Working Paper

Economic Dynamics Models with Innovations: A Probabilistic Approach

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WP-89-99
October 1989

IIASA

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Foreword

The aim of this paper is to include innovation processes with costly implementation into the classical theory of economic dynamics models. New stochastic optimization methods, developed to investigate these models, are discussed.

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Economic Dynamics Models with Innovations: A Probabilistic Approach

V.I. Arkin

The objective of this paper is twofold. First, to include innovation processes with costly implementation (emergence and propagation of new technologies) into the classical theory of economic dynamics models. Second, to show that the transition to the stochastic setting of the problem allows to partially eliminate difficulties due to the discrete nature of innovations' emergence, leading to, in the deterministic case, nonconvex extremal problems. This presentation is based on the classical Gale model in the simplest situation when the technology is extended only once. In this case, a non-standard, two-stage stochastic programming problem with controlled measure is shown to emerge.

The main results consist of a description of the structure of the dual variables (stimulating prices) and some related indicators of economic efficiency taking into account the probabilistic nature of the model. The major role in the system of economic indicators constructed is played by the new technology estimates arising due to the consideration of uncertainty and the lack of deterministic counterparts.

1 Presentation of the Approach

1.1 The deterministic case

The general model of economic dynamics can be described in terms of a multivalued mapping $Q(\cdot)$ translating the point $x \in R^n_+$, characterizing the stock of products at the beginning of the planning period, to one of the points of the set $Q(x) \subseteq R^n_+$ at the end of the period. The set $Q(x)$ describes the "technological possibilities" in the set x for one planning period, i.e., the set of all outputs that can be obtained from the resource vector x at the end of the period. The set $Q = \{x, z : z \in Q(x)\}$, which is the graph of $Q(\cdot)$, characterizes the body of our knowledge of all the ways the resources can be used. Given some initial resource vector $\hat{x}$, the system's dynamics can be described by:

$$x_{t+1} \in Q(x_t) , \quad x_0 = \hat{x} . \tag{1}$$

Let there exist the possibility to broaden our knowledge, i.e., to create a new technological mode. In other words, there exists the possibility of transition from the initial technology $Q^0 = Q$ to a new technology that is characterized by the multivalued mapping $Q^1(\cdot)$ with the graph $Q^1 \supset Q^0$.

Let Θ be the moment of emergence of a new technological mode, i.e., the moment from which the new technology can be used. Let us consider the moment Θ as depending on the choice of a system trajectory, i.e., being a controllable variable. This case will be referred to as the controlled "technological progress" (TP). The system's dynamics takes the form

$$x_{t+1} \in \begin{cases} Q^0(x_t) , & t < \Theta , \quad x_0 = \hat{x} , \\ Q^1(x_t) , & t \geq \Theta . \end{cases} \tag{2}$$

The function $\Theta = \Theta(x_1, x_2, \ldots)$ of integer values is considered be given. For (2) to be consistent, the function Θ should possess the nonanticipativity property:

$$\begin{aligned} &\text{if} \quad \Theta(x_1, \ldots, x_t, x_{t+1}, x_{t+2}, \ldots) = t \\ &\text{then} \quad \Theta(x_1, \ldots, x_t, x'_{t+1}, x'_{t+2}, \ldots) = t \\ &\text{for any sequence } (x'_{t+1}, x'_{t+2}, \ldots) \, . \end{aligned}$$

Example. Let A be some set in R^n, $x_0 \overline{\in} A$. Then the function $\Theta(x_1, x_2, \ldots) = \min\{t : x_t \in A\}$ will be nonanticipative.

Given a nonanticipative function $\Theta = \Theta(x_1, \ldots, x_\tau)$, $\tau < \infty$, one can formulate the extremal problem: to choose a trajectory (a plan) $\{x_t\}$ satisfying (2) and providing the maximum to the function

$$\sum_{0}^{\Theta-1} \varphi^0(x_t) + \sum_{\Theta}^{\tau} \varphi^1(x_t) \longrightarrow \max \tag{3}$$

where $\varphi^0(x)$, $\varphi^1(x)$ are given utility functions.

The problem (2)–(3) is, as a rule, nonconvex under "the best" assumptions concerning Q^0, Q^1, φ^0, φ^1, Θ.

Example. Let $\Theta = \min\{t : x_t \geq \xi\}$, where ξ is some fixed vector.

Let us introduce a new multivalued mapping

$$R(x) = \begin{cases} Q^1(x), & x \geq \xi \\ Q^0(x), & \text{otherwise} \, . \end{cases}$$

In this case, (2) is equivalent to the inclusion

$$x_{t+1} \in R(x_t) \, , \quad x_0 = \hat{x} \, . \tag{4}$$

Generally, the graph R of $R(\cdot)$ is not convex, even if the sets Q^1 and Q^0 are convex.

The situation noted creates one of the main difficulties when studying specific models of economic dynamics with endogenous discretely changing technologies.

1.2 Transition to the probabilistic setting

The main idea of the suggested approach is to consider Θ as a random variable with distribution dependent on the system trajectory. Running a little ahead, it should be noted that even though the formal probabilistic setting covers the deterministic case, the "nondegeneracy" conditions that will be imposed on the distributions of the corresponding random variables rule out the deterministic situation.

A plan in the probabilistic model will be represented by the two families of functions

$$Z^0 = \{x_t^0 \, , t \geq 0\} \, , \quad Z^1(\Theta) = \{x_t^1(\Theta) \, , t \geq \Theta\}$$

satisfying the conditions

$$\begin{aligned} &x_{t+1}^0 \in Q^0(x_t^0) \, , && t \geq 0 \, , \quad x_0 = \hat{x} \\ &x_{t+1}^1(\Theta) \in Q^1(x_t^1(\Theta)) \, , && t \geq \Theta \, , \quad x_\Theta^1(\Theta) = x_\Theta^0 \, . \end{aligned} \tag{5}$$

The distribution of the random integer-valued variable is given:

$$P(\Theta = t) = p(x_1^0, \ldots, x_t^0) \, . \tag{6}$$

The problem is to find a plan providing the maximum to the function

$$E^{Z^0} \left[\sum_{1}^{\Theta-1} \varphi^0(x_t^0) + \sum_{\Theta}^{\tau} \varphi^1\left(x_t^1(\Theta)\right) \right] \, . \tag{7}$$

Here, E^{Z^0} stands for the mathematical expectation showing that the distribution of the random variable Θ depends on the choice of the sequence $Z^0 = \{x_t^0, t \geq 0\}$.

In [1] a specific case of the above model has been considered when the distribution of the random variable Θ is of the form

$$P(x_1^0, \ldots, x_t^0) = q(x_t^0) \prod_{k=1}^{t-1} \left(1 - q(x_k^0) .\right)$$

This distribution corresponds to the situation in which the probability of transition to the new technology $q(x_t)$ at instant t is given, providing it did not emerge up to this instant.

This paper considers another way of forming the random variable Θ. Specifically, let ξ be some fixed vector characterizing the minimal costs necessary for creating a new technology. ξ can naturally be considered as a random vector with the given distribution function $\pi(x) = P(\xi \leq x)$. Then $\Theta = \Theta(\xi) = \min\{t : x_t^0 \geq \xi\}$ is a random variable, the distribution of which depends on the chosen trajectory $\{x_t^0\}$ and the distribution function of ξ.

Remark. In this paper we confine our discussion to a single extension of the initial technology. The above framework can be extended to the case of several technologies and also to the situation in which the parameters of new technological modes are not known in advance and can be defined only after the emergence of a corresponding technology.

2 Innovations Account in the Gale Model. Stimulating Prices

2.1 The Gale model "input–output"

The Gale model "input–output" is characterized by a technological set T, the elements of which are the pairs of nonnegative, n-dimensional vectors (a, b), and by an objective function φ defined on T. The set T is assumed to be convex and the function φ to be concave. The pairs (a, b) are treated as technological modes (production processes), a being input and b being output.

A sequence of production processes $Z_t = (a_t, b_{t+1}) \in T$ is called a "plan" and the following balance condition holds:

$$b_t \geq a_t\ , \quad t = k, \ldots, \tau - 1\ . \tag{8}$$

The vector of the initial stock $\hat{b}_k$ at instant k is assumed to be given. The problem is to find a plan providing the maximum to the expression

$$\sum_{k}^{\tau-1} \varphi(a_t, b_{t+1}) \longrightarrow \max\ . \tag{9}$$

A sequence of nonnegative n-dimensional vectors $\{\psi_t, t = k, k+1, \ldots, \tau\}$ will be called a "price system." The price system $\{\psi_t\}$, with the initial vector $\hat{\psi}_k$, is said to stimulate the plan $\{\hat{Z}_t\}$ with the initial vector $\hat{b}_k$ in the interval $[k, \tau]$ if the following conditions are satisfied:

A. For every $t \geq k$, the pair $\{\hat{a}_t, \hat{b}_{t+1}\}$ provides the maximum to the function

$$F_t(a, b) = \varphi(a, b) + \psi_{t+1} b - \psi_t a \tag{10}$$

for all $(a, b) \in T$.

B. For all $t \geq k$,

$$\psi_t(\hat{b}_t - \hat{a}_t) = 0\ . \tag{11}$$

The economic interpretation of these conditions is well-known. If some regularity condition holds, then the optimal plan is stimulated by a price system.

2.2 The deterministic Gale model with innovations

Let us assume that there exists the possibility of transition from the initial technology $T^0 = T$ to a technology T^1 possessing a larger set of production modes $T^1 \supset T^0$. To carry out the transition, some funds are necessary that are characterized by a vector ξ. The funds are created in the sphere of TP that is also described by the input–output model, i.e., a set of pairs $(c, d) \in Q \supseteq R_+^{2n}$ is given, where c represents the costs and d the output for one planning period. The vector d will be treated as the funds created during one planning period. The technology T^1 emerges when the funds accumulated in the sphere of TP reach some predetermined level $\xi \in R_+^n$ (for all the coordinates).

The model assumes the form

$$\begin{aligned} (a_t, b_{t+1}) &\in T^0, \quad 0 \le t \le \Theta - 1 \\ (a_t, b_{t+1}) &\in T^1, \quad \Theta \le t \le \tau - 1 \\ (c_t, d_{t+1}) &\in Q, \quad 0 \le t \le \tau - 1 \\ \Theta &= \min\left\{t : \sum_0^t d_k \ge \xi\right\} \\ b_t &\ge a_t + c_t\,, \quad 0 \le t \le \tau - 1\,. \end{aligned} \tag{12}$$

A plan is defined by a sequence $\{Z_t\} = \{(a_t, b_{t+1}), (c_t, d_{t+1})\}$ satisfying (12). The vectors of initially available resources $\hat{b}_0$ and initial funds $\hat{d}_0$ are considered given. The problem is to choose a plan providing the maximum to the function

$$\sum_0^{\Theta-1} \varphi^0(a_t, b_{t+1}) + \sum_\Theta^{\tau-1} \varphi^1(a_t, b_{t+1})\,. \tag{13}$$

Here, φ^0 and φ^1 are concave utility functions defined on T^0 and T^1, respectively. The sets T^0, T^1, and Q are assumed convex and satisfying $\varphi^1(a, b) \ge \varphi^0(a, b)$.

As already mentioned, the above economic dynamics model is not generally convex.

2.3 The transition to the stochastic model

Let us consider ξ as a random vector with a given distribution function $\pi(y) = P(\xi \le y)$. The function $\pi(y)$ is assumed to be continuously differentiable. $\Theta = \Theta(\xi)$ is an integer-valued random variable defined by

$$\Theta = \min\{t : y_t \ge \xi\}\,, \tag{14}$$

where $y_t = \sum_0^t d_k$.

In the stochastic case, a plan is defined by the two sequences $Z_t^0 = \{(a_t^0, b_{t+1}^0), (c_t^0, d_{t+1}^0)\}$, $t \ge 0$ and $Z_t^1(\Theta) = \{(a_t^1(\Theta), b_{t+1}^1(\Theta))\}$, $t \ge \Theta$ satisfying the constraints

$$\begin{aligned} (a_t^0, b_{t+1}^0) &\in T^0\,, \quad t \ge 0\,, \\ (c_t^0, d_{t+1}^0) &\in Q\,, \quad t \ge 0\,, \\ b_t^0 &\ge a_t^0 + c_t^0\,, \quad t \ge 0\,, \\ (a_t^1(\Theta), b_{t+1}^1(\Theta)) &\in T^1\,, \quad t \ge \Theta\,, \\ b_t^1(\Theta) &\ge a_t^1(\Theta)\,, \quad t \ge \Theta\,, \\ b_\Theta^1(\Theta) &= b_\Theta^0\,, \\ \Theta &= \min\left\{t : \sum_{k=0}^t d_k^0 \ge \xi\right\}\,. \end{aligned} \tag{15}$$

The vectors of the initial resources available $\hat{b}_0$ and of the initial funds $\hat{d}_0$ are assumed given, with $\pi(\hat{d}_0) = 0$.

The problem is to choose a plan providing a maximum to the function

$$E^{Z^0}\left[\sum_{t=0}^{\Theta-1} \varphi^0(a_t^0, b_{t+1}^0) + \sum_{t=0}^{\tau-1} \varphi^1\left(a_t^1(\Theta), b_{t+1}^1(\Theta)\right)\right] . \tag{16}$$

The expectation is taken with respect to the distribution of the random variable Θ, which is dependent on the plan $\{Z_t^0\}$.

2.4 Stimulating prices

The sequence of nonnegative vector functions $\{\psi_t^0\}$, $\{\alpha_t\}$, $\{\psi_t^1(\Theta)\}$ with values in R_n and the sequence of nonnegative scalar functions $\{R_t\}$ are said to stimulate the plan $\{Z_t^0\}$, $\{Z_t^1(\Theta)\}$ if the following conditions hold:

A. The price sequence $\{\psi_t^1(\Theta)\}$, $t \geq \Theta$ with the initial price vector $\psi_\Theta^1 = \psi_\Theta^1(\Theta)$ stimulates the plan $\{Z_t^1(\Theta)\}$ with the initial resource vector $b_\Theta^1(\Theta) = b_\Theta^0$ in the interval $[\Theta, \tau]$ (in the sense of definitions (10), (11)).

B. For every $t \geq 0$, the pair (a_t^0, b_{t+1}^0) provides the maximum to

$$F_t(a,b) = \varphi^0(a,b) + \bar{\psi}_{t+1} b - \psi_t^0 a \tag{17}$$

for all $(a,b) \in T^0$, where

$$\bar{\psi}_{t+1} = \frac{1-\pi(y_{t+1}^0)}{1-\pi(y_t^0)}\psi_{t+1}^0 + \frac{\pi(y_{t+1}^0)-\pi(y_t^0)}{1-\pi(y_t^0)}\psi_{t+1}^1 . \tag{18}$$

Here $\psi_{t+1}^1 = \psi_{t+1}^1(t+1)$ is the initial price vector for the sequence $\{\psi_k^1(t+1),\ k \geq t+1\}$.

C. For every $t \geq 0$, the pair (c_t^0, d_{t+1}^0) provides the maximum to

$$\Phi_t(c,d) = \alpha_{t+1} d - \psi_t^0 c \tag{19}$$

for all $(c,d) \in Q$.

D. The prices α_t satisfy the relation [1]

$$\alpha_t = \tilde{\alpha}_{t+1} + R_t \frac{\pi'(y_t^0)}{1-\pi(y_{t-1}^0)} . \tag{20}$$

Here,

$$\begin{aligned} R_t &= \sum_{k=t}^{\tau-1} \varphi^1\left(a_k^1(t), b_{k+1}^1(t)\right) - \\ &\quad -\left[\varphi^0(a_t^0, b_{t+1}^0) + \sum_{k=t+1}^{\tau-1} \varphi^1\left(a_k^1(t+1), b_{k+1}^1(t+1)\right)\right], \qquad (21)\\ \tilde{\alpha}_{t+1} &= \alpha_{t+1}\frac{1-\pi(y_t^0)}{1-\pi(y_{t-1}^0)} . \end{aligned}$$

E. For every $t \geq 0$,

$$\psi_t^0(b_t^0 - a_t^0 - c_t^0) = 0 . \tag{22}$$

[1] From now on $\pi'(y)$ denotes the vector $\left(\frac{\partial\pi}{\partial y^1}, \cdots, \frac{\partial\pi}{\partial y^n}\right)$.

Theorem 1 *Let the plan $\{Z_t^0\}$, $\{Z_t^1(\Theta)\}$ be optimal and the following regularity condition be satisfied: if there exist technological modes*

$$(\hat{a}_t, \hat{b}_{t+1}) \in T^0 \,, \quad (\hat{a}_t(\Theta), \hat{b}_{t+1}(\Theta)) \in T^1 \,, \quad t \geq 0$$

(not necessarily forming the plan) such that

$$\begin{aligned} b_t^0 &> \hat{a}_t + c_t^0 \,, \quad t \geq 0 \\ b_t^1(\Theta) &> \hat{a}_t(\Theta) \,, \quad t \geq \Theta \,, \end{aligned} \tag{23}$$

then there exist prices stimulating the plan $\{Z_t^0\}$, $\{Z_t(\Theta)\}$.

The proof of the theorem is postponed until Section 3.

Remark Recursion relations (20) can be resolved for α_t:

$$\alpha_t = \frac{1}{1 - \pi(y_{t-1}^0)} \sum_{k=t}^{\tau-1} R_k \pi_k'(y_k^0) \,. \tag{24}$$

2.5 A property of the optimal plan: the necessity of risk

Let $\{y_t^0, t \geq 0\}$ be a sequence which is a component of the optimal plan. Let us assume that

$$(0,0) \in Q \; ; \;\; (0,d) \overline{\in} Q, \; \forall\, d > 0 \; ; \;\; \psi_t^0 \neq 0, t \geq 0 \,.$$

Then $\pi(y_t^0) < 1$ for all $t \geq 0$.

In other words, the strategy of the investment to the new technology resulting in the emergence of the latter with probability one is not optimal.

Proof. Let us assume the contrary, i.e., there exists $0 < k \leq \tau - 1$ such that $\pi(y_k^0) = 1$. Then, by virtue of the smoothness of $\pi(y)$, $\pi'(y_k^0) = 0$ and, using (20), we obtain $\alpha_k = 0$. Then, using $\psi_{k-1}^0 \neq 0$ from (19), we have $c_{k-1}^0 = 0$. But, according to the assumption, $(0,d) \overline{\in} Q \forall d > 0$. Hence $d_k^0 = 0$. So we arrive at $y_k^0 = y_{k-1}^0 + d_k^0 = y_{k-1}^0$. Therefore, $\pi(y_{k-1}^0) = \pi(y_k^0) = 1$. Proceeding on in the same way, we obtain $\pi(y_k^0) = \pi(y_{k-1}^0) = \ldots = \pi(y_0^0) = 1$. But, by assumption, $\pi(y_0^0) = 0$, which is a contradiction.

2.6 Economic comment

First of all we note that, as in the deterministic Gale model, the stimulating prices make it possible to "untie" the balance constraints in time and to screen inefficient technological modes on the basis of the local information without recalculating the entire problem. In this sense the function $F_t(a,b)$ from (17) can be treated as a local efficiency criterion in the production sphere. The main difference from the deterministic situation is that the output in the production sphere at instant $t+1$ is evaluated in the prices $\bar{\psi}_{t+1}$ rather than in the prices ψ_{t+1}^0. As formula (18) shows, the prices $\bar{\psi}_{t+1}$ are the expected (forecasting) prices. Indeed, at instant t when a technological mode is being chosen for the planning period $(t, t+1)$, it is not known whether the new technology will appear at instant $t+1$. The probability of the new technology emerging at instant $t+1$, provided that it did not emerge before t, will be shown in Section 3 to equal

$$\frac{\pi(y_{t+1}^0) - \pi(y_t^0)}{1 - \pi(y_t^0)} \,.$$

The price of the resources output during the planning period $(t, t+1)$, provided that the new technology will appear at instant $t+1$ equals $\psi_{t+1}^1 = \psi_{t+1}^1(t+1)$. The price of the resources at instant $t+1$, provided that the old technology remained at instant $t+1$, equals ψ_{t+1}^0. Thus, $\bar{\psi}_t + 1$

is an average weighted (calculated) price at instant $t+1$ with weights equal to the conditional probabilities of emergence and nonemergence of the new technology at instant $t+1$.

Now we proceed to the analysis of relation (19). The function $\Phi_t(c,d)$ plays the role of a local optimality criterion in the sphere of TP. The quantity $\alpha_{t+1}d$ is the value of the funds, and the quantity $\psi_t^0 c$ characterizes, in terms of value, the input of the resources in the interval $(t, t+1)$ for the creation of the funds.

Equation (20), determining the quantity α_t, can be rewritten as

$$\alpha_t = \frac{1-\pi(y_t^0)}{1-\pi(y_{t-1}^0)}\left[\alpha_{t+1} + R_t\frac{\pi'(y_t^0)}{1-\pi(y_t^0)}\right] + \frac{\pi(y_t^0)-\pi(y_{t-1}^0)}{1-\pi(y_{t-1}^0)}\cdot 0\,. \tag{25}$$

The quantity $R_t\frac{\pi'(y_t^0)}{1-\pi(y_t^0)} + \alpha_{t+1}$ characterizes the estimate of the funds if the new technology did not appear at instant t; otherwise, this estimate equals zero since only a single technology extension is considered in our model. Thus α_t is the expected estimate of the funds in the sphere of TP. The quantity R_t can be treated as a characteristic (an estimate) of the new technology. By virtue of (21), it expresses the gain, from instant t onwards, of the system due to the emergence of the new technology at instant t compared to its emergence at instant $t+1$. The quantity R_t is determined in the optimal plan only by the parameters of the new technology T^1. If the fund increment Δy_t is sufficiently small, then it follows from (25) that

$$\alpha_t\Delta y_t \approx \frac{1-\pi(y_t^0)}{1-\pi(y_{t-1}^0)}\left[\frac{1-\pi(y_t^0+\Delta y_t)}{1-\pi(y_t^0)}\alpha_{t+1}\Delta y_t + \frac{\pi(y_t^0+\Delta y_t)-\pi(y_t^0)}{1-\pi(y_t^0)}\cdot R_t\right]\,. \tag{26}$$

The expression in brackets in (26) is the conditional expectation (conditioned on the nonemergence of the new technology at instant t) of the random variable taking the two values $\alpha_{t+1}\cdot\Delta y_t$ and R_t and representing the gain from the additional funds Δy_t. The quantity R_t corresponds to the gain from the emergence of the technology T^1 at instant t. The quantity $\alpha_{t+1}\Delta y_t$ corresponds to the value of the funds Δy_t at instant $t+1$, if the new technology did not emerge at instant t. Thus, relation (26) represents the balance in monetary form for the adjacent instants, which holds on the average.

The quantity $\alpha_t - \tilde{\alpha}_{t+1} = R_t\frac{\pi'(y_t^0)}{1-\pi(y_{t-1}^0)}$ naturally can be treated as a lease estimation of the funds at instant t. It shows an infinitesimal gain from one unit of the funds at instant t.

2.7 A simplified model of investment in the new technology

Let us assume that, in the model described by (12), the instant of emergence of the new technology is determined by the equation

$$\Theta = \min\left\{t : \sum_{k=0}^{t} c_k^0 \geq \xi\right\} + 1\,. \tag{27}$$

This model is a specific case of the initial model (12), when the set of pairs of the form (c,c), $c \geq 0$ is taken as a technological set Q. In this case it follows from (19) that if $c_t^0 > 0$, then $\alpha_{t+1} = \psi_t^0$, and the inequality $\alpha_{t_1} < \psi_t^0$ implies $c_t^0 = 0$. Thus, the value α_t determined by (20) or by (24) can be considered as the efficiency norm of the resource distribution between investment in production and the TP sphere, and ψ_t^0 can be considered as the limiting value of this norm.

In conclusion, we present some properties of the optimal plan and its stimulating prices. Let us denote $\gamma = \min\{t : \pi(y_t^0) > 0\}$. The time interval $[0, \gamma-1]$ will be called the "initial section" of the optimal plan. Let some components of c_t^0 be strictly positive in this section. Then it follows from the above that for these components the equality $\alpha_{t+1} = \psi_t^0$ holds for $0 \leq t \leq \gamma - 1$. In the interval $[0,\gamma-1]$, $\alpha_t = \sum_{k=\gamma}^{\tau-1} R_k\pi'(y_k^0) = \alpha$, where α is a constant vector. Therefore, the prices of the resources allocated to the TP sphere in the initial section of the optimal plan are constant.

Let us now show that if in the initial section $c_t^0 > 0$ for all the coordinates, i.e., every resource is used in the TP sphere in this section, then the optimal plan in this section involves the application of the constant technological mode (a^0, b^0).

Indeed, by virtue of the stationarity of the prices in the initial section, (17) implies

$$(a^0, b^0) = \arg \max_{(a,b) \in T^0} \left[\varphi^0(a,b) + \psi(b-a) \right] \quad . \tag{28}$$

If we additionally assume that for any i-th resource there exists an instant $m \geq \gamma$ such that $R_m \frac{\partial \pi}{\partial y_m^t}(y_m^0) > 0$ then $c_t^0 = c^0$ is a constant vector in the initial section and $b^0 = a^0 + c^0$. Indeed, the vector $\alpha = \psi$ is strictly positive in this case and (22) implies

$$\psi(b^0 - a^0 - c_t^0) = 0 \quad .$$

3 Proof of the Existence of Stimulating Prices

The proof is organized as follows. First, a general control problem for stochastic difference equations is formulated and the corresponding necessary optimality conditions (the maximum principle) are presented. Then, the initial model of economic dynamics is reformulated in terms of the optimal control problem. The maximum principle is applied to the resulting problem. Finally, the adjoint variables of the maximum principle are deciphered in terms of the economic dynamics model.

3.1 Smoothly convex control problem

Let (s_t, x_t) be a controlled process in which the first component, s_t, assumes its values in a finite set S, and the second component assumes its values in the n-dimensional Euclidean space R^n.

The dynamics of the process $\{x_t\}$ is described by the set of stochastic difference equations

$$\begin{aligned} x_{t+1} &= f^{t+1}(s_t, x_t, u_t, v_t, s_{t+1}) \\ x_0 &= x_0(s_0) \,. \end{aligned} \tag{29}$$

$x_0(s_0)$ is a given function. The controlling parameters $w = (u, v)$ are functions of the history of the process $\{s_t\}$, $u_t = u_t(s^t)$, $v_t = v_t(s^t)$, $s^t = (s_0, s_1, \ldots, s_t)$. The evolution of the process $\{s_t\}$ is defined by the initial distribution $\prod_0(s_0)$ and the set of the transition functions $\prod^{t+1}(s_t, x, v, s_{t+1})$, specifying the transition probabilities from the state s_t to the state s_{t+1} at instant $t+1$ and depending on the process values x_t and the controlling parameter v_t at instant t.

The choice of the controlling parameters w_t is restricted by the following constraints:

$$u_t \in U_t(s_t) \,, \quad v_t \in V_t(s_t) \tag{30}$$

$$g^t(s_t, x_t, u_t, v_t) \leq 0 \,. \tag{31}$$

Here $U_t(S)$, $V_t(s)$ are given sets depending on the parameter $s \in S$, $V_t(s) \in R^l$, and g_t is a vector function with values in the k-dimensional Euclidean space R^k. If some controls $W_t = (u_t, v_t)$ are chosen, then by virtue of system (29) the process $\{x_t\}$ is defined together with the transition probabilities $\prod^{t+1}(s_t, x_t, v_t, s_{t+1})$ that, together with the initial distribution $\prod_0(s_0)$ generate the probability measure $\prod^W$ in the space of all sequences $\{s^t\}$. The problem is to maximize the function

$$E^W \sum_0^{\tau-1} \varphi^t(s_t, x_t, u_t, v_t) \longrightarrow \max \tag{32}$$

in all controls $\{w_t\}$ satisfying constraints (30) and (31). Here, E^W stands for the expectation with respect to the measure $\prod_W$, and $\varphi(s, x, u, v)$ is a given function.

Let us formulate some assumptions for the problem under consideration.

A. The functions $f^{t+1}(s_t, x, u, v, s_{t+1})$, $\varphi^t(s_t, x, u, v)$, $g^t(s_t, x, u, v)$, and $\prod^{t+1}(s_t, x, v, s_{t+1})$ are jointly continuously differentiable with respect to the arguments (x, v).

B. For every set $s_t, x, v \in V_t(s_t)$; $u^1, u^2 \in U_t(s_t), 0 \leq \alpha \leq 1$ there exists an element $u \in U_T(s_t)$ such that

$$\begin{aligned}
\alpha\varphi^t(s_t, x, u^1, v) + (1-\alpha)\varphi^t(s_t, x, u^2, v) &\leq \varphi^t(s_t, x, u, v) \\
\alpha g^t(s_t, x, u^1, v) + (1-\alpha)g^t(s_t, x, u^2, v) &\leq g^t(s_t, x, u, v) \\
\alpha f^{t+1}(s_t, x, u^1, v, s_{t+1}) + (1-\alpha)f^{t+1}(s_t, x, u^2, v, s_{t+1}) &\leq f^{t+1}(s_t, x, u^2, v, s_{t+1}) \\
&= f(s_t, x, u, v, s_{t+1}) \forall s_{t+1} \in S
\end{aligned}$$

Let $\{x_t^*, u_t^*, v_t^*\}$ denote the solution of (29)–(32), $\prod^*$ be the corresponding measure in the space of all sequences $\{s_t\}$, $t = 1, 2, \ldots, \tau$, and E^* be the expectation with respect to the measure $\prod^*$.

Let us introduce the Hamiltonian

$$H^{t+1}(s_t, x, u, v, s_{t+1}) = \psi_0\varphi^t(s_t, x, u, v) + \psi_{t+1}f^{t+1}(s_t, x, u, v, s_{t+1}) - \lambda_t g^t(s_t, x, u, v). \quad (33)$$

Theorem 2 (The Maximum Principle) *Let $\{x_t^*, u_t^*, v_t^*\}$ be the solution to (29)-(32). Then there exist the functions $\psi_t = \psi_t(s^t)$ with values from R^n, $\lambda = \lambda(s^t) \geq 0$ with values from R^k, and $h_t = h_t(s^t)$ with values from R_1, $\psi \geq 0$ such that*

$$1)\; u_t^* = \arg\max_{u \in U_t(s_t)} E^*\left[H^{t+1}(s_t, x_t^*, u, v_t^*, s_{t+1})|s^t\right], \quad (34)$$

$$v_t^* = \arg\max_{v \in V_t(s_t)} \left\{E^*\left[H_V^{t+1}(s_t, x_t^*, u_t^*, v_t^*, s_{t+1})|s^t\right] + \sum_{s_{t+1} \in S} h_{t+1}\textstyle\prod_v^{t+1}(s_t, x_t^*, v_t^*, s_{t+1})\right\}v,$$

$$2)\; \psi_t = E^*\left[H_x^{t+1}(s_t, x_t^*, u, v_t^*, s_{t+1})|s^t\right] + \sum_{s_{t+1} \in S} h_{t+1}\textstyle\prod_x^{t+1}(s_t, x_t^*, v_t^*, s_{t+1}) \quad \psi_\tau = 0, \quad (35)$$

$$3)\; h_t = \psi_0\varphi^t(s_t, x_t^*, u_t^*, v_t^*) + \sum_{s_{t+1} \in S} h_{t+1}\textstyle\prod^{t+1}(s_t, x_t^*, v_t^*, s_{t+1}), \; h_\tau = 0, \quad (36)$$

$$4)\; \lambda_t g^t(s_t, x_t^*, u_t^*, v_t^*) = 0 \quad (37)$$

If, additionally, the following regularity condition holds:

$$\begin{array}{ll} \textit{there exists an element} & \hat{u}_t = \hat{u}_t(s_t) \in U_t(s_t) \forall s_t \in S \\ \textit{such that} & g^t(s_t, x_t^*, \hat{u}_t, v_t^*) < 0, \; t = 0, 1, \ldots, \tau - 1, \end{array} \quad (38)$$

then $\psi_0 > 0$ and one can specify $\psi_0 = 1$.

The above theorem is a "finite-dimensional" modification of the result from [2] and can be proved following the approach taken in [3].

3.2 The formulation of the economic dynamics model in terms of the control problem

Let us introduce into consideration the random process with two states $\{0\}$ and $\{1\}$, and a single transition from the state $\{0\}$ to the state $\{1\}$, $s_0 = 0$.

Let us consider the following control problem:

$$\begin{aligned} x_{t+1} &= b_{t+1} , \qquad x_0 = \hat{b}_0 , \\ y_{t+1} &= y_t + d_{t+1} , \quad y_0 = \hat{d}_0 , \\ & x_t \geq a_t + c_t , \\ & (a_t, b_{t+1}) \in T(s_t) , \\ & (c_t, d_{t+1}) \in Q . \end{aligned} \tag{39}$$

Here $T(s) = \begin{cases} T^0 , & s = 0 \\ T^1 , & s = 1 \end{cases}$. The control $u_t = (a_t, b_{t+1})$, $v_t = (c_t, d_{t+1})$ at instant t is chosen dependent on the history s^t; $u_t = u_t(s^t)$, $v_t = v_t(s^t)$.

The transition functions of the process s_t are defined by

$$\prod{}^{t+1}(s_t, y, d, s_{t+1}) = \begin{cases} \dfrac{\pi(y+d) - \pi(y)}{1 - \pi(y)} & \text{for } s_t = 0 , \quad s_{t+1} = 1 \\ \dfrac{1 - \pi(y+d)}{1 - \pi(y)} & \text{for } s_t = 0 , \quad s_{t+1} = 0 \end{cases} \tag{40}$$

$$\prod{}^{t+1}(s_t, y, d, s_{t+1}) = \begin{cases} 1 & \text{for } s_t = 1 , \quad s_{t+1} = 1 , \\ 0 & \text{for } s_t = 1 , \quad s_{t+1} = 0 . \end{cases}$$

The choice of the control $w_t = (u_t, v_t)$ satisfying constraints (39) generates the probability measure $\prod^W$ on the space of all sequences $\{s_t\}$, $t = 0, 1, \ldots, \tau - 1$.

One seeks to maximize the function

$$E^W \sum_{t=0}^{\tau-1} \varphi(s_t, a_t, b_{t+1}) \longrightarrow \max , \tag{41}$$

where E^W is the expectation with respect to the measure $\prod^W$ and

$$\varphi(s, a, b) = \begin{cases} \varphi^0(a, b) , & s = 0 , \\ \varphi^1(a, b) , & s = 1 . \end{cases}$$

The optimal control problem (39)–(41) is equivalent to the economic dynamics problem (15)–(16).

Indeed, every sequence of the controls $\{w_t\}$ is related to the plan $\{z_t^0, z_t^1(\Theta)\}$ in the economic dynamic model

$$W_t(s^t) = \begin{cases} z_t^0 , & s_k = 0 , \quad 0 \leq k \leq t , \\ z_t^1(\Theta) , & s_k = 0 , \quad 0 \leq k \leq \Theta - 1 ; \quad s_k = 1 , \quad \Theta \leq k \leq t \end{cases}$$

$$t = 1, 2, \ldots, \tau - 1 .$$

Obviously, the reverse is also true.

Let us show the validity of (40). Indeed, by virtue of the fact that the vector function y_t is not decreasing in all the coordinates,

$$\begin{aligned} P(s_{t+1} = 1 | s_t = 0) &= P\{y_{t+1} \geq \xi / \overline{y_t \geq \xi}\} \\ &= \frac{P\{y_{t+1} \geq \xi , \ \overline{y_t \geq \xi}\}}{P\{\overline{y_t \geq \xi}\}} \\ &= \frac{\pi(y_{t+1}) - \pi(y_t)}{1 - \pi(y_t)} \\ &= \frac{\pi(y_t + d_{t+1}) - \pi(y_t)}{1 - \pi(y_t)} \end{aligned}$$

Here, $\overline{y_t \geq \xi}$ is the complement of the set $\{y_t \geq \xi\}$.

3.3 Application of the maximum principle

It easily can be seen that, by virtue of the conditions of Theorem 1, problem (39)–(41) satisfies all the assumptions of Theorem 2.

By virtue of the regularity condition, the Hamiltonian in (39)–(41) has the form

$$H^{t+1}(s_t, x, y, u, v, s_{t+1}) = \varphi(s_t, a, b) + \psi_{t+1} b + \mu_{t+1}(y + d) - \lambda_t (a + c - x) .$$

It follows from the maximum principle that there exist vector functions $\psi_t = \psi_t(s^t)$, $\lambda_t = \lambda_t(s^t) \geq 0$, $\mu_t = \mu_t(s^t)$ with values in R^n, and a scalar function $h_t = h_t(s^t)$ such that

1) $(a_t^*, b_{t+1}^*) = \arg \max_{(a,b)\in T(s_t)} \left\{ \varphi^t(s_t, a, b) + E^*[\psi_{t+1}|s^t] b - \lambda_t a \right\}$,

2) $(c_t^*, d_{t+1}^*) = \arg \max_{(c,d)\in Q} \left\{ E^*[\mu_{t+1}|s^t] + \sum_{s_{t+1}\in S} h_{t+1}(s^{t+1}) \prod_d (s_t, y_t^*, d_{t+1}^*, s_{t+1}) \right\} d - \lambda_t c$,

3) $\mu_t = E^* \left[H_y^{t+1}(s_t, x_t^*, y_t^*, u_t^*, v_t^*, s_{t+1}) | s^t \right] + \sum_{s_{t+1}\in S} h_{t+1}(s^{t+1}) \prod_y (s_t, y_t^*, d_{t+1}^*, s_{t+1})$, $\mu_T = 0$,

4) $\psi_t = E^* \left[H_x^{t+1} s_t, x_t^*, y_t^*, u_t^*, v_t^*, s_{t+1}) | s^t \right] = \lambda_t$,

5) $h_t = \varphi(s_t, u_t^*) + \sum_{s_{t+1}\in S} h_{t+1}(s_{t+1}) \prod (s^t, y_t^*, d_{t+1}^*, s_{t+1})$,

6) $\lambda_t (b_t^* - a_t^* - c_t^*) = 0$.

3.4 Deciphering the adjoint variables

Let us denote

$$\begin{aligned}
&\psi_t(s^t) = \psi_t^0 && \text{for } s_k = 0\,,\ 0 \leq k \leq t\,, \\
&\psi_t(s^t) = \psi_t^1(\Theta) && \text{for } s_k = 0\,,\ k \leq \Theta - 1\,, \quad s_k = 1\,,\ \Theta \leq k \leq t\,, \\
&\mu_t(s^t) = \mu_t^0 && \text{for } s_k = 0\,,\ 0 \leq k \leq t-1\,,\ s_t = 1\,, \\
&h_t(s^t) = h_t^1 && \text{for } s_k = 0\,,\ 0 \leq k \leq t-1\,,\ s_t = 1\,, \\
&h_t(s^t) = h_t^0 && \text{for } s_k = 0\,,\ 0 \leq k \leq t\,, \\
&w_t = h_t^1 - h_t^0\ .
\end{aligned}$$

For $t \geq \Theta$ we obtain from relation 1) that the optimal plan $z_t'(\Theta) = (a_t^1(\Theta), b_{t+1}^1(\Theta))$ provides the maximum to the value

$$\varphi^1(a, b) + \psi_{t+1}^1(\Theta) b - \psi_t^1(\Theta) a \longrightarrow \max_{(a,b)\in T^1} .$$

It also follows from 1) that, for all $T \geq 0$, the optimal plan (a_t^0, b_{t+1}^0) provides the maximum to the value

$$\varphi^0(a, b) + \bar{\psi}_{t+1} b - \psi_t^0 a \longrightarrow \max_{(a,b)\in T^0} ,$$

where

$$\bar{\psi}_{t+1} = \psi_{t+1}^0 \frac{1 - \pi(y_{t+1}^0)}{1 - \pi(y_t^0)} + \psi_{t+1}^1 \frac{\pi(y_{t+1}^0) - \pi(y_t^0)}{1 - \pi(y_t^0)} .$$

Let us denote

$$\begin{aligned}
\bar{\mu}_t^0 &= \mu_t^0 \cdot \frac{1 - \pi(y_t^0)}{1 - \pi(y_{t-1}^0)} , \qquad (42) \\
\alpha_t &= \bar{\mu}_t^0 + w_t \frac{\pi'(y_t^0)}{1 - \pi(y_{t-1}^0)} .
\end{aligned}$$

It follows then from 2) that the value

$$\alpha_{t+1} d - \psi_t^0 c \longrightarrow \max_{c,d \in Q}$$

reaches its maximum on the optimal plan (c_t^0, d_{t+1}^0).

Our objective is to obtain a recursion equation for the vector α_t. For conciseness, let us denote

$$\pi_t = \pi(y_t^0) , \quad \pi_t' = \pi'(y_t^0) .$$

It follows from 3) that μ_t^0 satisfies the following recursion equation:

$$\mu_t^0 = \bar{\mu}_{t+1}^0 + w_{t+1} \left[\frac{\pi_{t+1}'}{1-\pi_t} - \frac{\pi_t'(1-\pi_{t+1})}{(1-\pi_t)^2} \right] , \quad \mu_\tau^0 = 0 . \tag{43}$$

Then, (42) together with (43) gives

$$\begin{aligned}
\alpha_t &= \left[\bar{\mu}_{t+1}^0 + w_{t+1} \left(\frac{\pi_{t+1}'}{1-\pi_t} - \frac{\pi_t'(1-\pi_{t+1})}{(1-\pi_t)^2} \right) \right] \frac{1-\pi_t}{1-\pi_{t-1}} + w_t \frac{\pi_t'}{1-\pi_{t-1}} \\
&= \mu_{t+1}^0 \frac{1-\pi_{t+1}}{1-\pi_{t-1}} - w_{t+1} \frac{\pi_{t+1}'}{1-\pi_{t-1}} + \frac{\pi_t'}{1-\pi_{t-1}} \left(w_t - w_{t+1} \frac{1-\pi_{t+1}}{1-\pi_t} \right) \\
&= \frac{1-\pi_t}{1-\pi_{t-1}} \left[\mu_{t+1}^0 \frac{1-\pi_{t+1}}{1-\pi_t} + w_{t+1} \frac{\pi_{t+1}'}{1-\pi_t} + R_t \frac{\pi_t'}{1-\pi_t} \right] \\
&= \frac{1-\pi_t}{1-\pi_{t-1}} \left[\alpha_{t+1} + \frac{\pi_t'}{1-\pi_t} R_t \right] \\
&= \tilde{\alpha}_{t+1} + R_t \frac{\pi_t'}{1-\pi_{t-1}}
\end{aligned}$$

Here,

$$R_t = w_t - w_{t+1} \frac{1-\pi_{t+1}}{1-\pi_t} .$$

On the other hand, one can immediately verify that

$$w_t - w_{t+1} \cdot \frac{1-\pi_{t+1}}{1-\pi_t} = h_t^1 - \varphi^0(a_t^0, b_{t+1}^0) - h_{t+1}^1 .$$

This results in

$$R_t = \sum_{k=t}^{\tau-1} \varphi^1 \left(a_k(t), b_{k+1}(t) \right) - \varphi^0(a_t^0, b_{t+1}^0) - \sum_{k=t+1}^{\tau-1} \varphi^1 \left(a_k(t+1), b_{k+1}(t+1) \right) .$$

Theorem 1 is completely proved.

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