Differential inclusions
4) Control systems with jumps.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences program

Contingent Cones to Reachable Sets of Control Systems

*Halina Frankowska**

1. Introduction

Consider the following optimal control problem in $\mathbf{R}^n$

$$\text{minimize } g(x(T)) \tag{1.1}$$

over the solutions to the control system

$$x'(t) = f(x(t), u(t)) \ \text{ a. e. in } [0, T] \tag{1.2}$$

$$u(t) \in U \text{ is a measurable selection} \tag{1.3}$$

$$x(0) \in C \tag{1.4}$$

Let $R(t,C)$ denote its reachable set at time t from the set of initial conditions $C \subset \mathbf{R}^n$ and $T_{R(t,C)}(x_0)$ the contingent cone to $R(t,C)$ at $x_0 \in \mathbf{R}^n$.

If a trajectory z of the control system (1.2) solves the above problem, then the derivative $g'(z(T))$ is non-negative in every tangent direction $w \in T_{R(T,C)}(z(T))$, i.e., $g'(z(T))$ belongs to the positive polar cone $T_{R(T,C)}(z(T))^+$ of $T_{R(T,C)}(z(T))$. This is the so-called Fermat rule. We thus obtain necessary conditions allowing to test whether a given trajectory z is optimal whenever we can characterize this positive polar cone. In this paper we study some necessary conditions which can be derived from the above Fermat rule. In the case of nonlinear system, the best we can hope is to characterize explicitly subsets Q of the tangent cone $T_{R(T,C)}(z(T))$, using variations of the solution $z(\cdot)$.

Then, by duality, $g'(z(T)) \in T_{R(T,C)}(z(T))^+ \subset Q^+$ and the inclusion $g'(z(T)) \in Q^+$ is a necessary condition of optimality. The larger is the set Q, the smaller is the set Q^+, so that necessary condition become stronger.

In particular, we prove that the reachable set at time T, $R^L(T)$, of the following linear control system

$$\begin{cases} w'(t) = & \frac{\partial f}{\partial x}(z(t), \bar{u}(t))\, w(t) + v(t) \text{ a.e.} \\ v(t) \in & T_{co\, f(z(t),U)}(z'(t)) \end{cases} \tag{1.5}$$

$$w(0) \in T_C(z(0))$$

(where $\bar{u}$ is a control corresponding to z) is contained in $T_{R(T,C)}(z(T))$. Hence whenever z is optimal, $g'(z(T)) \in R^L(T)^+$.

Such inclusion implies easily the celebrated Pontriagin's maximum principle: the solution q of the adjoint system

$$-q'(t) = \frac{\partial f}{\partial x}(z(t), \bar{u}(t))^* q(t) \text{ a.e. in } [0,T] \tag{1.6}$$

$$q(T) = g'(z(T)) \tag{1.7}$$

satisfies the minimum principle

$$< q(t), z'(t) > = \min_{u \in U} < q(t), f(z(t),u) > \quad \text{a.e. in } [0,T] \tag{1.8}$$

and the transversality condition

$$q(0) \in T_C(z(0))^+ \tag{1.9}$$

The aim of this paper is to go beyond the maximum principle and to provide some additional properties of the adjoint vector $q(\cdot)$ which can help to eliminate more candidates for optimality that the maximum principle does. Let us describe briefly the main ideas.

We introduce the "variations" $\{ W(t,z) : t \in [0,T]\}$ of $z(\cdot)$, defined by

$$W(t,z) := \{v : \exists\, h_i \to 0, h_i \geq 0, \mu_i \to 0+ \text{ such that} z(t+h_i) + \mu_i v \in R(t+h_i, C) + o(\mu_i)\}$$

(in particular $T_{R(t,C)}(z(t)) \subset W(t,z)$).

For all $0 \leq t \leq t+h \leq T$ define the reachable map $r(h,t) : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ of (1.5) by

$$r(h,t)\xi = \{w(t+h) : w \in W^{1,1}(t,t+h) \text{ is a solution of (1.5)}, w(t) = \xi\}$$

We shall prove that for all $t \in [0,T[$, $r(T-t,t)$ maps $W(t,z)$ into $T_{R(T,C)}(z(T))$ and, in particular,

$$r(T,0)T_C(z(0)) \subset T_{R(T,C)}(z(T))$$

Thus for all $t \in [0,T[, g'(z(T)) \in (r(T-t,t)\ W(t,z))^+$. If $r(T-t,t)$ was a linear operator, we would deduce from the bipolar theorem that $g'(z(T)) \in r(T-t,t)^{*-1}(W(t,z)^+)$, where $r(T-t,t)^*$ is the transpose of $r(T-t,t)$. But the reachable map $r(T-t,t)$ is not single-valued: it is positively homogeneous set-valued map (i.e. whose graph is a cone), which can also be transposed. We shall then prove two things: first that for all convex cone $Q \subset W(t,z)$

$$(r(T-t,t)Q)^+ = r(T-t,t)^{*-1}(Q^+) \tag{1.10}$$

and second that the transpose $r(T-t,t)^*$ can be computed in the following way

$$r(T-t,t)^*\pi = q(t) \tag{1.11}$$

where q is a solution to the system (1.6), (1.8) satisfying $q(T) = \pi$. By piecing together all these informations, we obtain the existence of a solution q of (1.6)-(1.9) satisfying

$$q(T) \in T_{R(T,C)}(z(T))^+ \tag{1.12}$$

$$q(t) \in W(t,z)^+ \text{ for all } t \in [0,T[\tag{1.13}$$

It also implies the following invariance property of reachable sets:

$$\text{If } T_{R(T,C)}(z(T))^+ \neq \{0\} \text{ then for all } t \in [0,T],\ T_{R(t,C)}(z(t))^+ \neq \{0\} \tag{1.14}$$

This result is of the same nature that a theorem of Ważewski saying that the boundary point of reachable set can be reached by only a boundary trajectory.

The inclusions (1.12)-(1.13) are an additional information described via *reachable sets*. For nonlinear systems the reachable sets and, consequently, the set of variations $W(t,z)$ are nor a priori known. But condition (1.13) still allows to eliminate some candidates for optimality among those satisfying the maximum principle. Let us emphasize that it is enough to know *one* element $w \in W(t,z)$ such that the solution q of (1.6), (1.7) satisfies $< q(t), w > <0$ to deduce that z is not optimal.

Inclusion (1.13) can also be seen as a *higher order* optimality condition since it deals with variations of $z(\cdot)$ of all orders. High order necessary conditions involving higher order derivatives of g are (of course) of an entirely different nature.

The high order necessary conditions in optimization have two features:

1) Necessary conditions involving the high order variations of constraints

2) Calculus of high order variations.

We shall not divide here any calculus of sets $W(t,z)$. The interested reader can find in [19] many examples of variation corresponding to piecewise C^{∞}-controls. They are constructed via Lie brackets of some vector fields. However, because of the Lavrentieff phenomenon, one should not expect such regularity from optimal trajectories. Still the results of [19] can be used at *regular* enough points of optimal control. The irregular points are much more difficult to address and require further investigations.

We shall study a more general dynamical system that the parametrized control system (1.2), (1.3), the so-called differential inclusion

$$x' \in F(x) \tag{1.15}$$

This is a generalized differential equation and the control system (1.2), (1.3) can be reduced to it by setting $F(x) = f(x,U)$. When f is continuous, the Filippov theorem (see [1, p.91] says that the solutions of (1.15) and (1.2), (1.3) do coincide.

In general the set-valued map F cannot be parametrized in a way to reduce the system (1.15) to (1.2), (1.3). The main reason for it being the restriction on admissible controls (1.3). Still this can be done when F has *convex* compact images and is continuous in the Hausdorff metric. But even in this case the parametrization would be only continuous and therefore not very useful because of the lack of differentiability of f.

The differential inclusions beside to be a description of more general dynamical systems provide a mathematical tool to carry the study of nonsmooth control systems, closed loop control systems:

$$x' = f(x,u), \quad u \in U(x) \tag{1.16}$$

and implicit dynamical systems

$$f(x,x') = 0 \tag{1.17}$$

We refer to [1], [9], [22], [6], and bibliographies contained therein for the corresponding examples of systems whose models are described by (1.16), (1.17).

Setting $F(x) = \bigcup_{u \in U(x)} f(x,u)$ and $F(x) = \{v : f(x,v) = 0\}$ we reduce (1.16) and (1.17) respectively to the differential inclusion (1.15).

Recall that the dynamical system (1.17) appears in the Lagrange problem (see [28]). In [28] two ways to treat (1.17) are described. One is an unjustified multiplier rule. The second is (again) an unjustified assumption that (1.17) can be rewritten as a control system (1.2), (1.3). In this paper we treat (1.17) via differential inclusion techniques.

Properties of the dynamical system given by (1.15) depend on the graph of the set-valued map F.

Actually the generalized differential equation (1.15) inherits many properties of ODE (see [1]). The one we exploit the most here is the variational inclusion, which is as useful as variational equation arising in ODE. It was extended to variational inclusions in [13], [12] and independently in [23]. Many results concerning inclusions can be found in [1], [9]-[16], [18], [23] (see also bibliographies contained therein).

The maximum principle for differential inclusions was proved in [9], [10], [12], [18], [23]. It involves either graphical derivatives of the set-valued map F ([12], [23]), or generalized Jacobians of selections from F [18], or the generalized gradient of Hamiltonian

$$H(x,p) = sup\{<p,e>: e \in F(x)\}$$

([9], [10]).

We prefer the "graphical" approach mainly for two reasons:

1. In general, even for smooth control systems, H is merely Lipschitz. Hence one is led to differentiate H in one or another generalized way. There is no yet any convenient notion of higher order generalized derivatives of H adequate for our purposes. Neither is it clear how one can solve the nonsmooth Hamiltonian inclusions. We rather deal with convex subcones of tangent cones to graph (F) and the associated convex processes. Convex process is a set-valued analogue of linear operator (see [25], [2]). In particular the Kalman rank condition can be extended to convex processes [3].

2. In the examples of applications we provide here, the Hamiltonian maximum principle is less powerfull than that involving the adjoint system (see Section 4, Remark 4.8 for a detailed discussion).

Tangent vectors to reachable sets are studied via local variations in Section 2. In Section 3 we investigate the adjoint of the reachable process, $r(T-t,t)^*$. The cone $T_{R(T,C)}(z(T))^+$ is studied in Section 4. Section 5 is devoted to necessary conditions for problem (1.1) for the (usual) control system (1.2), (1.3), the closed loop control system (1.16) and implicit dynamical system (1.17). In Section 6 we sketch how the same approach can be used to study control systems with jumps (deterministic impulse control systems). Examples are provided in Section 7.

We do not present here a thorough study of high order variation. Many results concerning smooth cases can be found in [19]. In the more general framework (1.15) one deals with the extended notion of Lie bracket for set-valued map. A second order result can be found in [14]. However the higher order variations require a further investigation.

2. Tangent Vectors to Reachable Sets

One of the main tools we use here is the following result due to Filippov [11].

Theorem (Filippov). Let $y:[a,b] \to \mathbf{R}^n$ be an absolutely continuous function and $G:[a,b] \times \mathbf{R}^n \to \mathbf{R}^n$ be a set-valued map with closed images such that

(i) for all $x \in \mathbf{R}^n$, the map $t \to G(t,x)$ is measurable

(ii) for some $\epsilon > 0, k \in L^1(a,b)$ and all t, $G(t,\cdot)$ has nonempty images and is $k(t)-$ Lipschitz on $y(t) + \epsilon B$.

Set $K = \exp(\int_a^b k(t)dt)$, $\rho := \int_a^b \text{dist}(y'(t), G(t,y(t)))dt$. If $\rho < \frac{\epsilon}{K}$, then there exists an absolutely continuous function $x:[a,b] \to \mathbf{R}^n$ satisfying $x(a) = y(a)$,

$$x'(s) \in G(s,x(s)) \text{ a.e. in}[a,b]$$

$\|x-y\|_{C(a,b)} \leq K\rho$ and for almost all $t \in [a,b]$

$$\|x'(t) - y'(t)\| \leq k(t)\rho \exp(\int_a^t k(s)ds) + \text{dist}(y'(t), G(t,y(t)))$$

Remark: The proof can be found in [1] under an additional assumption that G is continuous in t. In [9, p.115] the above theorem is stated in a weaker form but the proof allows to deduce the above stronger version. We provide a sketch of such deduction. The function x is constructed as the limit of a Cauchy sequence $x_i \in C(a,b;\mathbf{R}^n)$ $i = 0,1,...$ of absolutely continuous functions satisfying $x_i(a) = y(a)$ and for almost all $t \in [a,b]$ and all $i \geq 1$

$$\|x'_{i+1}(t) - x'_i(t)\| \leq k(t)\|x_i(t) - x_{i-1}(t)\| \leq k(t)\rho\frac{(\int_a^t k(s)ds)^{i-2}}{(i-2)!}$$

$$\|x'_1(t) - y'(t)\| = \text{dist}(y'(t), G(t,y(t)))$$

Hence for almost all $t \in [a,b]$ also the sequence $\{x'_i(t)\}$ is Cauchy. This and Lebesgue's dominated convergence theorem yield: the existence of $x \in C(a,b)$ such that for all $t \in [a,b]$

$$x(t) = x(a) + \int_a^t \lim_{i\to\infty} x'_i(s)ds$$

Hence x is absolutely continuous and we finally obtain that

$x_i'(s) \to x'(s)$ a.e. in $[a,b]$

Moreover for almost all $t \in [a,b]$

$$\|x_{i+1}'(t) - y'(t)\| \leq \sum_{j=1}^{i} \|x_{j+1}'(t) - x_j'(t)\| + \|x_1'(t) - y'(t)\| \leq$$

$$k(t)\rho \sum_{j=0}^{i} \left(\int_a^t k(s)ds\right)^j / j! + \|x_1'(t) - y'(t)\| \leq k(t)\rho \exp\left(\int_a^t k(s)ds\right) + \text{dist}(y'(t), G(t,y(t)))$$

Taking the limit we obtain that for almost all $t \in [a,b]$

$$\|x'(t) - y'(t)\| \leq k(t)\rho \exp\left(\int_a^t k(s)ds\right) + \text{dist}(y'(t), G(t,y(t))) \quad .$$

Consider a set-valued map F from $\mathbf{R}^n$ to $\mathbf{R}^n$ and a differential inclusion

$$x' \in F(x) \tag{2.1}$$

A function $x \in W^{1,1}(0,T)$, $T > 0$ (the Sobolev space) is called a trajectory of (2.1) if for almost all $t \in [0,T], x'(t) \in F(x(t))$. We denote by S_t the set of all trajectories of (2.1) defined on the time interval $[0,t]$. The reachable set of the inclusion (2.1) from a point $\xi \in \mathbf{R}^n$ at time $t \geq 0$ is given by

$$R(t,\xi) = \{x(t) : x \in S_t, x(0) = \xi\} \quad .$$

We observe that the reachable sets enjoy the semigroup property:

$$\begin{cases} R(t+h,\xi) = R(t,R(h,\xi)) & \text{for all } t,\ h \geq 0 \\ R(0,\xi) = \xi \end{cases} \tag{2.2}$$

Let $z \in S_T$ be a given trajectory. We study in this section tangent vectors to reachable set $R(T,C)$ at $z(T)$. We call a set $Q \subset \mathbf{R}^n$ a cone if for all $\lambda \geq 0$, $\lambda Q \subset Q$. Recall first

Definition 2.1. Let K be a subset of $\mathbf{R}^n$ and $x \in K$. The (Bouligand) contingent cone to K at x is given by

$$T_K(x) = \{v \in \mathbf{R}^n : \exists h_i \to 0+, v_i \to v \text{ such that } x + h_i v_i \in K\}$$

The intermediate tangent cone to K at x is defined by

$$I_K(x) = \{v \in \mathbf{R}^n : \forall h_i \to 0+ \exists v_i \to v \text{ such that } x + h_i v_i \in K\}$$

We refer to [2], [12] for properties of $T_K(x), I_K(x)$. Throughout the whole paper we assume that the set-valued map F in the right-hand side of the differential inclusion (2.1) satisfies the following assumption

$$(H_1)\begin{cases} \text{Dom } F := \{x : F(x) \neq \emptyset\} \text{ is open} \\ F \text{ has compact images and is Lipschitzian on Dom } F \end{cases} .$$

Definition 2.2. Let $F: \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ be a set-valued map locally Lipschitzian at x and $y \in F(x)$. The derivative of F at (x,y) is the set-valued map $dF(x,y): \mathbf{R}^n \rightrightarrows R^n$ given by: for all $u \in \mathbf{R}^n$

$$v \in dF(x,y)u \Longleftrightarrow \lim_{h \to 0+} \text{dist}(v, \frac{F(x+hu)-y}{h}) = 0 \quad .$$

Observe that graph $dF(x,y) := \{(u,v) : v \in dF(x,y)u\}$ is a closed cone equal to the intermediate tangent cone to graph (F) at (x,y). We refer to [12]-[14] for some properties and applications of the set-valued derivative.

We denote by $co\, F$ the convexified set-valued map, i.e. for all $x \in \mathbf{R}^n$, $co\, F(x)$ is the convex hull of $F(x)$.

Consider the "linearized inclusion"

$$w'(s) \in d\, co\, F(z(s), z'(s))\, w(s) \qquad \text{a.e.} \tag{2.3}$$

For all $h,t \geq 0, \xi \in \mathbf{R}^n$ define the reachable set $r(h,t)\xi$ of (2.3) by

$$r(h,t)\xi = \{w(t+h) : w \in W^{1,1}(t,t+h) \text{ satisfies (2.3)}, \, w(t) = \xi\}$$

Definition 2.3. Let $t \in [0,T[$. Set

$$W(t,z) = \{v : \exists h_i \geq 0, \mu_i \to 0+ \text{ such that } \lim_{i\to\infty} h_i = 0, z(t+h_i) + \mu_i v \in R(t+h_i,C) + o(\mu_i)B\}$$

$$\mathcal{W}(t,z) = \{v : \forall \mu_i \to 0+ \; \exists h_i \to 0, \, h_i \geq 0 \text{ such that } z(t+h_i) + \mu_i v \in R(t+h_i,C) + o(\mu_i)B\}$$

Observe that $W(t,z)$ and $\mathcal{W}(t,z)$ are closed cones. Moreover for all $t \in [0,T[$

$$T_{R(t,C)}(z(t)) \subset W(t,z), \, I_{R(t,C)}(z(t)) \subset \mathcal{W}(t,z) \subset W(t,z) \tag{2.4}$$

and, in particular, $T_C(z(0)) \subset W(0,z)$.

Remark. When for some integer $k \geq 1, \mu_i = h_i^k$, then the vector v can be seen as the $k-th$ order variation of $R(\cdot)$ at $(t,z(t))$.

Actually, variations of $R(\cdot,C)$ at (t,z) are mapped by $r(T-t,t)$ into the tangent vectors to $R(T,C)$.

Theorem 2.4. Assume that (H_1) is verified and let $t \in [0,T[$. Then for all $t < \tau \leq T$

$$r(\tau-t,t)\ W(t,z) \subset T_{R(\tau,C)}(z(\tau)), r(\tau-t,t)\mathcal{W}(t,z) \subset I_{R(\tau,C)}(z(\tau))$$

$$r(T-t,t)T_{R(t,C)}(z(t)) \subset T_{R(T,C)}(z(T))$$

To prove the above theorem, we need a consequence of the Filippov-Ważewski relaxation result (see [1], p. 124):

Consider the convexified inclusion

$$\begin{cases} x'(s) \in co\, F(x(s)) & \text{a.e.} \\ x(0) \in C \end{cases} \tag{2.5}$$

Proposition 2.5. Assume that (H_1) holds true. Then for all $t \in [0,T]$ the contingent (respectively intermediate) cones to the reachable sets of (2.1) and (2.5) at time t taken at the point $z(t)$ do coincide.

Proof (of Theorem 2.4). By Proposition 2.5, we may assume that F has convex images. Fix a solution w of (2.3) and let $h_i \geq 0, \mu_i \to 0+, v_i \to v = w(t)$ be such that $\lim_{i\to\infty} h_i = 0$, $z(t+h_i) + \mu_i v_i \in R(t+h_i,C)$. For all $s \in [t+h_i,\tau]$ set

$$y_i(s) = z(s) + \mu_i(v_i + \int_{t+h_i}^{s} w'(p)\,dp)$$

and let $L \geq 1$ denote the Lipschitz constant of F. Then for almost all $s \in [t+h_i,\tau]$ and all large i

$$\begin{cases} \text{dist}\,(y_i'(s), F(y_i(s))) \leq \text{dist}(z'(s) + \mu_i w'(s), F(z(s) + \mu_i w(s))) \\ + L\mu_i(\|v_i - v\| + \int_t^{t+h_i} \|w'(p)\|dp) \leq L\mu_i(\|w'(s)\| + \|w(s)\| + \|v_i - v\| + \int_t^{t+h_i} \|w'(p)\|dp) \end{cases} \tag{2.6}$$

Moreover,

$$\lim_{i\to\infty} (\|v_i - v\| + \int_t^{t+h_i} \|w'(p)\|dp) = 0$$

and, by definition of dF, for almost all $s \in [t,\tau]$

$$\lim_{i\to\infty} \text{dist}\,(z'(s) + \mu_i w'(s), F(z(s) + \mu_i w(s)))/\mu_i = 0$$

Thus, by the Lebesgue dominated convergence theorem and (2.6)

$$\lim_{i\to\infty} \int_{t+h_i}^{\tau} \text{dist}\,(y_i'(s), F(y_i(s)))\,ds/\mu_i = 0$$

From the Filippov theorem there exist

$$r_i \in R(\tau - t - h_i, z(t+h_i) + \mu_i v_i) \subset R(\tau, C)$$

such that $\|r_i - y_i(\tau)\| = o(\mu_i)$.

Since

$$\lim_{i\to\infty} (y_i(\tau) - z(\tau))/\mu_i = \lim_{i\to\infty} (v_i + \int_{t+h_i}^{\tau} w'(p)\,dp) = w(\tau)$$

we end the proof.

Theorem 2.6. Assume that (H_1) is verified and let $0 \le t \le \tau \le T$. Then the set

$$\{(w(t), w(\tau)) : w(t) \in T_{R(t,C)}(z(t)),\, w \in W^{1,1}(t,\tau) \text{ is a trajectory of (2.3)}\}$$

is contained in

$$T_{\{(x,y)\,:\, x \in R(t,C),\, y \in R(\tau - t, x)\}}(z(t), z(\tau)) \quad .$$

Proof. By the proof of Theorem 2.4 in the case when $h_i = 0$ for all $i \ge 1$, we know that there exist $v_i \to v$, $r_i \in R(\tau - t, z(t) + \mu_i v_i)$ such that $z(t) + \mu_i v_i \in R(t,C)$ and $\|r_i - z(\tau) - \mu_i(v_i + \int_t^{\tau} w'(p)\,dp)\| = o(\mu_i)$. Hence

$$\lim_{i\to\infty} (z(t) + \mu_i v_i - z(t), r_i - z(\tau))/\mu_i = (v, v + \int_t^{\tau} w'(p)\,dp) = (w(t), w(\tau)).$$

It was shown in [16] that under the hypothesis (H_1) the reachable map R has the following (first order) expansion: for all ξ near $z(t)$ and all small $h > 0$

$$R(h,\xi) = \xi + h\, co\, F(z(t)) + o(t,h) \tag{2.7}$$

where

$$\lim_{h\to 0+,\, \xi\to z(t)} \|o(t,h)\|/h = 0$$

and the equality in (2.7) has to be understood in the following way:

$$R(h,\xi) \subset \xi + h\, co\, F(z(t)) + o(t,h)B$$

$$\xi + h\, co\, F(z(t)) \subset R(h,\xi) + o(t,h)B$$

On the other hand, the function $z(\cdot)$ being absolutely continuous, for almost all $t \in [0,T]$ and all $h > 0$ we can write $z(t+h) = z(t) + hz'(t) + o(h)$. Applying (2.7) with $\xi = z(t)$ and using Definition 2.3 we obtain

$$co\, F(z(t)) - z'(t) \subset W(t,z) \text{ a.e. in } [0,T] \tag{2.8}$$

We have even a stronger result which we shall use in Theorem 2.9.

Theorem 2.7. Assume that (H_1) holds true. Then $\mathcal{W}(t,z) + T_{R(t,C)}(z(t)) \subset W(t,z)$, $\mathcal{W}(t,z) + I_{R(t,C)}(z(t)) = \mathcal{W}(t,z)$.

Proof. Fix $w \in \mathcal{W}(t,z), v \in T_{R(t,C)}(z(t))$ and let $\mu_i \to 0+, v_i \to v$ be such that $z(t) + \mu_i v_i \in R(t,C)$. Fix $h_i \to 0+$, $w_i \to w$, $y_i \in S_{t+h_i}$ such that $z(t+h_i) + \mu_i w_i \in R(h_i, z(t))$, $y_i(t) = z(t), y_i(t+h_i) = z(t+h_i) + \mu_i w_i$. Set $\bar{y}_i = y_i + \mu_i v_i$. Then dist $(\bar{y}_i'(s), F(\bar{y}_i(s))) \leq$ dist $(y_i'(s), F(y_i(s))) + L\mu_i\|v_i\| = L\mu_i\|v_i\|$, where L denotes the Lipschitz constant of F. This and Filippov's theorem imply the existence of $x_i \in S_{t+h_i}$ such that $x_i(t) = \bar{y}_i(t) = z(t) + \mu_i v_i \in R(t,C)$,

$$x_i(t+h_i) = \bar{y}_i(t+h_i) + o(\mu_i) = z(t+h_i) + \mu_i(w_i+v_i) + o(\mu_i) \in R(h_i,x_i(t)) \subset R(h_i, R(t,C))$$

Hence, from (2.2),

$$z(t+h_i) + \mu_i(w_i+v_i) \subset R(t+h_i,C) + o(\mu_i)$$

Definition 2.3 ends the proof of the first statement. The proof of the second one is analogous. We omit it.

In Section 4 we study "normal" cones to reachable sets along the trajectory z via a duality technique applied to convex subcones of the set $W(t,z)$. We introduce next an example of such subcone.

Definition 2.8. Let $t \in [0,T]$. A vector $v \in \mathbf{R}^n$ is called a smooth variation of order $k > 0$ at (t,z) if

$$\lim_{\substack{h\to 0+ \\ t'\to t+}} \text{dist}\left(v, \frac{R(h,z(t')) - z(t'+h)}{h^k}\right) = 0$$

The set of all variations of order k is denoted by $R^k(t,z)$. The closed cone spanned by all variations is called the expansion cone of the reachable map at (t,z) and is denoted by $R^{\infty}(t,z)$:

$$R^{\infty}(t,z) = cl \bigcup_{\substack{\lambda \geq 0 \\ k>0}} \lambda R^k(t,z)$$

The expansion cone at a stationary trajectory was introduced in [14] to study the problem of local controllability at a point of equilibrium. Clearly, whenever $v \in R^k(t,z)$ then for all $\mu_i \to 0+$ there exist $h_i \to 0+$ such that $z(t+h_i) + \mu_i v \in R(h_i, z(t)) + o(\mu_i)$. Hence Lemma 2.7 yields $T_{R(t,C)}(z(t)) + R^k(t,z) \subset W(t,z)$. Moreover

$$\begin{aligned} I_{R(t,C)}(z(t)) + R^{\infty}(t,z) &\subset \mathcal{W}(t,z) \\ \mathcal{T}_{R(t,C)}(z(t)) + R^{\infty}(t,z) &\subset W(t,z) \end{aligned} \tag{2.9}$$

Theorem 2.9. Assume that (H_1) holds true. Then $R^{\infty}(t,z)$ is a closed convex subcone of the cone of variations $W(t,z)$ satisfying (2.9).

This result is an immediate consequence of the closedness of $W(t,z)$ and

Lemma 2.10. If (H_1) holds true then

i) For all $K > k$, $0 \in R^k(t,z) \subset R^K(t,z)$

ii) For all $k > 0, (n+1)^{-k} co\, R^k(t,z) \subset R^k(t,z)$.

Proof. Clearly for all $k > 0$

$$0 \in R^k(t,z) \tag{2.10}$$

Fix $K > k > 0$ and observe that for all $v \in \mathbf{R}^n$, $t' \in [0,T[, h \in]0,1[$ we have $h^{K/k} < h$ and

$$\text{dist}\left[v, \frac{R(h,z(t'))-z(t'+h)}{h^K}\right] \leq \text{dist}\left[v, \frac{R(h^{K/k},z(t'+h-h^{K/k}))-z(t'+h-h^{K/k}+h^{K/k})}{(h^{K/k})^k}\right] .$$

This and Definition 2.8 imply i). To prove ii) fix $k > 0$, $\lambda_i \geq 0, v_i \in R^k(t,z), i = 0,...,m$ satisfying $\sum_{i=0}^{m} \lambda_i = 1$. We claim that

$$\sum_{i=0}^{m} \lambda_i^k v_i \in R^k(t,z) \quad . \tag{2.11}$$

Indeed consider $t_j \to t+, h_j \to 0+$. Then

$$z(t_j+\lambda_0 h_j)+h_j^k\lambda_0^k v_0 \in R(\lambda_0 h_j, z(t_j))+o(h_j^k)B \quad ,$$

where $\lim\limits_{j\to\infty} o(h_j^k)/h_j^k=0$. We proceed by the induction. Assume that we already proved that for some $0\le s<n$ and all j

$$z(t_j+h_j\sum_{i=0}^{s}\lambda_i) + h_j^k\sum_{i=0}^{s}\lambda_i^k v_i \in R(h_j\sum_{i=0}^{s}\lambda_i, z(t_j)) + o(h_j^k)B \tag{2.12}$$

with $\lim\limits_{j\to\infty} o(h_j^k)/h_j^k=0$. By Definition 2.8 applied with $t'=t_j+h_j\sum\limits_{i=0}^{s}\lambda_i$, $h=\lambda_{s+1}h_j$

$$z(t'+\lambda_{s+1}h_j) + h_j^k\lambda_{s+1}^k v_{s+1} \in R(h_j\lambda_{s+1},z(t')) + o(h_j^k)B.$$

This and the Filippov theorem yield

$$z(t'+\lambda_{s+1}h_j) + h_j^k\sum_{i=0}^{s+1}\lambda_i^k v_i \in R(h_j\lambda_{s+1},z(t') + h_j^k\sum_{i=0}^{s}\lambda_i^k v_i) + o(h_j^k)B \subset (\text{by } (2.12))$$

$$R(h_j\lambda_{s+1},R(h_j\sum_{i=0}^{s}\lambda_i,z(t_j))) + o(h_j^k)B = R(h_j\sum_{i=0}^{s+1}\lambda_i,z(t_j)) \ + o(h_j^k)B.$$

Hence (2.12) is valid also with s replaced by $s+1$. Applying (2.12) with $s=m$ we obtain that

$$\lim_{j\to\infty} \operatorname{dist}\left[\sum_{i=0}^{m}\lambda_i^k v_i, \frac{R(h_j,z(t_j))-z(t_j+h_j)}{h_j^k}\right]=0$$

and since $\{t_j\}$ and $\{h_j\}$ are arbitrary, Definition 2.8 implies (2.11). On the other hand, by the Carathéodory Theorem for all $v\in coR^k(t,z)$ there exist $\mu_i\ge 0, v_i\in R^k(t,z)$ such that $\sum\limits_{i=0}^{n}\mu_i=1$ and $\sum\limits_{i=0}^{n}\mu_i v_i=v$. Observe that $\sum\limits_{i=0}^{n}\sqrt[k]{\mu_i}/(n+1)\le 1$. Applying (2.11) with

$$\lambda_i=\sqrt[k]{\mu_i}/(n+1),\, v_{n+1}=0, \lambda_{n+1}=1-\sum_{i=0}^{n}\sqrt[k]{\mu_i}/(n+1)$$

we obtain that $(n+1)^{-k}v=\sum\limits_{i=0}^{n+1}\lambda_i^k v_i\in R^k(t,z)$. This proves ii).

3. The Adjoint Process $r(T-t,t)^*$

Recall that for a subset K of a Banach space E, its positive polar cone is given by

$$K^+ = \{p \in E^* : \forall u \in K, <p,u> \geq 0\}$$

We also recall

Definition 3.1. A set-valued $G : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is called a (closed) convex process if graph (G) is a closed convex cone.

We refer to Rockafellar [25] who introduced and studied this notion and to Aubin-Ekeland [2] for further properties.

Definition 3.2. Let $G : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ be a set-valued map. The adjoint map $G^* : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is given by $p \in G^*(q)$ if and only if for all $(x,y) \in \text{graph}(G)$, $<p,x> \leq <q,y>$. In other words $p \in G^*(q) \Longleftrightarrow (-p,q) \in \text{graph } (G)^+$.

Observe that the adjoint G^* is a closed convex process.

Let $\{A(s) : s \in [0,T]\}$ be a given family of closed convex processes from $\mathbf{R}^n$ to $\mathbf{R}^n$ satisfying

$$(H_2)\begin{cases} \text{i)} & \text{For all } w \in \mathbf{R}^n \text{ the map } s \to A(s)w \text{ is measurable} \\ \text{ii)} & \text{For all } s \in [0,T], \text{ the map } w \to A(s)w \text{ is } k(s) \text{-Lipschitzian, where } k \in L^\infty(0,T) \end{cases}$$

For all $0 \leq t \leq \tau \leq T$, we investigate the adjoint $r(\tau - t, t)^*$ by studying the inclusions

$$w'(s) \in A(s)w(s) \qquad \text{a.e.} \tag{3.1}$$

and

$$-q'(s) \in A(s)^* q(s) \qquad \text{a.e.} \tag{3.2}$$

in the case when

$$(H_3) \quad \text{graph } (A(s)) \subset \text{graph } (dco\, F(z(s), z'(s))) \text{ a.e. in } [0,T]$$

For a subset $Q \subset \mathbf{R}^n$ we denote by $r_Q(\tau - t, t)$ the restriction of r to Q, i.e.

$$r_Q(\tau - t, t)x = \begin{cases} r(\tau - t, t)x \text{ when } & x \in Q \\ \emptyset & \text{otherwise} \end{cases}$$

The main result of this section is

Theorem 3.3 If a family $\{A(s) : s \in [0,T]\}$ of closed convex processes from $\mathbf{R}^n$ to $\mathbf{R}^n$

satisfies (H_2) and (H_3) then for all $b \in \mathbf{R}^n$, convex cone $Q \subset \mathbf{R}^n$ and $0 \le t \le \tau \le T$

a) $r(\tau - t, t)^* b \subset \{q(t) : q \in W^{1,\infty}(t,\tau)$ satisfies (3.2), $q(\tau) = b\}$

b) $r_Q(\tau - t, t)^* b \subset \{q(t) : q \in W^{1,\infty}(t,\tau)$ satisfies (3.2), $q(\tau) = b\} - Q^+$

c) $(r(\tau - t, t)Q)^+ \subset \{q(\tau) : q \in W^{1,\infty}(t,\tau)$ satisfies (3.2), $q(t) \in Q^+\}$

To prove the above theorem we associate with all $0 \le t \le \tau \le T$ the convex process $\hat{r}(\tau - t, t) : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ defined by : for all $\xi \in \mathbf{R}^n$

$$\hat{r}(\tau-t,t)\xi = \{w(\tau) : w \text{ satisfies (3.1) on}[t,\tau],\ w(t) = \xi\} \tag{3.3}$$

Therefore, by the definition of the adjoint map, for all $b \in R^n$

$$r(\tau-t,t)^*\, b \subset \hat{r}(\tau - t,t)^*\, b \tag{3.4}$$

$$r_Q(\tau-t,t)^*\, b \subset \hat{r}_Q(\tau-t,t)^*\, b \tag{3.5}$$

$$(r(\tau-t,t)Q)^+ \subset (\hat{r}(\tau-t,t)Q)^+ \quad . \tag{3.6}$$

Theorem 3.3 follows from the above inclusions and the following two lemmas.

Lemma 3.4. If (H_2) holds true then for all $0 \le t \le \tau \le T$ and $b \in \mathbf{R}^n$

$$\hat{r}(\tau-t,t)^* b = \{q(t) : q \in W^{1,\infty}(t,\tau) \text{ satisfies (3.2)},\ q(\tau) = b\} \quad . \tag{3.7}$$

Lemma 3.5. If (H_2) holds true then for all convex cone $Q \subset \mathbf{R}^n$ and $b \in \mathrm{Dom}\,\hat{r}(\tau-t,t)^*$

$$\hat{r}_Q(\tau-t,t)^* b = \hat{r}(\tau-t,t)^* b - Q^+ \tag{3.8}$$

and

$$(\hat{r}(\tau-t,t)Q)^+ = \hat{r}(\tau-t,t)^{*-1}(Q^+)$$

Proof of Lemma 3.4. Fix $0 \le t \le \tau \le T$. Let us set

$X = W^{1,2}(t,\tau),\ Y = L^2(t,\tau) \times L^2(t,\tau)$

$L = \{(x,y) \in Y : y(s) \in A(s)x(s)$ a.e. in $[t,\tau]\}$

D, the differential operator on X, $Dx = x'$

γ, the trace operator on X, $\gamma(x) = (x(t),x(\tau))$.

Observe that L is a closed convex cone and, by the measurable selection theorem (see [26]),

$$L^+ = \{(-p,q) \in Y^* : p(s) \in A(s)^* q(s) \text{ a.e. in } [t,\tau]\} \quad . \tag{3.9}$$

We claim that

$$\mathrm{Im}(1 \times D) - L = Y \tag{3.10}$$

To prove it we have to verify that for all $(u,v) \in Y$ there exists $x \in X$ satisfying

$$x'(s) \in A(s)(x(s) - u(s)) + v(s) \text{ a.e. in } [t,\tau] \quad . \tag{3.11}$$

Fix $(u,v) \in Y$ and observe that, by (H_2), the set-valued map $[t,\tau] \times \mathbf{R}^n \ni (s,x) \to A(s)(x-u(s)) + v(s)$ is measurable in s and for almost all s it is Lipschitzian in x with the Lipschitz constant $k(s)$. Moreover $\mathrm{dist}(0, A(s)(-u(s)) + v(s)) \le k(s)\|u(s)\| + \|v(s)\|$. By the Filippov theorem there exist $M \ge 0$ and $x \in W^{1,1}(t,\tau)$ satisfying (3.11) and such that

$$\|x'(s)\| \le Mk(s) + k(s)\|u(s)\| + \|v(s)\| \text{ a.e. in } [t,\tau] \quad .$$

Thus $\|x'\| \in L^2(t,\tau)$ and, therefore $x \in X$. Hence we proved (3.10). By [3, Lemma 1.3] and (3.10) we obtain that

$$((1 \times D)^{-1}L)^+ = (1 \times D)^*(L^+) \quad . \tag{3.12}$$

Clearly $\gamma((1 \times D)^{-1}L) \subset graph(\hat{r}(\tau - t,t))$ and by (3.12), $\gamma^* graph(\hat{r}(\tau - t,t))^+ \subset ((1 \times D)^{-1}L)^+ = (1 \times D)^*(L^+)$. Hence for all $(a,b) \in graph\,\hat{r}(\tau - t,t)^+$ there exists $(-p,q) \in L^+$ such that

$$\gamma^*(a,b) = (1 \times D)^*(-p,q) \tag{3.13}$$

This implies that for all $w \in W_0^{1,2}(t,\tau)$,

$$0 = <(1 \times D)^*(-p,q),w> = \int_t^\tau (-pw + qw')(s)ds = \int_t^\tau w'(s)(q(s) + \int_t^s p(r)dr)ds \; .$$

Thus $q \in W^{1,2}(t,\tau)$ and $q' = -p$. By (3.9), $-q'(s) \in A(s)^* q(s)$ a.e. in $[t,\tau]$. From [3, Proposition 1.7b] we deduce that $q \in W^{1,\infty}(t,\tau)$. Moreover by (3.13) for all $x \in X, <(a,b),(x(t),x(\tau))> = <(q',q),(x,x')> = q(\tau)x(\tau) - q(t)x(t)$. Hence $(-a,b) = (q(t),q(\tau))$. and $q(t) \in \hat{r}(\tau - t,t)^* q(\tau)$. We proved that for all $b \in \mathbf{R}^n, \hat{r}(\tau - t,t)^* b$ is contained in the right-hand side of (3.7). On the other hand if q satisfies (3.2) then for all solution w of (3.1)

$$q(\tau)w(\tau) - q(t)w(t) = <(q',q),(w,w')> \ge 0 \quad .$$

This yields that $q(t) \in r(\tau - t,t)^* q(\tau)$ and ends the proof.

To prove Lemma 3.5 we apply some results from [2, pp. 142-143] concerning closed convex processes. Since in general $\hat{f}(\tau - t,t)$ is not closed we need the following

Lemma 3.6. If (H_2) holds true then $\hat{f}(\tau - t,t)$ is Lipschitzian on $\mathbf{R}^n$ and the set-valued map $cl\,\hat{f}(\tau - t,t)$ defined by : for all $u \in \mathbf{R}^n$, $cl\,\hat{f}(\tau - t,t)u = \overline{\hat{f}(\tau - t,t)u}$ is a Lipschitzian on $\mathbf{R}^n$ closed convex process. Moreover $(cl\,\hat{f}(\tau - t,t))^* = \hat{f}(\tau - t,t)^*$ is an upper semicontinuous set-valued map with compact images mapping bounded sets to bounded sets and Dom $\hat{f}(\tau - t,t)^* = \hat{f}(\tau - t,t)(0)^+$.

Proof of Lemma 3.6. Since $0 \in \hat{f}(\tau - t,t)0$, the set $\hat{f}(\tau - t,t)0$ is nonempty. Fix any $u \in \mathbf{R}^n$ such that $\hat{f}(\tau - t,t)u \neq \emptyset$ and let w be a solution of (3.1) on $[t,\tau]$ satisfying $w(t) = u$. Pick $v \in \mathbf{R}^n$ and set $y(\cdot) = w(\cdot) + v - u$. Then dist $(y'(s), A(s)y(s))$ $=$ dist$(w'(s),\ A(s)(w(s) + v - u)) \leq k(s)\|v - u\|$. This and the Filippov theorem imply the existence of a solution $\bar{w}$ of (3.1) defined on $[t,\tau]$ and satisfying $\bar{w}(t) = y(t) = w(t) + v - u = v$

$$\|\bar{w}(\tau) - y(\tau)\| \leq M\|v - u\|$$

where M does not depend on v,u. Thus $\hat{f}(\tau - t,t)v \neq \emptyset$ and

$$\|\bar{w}(\tau) - w(\tau)\| \leq \|\bar{w}(\tau) - y(\tau)\| + \|y(\tau) - w(\tau)\| \leq M\|v - u\| + \|v - u\| ,$$

i.e., $\hat{f}(\tau - t,t)$ is Lipschitz on $\mathbf{R}^n$ with the constant $M + 1$. Pick any $u,u_1 \in \mathbf{R}^n, v \in cl\,\hat{f}(\tau - t,t)u$ and consider $v_i \to v, v_i \in \hat{f}(\tau - t,t)u$. By the Lipschitz continuity of $\hat{f}(\tau - t,t)$ for some $w_i \in \hat{f}(\tau - t,t)u_1$, $\|w_i - v_i\| \leq (M + 1)\|u - u_1\|$. Taking a subsequence and keeping the same notations we may assume that w_i converges to some $w \in cl\,\hat{f}(\tau - t,t)u_1$. Then $\|w - v\| \leq (M + 1)\|u - u_1\|$ and this yields the Lipschitz continuity of $cl\,\hat{f}(\tau - t,t)$. Let $(u_i,v_i) \in$ graph $(\hat{f}(\tau - t,t))$ be a sequence converging to some (u,v). Then $v_i \in \hat{f}(\tau - t,t)u_i$ and, by Lipschitz continuity, for some $w_i \in \hat{f}(\tau - t,t)u$ we have $\|w_i - v_i\| \leq (M + 1)\|u - u_i\|$. Hence $w_i \to v$ and $v \in cl\,\hat{f}(\tau - t,t)u$. This implies that

$$\overline{\text{graph}\,(\hat{f}(\tau - t,t))} = \text{graph}\,(cl\,\hat{f}(\tau - t,t)) \tag{3.14}$$

and therefore graph $(cl\,\hat{f}(\tau - t,t))$ is a closed convex cone. Hence $cl\,\hat{f}(\tau - t,t)$ is a closed convex process and

$$\text{graph}\,(\hat{f}(\tau - t,t))^+ = \text{graph}\,(cl\,\hat{f}(\tau - t,t))^+ \quad .$$

From Definition 3.2 we deduce that $\hat{f}(\tau - t,t)^* = (cl\,\hat{f}(\tau - t,t))^*$. The last statements follow from [3, Proposition 1.7].

Proof of Lemma 3.5. We prove first that

$$\hat{r}_Q(\tau - t,t)^* = (cl\,\hat{r}_{\bar{Q}}(\tau - t,t))^* \tag{3.15}$$

Indeed fix $u_i \in Q$, $v_i \in \hat{r}(\tau - t,t)u_i$ such that $\lim_{i\to\infty}(u_i,v_i) = (u,v)$. Then $u \in \bar{Q}$ and $(u,v) \in \overline{graph(\hat{r}(\tau - t,t))} =$ (by (3.14)) graph $(cl\,\hat{r}(\tau - t,t))$. Hence $v \in cl\,\hat{r}_{\bar{Q}}(\tau - t,t)$ and we proved that $\overline{graph(\hat{r}_Q(\tau - t,t))} =$ graph $(cl\,\hat{r}_{\bar{Q}}(\tau - t,t))$. This yields (3.15). We also know that Dom $(cl\,\bar{\hat{r}}(\tau - t,t)) = \mathbf{R}^n$. Hence using [2, pp. 142-143] we obtain (3.8).

To prove the second statement we observe that the Lipschitz continuity of $cl\,\hat{r}(\tau - t,t)$ yields

$$\overline{cl\,\hat{r}(\tau - t,t)\bar{Q}} = \overline{cl\,\hat{r}(\tau - t,t)Q}$$

Hence $(\hat{r}(\tau - t,t)Q)^+ = (cl\,\hat{r}(\tau - t,t)Q)^+ = (cl\,\hat{r}(\tau - t,t)\bar{Q})^+ =$ (by [2,pp.142-143]) $cl\,\hat{r}(\tau - t,t)^{*-1}(Q^+) =$ (by Lemma 3.6) $\hat{r}(\tau - t,t)^{*-1}(Q^+)$. The proof is complete.

4. The Cone $T_{R(\tau,C)}(z(\tau))^+$.

In this section we assume that (H_1) holds true and that there exists a family of closed convex processes $\{A(s)\}_{s\in[0,T]}$ satisfying (H_2) and (H_3).

Observe that the dual form of Theorem 2.4 is : for all $0 \le t < \tau \le T$

$$T_{R(\tau,C)}(z(\tau))^+ \subset (r(\tau-t,t)\ W(t,z))^+ \tag{4.1}$$

Hence we can "estimate" $T_{R(\tau,C)}(z(\tau))^+$ using the set $(r(\tau-t,t)\ W(t,z))^+$. We study this last set via a duality technique.

Consider again the adjoint differential inclusion

$$-q'(s) \in A(s)^*q(s) \text{ a.e.} \tag{4.2}$$

Theorem 4.1. Assume that $(H_1),(H_2),(H_3)$ hold true. Let $Q(t) \subset W(t,z)$ be a family of convex cones such that for all $0 \le t \le t_1 \le T, \hat{r}(t_1-t,t)Q(t) \subset Q(t_1)$. Then for all $\tau \in [0,T]$

$$T_{R(\tau,C)}(z(\tau))^+ \subset \{q(\tau) : q \in W^{1,\infty}(0,\tau) \text{ satisfies (4.2), } q(t) \in Q(t)^+ \text{ on } [0,\tau[\}$$

Consider next the differential inclusion

$$\begin{cases} -q'(s) \in A(s)^*q(s) & \text{a.e.} \\ <q(s),z'(s)> = \min\{<q(s),e>: e\in F(z(s))\} & \text{a.e.} \end{cases} \tag{4.3}$$

Theorem 4.2. Assume that (H_1),(H_2),(H_3) hold true and let $Q(t) \subset W(t,z)$ be any family of convex cones. Then for all $\tau \in [0,T]$

$$T_{R(\tau,C)}(z(\tau))^+ \subset \{q(\tau): q \in W^{1,\infty}(0,\tau) \text{ satisfies (4.3), } q(t) \in Q(t)^+ \text{ on } [0,\tau[\}$$

In particular

$$T_{R(\tau,C)}(z(\tau))^+ \subset \{q(\tau): q \in W^{1,\infty}(0,\tau) \text{ satisfies (4.3), } q(t) \in R^\infty$$

Observe that the statements of the above theorems depend on the choice of $\{A(s)\}$ and $\{Q(s)\}$. From (4.1) and Theorem 3.3c) we obtain

Lemma 4.3. If $(H_1),(H_2),(H_3)$ hold true, then for any $0 \le t < \tau \le T$ and any convex cone $Q \subset W(t,z)$

$$T_{R(\tau,C))}(z(\tau))^+ \subset \{q(\tau) : q \in W^{1,\infty}(t,\tau) \text{ satisfies (4.2), } q(t) \in Q^+\} \quad .$$

Proof of Theorem 4.1. We shall apply the above lemma. Fix $\tau \in]0,T]$ and $b \in T_{R(\tau,C)}(z(\tau))^+$.

Step 1. Fix any $0 \leq t_1 < \cdots < t_m < \tau$. We first prove the existence of $q \in W^{1,\infty}(0,\tau)$ satisfying (4.2) such that

$$q(\tau) = b \tag{4.4}$$

$$q(t_i) \in Q(t_i)^+ \quad \forall\, i = 1,\dots,m \tag{4.5}$$

By the assumptions of theorem, inclusion (4.5) implies that

$$q(t_i) \in (\hat{r}(t_i - t_{i-1},t_{i-1})\, Q(t_{i-1}))^+ \tag{4.6}$$

We proceed by the induction. By Lemma 4.3 there exists $q \in W^{1,\infty}(t_m,\tau)$ satisfying (4.2) (4.4), (4.5) with $i = m$. Assume that we already know that for some $2 \leq j \leq m$ there exists $q \in W^{1,\infty}(t_j,\tau)$ such that (4.2), (4.4), (4.5) hold true with $i \geq j$. From (4.6) we deduce that $q(t_j) \in (\hat{r}(t_j - t_{j-1},t_{j-1})\, Q(t_{j-1}))^+$. Applying Lemmas 3.4, 3.5 with $\tau = t_j$, $b = q(t_j)$ and $t = t_{j-1}$ we prove the existence of $\hat{q} \in W^{1,\infty}(t_{j-1},t_j)$ satisfying (4.2) such that $\hat{q}(t_j) = q(t_j), \hat{q}(t_{j-1}) \in Q(t_{j-1})^+$. Setting

$$q(s) = \begin{cases} q(s) \text{ when } s \in [t_j,\tau] \\ \hat{q}(s) \text{ when } s \in [t_{j-1},t_j] \end{cases}$$

we end the proof of Step 1.

Step 2.. Let $t_i \in [0,\tau], i = 1,2,\dots$ be a dense subset of $[0,\tau]$. Set

$$L = \{(x,y) \in L^2(0,\tau) \times L^2(0,\tau) : x(s) \in A(s)^* y(s) \text{ a.e.}\}$$

since $A(s)^*$ are closed convex processes, by Mazur's lemma, L is weakly closed in $L^2(0,\tau) \times L^2(0,\tau)$. By Step 1, for all $j \geq 1$ there exists $q_j \in W^{1,\infty}(0,\tau)$ satisfying (4.2) and such that $q_j(\tau) = b$ and for all $1 \leq i \leq j$

$$q_j(t_i) \in Q(t_i)^+ \tag{4.7}$$

By [3, Proposition 1.6 b)] for all j and almost all $s \in [0,\tau], \|q_j'(s)\| \leq k(s)\|q_j(s)\|$. This and Gronwall's lemma imply that $\{q_j\}$ is bounded in $W^{1,2}(0,\tau)$ and, by reflexivity, it has a weak cluster point q. Since L is weakly closed, q satisfies (4.2) and, by (4.7), for all $i, q(t_i) \in Q(t_i)^+$. Fix $t \in [0,\tau]$, $w \in Q(t)$ and let $\{t_{i_k}\}$ be a subsequence converging to t from the right. Since $\{A(s)\}$ satisfy (H_2), by the Filippov theorem, there exist

$w_k \in \hat{r}(t_{i_k}-t,t)w$ converging to w. Moreover for all k, $<q(t_{i_k}), w_k> \geq 0$. Therefore, taking the limit, we get $q(t) \in Q(t)^+$ for all $t \in [0,\tau]$. This ends the proof.

To prove Theorem 4.2 we need two lemmas.

The next one shows how a given family $\{A(s)\}$ can be "increased" to a larger family of closed convex processes still satisfying $(H_2),(H_3)$.

Lemma 4.4. For all $s \in [0,T]$ such that $z'(s) \in F(z(s))$ and for all $x \in \mathbf{R}^n$ set

$$G(s)x = \overline{A(s)x + T_{coF(z(s))}(z'(s))}$$

and set $G(s) = A(s)$ for all other s. Then $\{G(s)\}_{s\in[0,T]}$ are closed convex processes satisfying $(H_2),(H_3)$ and $A(s) \subset G(s)$. Moreover for almost all $s \in [0,T]$ and all $q \in \mathbf{R}^n$

$$G(s)^*q = \begin{cases} A(s)^*q & \text{when } q \in (F(z(s)) - z'(s))^+ \\ \emptyset & \text{otherwise} \end{cases} \tag{4.8}$$

Proof. From the definition of $G(s)$, exactly as in the proof of Lemma 3.6, we deduce that $G(s)(\cdot)$ is $k(s)$ - Lipschitz on $\mathbf{R}^n$. By [12, Lemma 2.8] we know that $\{G(s)\}$ satisfy (H_3). Since $G(s)(\cdot)$ is continuous and has closed images, graph $(G(s))$ is closed. It is also clear that graph $(G(s))$ is a cone. To prove its convexity it is enough to consider only those $s \in [0,T]$ that satisfy $z'(s) \in F(z(s))$. Fix such s and $u,v \in \mathbf{R}^n$. Since $A(s)$ is a convex process and $T_{coF(z(s))}(z'(s))$ is a convex cone we obtain

$$A(s)u + T_{coF(z(s))}(z'(s)) + A(s)v + T_{coF(z(s))}(z'(s)) \subset A(s)(u+v) + T_{coF(z(s))}(z'(s))$$

This yields that

$$G(s)u + G(s)v \subset \overline{A(s)(u+v) + T_{coF(z(s))}(z'(s))} = G(s)(u+v)$$

Hence $G(s)$ is a closed convex process. Moreover, by [25], for all $q \in \mathbf{R}^n$,

$$G(s)^*q = \begin{cases} A(s)^*q & \text{when } q \in T_{coF(z(s))}(z'(s))^+ \\ \emptyset & \text{otherwise} \end{cases} \tag{4.9}$$

Since $co\ F(z(s))$ is a convex set we also have

$$T_{coF(z(s))}(z'(s)) = \overline{\bigcup_{i=1,2,\ldots} i(co\ F(z(s)) - z'(s))} \tag{4.10}$$

and therefore

$$T_{coF(z(s))}(z'(s))^+ = (coF(z(s)) - z'(s))^+ \ .$$

Using (4.9) we deduce from the last equality that for almost all $s \in [0,T]$, (4.8) holds true. To end the proof it remains to show that for all $x \in \mathbf{R}^n$, the map $s \to G(s)x$ is measurable. Since the map $s \to F(z(s))$ is continuous it is also measurable. By Castaing's representation theorem [8] and the assumption $(H_2)i)$ there exist measurable selections

$$f_n(s) \in F(z(s)), \quad g_n(s) \in A(s)x \quad n = 1,2,\ldots$$

such that for all s

$$\overline{\bigcup_{n\geq 1} f_n(s)} = F(z(s)), \quad \overline{\bigcup_{n\geq 1} g_n(s)} = A(s)x$$

Hence, using (4.10) we obtain

$$G(s)x = \overline{\bigcup_{n\geq 1} g_n(s) + \bigcup_{i\geq 1} i \bigcup_{n\geq 1} (f_n(s) - z'(s))} = \overline{\bigcup_{\substack{n\geq 1\\ i\geq 1}} g_n(s) + i(f_n(s) - z'(s))}$$

Since the functions $s \to g_n(s) + i(f_n(s) - z'(s))$ are measurable the last equality and Castaing's theorem imply that $s \to G(s)x$ is a measurable set-valued map.

In Theorem 4.1 we deal with convex cones $Q(t) \subset W(t,z)$ which have the invariance property:

$$\forall 0 \leq t < t_1 \leq T, r(t_1 - t,t)Q(t) \subset Q(t_1) \tag{4.11}$$

The next result shows how such cones can be constructed.

Lemma 4.5 Let $\{A(s)\}_{s\in[0,1]}$ be any family of closed convex processes satisfying (H_2),(H_2) and $\hat{Q}(t) \subset W(t,z)$ be convex cones. Then there exist convex cones $Q(t) \supset \hat{Q}(t)$ satisfying (4.11).

Proof. For all $0 \leq t_1 \leq \ldots \leq t_m \leq T$ define recursively cones $P(t_1) = \hat{Q}(t_1) + \hat{r}(t_1,0)\hat{Q}(0)$ $,\ldots,P(t_1,\ldots,t_{i+1}) = \hat{Q}(t_{i+1}) + \hat{r}(t_{i+1} - t_i,t_i)P(t_1,\ldots,t_i)$. By Theorems 2.4, 2.7 using an induction argument we prove that for all $i \geq 1, P(t_1,\ldots,t_i) \subset W(t,z)$. Set

$$Q(t) = \bigcup_{\substack{0\leq t_1\leq\ldots\leq t_m=t\\ m\geq 1}} P(t_1,\ldots,t_m)$$

Clearly $Q(t)$ is a cone containing $\hat{Q}(t)$ and, by definition of $Q(t)$, for all $0 \leq t \leq t_1 < T$, $r(t_1 - t,t)Q(t) \subset Q(t_1)$. It remains to prove that $Q(t)$ is convex, i.e. we have to check that for all $0 \leq t_1 \leq \cdots \leq t_m = t, 0 \leq t_1' \leq \cdots \leq t_k' = t$

$$P(t_1,\ldots,t_m) + P(t_1',\ldots,t_k') \subset Q(t) \tag{4.12}$$

We proceed by the induction with respect to $m+k$. Observe that for all $t \in [0,T]$, $P(t)$ is a convex cone. Fix $t \in [0,T]$. Assume that for some $j \geq 2$ and all $m \geq 1, k \geq 1, 0 \leq t_1 \leq \cdots \leq t_m = t, 0 \leq t_1' \leq \cdots \leq t_k' = t$ satisfying $m + k \leq j$ the relation (4.12) holds true. Fix $0 \leq t_1 \leq \cdots \leq t_{m+1} = t, 0 \leq t_1' \leq \cdots \leq t_k' = t$ such that $m + k = j$, $t_{k-1} \leq t_m$. Then $P(t_1,\cdots,t_m) + P(t_1',\cdots,t_{k-1}',t_m) \subset Q(t_m)$. Moreover by definition of $P(\cdot)$, using that $\hat{r}$ is a convex process we obtain

$$\hat{r}(t_m - t_{k-1}', t_{k-1}')P(t_1',\ldots,t_{k-1}') \subset P(t_1',\ldots,t_{k-1}',t_m)$$

This and definition of $Q(t)$ imply:

$$\begin{aligned} P(t_1,\ldots,t_{m+1}) + P(t_1',\ldots,t_k') &= \hat{Q}(t) + \hat{r}(t-t_m,t_m)P(t_1,\ldots,t_m) \\ &+ \hat{Q}(t) + \hat{r}(t-t_m,t_m)\hat{r}(t_m - t_{k-1}',t_{k-1}')P(t_1',\ldots,t_{k-1}') \\ &\subset \hat{Q}(t) + \hat{r}(t-t_m,t_m)(P(t_1,\ldots,t_m) + P(t_1,\ldots,t_{k-1}',t_m)) \\ &\subset \hat{Q}(t) + \hat{r}(t-t_m,t_m)Q(t_m) \subset Q(t) \end{aligned}$$

Proof of Theorem 4.2 By Lemma 4.4 we replace the family $\{A(s)\}$ by the new family $\{G(s)\}$ satisfying (H_2),(H_3) and (4.8). From Lemma 4.5 ot is not restrictive to assume that the family $\{Q(s)\}$ satisfies (4.11). Theorem 4.1 applied with $\{G(s)\}$ yields the result.

Corollary 4.6. Assume that $(H_1),(H_2),(H_3)$ hold true and let Q be a convex subcone of $T_C(z(0))$. Then for all $\tau \in [0,T]$

$$T_{R(\tau,C)}(z(\tau))^+ \subset \{q(\tau) : q \in W^{1,\infty}(0,\tau) \text{ satisfies (4.3)}, \ q(0) \in Q^+\}$$

Proof. Setting $Q(t) = \hat{r}(t,0)Q$ and applying Theorem 4.1 with closed convex processes $\{G(s)\}$ of Lemma 4.4 we deduce from (4.8) our statement.

Theorem 4.7. Assume that $(H_1),(H_2),(H_3)$ hold true and that for any $t \in [0,T], q_1, q_2, \in W^{1,\infty}(\tau,t)$ satisfying (4.3) and equal at t we have $q_1/\|q_1\| = q_2/\|q_2\|$ on $[0,t]$. Then for all $\tau \in [0,T]$

$$T_{R_{(\tau,C)}}(z(\tau))^+ \subset \{q(\tau) : q \in W^{1,\infty}(0,\tau) \text{ satisfies (4.3) and } q(t) \in W(t,z)^+ \text{ on } [0,\tau[\} \quad .$$

In particular the above happens when for almost all $s \in [0,T]$, the adjoint $A(s)^*$ is single valued on its domain of definition.

Proof. Fix $\tau \in [0,T]$, $b \in T_{R(\tau,C)}(z(\tau))^+$, $t \in [0,\tau[$, $c \in W(t,z)$. By Theorem 4.1 applied with the family of closed convex processes $\{G(s)\}$ and convex cones

$$Q(s) = \begin{cases} \emptyset & \text{for } s < t \\ \mathbf{R}_+ c & \text{for } s = t \\ \hat{r}(s-t,t)Q(t) & \text{for } s > t \end{cases}$$

using (4.8) we prove the existence of $q \in W^{1,\infty}(0,\tau)$ satisfying (4.3) such that $q(\tau) = b$, $<q(t),c> \geq 0$, Since $c \in W(t,z)$ and $t \in [0,\tau[$ are arbitrary, by the assumptions of theorem $q(t) \in W(t,z)^+$ on $[0,\tau[$.

Corollary 4.8 Assume that (H_1) holds true and that there exist linear operators $A(s) \in L(\mathbf{R}^n, \mathbf{R}^n)$ satisfying $(H_2), (H_3)$. Then for all $\tau \in [0,T]$

$$T_{R(\tau,C)}(z(\tau))^+ \subset \{q(\tau) : -q'(s) = A(s)^* q(s),\ <q(s),z'(s) = \min_{e \in F(z(s))} <q(s),e>, q(s) \in W(s,z)^+ \text{ in } [0,\tau]\}$$

Proof. The transposed linear operator $A(s)^*$ is equal to the adjoint process in the sense of Definition 3.1 (see Rockafellar [25]). Since for all $b \in T_{R(\tau,C)}(z(\tau))^+$, the solution of the linear equation $-q'(s) = A(s)^* q(s)$; $q(\tau) = b$ is unique the proof follows from Theorem 4.7.

Theorem 4.9. Let $R_C(T,\cdot)$ denote the restriction of a reachable map $R(T,\cdot)$ to the set C. Then for every convex cone $Q \subset T_C(z(0))$

$$T_{\text{graph}\, R_C(T,\cdot)}(z(0),z(T))^+ \subset \{(\pi - q(0), q(T)) : q \in W^{1,\infty}(0,T) \text{ satisfies (4.3) and } \pi \in Q^+\}$$

Proof. By Theorem 2.6

$$T_{\text{graph}\, R_C(T,\cdot)}(z(0),z(T))^+ \subset \{(w(0), r(T,0)w(0)) : w(0) \in T_C(z(0))\}^+ \tag{4.13}$$

We replace closed convex processes $\{A(s)\}$ by $\{G(s)\}$ from Lemma 4.4 and keep the same notation $\hat{r}$ for the reachable map of the inclusion

$$w'(s) \in G(s)w(s) \quad \text{a.e.}$$

Then by (3.4), (4.13) we obtain

$$T_{\text{graph}\, R_C(T,\cdot)}(z(0),z(T))^+ \subset \{(a, \hat{r}(T,0)a) : a \in Q\}^+$$

and from Lemma 3.5 we deduce that for all $(p,q) \in T_{\mathrm{graph} R_C(T,\cdot)}(z(0),z(T))^+$ we have $p + \hat{r}(T,0)^* q \in Q^+$. Lemma 3.4 ends the proof.

Remark 4.10. (On the Hamiltonian inclusions):

For all $x,p \in \mathbf{R}^n$ the Hamiltonian of F is defined by

$$H(x,p) = \sup_{e \in F(x)} <p,e> = \sup_{e \in coF(x)} <p,e>$$

If (H_1) holds true, then H is locally Lipschitz on Dom $F \times \mathbf{R}^n$ (see for example [9]). Let us assume that for all s, Dom $A(s)^*$ is a subspace of $\mathbf{R}^n$ and $A(s)^*$ is *linear on* Dom $A(s)^*$.

Consider an absolutely continuous solution q of (4.3) defined on the time interval $[0,T]$. Pick any $s \in]0,1[$ such that $<q(s),z'(s)> = \min_{e \in F(z(s))} <q(s),e>, -q'(s) = A(s)^* q(s)$. Set $\bar{q} = -q$ and fix any u. Let $v = A(s)u$ and $v_h \to v$ (when $h \to 0+$) be such that $z'(s) + hv_h \in coF(z(s)+hu)$. Then for all $w \in \mathbf{R}^n$ we have

$$\limsup_{h \to 0+} \frac{H(z(s)+hu, \bar{q}(s)+hw) - H(z(s),\bar{q}(s))}{h} \geq$$

$$\limsup_{h \to 0+} \frac{<\bar{q}(s)+hw, z'(s)+hv_h> - <\bar{q}(s),z'(s)>}{h} = <w,z'(s)> + <\bar{q}(s),v> =$$

$$<w,z'(s)> + <q'(s),u> = <(q'(s),z'(s)),(u,w)> .$$

In particular this yields that

$$(q'(s),z'(s)) \in \partial H(z(s),q(s)) \tag{4.14}$$

where ∂H denotes the generalized gradient of H (see [9]). Hence in this particular case every solution of (4.3) is also a solution of the Hamiltonian inclusion (4.14). It may happen that for a family of closed convex processes satisfying (H_2), (H_3) the only solution of (4.3) is $q \equiv 0$ and in the same time the Hamiltonian inclusion (4.14) has solutions different from zero (see the example from [18]). Hence in this particular case it is more convenient to use the adjoint inclusion (4.3) that the Hamiltonian inclusion (4.14) to estimate the cone $T_{R(T,C)}(z(T))^+$. In a more general case it is not known how to compare solutions of (4.3) and (4.14).

5. Application: High Order Maximum Principles

1) Minimization with respect to the final state

Let U be a compact metric space and $f\colon \mathbf{R}^n \times U \to \mathbf{R}^n$ be a continuous function, $g : \mathbf{R}^n \to \mathbf{R}, C \subset \mathbf{R}^n$. Consider the following optimal control problem

$$\text{minimize } g(x(1)) \tag{5.1}$$

Over the solutions of the control system

$$\begin{cases} x'(t) = f(x(t), u(t)) & \text{a.e in } [0,1] \\ x(0) \in C & u(t) \in U \text{ is measurable.} \end{cases} \tag{5.2}$$

Set $F(x) = f(x, U)$ for all $x \in \mathbf{R}^n$. By the Filippov Theorem [1, p. 91] solutions of the control system (5.2) and the differential inclusion

$$\begin{cases} x'(t) \in F(x(t)) \text{ a.e. on } [0,1] \\ x(0) \in C \end{cases} \tag{5.3}$$

do coincide.

Theorem 5.1. Assume that a trajectory control pair $(z, \bar{u})$ solves the above problem and for a constant L and all $u \in U, f(\cdot,u)$ is L-Lipschitzian on a neighborhood of $z([0,1])$. If g is differentiable at $z(1)$ and for almost all $t, f(\cdot,u(t))$ is differentiable at $z(t)$ then there exists $q \in W^{1,\infty}(0,1)$ such that

$$\begin{cases} -q'(t) = q(t)\dfrac{\partial f}{\partial x}(z(t),\bar{u}(t)) & \text{a.e.} \\ < q(t), z'(t) > = \min\limits_{u\in U} < q(t), f(z(t),u) > & \text{a.e.} \end{cases} \tag{5.4}$$

$$q(1) = g'(z(1)), \quad q(0) \in T_C(z(0))^+ \tag{5.5}$$

$$q(t) \in W(t,z)^+ \text{ for all } t \in [0,1[\;. \tag{5.6}$$

Proof. By the assumptions, the set-valued map F defined above satisfies (H_1). Moreover for almost all $s \in [0,1], \dfrac{\partial}{\partial x} f(z(s), \bar{u}(s)) \subset dF(z(s), z'(s)) \subset d\, coF(z(s), z'(s))$. Set $A(s) = \dfrac{\partial}{\partial x} f(z(s), \bar{u}(s))$. Since $\|A(s)\| \le L$, $A(s)$ is L-Lipschitz. Hence (H_2), (H_3) hold true. On the other hand for every solution x of (5.3) we have $g(x(1)) - g(z(1)) \ge 0$ and this yields

$$\forall w \in T_{R(1,C)}(z(1)), \quad g'(z(1))w \ge 0$$

i.e.

$$g'(z(1)) \in T_{R(1,C)}(z(1))^+ \ .$$

Corollary 4.8 ends the proof.

Corollary 5.2. Under all assumptions of Theorem 5.1, assume that for some $t \in [0,1[, W(t,z)^+ = \{0\}$. Then $z(1)$ is a critical point of g and if g is locally C^2 at $z(1)$ then $g''(z(1)) \geq 0$ on $T_{R(1,C)}(z(1))$. In particular this happens when $T_C(z(0))^+ = \{0\}$.

Proof. Let q be as in Theorem 5.1 and t be such that $W(t,z)^+ = \{0\}$. Then $q(t) = 0$ and, by the uniqueness of q, $q(1) = 0$. Hence, by (5.5), $g'(z(1)) = 0$. Assume next that g is locally C^2 and fix $w \in T_{R(1,C)}(z(1))$. Then for some $h_i \to 0+$, $w_i \to w$, $z(1) + h_i w_i \in R(1,C)$ and since z solves the problem (5.1), (5.2) $g(z(1) + h_i w_i) - g(z(1)) = \frac{1}{2} g''(z(1)) w_i w_i h_i^2 + o(h_i^2) \geq 0$. Taking the limit we end the proof.

2) Minimization with respect to the both end points

Let f, U be as in example 1) and $\varphi : \mathbf{R}^{2n} \to \mathbf{R}$ be a given function. Consider the problem

$$\text{minimize } \varphi(x(0),x(1)) \tag{5.7}$$

over the solutions of the control system (5.2). If a trajectory-control pair $(z,\bar{u})$ solves the problem (5.7), (5.2) and g is differentiable at $(z(0),z(1))$ then

$$\forall (w(0),w(1)) \in T_{\text{graph} R_C(1,\cdot)}(z(0),z(1)), \varphi'(z(0), z(1))\,(w(0),\, w(1)) \geq 0 \quad ,$$

i.e. $\varphi'(z(0), z(1))$ is in the positive polar of the tangent cone. Let $W(t,z)$ denote the cone of variations of reachable sets $R(\cdot,z(0))$.

Theorem 5.3. Assume that a trajectory-control pair $(z,\bar{u})$ solves the above problem, f satisfies all the assumptions of Theorem 5.1 and φ is differentiable at $(z(0),z(1))$. Then there exists $q \in W^{1,\infty}(0,1)$ satisfying (5.4), (5.6) and such that

$$q(1) = \frac{\partial}{\partial x_2} \varphi(z(0),z(1)) \ , \qquad q(0) \in T_C(z(0))^+ - \frac{\partial}{\partial x_1} \varphi(z(0),z(1))$$

Proof. By the proof of Theorem 5.1 the family of maps $A(s) = \frac{\partial}{\partial x} f(z(s),\bar{u}(s)), s \in [0,1]$ satisfies $(H_2), (H_3)$. We already know that $\varphi'(z(0),z(1)) \in T_{\text{graph} R_C(1,\cdot)}(z(0),z(1))^+$. Fix $b \in T_C(z(0))$. Applying Theorem 4.9 with $Q = \mathbf{R}_+ b$ we deduce that the solution q of

(5.4) satisfying $q(1) = \frac{\partial}{\partial x_2}\varphi(z(0),z(1))$ verifies

$$\frac{\partial}{\partial x_1}\varphi(z(0),z(1)) \in (\mathbf{R}_+ b)^+ - q(0)$$

Hence $< q(0) + \frac{\partial}{\partial x_1}\varphi(z(0),z(1)), b> \geq 0$. Since q does not depend on b we obtain that $q(0) + \frac{\partial}{\partial x_1}\varphi(z(0),z(1)) \in T_C(z(0))^+$. It remains to show that q satisfies (5.6). Set $g(x) = \varphi(z(0),x)$. Then $g'(z(1)) = \frac{\partial}{\partial x_2}\varphi(z(0),z(1))$. Clearly, $(z,\bar{u})$ is an optimal solution of problem (5.1), (5.2) with $C = \{z(0)\}$. Applying Theorem 5.1 with $C = \{z(0)\}$ we end the proof.

Corollary 5.4. Under all assumptions of Theorem 5.3 assume that for some $t \in [0,1[$, $W(t,z)^+ = \{0\}$. Then $\frac{\partial}{\partial x_1}\varphi(z(0),z(1)) \in T_C(z(0))^+$. Moreover if $T_C(z(0))^+ = \{0\}$ then $(z(0),z(1))$ is a critical point of φ and if φ is locally C^2 at $(z(0),z(1))$, then $g''(z(0),z(1)) \geq 0$ on $T_{\text{graph}\, R_C(1,\cdot)}(z(0),z(1))$.

The proof follows by the same arguments as in Corollary 5.2.

3) Closed loop control systems.

Let $U : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ be a set-valued map with compact nonempty images, C be a nonempty subset of $\mathbf{R}^n$ and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be a locally Lipschitzian function, $g : \mathbf{R}^n \to \mathbf{R}$. Consider the following control problem

$$\text{minimize } g(x(1)) \tag{5.8}$$

over trajectories of the control system

$$\begin{cases} x'(t) = f(x(t),u(t)) & \text{a.e. in } [0,1] \\ x(0) \in C & u(t) \in U(x(t)) \text{ is measurable} \end{cases} \tag{5.9}$$

Set $F(x) = \{f(x,u) : u \in U(x)\}$. It is clear that every trajectory of (5.9) is a trajectory of the differential inclusion

$$\begin{cases} x'(t) \in F(x(t)) \\ x(0) \in C \end{cases} \quad \text{a.e. in } [0,1] \tag{5.10}$$

Lemma 5.5. If U is upper semicontinuous then the set of trajectories of the closed loop control system (5.9) do coincide with the set of trajectories of the differential inclusion

(5.10).

Proof. We have to show that with every trajectory $x \in W^{1,1}(0,1)$ of the inclusion (5.10) we can associate a measurable function $u : [0,1] \to \mathbf{R}^m$ satisfying

$$x'(t) = f(x(t), u(t)), \quad u(t) \in U(x(t)) \text{ a.e. in}[0,1]$$

For all $t \in [0,1]$ set $\hat{U}(t) = \{u \in U(x(t)) : x'(t) = f(x(t),u)\}$. Then for almost all $t \in [0,1]$, $\hat{U}(t)$ is a closed, nonempty set. We claim that $\hat{U}$ is a measurable set-valued map. Indeed fix a closed subset $\hat{C} \subset \mathbf{R}^m$ and observe that the set

$$D := \{(t,f(x(t),u)) : t \in [0,1], \; u \in U(x(t)) \cap \hat{C}\}$$

is closed. Moreover

$$\{t : \hat{U}(t) \cap \hat{C} \neq \emptyset\} = \{t : (t,x'(t)) \in D\}$$

Thus $\{t : \hat{U}(t) \cap \hat{C} \neq \emptyset\}$ is a Lebesgue measurable set and, since $\hat{C}$ is an arbitrary closed subset of $\mathbf{R}^m$, we proved that $\hat{U}$ is measurable. From the measurable selection theorem (see for example [26]) follows the existence of a measurable selection $u(t) \in \hat{U}(x(t))$, $t \in [0,1]$. The very definition of the map $\hat{U}$ ends the proof.

In the theorem below we assume that $f(x,U(x))$ is *regular* in the following sense: If for some x and $\bar{u} \in U(x)$, $q \neq q_1 \neq 0$ we have

$$\sup_{u \in U(x)} <q,f(x,u)> = <q,f(x,\bar{u})>; \; \sup_{u \in U(x)} <q_1,f(x,u)> = <q_1,f(x,\bar{u})>$$

then for some $\lambda \geq 0$ $q = \lambda q_1$. Geometrically this means that every boundary point of $\text{co}f(x,U(x))$ has at most one normalized outer normal.

Theorem 5.6. Assume that a trajectory control pair $(z,\bar{u})$ solves the above problem, that f is differentiable at $(z(t),\bar{u}(t))$, g is differentiable at $z(1)$, U is Lipschitzian on a neighborhood of $z([0,1])$ and $f(x,U(x))$ is regular. Further assume that there exist closed convex processes $B(s) \subset dU(z(s),\bar{u}(s))$ satisfying (H_2). Then there exists a solution $q \in W^{1,\infty}(0,1)$ of the inclusion

$$-q' \in \frac{\partial f}{\partial x}(z(t),\bar{u}(t))^* q + B(t)^* \frac{\partial f}{\partial u}(z(t),\bar{u}(t))^* q$$

satisfying (5.5), (5.6) and the minimum principle

$$<q(t),z'(t)> = \min_{u \in U(z(t))} <q(t),f(z(t),u)> \text{ a.e.}$$

Proof. From differentiability of f at $(z(t),\bar{u}(t))$ we deduce that for almost all t and for all $w \in \mathbf{R}^n$

$$\frac{\partial f}{\partial x}(z(t),\bar{u}(t))w + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))dU(z(t),\bar{u}(t))w \subset dF(z(t),z'(t))w$$

Hence closed convex processes

$$A(t) := \frac{\partial f}{\partial x}(z(t),\bar{u}(t)) + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))B(t)$$

satisfy (H_2), (H_3). Since z is the minimizing trajectory for all $w \in T_{R(1,C)}(z(1))$, $g'(z(1))w \geq 0$. Thus $g'(z(1)) \in T_{R(1,C)}(z(1))^+$. We apply Theorem 4.7. Let q_1,q_2 be two solutions of (4.3) such that $q_1(t) = q_2(t) \neq 0$. Then $q_i \neq 0$ on $[0,t]$ and

$$<q_1(s), z'(s)> = \min_{e \in F(z(s))} <q_1(s),e> \quad \text{a.e.}$$

$$<q_2(s), z'(s)> = \min_{e \in F(z(s))} <q_2(s),e> \quad \text{a.e. .}$$

Since $f(x,U(x))$ is regular $\dfrac{q_1(s)}{\|q_1(s)\|} = \dfrac{q_2(s)}{\|q_2(s)\|}$ a.e. in $[0,1]$ and, by continuity of $q(\cdot)$ we obtain $q_1/\|q_1\| = q_2/\|q_2\|$. Hence the result will follow from Theorem 4.7 if we show that

$$A(t)^* \subset \frac{\partial f}{\partial x}(z(t),\bar{u}(t))^* + B(t)^* \frac{\partial f}{\partial u}(z(t),\bar{u}(t))^*$$

Fix $p \in A(t)^*q$. Then for all $w \in \mathbf{R}^n$, $v \in B(t)w$

$$<p,w> \leq <q, \frac{\partial f}{\partial x}(z(t),\bar{u}(t))w + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))v> = <\frac{\partial f}{\partial x}(z(t),\bar{u}(t))^*q,w> + <\frac{\partial f}{\partial u}(z(t),\bar{u}(t))^*q,v>$$

and therefore

$$<p - \frac{\partial f}{\partial x}(z(t),\bar{u}(t))^*q,w> \leq <\frac{\partial f}{\partial u}(z(t),\bar{u}(t))^*q,v>$$

By the definition of the adjoint process

$$p - \frac{\partial f}{\partial x}(z(t),\bar{u}(t))^*q \in B(t)^* \frac{\partial f}{\partial u}(z(t),\bar{u}(t))^*q$$

and we finally obtain

$$p \in \frac{\partial f}{\partial x}(z(t),\bar{u}(t))^*q + B(t)^* \frac{\partial f}{\partial u}(z(t),\bar{u}(t))^*q$$

The proof is complete.

The next result is an extension of the main theorem from [22].

Theorem 5.7. Assume that a trajectory control pair $(z,\bar{u})$ solves the above problem, that f is differentiable at $(z(t),\bar{u}(t))$, g is differentiable at $z(1)$ and U is Lipschitzian on a neighborhood of $z([0,1])$. Further assume that for almost all t there exists a differentiable at $z(t)$ selection $u_t(x) \in U(x)$ satisfying $u_t(z(t)) = \bar{u}(t)$. Then there exists a solution $q \in W^{1,\infty}(0,1)$ of the equation

$$\begin{cases} -q' = q\left[\frac{\partial f}{\partial x}(z(t),\bar{u}(t)) + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))\frac{\partial u_t}{\partial x}(z(t))\right] \\ <q(t),z'(t)> = \min\limits_{u\in U(z(t))} <q(t),f(z(t),u)> \text{ a.e.} \end{cases} \tag{5.11}$$

satisfying (5.5) and (5.6). The above theorem was proved by Leitmann in [22] without the inclusion (5.6).

Proof. The set-valued map $F(x) = f(x,U(x))$ satisfies the hypothesis (H_1) on a neighborhood of $z([0,1])$. Moreover the linear operators

$$A(t) = \frac{\partial f}{\partial x}(z(t),u(t)) + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))\frac{\partial u_t}{\partial x}(z(t)), \quad t \in [0,1] \ .$$

verify (H_2) and (H_3). Since z is the minimizing trajectory for all $w \in T_{R(1,C)}(z(1))$, $g'(z(1))w \geq 0$. Thus $g'(z(1)) \in T_{R(1,C)}(z(1))^+$ and the result follows from Corollary 4.8 and the inclusion $W(0,z)^+ \subset T_C(z(0))^+$.

4) An implicit dynamical system

Consider a continuously differentiable function $f: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$ and a function $g: \mathbf{R}^n \to \mathbf{R}$, $C \subset \mathbf{R}^n$.

We study here the problem

$$\text{minimize } g(x(1)) \tag{5.12}$$

over the absolutely continuous solutions of the implicit dynamical system

$$f(x(t),\, x'(t)) = 0 \text{ a.e. in } [0,1] \tag{5.13}$$

satisfying the initial point constraint

$$x(0) \in C \tag{5.14}$$

Such systems arise as models for nonlinear circuits. In general they can not be reduced to the state variable form, $z' = f(z,t)$ or to the control system (5.2) (see [6], bibliographical comments on p. 147).

Set $F(x) = \{v : f(x,v) = 0\}$ and consider the differential inclusion

$$x'(t) \in F(x(t)) \text{ a.e. in } [0,1] \tag{5.15}$$

Clearly solutions of (5.13) and (5.15) do coincide. Moreover, by continuity of f, graph (F) is a closed set. The following result was proved in [15]:

Lemma 5.8. Assume that for all $x \in \mathbf{R}^n$

$$\liminf_{\|v\| \to \infty} \|f(x,v)\| > 0 \tag{5.16}$$

Then F has compact images. If moreover for all $(x,v) \in$ graph (F) the derivative $\frac{\partial}{\partial v} f(x,v)$ is surjective then $DomF$ is open and F is locally Lipschitzian on it, and

$$ker\, f'(x,v) = \text{graph}\,(dF(x,v))$$

In particular this implies that $dF(x,v)$ is a closed convex process.

Lemma 5.9. Under all assumptions of Lemma 5.8 for every solution x of (5.13) there exist $L > 0$ such that for almost all $s \in [0,1]$, $dF(x(s), x'(s))$ is L-Lipschitz on $\mathbf{R}^n$ and

$$dF(x(s),x'(s))^* q = \begin{cases} \frac{\partial f}{\partial x}(x(s),x'(s))^* \frac{\partial f}{\partial v}(x(s),x'(s))^{*-1} q & \text{if } q \in ker\ \frac{\partial f}{\partial v}(x(s),x'(s))^{\perp} \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. Fix a solution x of (5.13). Since the derivative $\frac{\partial f}{\partial v}$ is surjective on graph (F), for all $(x,y) \in$ graph (F) there exists $\rho > 0$ such that

$$\{v \in \mathbf{R}^m : \|v\| \le \rho\} \subset \frac{\partial f}{\partial v}(x,y)\ (\{u \in \mathbf{R}^n : \|u\| \le 1\}) \tag{5.17}$$

Since $f \in C^1$, the assumption (5.16) implies that there exists a compact set K such that for almost all $s \in [0,1], (x(s),x'(s)) \in K$. This, (5.17) and continuity of $\frac{\partial f}{\partial v}$ imply that for some $\rho > 0$ and almost all $s \in [0,1]$

$$\{v \in \mathbf{R}^m : \|v\| \le \rho\} \subset \frac{\partial f}{\partial v}(x(s),x'(s))(\{u \in \mathbf{R}^n : \|u\| \le 1\})$$

Using again [15, Theorem 10.1] we deduce that for some $L > 0$ and almost all $s \in [0,1], dF(x(s),x'(s))$ is L-Lipschitz on a neighborhood of zero. Since $dF(x(s),x'(s))$ is a convex process we finally obtain that it is L-Lipschitz on $\mathbf{R}^n$. By the definition of the adjoint process

$$\text{graph}\,(F(x(s),x'(s))^*) = (\ker f'(x(s),x'(s)))^{\perp} = Im f'(x(s),x'(s))^*$$

Hence for all $(p,q) \in \text{graph}\,(dF(x(s),x'(s))^*)$ there exist $\alpha \in \mathbf{R}^m$ such that

$$p = \frac{\partial f}{\partial x}(x(s),x'(s))^*\alpha \ , \quad q = \frac{\partial f}{\partial v}(x(s),x'(s))^*\alpha$$

Since $\frac{\partial f}{\partial v}(x(s),x'(s))$ is surjective the adjoint linear operator $\frac{\partial f}{\partial v}(x(s),x'(s))^*$ is injective and hence invertible on

$$\text{Im}\frac{\partial f}{\partial v}(x(s),x'(s))^* = (\ker\frac{\partial f}{\partial v}(x(s),x'(s)))^{\perp} \ .$$

Thus

$$q \in (\ker\frac{\partial f}{\partial v}(x(s),x'(s)))^{\perp} \ , \ p = \frac{\partial f}{\partial x}(x(s),x'(s))^*\frac{\partial f}{\partial v}(x(s),x'(s))^{*-1}q$$

Theorem 5.10. Assume that z solves the problem (5.12) - (5.14), f satisfies all the assumptions of Lemma 5.8 and g is differentiable at $z(1)$. Then there exists $q \in W^{1,\infty}(0,1)$ satisfying

$$-q'(s) = \frac{\partial f}{\partial x}(z(s),z'(s))^*\frac{\partial f}{\partial v}(z(s),z'(s))^{*-1}q(s) \text{ a.e.} \tag{5.18}$$

$$q(1) = g'(z(1)),\ q(s) \in (\ker\frac{\partial f}{\partial v}(x(s),x'(s)))^{\perp} \tag{5.19}$$

$$\min\{<q(s),e> : f(z(s),e) = 0\} = <q(s),z'(s)> \text{ a.e.} \tag{5.20}$$

$$q(s) \in W(s,z)^+ \text{ for } s \in [0,1[\tag{5.21}$$

Proof. For all $w \in T_{R(1,C)}(z(1))$, $g'(z(1))w \geq 0$. Hence $g'(z(1)) \in T_{R(1,C)}(z(1))^+$. Since the solution of (5.18) is uniquely defined we may apply Theorem 4.7 with closed convex processes $\{dF(x(s),x'(s))\}_{s \in [0,1]}$. Lemma 5.9 ends the proof.

6. An Impulse Closed Loop Deterministic Control Problem

Let $U: \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ be a set-valued map with compact nonempty images, C be a nonempty subset of $\mathbf{R}^n$ and $f: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be a locally Lipschitzian function, $g: \mathbf{R}^n \to \mathbf{R}$.

Further let $V: \mathbf{R}^n \to \mathbf{R}^p$ be a set-valued map of shift parameters and $\varphi: \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ be a given function.

Consider the closed loop control system

$$\begin{cases} x'(t) = f(x(t), u(t)) \ , \quad u(t) \in U(x(t)) \text{ a.e. in } [0,1] \\ x(0) \in C \end{cases} \tag{6.1}$$

A sequence $\{(t_i, v_i) : i = 1,\dots,j\}$ is called an impulse strategy of a left-continuous trajectory $x: [0,1] \to \mathbf{R}^n$, if $0 = t_1 \leq \cdots \leq t_j = 1$ and for all i

$$v_i \in V(x(t_i)) \tag{6.2}$$

$$x \in W^{1,1}(t_i, t_{i+1}) \tag{6.3}$$

$$x(t_i+) = x(t_i) + \varphi(x(t_i), v_i) \tag{6.4}$$

and x satisfies (6.1) with a measurable control u. Such trajectory x is called admissible.

This type of systems is met in a number of optimal control problems in economics and management (see for example [7, pp. 281-285]). We refer to [5], [24] and the bibliographies contained therein for previous results on discontinuous optimal trajectories.

Consider a function $g: \mathbf{R}^n \to \mathbf{R}$. The problem we study here consists in characterization of a solution z to the problem

$$\min\{g(x(1)) : x \text{ is an admissible trajectory}\} \ . \tag{6.5}$$

The approach is essentially the same. So we shall only stress the main points. For all $x \in \mathbf{R}^n$ set $F(x) = f(x, U(x))$. Exactly as Lemma 5.5, we prove

Lemma 6.1. If U is upper semicontinuous then the set of admissible trajectories coincide with the set of left-continuous functions $x: [0,1] \to \mathbf{R}^n$ satisfying for some $0 = t_1 \leq \cdots \leq t_j = 1$ and $v_i \in V(x(t_i))$ the following relations

$$\begin{cases} x \in W^{1,1}(t_i, t_{i+1}) \\ x'(t) \in F(x(t)) \qquad \text{a.e.} \\ x(0) \in C \\ x(t_i+) = x(t_i) + \varphi(x(t_i), v_i) \end{cases} \tag{6.6}$$

Theorem 6.2. Assume that a trajectory-control pair (z,u) solves the above problem and let $\{(t_i,v_i): i = 1,...,l\}$ be a corresponding strategy. Further assume that $U,\bar{u},g,f$ satisfy all the assumptions of Theorem 5.7, that φ is differentiable at $(z(t_i),v_i)$ and for all i there exists a differentiable at $z(t_i)$ selection $\nu_i(x) \in V(x)$ such that $\nu_i(z(t_i)) = v_i$. Then there exists a (left-continuous) function $q:[0,1] \to \mathbf{R}^n$ satisfying (5.5), (5.11) and such that for all i

$$q \in W^{1,\infty}(t_i,t_{i+1}) \tag{6.7}$$

$$q(t_i) = q(t_i+)[id + \frac{\partial \varphi}{\partial x}(z(t_i),v_i) + \frac{\partial \varphi}{\partial v}(z(t_i),v_i)\frac{\partial \nu_i}{\partial x}(z(t_i))] \tag{6.8}$$

$$q(t_i+) \in T_{\varphi(z(t_i),\,V(z(t_i)))}(\varphi(z(t_i),v_i))^+ \tag{6.9}$$

Furthermore

a) If the right derivative $z'(t_i+)$ does exist then

$$\min_{u \in U(z(t_i))} < q(t_i),f(z(t_i),u)> \ \geq \ < q(t_i+),z'(t_i+)>$$

b) If the left derivative $z'(t_i-)$ does exist then

$$\min_{u \in U(z(t_i+))} < q(t_i+),f(z(t_i+),u)> \ \geq \ < q(t_i),z'(t_i-)>$$

c) If z has the right and left derivative at t_i then

$$\begin{cases} \min_{u \in U(z(t_i+))} < q(t_i+),f(z(t_i+),u)> = \\ \min_{u \in U(z(t_i))} < q(t_i),f(z(t_i),u)> \\ + < q(t_i),z'(t_i-)> = < q(t_i+),z'(t_i+)> \ . \end{cases} \tag{6.10}$$

When U does not depend on x the assumption that $f(x,\cdot)$ is locally Lipschitzian can be omitted and we have

Theorem 6.3. Let U be a compact metric space of controls, V be a set of shift parameters, $f: \mathbf{R}^n \times U \to \mathbf{R}^n$ be a continuous function and $\varphi: \mathbf{R}^n \times V \to \mathbf{R}^n$. Assume that a trajectory-control pair (z,u) solves the problem

$$\text{minimize } g(x(1)) \tag{6.11}$$

over the solution of the system

$$\begin{cases} x'(t) = f(x(t),u(t)) \ , \ \ u(t) \in U \ \text{ a.e. in}[0,1] \\ x(0) \in C \ \text{ and for some } \ 0 = t_1 \leq \cdots \leq t_j = 1 \\ v_i \in V \ \text{ and all } i, \ \ x \in W^{1,1}(t_i,t_{i+1}) \\ x(t_i+) = x(t_i) + \varphi(x(t_i),v_i) \end{cases} \tag{6.12}$$

and let $\{(t_i,v_i) : i = 1,...l\}$ be a strategy of z. If f,g satisfy all the assumptions of Theorem 5.1 and $\varphi(\cdot,v_i)$ is differentiable at $z(t_i)$ then there exists a left-continuous function $q:[0,1] \to \mathbf{R}^n$ satisfying (5.4), (5.5), (6.7), a), b), c) and

$$q(t_i) = z(t_i+)(id + \frac{\partial \varphi}{\partial x}(z(t_i),v_i)) \tag{6.8'}$$

$$q(t_i+) \in T_{\varphi(z(t_i),\, V)}(\varphi(z(t_i),v_i))^+ \tag{6.9'}$$

As in section 2 we associate the reachable set $R(t,C)$ at time t with the differential inclusion (6.6).

To prove the Theorem 6.2 we need the following (simple) lemmas.

Lemma 6.4. For all $i = 1,...,l-1$ set

$$C_i = R(t_i,C) \cup \{x + \varphi(x,v) : x \in R(t_i,C) \ , \ \ v \in V(x)\}$$

Then

$$T_{\varphi(z(t_i),\, V(z(t_i)))}(\varphi(z(t_i),v_i)) \subset T_{C_i}(z(t_i+))$$

The proof follows from the inclusion $z(t_i) + \varphi(z(t_i), V(z(t_i))) \subset C_i$ and the definition of the contingent cone.

Lemma 6.5. For all $i = 1,...,l-1$ set

$$A_i = id + \frac{\partial \varphi}{\partial x}(z(t_i),v_i) + \frac{\partial \varphi}{\partial v}(z(t_i),v_i)\frac{\partial \nu_i}{\partial x}(z(t_i))$$

Then

$$A_i(T_{R(t_i,C)}(z(t_i))) \subset T_{C_i}(z(t_i+)) \quad .$$

Proof. Fix $1 \leq i \leq l-1$, $w \in T_{R(t_i,C)}(z(t_i))$ and let $h_j \to 0+$, $w_j \to w$ be such that $z(t_i) + h_j w_j \in R(t_i,C)$. Then

$$z(t_i) + h_j w_j + \varphi(z(t_i) + h_j w_j,\ \nu_i(z(t_i) + h_j w_j)) =$$

$$z(t_i+) + h_j w_j + \frac{\partial\varphi}{\partial x}(z(t_i),v_i)h_j w_j + \frac{\partial\varphi}{\partial v}(z(t_i),v_j)\frac{\partial\nu_i}{\partial x}(z(t_i))h_j w_j + o(h_j) \in C_i$$

The definition of the contingent cone ends the proof.

Lemma 6.6. Assume that z has the right derivative $z'(t_i+)$ at t_i and let $u \in U(z(t_i))$. Then the solution w of the linear system

$$\begin{cases} w' = [\frac{\partial f}{\partial x}(z(t),\bar{u}(t)) + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))\frac{\partial u_t}{\partial x}(z(t))]w \\ w(t_i) = A_i f(z(t_i),u) - z'(t_i+) \end{cases}$$

satisfies

$$w(t_{i+1}) \in T_{R(t_{i+1},C)}(z(t_{i+1}))$$

Proof. Fix $h_j \to 0+$ and let x be a solution of the inclusion

$$\begin{cases} x'(t) \in F(x(t)) \text{ a.e. in } [t_i,t_{i+1}] \\ x(t_i) = z(t_i)\ ; \quad x'(t_i) = f(z(t_i),u) \end{cases}$$

Then

$$x(t_i + h_j) = z(t_i) + h_j f(z(t_i),u) + o(h_j) \in R(t_i + h_j,C)$$

and therefore

$$x(t_i + h_j) + \varphi(x(t_i + h_j),\nu_i(x(t_i + h_j))) = z(t_i+) + h_j A_i f(z(t_i),u) + o(h_j)$$

Thus

$$x(t_i + h_j) + \varphi(x(t_i + h_j),\nu_i(x(t_i + h_j))) = z(t_i + h_j) + h_j[A_i f(z(t_i),u) - z'(t_i+)] + o(h_j)$$

and $A_i f(z(t_i),u) - z'(t_i+)$ can be seen as a variation of $R(\cdot,C)$ at $(t_i,x(t_i+))$. The proof then follows by the same arguments as Theorem 2.4.

Lemma 6.7. Assume that z has the left derivative $z'(t_i-)$ at t_i. Then for all $u \in U(z(t_i+))$

$$f(z(t_i+),u) - A_i z'(t_i-) \in T_{C_i}(z(t_i+))$$

Proof. Fix $h_j \to 0+$, $u \in U(z(t_i+))$ and set

$$x_j := z(t_i - h_j) + \varphi(z(t_i - h_j),\ \nu(z(t_i - h_j)))$$

Since F is locally Lipschitzian there exists $M > 0$ such that for all j and $t \in [t_i - h_j, t_i]$

$$\text{dist}\,(f(z(t_i+),u),\, F(x_j + (t - t_i + h_j)f(z(t_i+),u))) \leq M(\|x_j - z(t_i+)\| + h_j\|f(z(t_i+),u)\|) = 0(h_j)$$

This and Filippov's theorem imply that

$$x_j + h_j f(z(t_i+),u) \in R(t_i,C) + o(h_j)$$

The definitions of x_j and of the contingent cone end the proof.

Lemma 6.8. For all $p \in T_{R(t_{i+1},C)}(z(t_{i+1}))^+$ there exists $q \in W^{1,\infty}(t_i,t_{i+1})$ satisfying (5.11), such that

$$q(t_{i+1}) = p \ , \ \ q(t_i+) \in T_{\varphi(z(t_i),\,V(z(t_i)))}(\varphi(z(t_i),v_i))^+ \tag{6.13}$$

$$q(t_i+)A_i \in T_{R(t_i,C)}(z(t_i))^+ \tag{6.14}$$

Moreover q satisfies a), b), c) of Theorem 6.2 with $q(t_i) = q(t_i+)A_i$.

Proof. Consider the differential inclusion

$$\begin{cases} x'(t) \in f(x(t),U(x(t))) \ \text{ a.e. in } [t_i,t_{i+1}] \\ x(t_i) \in C_i \end{cases} \tag{6.15}$$

and observe that its reachable set $\hat{R}(t_{i+1},C_i)$ at time t_{i+1} is contained in $R(t_{i+1},C)$. Thus $p \in T_{\hat{R}(t_{i+1},C_i)}(z(t_{i+1}))^+$. By Corollary 4.8 applied on the time interval $[t_i,t_{i+1}]$ to (6.15) and linear operators

$$A(t) = \frac{\partial f}{\partial x}(z(t),\bar{u}(t)) + \frac{\partial f}{\partial u}(z(t),\bar{u}(t))\frac{\partial u_t}{\partial x}(z(t))$$

there exists $q \in W^{1,\infty}(t_i,t_{i+1})$ satisfying (5.11) such that $q(t_{i+1}) = p$ and

$$q(t_i+) \in T_{C_i}(z(t_i+))^+ \tag{6.16}$$

Then (6.13) follows from (6.16) and Lemma 6.4 and (6.14) results from (6.16) and Lemma 6.5. Lemma 6.7 and (6.16) imply b). Since q solves the linear equation (5.11), Lemma 6.6 implies that for all $u \in U(z(t_i))$

$$< q(t_i),\ A_i f(z(t_i),u) - z'(t_i+) > \ \geq 0$$

Hence a). On the other hand by [13]

$$z'(t_i-) \in \overline{co}F(z(t_i-))$$

$$z'(t_i+) \in \overline{co}F(z(t_i+))$$

This and a), b) imply that

$$<q(t_i),z'(t_i-)> \leq \min_{u \in U(z(t_i+))} <q(t_i+),f(z(t_i+),u)> \leq$$

$$<q(t_i+),z'(t_i+)> \leq \min_{u \in U(z(t_i))} <q(t_i),f(z(t_i),u)> \leq <q(t_i),z'(t_i-)>$$

and the claim c) follows.

Proof of Theorem 6.2. Since z is an optimal trajectory $g'(z(1))w \geq 0$ for all $w \in T_{R(1,C)}(z(1))$. Thus $g'(z(1)) \in T_{R(1,C)}(z(1))^+$ and we may apply Lemma 6.8. with $p = g'(z(1))$. Set

$$q(t_{l-1}) = q(t_{l-1}+)A_{l-1} \in T_{R(t_{l-1},C)}(z(t_{l-1}))^+ \quad .$$

Then Lemma 6.8 can be applied again with $p = q(t_{l-1})$. We complete the proof using an induction argument and Lemma 6.8.

Observe that the Lipschitz continuity of $f(x,\cdot)$ is needed to prove the local Lipschitzianity of the map $x \to f(x,U(x))$. When the control map U does not depend on x, the set-valued map $x \to f(x,U)$ is locally Lipschitzian and therefore the same proof implies Theorem 6.3.

Remark. Theorems 6.2 and 6.3 can be stated together with a higher order condition on the adjoint vector q. However we do not do it here in order to simplify the presentation of the result.

7. Examples

Example 1: Smooth control system.

Consider the following optimal control problem in $\mathbf{R}^2$:

$$\text{minimize } y(1)$$

over the solutions of control system

$$\begin{cases} x' = 1 + u(x + y^2), \; u \in \{0,1\} \\ y' = u(2y - x) \\ x(0) = y(0) = 0 \end{cases} \tag{7.1}$$

Set $\bar{u} \equiv 0$. Then $z(t) = (t,0)$ is a solution of (7.1). Moreover $q \equiv (0,1)$ verifies the maximum principle (5.4). On the other hand, setting $u \equiv 1$ we obtain the following Taylor expansion of the corresponding solution (x,y) of (7.1)

$$x(t) = tx'(0) + \frac{t^2}{2}x''(0) + o(t^2) = t + \frac{t^2}{2} + o(t^2)$$

$$y(t) = ty'(0) + \frac{t^2}{2}y''(0) + o(t^2) = -\frac{t^2}{2} + o(t^2)$$

Hence $z(t) + t^2(\frac{1}{2},-\frac{1}{2}) \in R(t,0) + o(t^2)$ and therefore $(\frac{1}{2},-\frac{1}{2}) \in W(0,z)$. But $<(0,1),(\frac{1}{2},-\frac{1}{2})> < 0$. Comparing with (5.6) we deduce that the pair $(z,\bar{u})$ is not optimal.

Example 2: Implicit dynamical system.

Consider the following problem in $\mathbf{R}^2$:

$$\text{minimize}\, 2\sin y(1) - x(1)$$

Over the solutions of the implicit system

$$\begin{cases} \dot{x}^4 + \exp(\dot{y} - 2\dot{x}) - 16x^2 - \exp(4x^2 - y^2) = 0 \\ x(0) = 0, y(0) = 0 \end{cases} \tag{7.2}$$

Then (7.2) satisfies all the assumptions of Lemma 5.8. Observe that $z = (x,y) \equiv 0$ is a solution of (7.2). Set $q \equiv (-1,2)$ and

$$F(0) = \{(u,v) \colon u^4 + \exp(u - 2v) - 1 = 0\}$$

Then for all $(u,v) \in F(0), u - 2v \leq 0$. Hence $\min\{<q,e>: e \in F(0)\} \geq 0$. Therefore q verifies the maximum principle (5.18)-(5.20). On the other hand the trajectory $t \to (-t^2,-2t^2)$ is a solution of (7.2). Hence $(-1,-2) \in W(0,z)$ and $<(-1,2)(-1,-2)> = -3 < 0$. Consequently (5.21) does not hold and therefore the zero trajectory is not optimal.

Example 3: Differential inclusion.

Consider the problem

$$\text{minimize}\, g(x(1))$$

over the solutions of the differential inclusion

$$x' \in F(x), \;\; x(0) = x_0 \tag{7.3}$$

where $F: \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is a set-valued map with convex images satisfying (H_1) and $g: \mathbf{R}^n \to \mathbf{R}$ is a differentiable function.

The high order variations for this problem can be studied via an extension of Lie brackets to set-valued maps. Although, repeating arguments from [14], we can do it for a general trajectory z of (7.3) at every point t where z is twice continuously differentiable, the calculations are quite lenghtly. This is why in this example we only treat the case

$$0 \in F(x_0)$$

and the constant trajectory $z \equiv 0$ using the ready results from [14].

From now on we assume that $0 \in F(x_0)$. To state a second order condition for optimality we recall

Definition 7.1 Let $Q \subset F(x)$. We set

$$[F,F]_Q(x) = \{dF(x,a)b - dF(x,b)a : a,b \in Q\}$$

The following theorem tests for optimality the constant trajectory $z \equiv x_0$.

Theorem 7.2 Let $A \subset dF(x_0,0)$ be a Lipschitzian closed convex process, $Q \subset F(x_0)$ be a convex set such that

(i) $0 \in \operatorname{rint} Q$

(ii) F is lower semicontinuously differentiable on $x_0 \times Q$ (see [14]).

If $z \equiv x_0$ is optimal then there exists a solution q of the differential inclusion

$$-q' \in A^*q \; ; \quad q(1) = g'(x_0)$$

Satisfying the minimum principle

$$\min_{e \in F(x_0)} \langle q(t), e \rangle = 0 \text{ for all } t \in [0,1]$$

and the second order condition

$$q(t) \in (dF(x_0,0)Q)^+, \quad q(t) \in ([F,F]_Q(x_0))^+$$

for all $t \in [0,1]$.

Proof. Fix $t \in [0,1]$. By [14, Theorem 5.2], $dF(x_0,0)Q \subset R^\infty(t,x_0)$. From [14, Proof of Theorem 6.1] we deduce that $[F,F]_Q(x_0) \subset R^\infty(t,x_0)$. Since $z \equiv x_0$ is optimal, $g'(x_0) \in T_{R(1,x_0)}(x_0)^+$. Theorem 4.2 ends the proof.

Final remark. It is clear that the creation of a differential and "variational" calculus of set-valued maps (applied to reachable sets) is needed to make the field of applications broader. Special difficulties do arise at all points where the trajectory tested for optimality is not continuously differentiable. This difficulty was not overcomed up to now in the literature by any theorem concerning high order necessary conditions. It is usually assumed that the optimal trajectory is C^{∞} (or piecewise C^{∞}) (see for example [20], [19], [4]). But, because of the Lavrentieff phenomenon, such assumption is not reasonable. This is also the reason why we state here necessary conditions using "general" variations of reachable sets.

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