Working Paper

Probabilistic Models of Economic Dynamics with Endogenous Changes of Technology

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IIASA

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Foreword

The probabilistic counterparts of classic models of economic dynamic models with discrete emergence innovations are discussed. Stochastic discrete maximum principles are used to prove the existence of dual variables (stimulating prices).

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Probabilistic Models of Economic Dynamics with Endogenous Changes of Technology

V.I. Arkin

We study economic dynamics models in which technological changes (the emergence of new technological modes) are related to the expenditures of resources taken from the sphere of material production. The production sphere is described by the dynamic model "input–output," which in turn is defined in terms of technological sets or production functions. New technologies arise from the sphere of "technological progress" (TP), described by a similar model. The instants at which new technologies emerge are taken to be random variables, whose characteristics depend upon the functioning of the TP sphere. So, we have the optimization problem of allocating resources between the production and the TP spheres and choosing the corresponding technological modes in the respective spheres. This approach was proposed in [1], in which a general scheme for describing economic dynamics models with endogenous TP was formulated in terms of the controlled random processes theory.

Two models are considered in this paper. The first is a generalization of the classical Gale model where the probability of technological change at instant t is determined by "funds" accumulated up to that time in the TP sphere. The second model is the stochastic analog of the multisectoral macroeconomic model that was discussed in [2] for the case of continuous time. As in the first model, the production function change is random and is determined by TP funds.

The main results discussed in this paper are a description of dual variables (stimulating prices) and the establishment of the related indicators of economic efficiency. In the system of obtained stimulating prices, the estimates of new technologies related to the stochastic nature of the models should be singled out. They have no deterministic analogs.

1 Presentation of the Approach

The general model of economic dynamics can be described in terms of a multivalued mapping $Q(\cdot)$ translating the point $x \in R^n_+$, characterizing the stock of products at the beginning of the planning period, to one of the points of the set $Q(x) \subseteq R^n_+$ at the end of the period. The set $Q(x)$ describes the "production possibilities" of the economy in the state x for one period, i.e., the set of all outputs that can be obtained from the resource vector x at the end of the period. The graph of the mapping $Q = \{x, Z : Z \in Q(x)\}$ characterizes the body of our knowledge of the system's technology as a whole. So, with the given initial resource vector $\hat{x}$, the dynamics of the system can be described by:

$$x_{t+1} \in Q(x_t), \qquad x_0 = \hat{x}_0 \tag{1}$$

Suppose that there exists the possibility of transition to a new technology Q^1 with larger production possibilities than the technology $Q, Q^1 \supset Q$. For this possibility to be realized, some funds should be made, i.e., some resources should be spent. Assume that these expenditures are characterized by a vector y, some components of which may be zero. Thus the new technology Q^1 may be used in states x, satisfying the relation $x \geq y$ (for each coordinate). If we introduce a new multivalued mapping

$$R(x) = \begin{cases} Q^1(x)\,, & x \geq y \\ Q(x)\,, & \text{otherwise} \end{cases} \tag{2}$$

then, the dynamics of the system, taking into account the new technology, can be described again by a relation of type (1). Note, however, that even if the sets Q and Q^1 are convex, the graph of the multivalued mapping (2) (the set R) is not necessarily convex. Let x_t satisfy the inclusion $x_{t+1} \in R(x_t)$, $x_0 = \hat{x}_0$ and τ be a finite planning horizon. The instant of transition to the new technology is denoted by Θ

$$\Theta = \min\{t : x_t \geq y\} \tag{3}$$

If the inequality associated with minimum is not satisfied for any $t \geq \tau - 1$, we assume $\Theta = \tau$. In other words, Θ is the first instant when the technology Q^1 may be used. Then the system's trajectory satisfies the evident relations

$$\begin{aligned} x_{t+1} &\in Q(x_t), & t &< \Theta, \\ x_{t+1} &\in Q^1(x_t), & t &\geq \Theta. \end{aligned} \tag{4}$$

In real situations, the expenditures necessary for transition to the new technology are not know exactly and we may speak only about the influence of these expenditures on the probability of transition to the new technology Q^1 during one planning period. Assume that the emergence of a new technology is a random event, at the interval $(t, t+1)$ the probability of this event is characterized by a given non-increasing function $\pi(x_t)$. In the deterministic situation

$$\pi(x) = \begin{cases} 1, & x \geq y \\ 0, & \text{otherwise} \end{cases}$$

In a stochastic model we shall call "plan" the sequence satisfying the relations (4), where Θ takes values $1, 2, \ldots, \tau - 1, \tau$.

Before instant Θ, the plan $\{x_t\}$ is a function of time, after this instant, it is a function of the values Θ, x_Θ and a current instant $t \geq \Theta$, $x_t = x_t(\Theta, x_\Theta)$. The probability distribution of the instant Θ is define by the formulae:

$$\begin{aligned} P(\Theta = t+1) &= \pi(x_t) \prod_{k=0}^{t-1} (1 - \pi(x_k))\,, \qquad t = 1, \ldots, (t-2) \\ P(\Theta = 1) &= \pi(x_0) \end{aligned} \tag{5}$$

Let $\varphi(x)$, $\varphi^1(x)$ be utility functions. It is required to find a plan providing the maximum to the relation:

$$E\left[\sum_1^{\Theta-1} \varphi(x_t) + \sum_\Theta^{\tau-1} \varphi^1(x_t)\right] \longrightarrow \max \tag{6}$$

This formulation of the problem may be extended for the case when there are N technologies, $Q^1 \subset Q^2 \subset \ldots \subset Q^N$, widening sequentially the initial technological set $Q^0 = Q$.

Let Θ_k be the instant of emergence of the technology Q^k, $\Theta_0 = 0 < \Theta_1 < \Theta_2 < \ldots < \Theta_N \leq \tau$, $\{x_t\}$ be the system's trajectory (plan), and satisfy the inclusions:

$$x_{t+1} \in Q^k(x_t), \qquad \Theta_k \leq t \leq \Theta_{k+1}, \qquad k = 0, \ldots, N, \qquad x_0 = \hat{x}_0\,. \tag{7}$$

In the interval $\Theta_k \leq t < \Theta_{k+1}$ the plan $\{x_t\}$ is a function of the instant t and values Θ_k, x_{Θ_k}. In other words, if it is known that there is the technology Q^k in the system at the instant t, then to find the plan at this instant we only need to know when the technology Q^k first appeared and what the stock of the resources $x(\Theta_k)$ was at that time. The probabilities of transition from the technology Q^k to the technology Q^{k+1} at the interval $(t, t+1)$ are given. A joint distribution of random variables $(\Theta_1, \dots, \Theta_N)$ is calculated with the help of the conditional probabilities

$$P(\Theta_{k+1} = i+1 \mid \Theta_k = j) = \pi^k(x_i) \prod_{k=j}^{i-1} \left(1 - \pi^k(x_t)\right), \qquad j < i \tag{8}$$

and the initial distribution of the instant Θ_1 given by formula (5).

Let the target functions $\varphi^k(x)$ characterizing the income of the system with the technology Q^k in the state x be given. Let us introduce the random variables:

$$J_K = \sum_{t=\Theta_k}^{\Theta_{k+1}-1} \varphi^k(x_t), \qquad k = 0, \dots, N, \qquad \Theta_0 = 0, \qquad \Theta_{N+1} = \tau \tag{9}$$

expressing the income of the system from the technology Q^k (see footnote 1). It is required to maximize the summary income

$$E \sum_{k=0}^{N+1} J_k \longrightarrow \max \tag{10}$$

for all plans satisfying the relations (7).

This scheme can be used when the parameters of future technologies are not completely known. In this case, we consider that every technology Q^k depends on a random parameter s_k with a known probability distribution. The random parameters s_k are independent and their values become known in the instant of transition to a corresponding technology. Thus, the plan at the instant $t, \Theta_k \leq t < \Theta_{k+1}$ is completely determined by the values of random variables $\Theta_k, s_k, x_{\Theta_k}$. The general theory (the theorem of sufficiency of the Markovian controls) implies that consideration of the plans which, at instant t, depend on the whole trajectory up to instant t (general nonanticipative plans) does not increase the value of the problem.

2 Account of Technological Progress in the Gale Model: Stimulating Prices

2.1 Presentation of the model

The Gale model "input–output" is characterized by a technological set T, the elements of which are the pairs of non-negative n-dimensional vectors (a, b) and by a target function $\varphi(a, b)$. The set T is assumed to be convex and the function φ to be concave. The pairs (a, b) are treated as technological processes (modes of production), a being input and b being output in a sufficiently wide sense. A sequence of the technological processes $z_t = (a_t, b_{t+1}) \in T$ is called a plan if the following condition holds:

$$b_t \geq a_t, \qquad t = 0, 1, \dots, \tau - 1 \ . \tag{11}$$

Vector b_0 is called the vector of the initial stocks and is assumed to be fixed. It is required to find a plan providing the maximum to the expression

$$\sum_0^{\tau-1} \varphi(z_t) \longrightarrow \max \tag{12}$$

A sequence of non-negative n-dimensional vectors $\{\psi_t\}, t = 0, \dots, \tau$ will be called a price system. The price system is said to stimulate the plan $\{\hat{z}_t\}$ if the following conditions hold:

[1]Without restricting generality, we may assume that all the random variables Θ_k are concentrated in the interval $[0, \tau]$. Otherwise, we can always take the new variables $\Theta_k = \min(\Theta_k, \tau)$.

1. For every $t \geq 1$ the pair $(\hat{a}_t, \hat{b}_{t+1})$ brings the maximum to the function

$$F_t(a,b) = \varphi(a,b) + \psi_{t+1}b - \psi_t a \tag{13}$$

for all $(a,b) \in T$.

2. For all $t \geq 0$,

$$\psi_t(\hat{b}_t - \hat{a}_t) = 0 \tag{14}$$

The economic interpretation of these conditions is well-known. If some regularity condition holds, then the optimal plan $\{\hat{z}_t\}$ is stimulated by some price system. We give a variant of a regularity condition: for every $t \geq 0$, there exists a technological mode $(\hat{a}_t, \hat{b}_{t+1}) \in T$ such that

$$\hat{b}_t > \hat{a}_t, \qquad t = 0, \ldots, \tau - 1 \ . \tag{15}$$

It is clear that if T contains the point $(0,0)$, condition (15) means producing all products on the optimal plan.

Let us assume that, together with the technology $T = T_0$, there are possibilities of transition to other, more progressive technologies $T_1 \subset T_2 \subset \ldots \subset T_N$, $T_1 \supset T_0$. These transitions may be only sequential, i.e., from T_1 it is impossible to move to T_3 without T_2. As mentioned above, to realize this possibility, expenditures of resources are necessary. These resources are taken from the sphere of material production which is described by the Gale model formulated above. These expenditures are realized in the TP sphere which is also described by the model "input-output", i.e., the set of pairs $(c,d) \in Q \subseteq R_+^{2m}$ is given, c being input and d being output for one planning period. A vector d is interpreted as funds made in the TP sphere. The main products of the TP sphere are the new technologies developed on the funds d. It is assumed that if the technology T_k was in the production sphere at instant t, then the probability of emergence of the technology T_{k+1} at instant $t+1$ is $\pi_k(d_t)$, where a given function $\pi_k(d)$ is non-decreasing and differentiable with respect to all its arguments. Let $\Theta_1 < \Theta_2 < \cdots < \Theta_N$ be instants of emergence of the technologies $T_1, T_2, \ldots, T_N$. We consider the time interval $\Theta_k \leq t < \Theta_{k+1}$. At this interval the technology T_k is used. We denote by V_t the part of the output of the material production sphere given to the TP sphere at instant t. Then, in the interval $\Theta_k \leq t < \Theta_{k+1}$ the plan $z_t = \{(a_t, b_{t+1}), V_t, (c_t, d_{t+1})\}$ satisfies the relations:

$$\begin{gathered} b_t - V_t \geq a_t \ , \\ d_t + V_t \geq c_t \ , \\ (a_t, b_{t+1}) \in T_k \ , \ (c_t, d_{t+1}) \in Q \\ V_t \geq 0 \\ k = 0, 1, \ldots, N \quad \Theta_{N+1} = \tau \end{gathered} \tag{16}$$

Thus, in the interval $\Theta_k \leq t < \Theta_{k+1}$ the plan is found based on information about the instant Θ_k when the new technology T_k emerged, about the stocks of resources b_{Θ_k}, and the TP funds d_{Θ_k} at this instant. Vectors of the initial resources $\hat{b}_0$ and the initial funds $\hat{d}_0$ are given. When $x_t = d_t$, the conditional distribution of instant Θ_k is determined by formula (8). We assume that the corresponding target function $\varphi_k(a,b)$ is defined for each technological set T_k. The sets T_k are assumed to be convex and the functions φ_k to be concave. Moreover, we assume $\varphi^k(a,b) \geq \varphi^{k-1}(a,b)$, $(a,b) \in T_{k-1}$. It is required to maximize the function

$$E \sum_{k=0}^{N} J_k \longrightarrow \max \ , \tag{17}$$

where

$$J_k = \sum_{t=\Theta_k}^{\Theta_{k+1}-1} \varphi^k(a_t, b_{t+1}) \qquad k = 0, 1, \ldots, N \qquad \Theta_{N+1} = \tau \tag{18}$$

2.2 Stimulating prices

We say that the system of prices

$$\psi_t^k = \psi_t^k(\theta_1, \ldots, \Theta_k) \ , \quad \mu_t^k = \mu_t^k(\Theta_1, \ldots, \Theta_k) \ ,$$
$$h_t^k = h_t^k(\theta_k, \hat{b}_{\Theta_k}, \hat{d}_{\Theta_k}) \ , \quad \Theta_k \leq t < \Theta_{k+1}$$

stimulates the plan $\{(\hat{a}_t, \hat{b}_{t+1}), \hat{V}_t, (\hat{c}_t, \hat{d}_{t+1})\}$, if the following conditions hold:

1. For every t, $\Theta_k \leq t < \Theta_{k+1}$ the technological mode $(\hat{a}_t, \hat{b}_{t+1})$ provides the maximum to the value
$$F_t^k(a, b) = \varphi^k(a, b) + \bar{\psi}_{t+1}^k b - \psi_t^k a \tag{19}$$
for all the pairs $(a, b) \in T_k$. Here
$$\bar{\psi}_{t+1}^k = \left(1 - \pi^k(\hat{d}_t)\right) \psi_{t+1}^k + \pi^k(\hat{d}_t)\psi_{t+1}^{k+1} \qquad \psi_\tau^k = 0 \qquad k = 0, \ldots, N \tag{20}$$
$$\psi_t^k(\hat{b}_t - \hat{V}_t - \hat{a}_t) = 0 \tag{21}$$

2. For every $\Theta_k \leq t < \Theta_{k+1}$ the process $(\hat{c}_t, \hat{d}_{t+1})$ maximizes the value
$$\Phi_t^k(c, d) = \bar{\mu}_{t+1}^k \cdot d - \left(\mu_t^k - h_{t+1}^k \frac{\partial \pi_k}{\partial d}(\hat{d}_t)\right) c \tag{22}$$
for all the technological modes $(c, d) \in Q$. Here
$$\bar{\mu}_{t+1}^k = \left(1 - \pi^k(\hat{d}_t)\right) \mu_{t+1}^k + \pi^k(\hat{d}_t)\mu_{t+1}^{k+1} \qquad \mu_\tau^k = 0 \ . \tag{23}$$
The scalar value $h_{t+1}^k \geq 0$ is defined by the formula:
$$h_{t+1}^k = E\left[\hat{W}_{t+1}^k | \theta_{k+1} = t, \Theta_k, \hat{b}_{\Theta_k}, \hat{d}_{\Theta_k}\right] - E\left[\hat{W}_{t+1}^k | \theta_{k+1} > t, \Theta_k, \hat{b}_{\Theta_k}, \hat{d}_{\Theta_k}\right] \ , \tag{24}$$
where
$$\hat{W}_{t+1}^k = \sum_{\nu=t+1}^{\Theta_{k+1}-1} \varphi^k(\hat{a}_\nu, \hat{b}_{\nu+1}) + \sum_{k+1}^{N} \hat{J}_\nu$$
$$\hat{J}_\nu = \sum_{t=\Theta_\nu}^{\Theta_{\nu+1}-1} \varphi^\nu(\hat{a}_t, \hat{b}_{t+1}) \ .$$
The following inequality takes place:
$$\mu_t^k \geq h_{t+1}^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \ . \tag{25}$$
If the following strict inequality holds for some coordinates:
$$\hat{d}_t + \hat{V}_t > \hat{c}_t \ , \tag{26}$$
then for these coordinates the following relation holds:
$$\mu_t^k = h_t^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \ . \tag{27}$$

3. For the resources which are taken from the production sphere to the TP sphere $\hat{V}_t > 0$ (for all the coordinates)

$$\psi_t^k = \mu_t^k - h_{t+1}^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \ . \tag{28}$$

For other resources, $\hat{V} = 0$

$$\psi_t^k \geq \mu_t^k - h_{t+1}^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \ . \tag{29}$$

Theorem 1 *Let the plan $\{\hat{z}_t\}$ be optimal and the following regularity condition hold: There exist non-negative vectors $\bar{V}_t^k \geq 0$ and technological modes $(\tilde{a}_t^k, \tilde{b}_{t+1}^k) \in T_k$, $(\tilde{c}_t^k, \tilde{d}_{t+1}^k) \in Q$, $k = 0, 1, \dots, N$, $\Theta_k \leq t < \Theta_{k+1}$ (not necessarily forming the plan) such that*

$$\begin{aligned} \hat{b}_t + \bar{V}_t^k &> \tilde{a}_t^k \ , \\ \hat{d}_t + \bar{V}_t^k &> \tilde{c}_t^k \ . \end{aligned} \tag{30}$$

Then, there exist prices stimulating the plan $\hat{z}$.

Proof. We use the maximum principle for a stochastic control problem (Theorem 2 in [4]). For this we reformulate the initial model of economic dynamics as the optimal control problem. We consider the random process S_t taking values $T_0, T_1, \dots, T_N$.

We introduce subsidiary variables (phase coordinates) $x_t = x_t(S^t)$, $y_t = y_t(S^t)$, where $S^t = (S_0, \dots, S_t)$ is the history of the process S_t up to instant t.

$$\begin{gathered} x_{t+1} = b_{t+1} \ , \qquad x_0 = \hat{b}_0 \ , \\ y_{t+1} = d_{t+1} \ , \qquad k_0 = \hat{d}_0 \ , \\ x_t - V_t \geq a_t \ , \\ y_t - V_t \geq c_t \ , \\ V_t \geq 0 \ , \ \ (a_t, b_{t+1}) \in S_t \ , \ \ (c_t, d_{t+1}) \in Q \end{gathered} \tag{31}$$

The control $U_t = \{V_t, (a_t, b_{t+1}), (c_t, d_{t+1})\}$ is looked up in the class of Markovian controls $U_t = U_t(S_t, x_t, y_t)$. Transition probabilities of the process S_t are given by the formulae:

$$\begin{aligned} P(S_{t+1} = T_{k+1}/S_t = T_k) &= \pi^k(y_t) \ , \\ P(S_{t+1} = T_k/S_t = T_k) &= 1 - \pi^k(y_t) \ . \end{aligned}$$

The other transition probabilities are zero. The initial distribution $P(S_0 = T_0) = 1$ is also given. It is required to maximize the function:

$$E \sum_{t=0}^{\tau-1} \varphi^{S_t} (a_t(x_t, y_t, S_t), , b_{t+1}(x_t, y_t, S_t)) \longrightarrow \max \tag{32}$$

The mathematical expectation in formula (32) is taken in the measure (in the space of sequences $\{S^\tau\}$) generated by the transition probabilities and the initial distribution mentioned above.

This optimal control problem is equivalent to the initial model of economic dynamics. Let every trajectory in the space of technologies correspond to the sequence $\Theta_1, \Theta_2, \dots, \Theta_m$ where Θ_i is the instant of emergence of technology T_i. This is a one-to-one correspondence. In the interval $\Theta_k \leq t < \Theta_{k+1}$ the Markovian control is $u_t(x_t, y_t, S_t) = u_t(x_t(b_{\Theta_k}, d_{\Theta_k}, \Theta_k), y_t(b_{\Theta_k}, d_{\Theta_k}, \Theta_k), T_k) = w_t(\Theta_k, b_{\Theta_k}, d_{\Theta_k})$, i.e., it is a plan in the problem of economic dynamics. The probability of the sequence S^τ generated by the Markovian plan is equal to the probability of the corresponding sequence $\Theta_1, \dots, \Theta_N$. Thus, by the optimal plan of the control problem, we can construct an optimal plan for modeling of economic dynamics. It is obvious that the reverse statement is true.

Problem (31–32) with the assumptions made and the regularity condition taken into account is completely described by Theorem 2 in [2].

Compose the Hamiltonian:

$$\begin{aligned} H_{t+1} &= \varphi^{S_t}(a_t, b_{t+1}) + \psi_{t+1} b_{t+1} + \mu_{t+1} d_{t+1} - \lambda_t(a_t + V_t - x_t) - \\ &\quad - \alpha_t(c_t - V_t - y_t) + q^1_{t+1}\pi^{S_t}(y_t) + q^1_{t+2}(1 - \pi^{S_t}(y_t)) \ . \end{aligned} \tag{33}$$

in accordance with the maximum principle there are adjoint variables $\psi_t = \psi_t(S^t)$, $\mu_t = \mu_t(S^t)$, $\lambda_t = \lambda_t(S^t)$, $\alpha_t = \alpha_t(S^t)$, $q^1_{t+1} = q^1_{t+1}(\hat{x}_t, \hat{y}_t, S_t)$, $q^2_{t+1} = q^2_{t+1}(\hat{x}_t, \hat{y}_t, S_t)$, such that the value

$$\varphi^{S_t}(a,b) + \bar{E}\left[\psi_{t+1}/S^t\right] b - \lambda_t a + \bar{E}\left[\mu_{t+1}/S^t\right] d - \mu_t c + (\alpha_t - \lambda_t)V \tag{34}$$

reaches its maximum for variables $(a,b) \in S_t$, $(c,d) \in Q$, $V \geq 0$ on the optimal plan. For dual variables from the adjoint system of the maximum principle we obtain the following relations:

$$\begin{aligned} &\psi_t = \lambda_t \ , \quad d_t = \mu_t - h_{t+1}\frac{\partial \pi^{S_t}}{\partial y_t}(\hat{y}_t) \ , \\ &\psi_\tau(S^\tau) = 0 \ , \quad \mu_\tau(S^\tau) = 0 \ , \\ &h_{t+1} = q^1_{t+1} - q^2_{t+1} \ , \\ &q^1_{t+1} = \hat{E}\left[\sum_{t+1}^{\tau-1} \varphi^{S_\nu}(\hat{a}_\nu, \hat{b}_{\nu+1})/S_{t+1} \subset S_t, \hat{x}_{t+1}, \hat{y}_{t+1}\right] \ , \\ &q^2_{t+1} = \hat{E}\left[\sum_{t+1}^{\tau-1} \varphi^{S_\nu}(\hat{a}_\nu, \hat{b}_{\nu+1})/S_{t+1} = S_t, \hat{x}_{t+1}, \hat{y}_{t+1}\right] \ . \end{aligned} \tag{35}$$

Taking into account (35), (33) implies that on the optimal plan the following expressions reach their maxima:

$$\begin{aligned} F_{t+1}(a,b,S_t) &= \varphi^{S_t}(a,b) + E[\psi_{t+1}|S_t]\cdot b - \psi_t a \longrightarrow_{(a,b)\in S_t} \max \\ \Phi_{t+1}(c,d,S_t) &= \hat{E}[\mu_{t+1}/S^t]d - \left(\mu_t - h_{t+1}\frac{\partial \pi^{S_t}}{\partial y}(\hat{y}_t)\right)c \longrightarrow_{(c,d)\in Q} \max \ . \end{aligned} \tag{36}$$

Further, (33) implies that if $\hat{V}_t > 0$ for some coordinates, then for these coordinates:

$$\psi_t = \mu_t - h_{t+1}\frac{\partial \pi^{S_t}}{\partial y}(\hat{y}_t) \tag{37}$$

If $\hat{V}_t = 0$ for some coordinates, then, for these coordinates:

$$\psi_t \geq \mu_t - h_{t+1}\frac{\partial \pi^{S_t}}{\partial y}(\hat{y}_t) \tag{38}$$

Let every trajectory S^t in the space of technologies correspond to the sequence $\Theta_1, \ldots, \Theta_k$ where Θ_i is the instant of emergence of the technology T_i $(i = 1, \ldots, k)$. This is a one-to-one correspondence. Let

$$\left.\begin{aligned}\psi_t^k(\Theta_1,\ldots,\Theta_k) &= \psi_t(S^t)\ ,\\ \mu_t^k(\Theta_1,\ldots,\Theta_k) &= \mu_t(S^t)\ ,\end{aligned}\right\} \quad \Theta_k \le t < \Theta_{k+1}$$

$$q_{t+1}^1\left(\hat{x}_{t+1}(\Theta_k,\hat{x}_{\Theta_k},\hat{y}_{\Theta_k}),\hat{y}_{t+1}(\Theta_k,\hat{y}_{\Theta_k})\right) = \hat{E}\left[\sum_{k+1}^{N} \hat{J}_\nu|\Theta_{k+1} = t+1,\Theta_k,\hat{x}_{\Theta_k},\hat{y}_{\Theta_k}\right]\ ,$$

$$q_{t+1}^2\left(\hat{x}_{t+1}(\Theta_k,\hat{x}_{\Theta_k},\hat{y}_{\Theta_k}),\hat{y}_{t+1}(\Theta_k,\hat{y}_{\Theta_k})\right) = \hat{E}\left[\sum_{t+1}^{\Theta_{k+1}-1} \varphi^k(\hat{a}_\nu,\hat{b}_{\nu+1}) + \right.$$
$$\left. + \sum_{k+1}^{N} \hat{J}_\nu|\Theta_{k+1} > t+1,\Theta_k,\hat{x}_{\Theta_k},\hat{y}_{\Theta_k}\right]\ ,$$

The expressions under the sign of mathematical expectation depend on the values $\hat{x}_{\Theta_k}$, $\hat{y}_{\Theta_k}$, and Θ_k as parameters since

$$\hat{x}_{t+1} = \hat{y}_{t+1}\ \ ,\ \ \hat{y}_{t+1} = \hat{d}_{t+1}$$

depend on

$$\Theta_k, \hat{x}_{\Theta_k} = \hat{b}_{\Theta_k}\ ,\ \hat{y}_{\Theta_k} = \hat{d}_{\Theta_k}$$

by virtue of the definition of the plan for $t \in [\Theta_k,\Theta_{k+1})$ The transversality condition of the maximum principle implies immediately the relations (21), (26), (27). Thus, we have proved the existence of stimulating prices with the properties mentioned above.

2.3 Economic interpretation

First of all we remark that as well as in the deterministic Gale model the stimulating prices take off time balance constraints and allow to screen unefficient technological modes on the basis of the local information without recalculating the entire problem. In this sense expression (19) may be treated as a local efficiency criterion in the sphere of material production. The main difference from the deterministic situation is that the output in the production sphere at the instant $t+1$ is evaluated in the prices $\bar{\psi}_{t+1}^k$ rather than in the prices ψ_{t+1}^k. As formula (20) shows, the prices are expected (forecasting) prices, because at instant $t+1$, the emergence of a new technology T_{k+1} is possible with the probability $\pi^k(\hat{d}_t)$ and transition to the plan corresponding to the technology T_{k+1} with the vector of initial resources $\hat{b}_{t+1}$ and their prices ψ_{t+1}^{k+1}. The condition (21) has a traditional interpretation meaning that the prices of unused resources are zero.

Now we proceed to the analysis of relation (22). The function $\Phi_t^k(c,d)$ plays the part of a local optimality criterion in the TP sphere. The values μ_t^k are treated as estimates of resources spent at instant t and the values $\bar{\mu}_{t+1}^k$ as well as the prices $\bar{\psi}_{t+1}^k$ already considered, are forecasting estimates of the funds which are turned out in the TP sphere at the planning period $(t,t+1)$.

Next we turn to the analysis of the value h_t^k. This estimate has no deterministic analogs and it is wholly stipulated by the probabilistic structure of the problem. Formula (35) implies that this estimate is calculated explicitly by the optimal plan and shows increment of the global criterion if the technology T_{k+1} emerges at instant $t+1$ in comparison with its emergence at the instants following $t+1$. The funds $\hat{d}_t$ which are in the TP sphere at instant t carry out two functions at the interval $(t,t+1)$. Firstly, they are used for creating a new technology T_{k+1}; secondly, they are used as inputs for the creation of new funds $\hat{d}_{t+1}$ (the work done in anticipation of the future). The value $h_{t+1}^k\pi(\hat{d}_t)$ may be treated as the gain expected from the transition to the technology T_{k+1} at the instant $t+1$. The derivative of this value $h_{t+1}^k\frac{\partial\pi^k}{\partial d}(\hat{d}_t)$ characterizes the increment of this gain for the unit of the resources spent. The value μ_t is the complete estimate of resources in the TP sphere at instant t. Since the resources spent at instant

t are used in the production of new funds and do not influence the emergence of the technology T_{k+1} at the instant $t+1$, their estimate is expressed by the value

$$\mu_t^k - h_{t+1}^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \geq 0$$

As to the resources which are not spent for the creation of new funds in the interval $(t, t+1)$ but influence the emergence of the technology T_{k+1}, their complete estimate is as follows (see relation (27)):

$$\mu_t^k = h_{t+1}^k \frac{\partial \pi^k}{\partial d}(\hat{d}_t) \ .$$

3 A Resource Allocation Model: The Dynamics of Stimulating Prices

3.1 A resource allocation model

Let the state of the economy be characterized by a set of production funds (factors of production) $x = (x^1, \ldots, x^n)$. For simplicity we assume that only one product is produced and consumed in this system. A production function $f(x)$ is given that shows what quantity of a product can be produced with the help of this set of production funds x. The dynamics of the system is described in the following way. If at instant t the set of production funds is characterized by the vector x_t, then, by instant $t+1$ the product $f(x_t)$ is produced. This product is distributed among the production funds. We denote by u^i the part of the output which is used for the increase of the funds of the i-th type. Then the dynamics of the system can be described by the following equations:

$$x_{t+1}^i = x_t^i + u_t^i f(x_t) \ ; \qquad t = 0, 1, \ldots, \tau - 1 \ ; \qquad i = 1, \ldots, n \ ; \qquad x_0 = \hat{x}_0 \ . \tag{39}$$

The set of control parameters $u_t = (u_t^1, \ldots, u_t^n)$,

$$u_t^i \geq 0 \ , \ \sum_{i=1}^{n} u_t^i = 1 \tag{40}$$

we call "the plan of allocation of resources" or simply "the plan".

If the initial vector of production funds $\hat{x}_0$ and some plan $\{u_t\}$ are given, then equation (39) determines the trajectory $\{x_t\}$. It is required to find the plan maximizing the function:

$$\sum_{1}^{\tau-1} \varphi(x_t) \longrightarrow \max \tag{41}$$

Further we shall assume that $f(x)$, $\varphi(x)$ are continuously differentiable, non-decreasing functions.

Let $\{\hat{u}_t\}$ be a plan of allocation of resources and $\{\hat{x}_t\}$ be the trajectory corresponding to it. We shall say that the system of estimates $\{\psi_t\}$ meets the plan $\{\hat{u}_t\}$ if

$$\psi_t^i = \psi_{t+1}^i + \max_j(\psi_{t+1}^j) \frac{\partial f}{\partial x^i}(\hat{x}_t) + \frac{\partial \varphi}{\partial x^i}(\hat{x}_t) \ , \qquad t = 0, 1, \ldots, \tau - 1 \ , \qquad \psi_\tau = 0 \ . \tag{42}$$

We denote $\alpha_t = \max_j\{\psi_t^j\}$. We shall say that the system of estimates meeting the plan $\{\hat{u}_t\}$ stimulates the plan $\{\hat{u}_t\}$ if

$$\max_{\sum u^i = 1, u^i \geq 0} \sum_{i=1}^{n} \psi_t^i u^i = \sum_{i=1}^{n} \psi_t^i \hat{u}_t^i = \alpha_t \tag{43}$$

Under these assumptions if the plan $\hat{u}_t$ is optimal, then the system of estimates meeting it stimulates this plan. This assertion follows from the maximum principle (see [3]).

3.2 Account of technological changes

We assume that from the technology given by the production function $f(x) = f^0(x)$, the sequential transition to the technologies given by the production functions $f^1(x) \le f^2(x) \le \cdots \le f^N(x)$ is possible. These transitions, as well as those in the model of the previous section, are given by the function $\pi^k(y_t)$ characterizing the probability of transition from the production function $f^k(x)$ to $f^{k+1}(x)$ in the time interval $(t, t+1)$. Here y_t represents the funds created in the TP sphere at the instant t.

Let $\Theta_0 = 0, \Theta_1, \ldots, \Theta_N$ be the instants of emergence of the technologies. Let $\Theta_{N+1} = \tau$.

The dynamics of the system is described in the following way:

$$\begin{aligned}
&x^i_{t+1} = x^i_t + u^i_t f^k(x)\,, \\
&y^j_{t+1} = y^j_t + v^j_t f^k(x)\,, \qquad \Theta_k \le t < \Theta_{k+1} \\
&\sum_{i=1}^{n} u^i_t + \sum_{j=1}^{m} v^j_t = 1 \\
&u^i_t \ge 0\,, \quad v^j_t \ge 0
\end{aligned} \tag{44}$$

The plan of allocation of the resources $w_t = \{u^i_t, v^j_t,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ at the interval $\Theta_k \le t < \Theta_k + 1$ is looked up as a function of the instant Θ_k and the parameters $z_{\Theta_k} = (x_{\Theta_k}, y_{\Theta_k})$, $w_t = w_t(\Theta_k, z_{\Theta_k})$. The distribution of the random variables $\Theta_1, \ldots, \Theta_N$ exactly as in the Gale model is given by the conditional probabilities (5) substituting x for y.

Let $\varphi^0(x) \le \varphi^1(x) \le \cdots \le \varphi^N(x)$ be the utility functions corresponding to the technologies $f^0(x), f^1(x), \ldots, f^N(x)$. We denote

$$J_k = \sum_{t=\Theta_k}^{\Theta_{k+1}-1} \varphi^k(x_t)\,.$$

It is required to maximize the function

$$E \sum_{k=0}^{N} J_k \longrightarrow \max \tag{45}$$

3.3 Stimulating estimates

We assume that $f^k(x), \varphi^k(x), \pi^k(y)$ are continuously differentiable and non-decreasing functions. Let $\{w_t\}$ be a resource allocation plan and $\{\hat{z}_t\}$ be the trajectory corresponding to it. We shall say that the system of estimates $\lambda^k(t) = \{\psi^k_i(t), \mu^k_j(t), h^k(t)\}$, $\psi^k_i = \psi^k_i(t, \Theta_k, \hat{z}_{\Theta_k})$, $\mu^k_j(t) = \mu^k_j(t, \Theta_k, \hat{z}_{\Theta_k})$, $h^k(t) = h^k(t, \Theta_k, \hat{z}_{\Theta_k})$ meets the plan $\{\hat{W}_t\}$ if

$$\begin{aligned}
\psi^k_i(t) &= \bar{\psi}^k_i(t+1) + \alpha^k(t+1)\frac{\partial f^k}{\partial x_i}(\hat{x}_t) + \frac{\partial \varphi^k}{\partial x_i}(\hat{x}_t)\,, \\
\mu^k_j(t) &= \bar{\mu}^k_j(t+1) + h^k(t+1)\frac{\partial \pi^k}{\partial y_i}(\hat{y}_t)\,, \\
\psi^k_i(\tau) &= \mu^k_j(\tau) = 0\,, \qquad \Theta_k \le t < \Theta_{k+1}\,,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&\bar{\psi}^k_i(t+1) = \psi^k_i(t+1)(1 - \pi^k(\hat{y}_t)) + \psi^{k+1}_i(t+1)\pi^k(\hat{y}_t)\,, \\
&\bar{\mu}^k_j(t+1) = \mu^k_j(t+1)(1 - \pi^k(\hat{y}_t)) + \mu^{k+1}_j(t+1)\pi^k(\hat{y}_t)\,, \\
&\alpha^k(t+1) = \max_{i,j}\left\{\bar{\psi}^k_i(t+1), \bar{\mu}^k_j(t+1), i = 1, \ldots, n\,, \quad j = 1, \ldots, m\right\}\,.
\end{aligned} \tag{47}$$

The scalar variable $h^k(t+1)$ is defined by the formula

$$h^k(t+1) = E\left[W_{t+1}^k/\Theta_{k+1} = t+1, \Theta_k, \hat{z}_{\Theta_k}\right] - E\left[W_{t+1}^k/\Theta_{k+1} > t+1, \Theta_k, \hat{z}_{\Theta_k}\right], \tag{48}$$

where [2]

$$W_{t+1}^k = \sum_{\nu=t+1}^{\Theta_{k+1}-1} \varphi^k(\hat{x}_\nu) + \sum_{\nu=k+1}^{N} \hat{J}_\nu$$

We shall say that the system of estimates $\{\lambda^k(t)\}$ meeting the plan $\{\hat{W}_t\}$ has the property of stimulation of this plan if

$$\max_{\substack{\sum_i u^i + \sum_j v^j = 1 \\ u^i \geq 0,\ v^j \geq 0}} \left[\sum_i \bar{\psi}_i^k(t+1)u^i + \sum_j \bar{\mu}_j^k(t+1)v^i\right] = \sum_i \bar{\psi}_i^k(t+1)\hat{u}_t^j + \sum_j \bar{\mu}_j^k(t+1)\hat{v}_t^j = \alpha^k(t+1) \tag{49}$$

Theorem 2 *If the plan $\{\hat{w}_t\}$ is optimal, then the system of estimates meeting it has the property of stimulation.*

The proof of Theorem 2 follows from the maximum principle for a stochastic control problem (Theorem 1 in [4]) and is carried out by the same scheme as the proof of stimulating prices existence in the Gale model.

3.4 Economic interpretation

First of all we remark that the estimates $\bar{\psi}_i^k(t+1)$, $\bar{\mu}_j^k(t+1)$ are the forecasts of estimates of the corresponding funds at the instant $t+1$ provided that there was the technology $f^k(x)$ at the instant t and at the next instant transition to the technology $f^{k+1}(x)$ is possible with the probability $\pi^k(\hat{y}_t)$. It would be reasonable to interpret the maxima of these estimates $\alpha^k(t+1)$ as the expected (forecasting) price of the product which will be turned out by the instant $(t+1)$. Really, the product turned out "is sold" only to the funds for which the forecasting estimate coincides with the expected price of the product (relation (49)). From this point of view, the value $R_{t+1}^k(x) = \alpha^k(t+1)f^k(x)$ may be treated as the production function which is calculated in forecasting prices. Thus, the relation

$$\psi_i^k(t) - \bar{\psi}_i^k(t+1) = \frac{\partial R_{t+1}^k}{\partial x}(\hat{x}_t) + \frac{\partial \varphi^k}{\partial x}(\hat{x}_t)$$

means that the expected least estimation of funds is equal to the sum of the gradient of utility and the factor of efficiency of capital.

As we remarked in Section 2, the quantity $R^k(t+1)$ may be interpreted as the estimate of the technology f^{k+1} at the instant $t+1$ or, to be more exact, as the estimate of the probability of emergence of this technology. Thus, the expression $Q^k(y) = h^k(t+1)\pi^k(y)$ may be comprehended as a production function for the TP sphere. Then, (46) implies

$$\mu_j^k(t) - \bar{\mu}_j^k(t+1) = \frac{\partial Q^k}{\partial y_j}(\hat{y}_t)\,.$$

In other words, the forecasting lease estimation of the funds in the TP sphere should coincide with the factor of efficiency of capital for these funds.

[2] We agree to assume that $\sum_{i=m}^{n} a_i = 0$ if $n < m$.

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