WORKING PAPER

DYNAMIC REGULATION OF CONTROLLED SYSTEMS, INERTIA PRINCIPLE AND HEAVY VIABLE SOLUTIONS

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FOREWORD

Existence of viable (controlled invariant) solutions of a control problem regulated by absolutely continuous open loop controls is proved by using the concept of ***viability kernels*** of closed subsets (largest closed controlled invariant subsets). This is needed to provide ***dynamical feedbacks***, i.e., differential equations governing the evolution of viable controls. Among such differential equations, the differential equation providing ***heavy solutions*** (in the sense of heavy trends), i.e., governing the evolution of controls with minimal velocity is singled out.

Among possible applications, these results are used to define ***global contingent subsets*** of the contingent cones which allow to prove the convergence of a modified version of the ***structure algorithm*** to a closed viability domain of any closed subset.

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Dynamic Regulation of Controlled Systems, Inertia Principle and Heavy Viable Solutions

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Introduction

Let us consider two finite dimensional vector-spaces X and Z, X being the state space and Z the control space and a closed subset K of the state space X.

We define the control system (f, U) by a set-valued map $U : K \rightsquigarrow Z$ associating with each state x the set $U(x)$ of feasible controls (subject to state-dependent constraints) and by a single-valued map $f : \text{Graph}(U) \mapsto X$ describing the dynamics of the system

$$(i) \quad \text{for almost all } t \geq 0, \ x'(t) = f(x(t), u(t)) \ \text{ where } \ u(t) \in U(x(t))$$

Viable solutions are the ones which satisfy

$$\forall\, t \geq 0, \ x(t) \in K$$

We recall that the *contingent cone to K at $x \in K$* is the set

$$T_K(x) = \left\{ v \in X \ \mid \ \liminf_{h \to 0+} \frac{d_K(x + hv)}{h} = 0 \right\}$$

We introduce the *regulation map* R_U associating with every state $x \in K$ the subset of controls $u \in U(x)$ such that the corresponding velocity is contingent to K at x:

$$\forall\, x \in K, \ \ R_U(x) \ := \ \{u \in U(x) \mid f(x, u) \in T_K(x)\}$$

The Viability Theorem states in essence that under adequate assumptions, for any initial state $x_0 \in K$, there exists a viable solution to the control problem if and only if $R_U(x) \neq \emptyset$ for any $x \in K$. (This property enjoyed by K is called *controlled invariance*.) Furthermore, if this is the case, the viable solutions are regulated by controls satisfying the *regulation law*

$$\text{for almost all } t \geq 0, \ \ u(t) \ \in \ R_U(x(t))$$

In this paper, we are looking for a system of differential equations or a differential inclusion governing *the evolution of both viable states and controls*, so that we can look for

— *heavy solutions*, which are evolutions where the controls evolve with minimal velocity

— *punctuated equilibria*, i.e., evolutions in which the control $\bar{u}$ remains constant whereas the state may evolve in the associated *viability cell*, which is the viability domain of $x \mapsto f(x, \bar{u})$,

The idea which allows to achieve these aims is quite simple: *we differentiate the regulation law.* This is possible since we know how to differentiate set-valued maps. The idea is very simple, and goes back to the prehistory of the differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function:

We regard the contingent cone to the graph of the set-valued map $F : X \rightsquigarrow Y$ at some point (x, y) of its graph as the graph of the associated "contingent derivative" of F at this point (x, y):

$$\text{Graph}(DF(x,y)) \;:=\; T_{\text{Graph}(F)}(x,y) \tag{1}$$

If a viable control $u(\cdot)$ is absolutely continuous, we deduce then from the regulation law that

$$(ii) \quad \text{for almost all } t \geq 0, \;\; u'(t) \;\in\; DR_U(x(t), u(t))(f(x(t), u(t)))$$

This is the second half of the system of differential inclusions we are looking for.

We observe that this new differential inclusion has a meaning whenever the state-control pair $(x(\cdot), u(\cdot))$ remains in the graph of R_U. Fortunately, by the very definition of the contingent derivative, the graph of R_U is a viability domain of the new system (i),(ii).

Unfortunately, as soon as viability constraints involve inequalities, there is no hope for the graph of the contingent cone, and thus, for the graph of the regulation map, to be closed, so that, the Viability Theorem cannot apply.

We also observe that if the contingent derivative of U obeys a growth condition of the type[1]

$$(\mathcal{G}) \;\; \forall\, (x,u) \in \text{Graph}(U), \quad \inf_{v \in DU(x,u)(f(x,u))} \|v\| \;\leq\; c(\|u\| + \|x\| + 1)$$

[1]which follows for instance from the boundedness of the contingent derivative: $\|DU(x,u)\| \leq c$ and the linear growth of f.

then absolutely continuous controls verify the growth condition

$$(iii) \quad \|u'(t)\| \le c(\|u(t)\| + \|x(t)\| + 1)$$

So, a strategy to overcome the above difficulty is to introduce[2] the a priori growth condition (iii) and to look for the viability kernel of (i.e., the largest *closed* viability domain contained in) Graph(U) of the system of differential inclusions (i),(iii). Such a viability kernel does exist (see Theorem 1.5 below).

If we regard this viability kernel as the closed graph of a (possibly empty) set-valued map denoted by $R_U^c : X \rightsquigarrow Z$, then we infer from Theorem 1.5 that *whenever the initial state* x_0 *is chosen in* $\mathrm{Dom}(R_U^c)$ *and the initial control* u_0 *in* $R_U^c(x_0)$, *there exists a solution to the system of differential inclusions (i) and*

$$(iv) \quad u'(t) \in G_c(x(t), u(t)) := DR_U^c(x(t), u(t))(f(x(t)), u(t))$$

This is how we shall obtain ***absolutely continuous viable state-control solutions*** to our regulation problem[3].

As an example, we shall compute the regulation maps R^c for one dimensional affine system in section 2.

We observe for instance that by taking $c = 0$, inequalities (iii) provide constant controls u_0, and thus solutions $x(\cdot)$ to the problem $x'(t) = f(x(t), u_0)$ which are viable in the closed subset $U^{-1}(u_0)$ whenever this subset is not empty. If this is the case, we shall say that u_0 is a *punctuated equilibrium* and that $R_U^{0^{-1}}(u_0)$ is its associated *viability cell, which is the closed subset of states regulated by the constant control* u_0.

Instead of looking for closed loop control selections of the regulation map R_U, we shall look for selections $g(\cdot, \cdot)$ of the set-valued map $G_c(\cdot, \cdot)$ defined above, which we shall call a *dynamical closed-loop*.

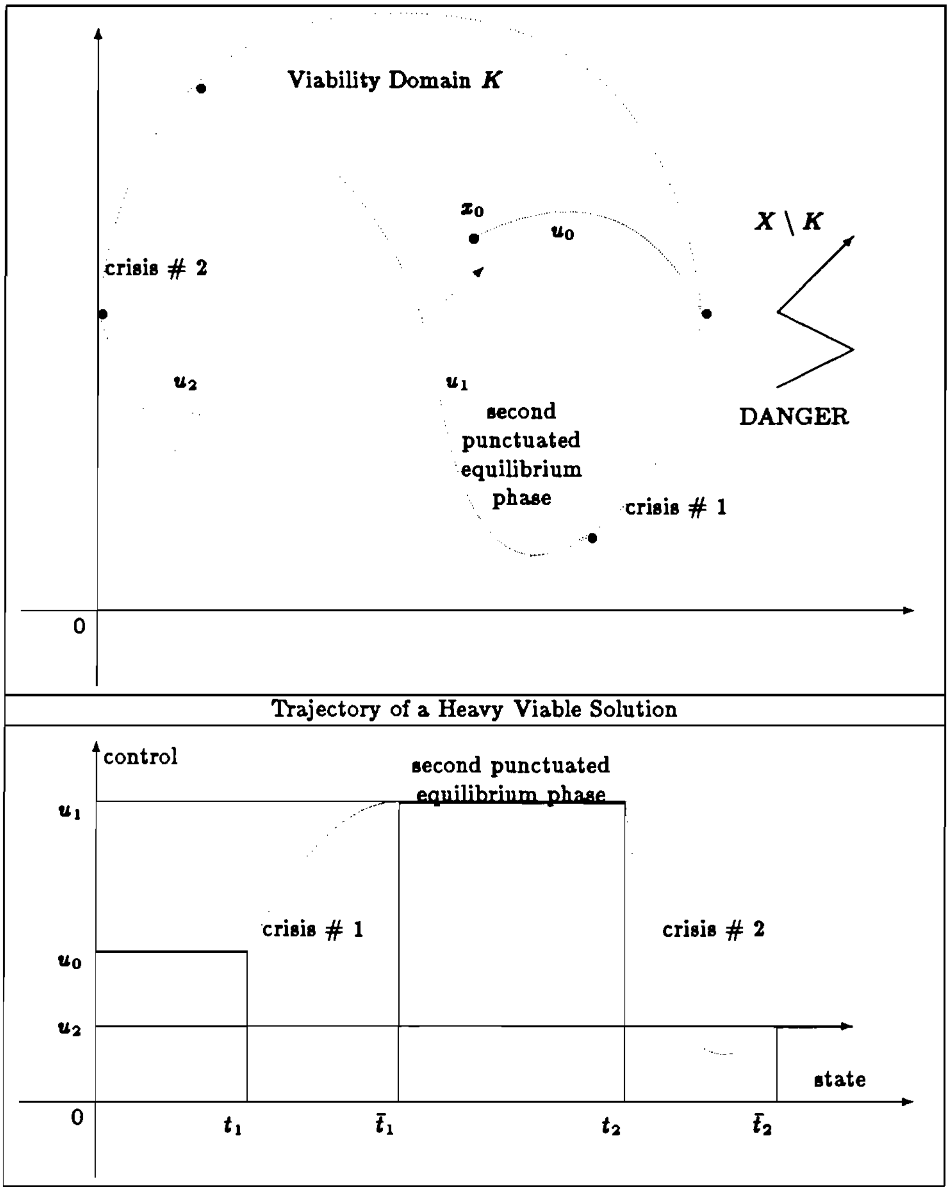
Naturally, under adequate assumptions, Michael's Theorem implies the existence of a continuous dynamical closed loop. But under the same assumptions, we shall show that we can take as dynamical closed-loop the

[2] even if growth conditions on the contingent derivative of U are absent.

[3] We remark that the above growth condition ($\mathcal{G}$) means that the graph of U is a viability domain of the system of differential inclusions (i),(iii), and consequently, that it coincides with its viability kernel, i.e., that $R_U^c = U$.

Therefore, growth condition ($\mathcal{G}$) implies that *absolutely continuous viable state-controls do exist for every initial state* $x_0 \in \mathrm{Dom}(U)$ *and initial control* $u_0 \in U(x_0)$. But this property (and thus, condition ($\mathcal{G}$)) is too strong in the framework of viability (or controlled invariance) problems, where we look only for the existence of *at least a control* providing a given condition.

Figure 1: Heavy Viable Solutions



Punctuated Evolution of the Controls. **Starting from x_0 with the constant control u_0, the solution evolves in K until at time t_1, (first *punctuated equilibrium phase*) the velocity $f(x(t_1), u(t_1))$ is outward. Then a *crisis* happens during which controls also evolve (as slowly as possible) to maintain viability, until time $\bar{t}_1$ where the velocities $f(x(t), u(\bar{t}_1))$ are contingent to K during the *second punctuated equilibrium phase*.**

minimal selection $g^0(\cdot,\cdot)$ defined by

$$g^0(x,u) \in G_c(x,u) \ \& \ \|g^0(x,u)\| := \min_{v\in G_c(x,u)} \|v\|$$

We shall call the smooth viable control-state solutions to the system of differential equations

$$x'(t) = f(x(t),u(t)) \ \& \ u'(t) = g^0(x(t),u(t))$$

heavy viable solutions to the control problem, heavy in the sense of heavy trends. They are the ones for which ***the control evolves with minimal velocity***. In the case of usual differential inclusions[4] $x' \in F(x)$, where the controls are the velocities, they are the solutions with minimal acceleration, or maximal inertia.

They obey the ***inertia principle***:

"keep the controls constant as long as they provide viable solutions"

because $g^0(x,u) = 0$ when $0 \in G_c(x,u)$. Indeed, if the velocity 0 belongs to $G_c(x(t_1),u(t_1))$, then the control will remain equal to $u(t_1)$ as long as for $t \geq t_1$, a solution $x(\cdot)$ to the differential equation $x'(t) = f(x(t),u(t_1))$ satisfies the condition $0 \in G_c(x(t),u(t_1))$.

If at some time t_f, $u(t_f)$ is a punctuated equilibrium, then the solution enters the viability cell associated to this control and may remain in this viability cell forever[5] and the control will remain equal to this punctuated equilibrium. Viable heavy solutions are studied in the general case in section 3 and in the case of smooth viability constraints, in section 4.

We already mentioned that in general, the graph of the contingent cone map $T_K(\cdot)$ is not closed. In order to obtain this property, we suggest in section 5 to replace the contingent cone $T_K(x)$ by the subset $T_K^c(x)$ of directions $v \in T_K(x)$ such that there exist a measurable function $x''(\cdot)$ bounded by the constant c satisfying

$$\forall\, t \geq 0, \ \ x(t) := x + tv + \int_0^t (t-\tau)x''(\tau)d\tau \ \in \ K$$

We shall see that the graph of the set-valued map $T_K^c(\cdot)$ is the viability kernel of the closure of the graph of the contingent cone map $T_K(\cdot)$ of the

[4]we tker $U(x) = F(x)$ and $f(x,u) = u$.

[5]as long as the viability domain does not change for external reasons which are not taken into account here.

map $(x,v) \rightsquigarrow \{v\} \times cB$. *It is therefore closed*. These subsets, which can be interpreted as ***global contingent sets***, enjoy properties that the contingent cones may not have.

These properties are used in section 6 to prove the convergence of a modified version of the ***Byrnes-Isidori zero dynamics algorithm***[6] to a closed viability domain (instead of the viability kernel).

In this paper, X, Y, Z denote finite dimensional vector-space and B the unit ball of any of these spaces.

1 Smooth State-Control Solutions

Let us consider a control system (U, f) defined by a set-valued map $U : Z \rightsquigarrow X$ and a single-valued map $f : \text{Graph}(U) \mapsto X$, where X is regarded as the state space of the system, Z the control space, f as describing the dynamics and U the a priori feedback. The evolution of a ***viable state-control pair*** $(x(\cdot), u(\cdot))$ is governed by

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & \forall\, t \geq 0,\ u(t) \in U(x(t)) \end{cases} \tag{2}$$

We shall say that it is ***smooth*** is both $x(\cdot)$ and $u(\cdot)$ are absolutely continuous and that they are ***φ-smooth*** if they are smooth and satisfy

$$\text{for almost all } t \geq 0,\ \|u'(t)\| \leq \varphi(x(t), u(t))$$

We can obtain smooth viable solutions by setting a bound to the growth to the evolution of controls. For that purpose, we shall associate with this control system and with any non negative continuous function $(x, u) \mapsto \varphi(x, u)$ with linear growth[7] the system of differential inclusions

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in \varphi(x(t), u(t))B \end{cases} \tag{3}$$

We observe that ***any solution $(x(\cdot), u(\cdot))$ to the system of differential inclusions (3) which is viable in*** Graph(U) ***is a φ-smooth solution to the control system (2).***

[6] which is a generalization of the structure algorithm introduced by Silverman in [27] and Basile & Marro in [7] for linear control systems.

[7] which can be a constant $\rho > 0$, or the function $c\|u\|$, or the function $(x, u) \to c(\|u\| + \|x\| + 1)$.

Let us recall the statement of the Viability Theorem. We say that a set-valued map is a Peano map if it is upper semicontinuous with nonempty compact convex images and with linear growth[8].

A subset $K \subset \text{Dom}(F)$ is called a *viability domain* of F if and only if

$$\forall\, x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$$

Theorem 1.1 (Viability Theorem) *Let us consider a* Peano map$F : X \rightsquigarrow X$ *and a* closed *subset* $K \subset \text{Dom}(F)$ *of a finite dimensional vector space* X.

If K *is a* viability domain , *then for all initial state* $x_0 \in K$, *there exists a* viable *solution on* $[0, \infty[$ *to differential inclusion*

$$x'(t) \in F(x(t))$$

We thus deduce from this Viability Theorem applied to the system (3) on the graph of U the following Regularity Theorem:

Theorem 1.2 *Let us assume that the graph of U is closed and that f is continuous and has linear growth.*

Then for any initial state $x_0 \in \text{Dom}(U)$ and any initial control $u_0 \in U(x_0)$, there exists a φ-smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (2) starting at (x_0, u_0) if and only if the set-valued map U satisfies

$$\forall\, (x, u) \in \text{Graph}(U), \; DU(x, u)(f(x, u)) \cap \varphi(x, u)B \neq \emptyset \tag{4}$$

Proof — The conclusion of the theorem amounts to saying that the closed subset Graph(U) enjoys the viability property. By Viability Theorem 1.1, which we can apply since the set-valued map $(x, u) \rightsquigarrow \{f(x, u)\} \times \varphi(x, u)B$ is upper semicontinuous with compact convex values and has linear growth, this is the case if and only if it is a viability domain, i.e., if and only if

$$\forall\, (x, u) \in \text{Graph}(U), \; T_{\text{Graph}(U)}(x, u) \cap (\{f(x, u)\} \times \varphi(x, u)B) \neq \emptyset$$

By the very definition of the contingent derivative of U, this is the necessary and sufficient condition of the theorem. □

We know that whenever the right-hand side of an ordinary differential equation is differentiable, its solutions are twice differentiable. The extension of this property to the case of differential inclusions is just a consequence of the above theorem when we take $f(x, u) = u$:

[8]or equivalently, in the case of finite dimensional state spaces, closed set-valued maps with convex values and linear growth.

Corollary 1.3 *Let $F : X \rightsquigarrow X$ be a closed set-valued map such that*

$$\forall\, x \in \mathrm{Dom}(F),\ \forall\, v \in F(x),\ \ DF(x,v)(v) \cap \varphi(x,v)B \neq \emptyset$$

Then, for any $x_0 \in \mathrm{Dom}(F)$ and $v_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to the differential inclusion

$$x'(t) \in F(x(t)),\ x(0) = x_0 \ \&\ x'(0) = v_0$$

such that both $x(\cdot)$ and $x'(\cdot)$ are absolutely continuous.

The assumption of the above theorem is too strong, since it requires that property (4) is satisfied for all controls u of $U(x)$ (so that we have a solution for every initial control chosen in $U(x_0)$). We may very well be content with the existence of a smooth solution for only some initial control in $U(x_0)$.

So, we can relax the problem by looking for the largest closed set-valued feedback map contained in U in which we can find the initial state-controls yielding smooth viable solutions to the control system. This amounts to studying the viability kernels of Graph(U) for the system of differential inclusions (3), where the viability kernel is defined as follows:

Definition 1.4 (Viability Kernel) *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$. We shall say that the largest closed viability domain contained in K (which may be empty) is the* viability kernel of K *and denote it by* $\mathrm{Viab}_F(K)$ *or, simply,* $\mathrm{Viab}(K)$.

We recall that such a viability kernel does exist and can be characterized.

Theorem 1.5 *Let us consider a nontrivial Peano map $F : X \rightsquigarrow X$. Let $K \subset \mathrm{Dom}(F)$ be closed. Then the viability kernel of K exists (possibly empty) and is the subset of initial states such that at least one solution starting from them is viable in K.*

Remark — When $K := h^{-1}(0)$ is defined by equality constraints (where $h : X \mapsto Y$ is an observation map), the restriction of the control system to the viability kernel of $h^{-1}(0)$ is called *zero dynamics*. See the series of papers [21,9,10,11,13] devoted to this question. In this case, the viability kernel is obtained by the zero dynamics algorithm described in section 6. □

This leads us to introduce the following

Definition 1.6 (**φ-growth regulation map**) *Let us consider the control system (2). We shall denote by $R^{\varphi} := R_U^{\varphi}$ the set-valued map whose graph is the viability kernel of Graph(U) for the system of differential inclusions (3). We shall call it the* φ-growth regulation map *to the control system (2). If $\varphi \equiv 0$, we shall say that R_U^0 is the* punctuated regulation map. *Controls u such that $(R^0)^{-1}(u)$ are not empty are called* punctuated equilibria.

We thus deduce from Theorem 1.5 the following result on the existence of smooth viable solutions.

Theorem 1.7 *Let us assume that the graph of U is closed and that f is continuous and has linear growth.*

Then for any initial state $x_0 \in \mathrm{Dom}(R^{\varphi})$ and any initial control $u_0 \in R^{\varphi}(x_0)$, there exists a smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (2) starting at (x_0, u_0), where the solution $x(\cdot)$ is regulated by a control $u(\cdot)$ starting at u_0 through the smooth regulation law:

$$\forall\, t \geq 0, \;\; u(t) \;\in\; R^{\varphi}(x(t)) \tag{5}$$

Remark — We observe that the graph of R_U^{φ} is also the viability kernel of the graph of the regulation map R_U and that the regulation maps R^{φ} are increasing with φ. □

The case when the growth φ is equal to 0 is particularly interesting, because it determines areas where the evolution of the control is constant.

Proposition 1.8 *The subset $(R^0)^{-1}(u)$ is the viability kernel of $U^{-1}(u)$ for the differential equation*

$$x'(t) = f(x(t), u)$$

parametrized by the constant control u.

Proof — Indeed, $(R^0)^{-1}(u)$ describes the subset of Dom(U) which is controlled by the constant control u because for any initial state x_0 given in $(R^0)_U^{-1}(u)$, there exists a solution $x(\cdot)$ to the differential inclusion

$$\begin{cases} i) & x'(t) \;=\; f(x(t), u) \quad u \text{ remains constant} \\ ii) & u'(t) \;=\; 0 \end{cases}$$

i.e., of the differential equation $x'(t) = f(x(t), u)$ which is *viable in* $(R^0)_U^{-1}(u)$. □

Naturally, *when $(R^0)_U^{-1}(u)$ is reduced to a point, this point is an equilibrium.*

2 Example

We illustrate these concepts of regulation maps in the case of the simplest dynamical economic model (one commodity, one consumer).

Let $K := [0, b]$ the subset of a scarce commodity x. Assume that the consumption rate of a consumer is equal to $a > 0$, so that, without any further restriction, its exponential consumption will leave the viability subset $[0, b]$. Hence its consumption is slowed down by a price which is used as a control. In summary, the evolution of its consumption is governed by the control system

$$\text{for almost all } t \geq 0, \quad x'(t) = ax(t) - u(t), \quad \text{where } \; u(t) \geq 0$$

subjected to the constraints

$$\forall\, t \geq 0, \quad x(t) \in [0, b]$$

The a priori feedback map U is defined by $U(x) := \mathbf{R}_+$. Hence the regulation map is given by the formula

$$R_K(0) \; = \; \{0\}, \;\; R_K(x) = \mathbf{R}_+ \;\text{ when }\; x \in]0, b[\;\; \& \;\; R_K(b) = [ab, +\infty[$$

Its graph is not closed, and its closure is the graph of U, equal to $[0, b] \times \mathbf{R}_+$

We see at once that the viable equilibria of the system range over the ***equilibrium line*** $u = ax$. Viability is guaranteed each time that the price $u(t)$ is chosen in $R(x(t))$, i.e., $u = 0$ when $x = 0$ (and thus, the system cannot leave the equilibrium because negative prices are not allowed "to start" the system) and $u \geq ab$ when $x = b$, so that the price is large enough to stop or decrease consumption.

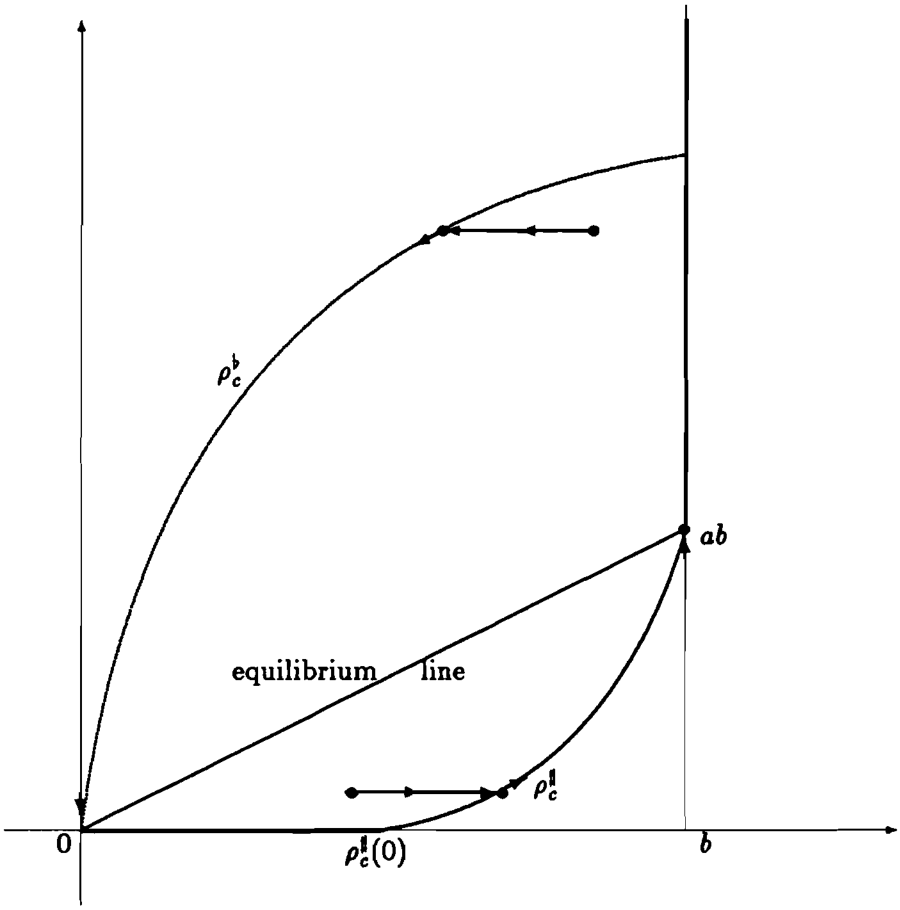
Assume that the system obeys the inertia principle: *it keeps the price constant as long as it works*. Take for instance $x_0 > 0$ and $u_0 \in [0, ax_0[$. Then the consumption increases[9] and when it reaches the boundary b of the interval, the system has to switch very quickly to a velocity large enough to slow down the consumption for the solution to remain in the interval $[0, b]$.

But there is a bound to growth of prices (and inflation rates), so that we should set a bound[10] on price velocities: $|u'(t)| \leq c$. We shall associate with such a bound a "last warning" threshold to modify the price: there is a level of consumption after which it will be impossible to slow down the consumption with a velocity smaller than or equal to c to forbid it to increase beyond the boundary b.

[9] it is equal to $(e^{at}(ax_0 - u_0) + u_0)/a$.

[10] we take $\varphi(x, u) \equiv c$.

Figure 2: Evolution of a Heavy Solution



We shall find this bound[11] and introduce *heavy solutions* which will be studied in full generality later for building this regulation law. They are the one whose controls evolve with the "smallest velocity". It may be useful to be acquainted with this concept on an example, and this one illustrates well how heavy solutions evolve.

We thus consider the c-bounded state-control solutions, which are the solutions to the system

$$\left\{\begin{array}{ll} i) & \text{for almost all } t \geq 0,\ \ x'(t) = ax(t) - u(t) \\ ii) & \text{and } \ -c \leq u'(t) \leq c \end{array}\right. \tag{6}$$

which are viable in Graph(U).

We introduce the functions $\rho^\sharp$ and $\rho^\flat$ defined on $[0,\infty[$ by

$$\left\{\begin{array}{lll} i) & \rho_c^\flat(u) & := \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u) \approx \frac{u^2}{2c} \\ ii) & \rho_c^\sharp(u) & := -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2 \end{array}\right.$$

and the functions $r^\sharp$ and $r^\flat$ defined on $[0,b]$ by

$$\left\{\begin{array}{ll} i) & r^\flat(x) = u \ \text{ if and only if } \ u = \rho_c^\flat(x) \\ ii) & r^\sharp(x) = 0 \ \text{ if } \ x \in [0, \rho_c^\sharp(0)] \ \ (\rho^\sharp(0) = \frac{c}{a^2}(1 - e^{-a^2 b/c})) \\ iii) & r^\sharp(x) = u \ \text{ if and only if } \ u = \rho^\sharp(x) \ \text{ when } \ x \in [\rho_c^\sharp(0), b] \end{array}\right.$$

Proposition 2.1 *The c-bounded growth regulation map of system (6) is defined by*

$$\forall\, x \in [0,b],\ \ R^c(x) = [r^\sharp(x), r^\flat(x)] \tag{7}$$

Proof — Indeed, set $u^\sharp(t) := u_0 + ct$ and $u^\flat := u_0 - ct$ and denote by $x^\sharp(\cdot)$ and $x^\flat(\cdot)$ the solutions starting at x_0 to differential equations

$$x' = ax - u^\sharp(t)$$

and

$$x' = ax - u^\flat(\cdot)$$

respectively. Then any solution $(x(\cdot), u(\cdot))$ to the system (6) satisfies $u^\flat(\cdot) \leq u(\cdot) \leq u^\sharp(\cdot)$ and thus, $x^\sharp(\cdot) \leq x(\cdot) \leq x^\flat(\cdot)$ because

$$x(t) = e^{at}x_0 - \int_0^t e^{a(t-s)}u(s)ds$$

[11]provided by the c-regulation map R_c.

We also observe that the equations of the curves $t \mapsto (x^\sharp(\cdot), u^\sharp(\cdot))$ and $t \mapsto (x^\flat(\cdot), u^\flat(\cdot))$ passing through (x_0, u_0) are solutions to the differential equations

$$d\rho^\sharp = \frac{1}{c}(a\rho^\sharp - u)du \ \& \ d\rho^\flat = -\frac{1}{c}(a\rho^\flat - u)du$$

the solutions of which are

$$\begin{cases} i) & \rho^\sharp(u) = e^{a(u-u_0)/c}(x_0 - u_0/a - c/a^2) + u/a + c/a^2 \\ ii) & \rho^\flat(u) = e^{a(u_0-u)/c}(x_0 - u_0/a + c/a^2) + u/a - c/a^2 \end{cases}$$

Let $\rho_c^\flat$ be the solution passing through $(0,0)$, which is equal to

$$\rho_c^\flat(u) = \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u)$$

and

$$\rho_c^\sharp(u) = -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2$$

be the solution passing through the pair (ab, b).

— We check that the viability kernel is contained in the graph of R^c by contraposition.

If $u_0 > r^\flat(x_0)$, then any solution $(x(\cdot), u(\cdot))$ starting from (x_0, u_0) satisfies

$$x(t) \leq x^\flat(t) = \rho_c^\flat(u^\flat(t)) \leq \rho_c^\flat(u(t))$$

because $\rho_c^\flat(\cdot)$ is nondecreasing. Hence, when $x(t_1) = 0$, we deduce that $u(t_1) > 0$, so that such solution is not viable, and thus, (x_0, u_0) does not belong to the viability kernel.

If $0 \leq u_0 < r^\sharp(x_0)$, any solution $(x(\cdot), y(\cdot))$ satisfies inequalities

$$x(t) \geq x^\sharp(t) = \rho_c^\sharp(u^\sharp(t)) \geq \rho_c^\sharp(u(t))$$

Therefore, when $x(t_1) = b$ for some time t_1, its velocity $x'(t_1) = ab - u(t_1)$ is positive, so that the solution is not viable.

— It remains to prove that the viability kernel is equal to the graph of R^c by constructing particular viable solutions starting from any point (x_0, u_0) of this graph. We choose the ***heavy solutions***.

The equilibrium line $u = ax$ is contained in the viability kernel: if we start from an equilibrium, both the state and the controls can be kept constant.

We shall now investigate the cases when the initial control u_0 is below or above the equilibrium line.

Consider the case when $x_0 > 0$ and the price $u_0 \in [r^\sharp(x_0), ax_0[$. Since we want to choose the price velocity with minimal norm, we take[12] $u'(t) = 0$ as long as the solution $x(\cdot)$ to the differential equation $x' = ax - u_0$ yields a consumption $x(t) < \rho_c^\sharp(u_0)$. When for some time t_1, the consumption $x(t_1) = \rho_c^\sharp(u_0)$, it has to be slowed down. Indeed, otherwise $(x(t_1 + \varepsilon), u_0)$ will be below the curve $\rho_c^\sharp$ and we saw that in this case, any solution will eventually cease to be viable. Therefore, prices should increase to slow down the consumption growth. The idea is to take the smallest velocity u' such that the vector $(x'(t_1), u')$ takes the state inside the graph of R^c: they are the velocities $u' \geq x'(t_1)/\rho_c^{\sharp\prime}(u_0)$. By construction, it is achieved by the velocity of $x^\sharp(\cdot)$, which is the highest one allowed to increase prices. Therefore, by taking

$$x(t) := x^\sharp(t) := e^{a(t-t_1)}(x(t_1) - u_0/a - c/a^2) + c(t - t_1)/a + u_0/a + c/a^2$$

and $u(t) := u_0 + c(t - t_1)$ for $t \in [t_1, t_1 + (ab - u_0)/c]$, we get a solution which ranges over the curve $x^\sharp(t) = \rho_c^\sharp(u^\sharp(t))$. This a heavy solution because, for the same reason than above, the smallest velocity of the price (which is unique along this curve) is chosen. According to the above differential equation, we see that $x(t)$ increases to b where it arrives with velocity 0 and the price increases linearly until it arrives to the equilibrium price ab. Since (b, ab) is an equilibrium, the heavy solution stays there: we take $x(t) \equiv b$ and $u(t) \equiv 0$ when $t \geq t_1 + u_0/c$. So we have built a viable solution starting from (x_0, u_0), so that the region between the "curve $\rho^\sharp$" and the equilibrium line is contained in the viability kernel, i.e., the graph of R^c.

Consider now the case when $u_0 \in [ax_0, r^\flat(x_0)]$, where we follow the same construction of the heavy viable solution. We start by taking $u'(t) = 0$, and thus, $u(t) = u_0$, as long as the solution $x(\cdot)$ to the differential equation $x' = ax - u_0$, which decreases, satisfies $x(t) > \rho_c^\flat(u_0)$. Then, when $x(t_1) = \rho_c^\flat(u_0)$ for some t_1, we take

$$x(t) = x^\flat(t) := e^a(t - t_1)(x(t_1) - u_0/a + c/a^2) - c(t - t_1)/a + u_0/a - c/a^2$$

and $u(t) := u_0 - c(t - t_1)$ for $t \in [t_1, t_1 + u_0/c]$ in order to avoid leaving the viability kernel. Finally, for $t \geq t_1 + u_0/c$, we take $x(t) \equiv 0$ and $u(t) \equiv 0$. This particular solution, is viable, so that the pairs (x_0, u_0) where $u_0 \in [ax_0, r^\flat(x_0)]$ belong to the viability kernel. □

[12] and realize in this case the dream of economists, which, despite the teachings of history, are looking for constant prices and commodities ...

Remark — We observe that for any $x \in]0, b[$,

$$\lim_{c \to 0+} r^\flat(x) = \lim_{c \to 0+} r^\sharp(x) = ax, \ \lim_{c \to \infty} r^\sharp(x) = 0 \ \& \ \lim_{c \to \infty} r^\flat(x) = +\infty$$

In other words, the graph of R^c starts from the equilibrium line when $c = 0$ and converges in some sense to the graph of U when $c \to +\infty$. □

3 Heavy Viable Solutions

Let us consider a control system (U, f) which has a nontrivial φ-growth regulation map R_U^φ for some $\varphi \geq 0$.

Proposition 3.1 *The smooth viable state-control pairs $(x(\cdot), u(\cdot))$ to the control system (2) are also solutions to the system of differential inclusions*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in DR_U^\varphi(x(t), u(t))(f(x(t), u(t))) \end{cases} \tag{8}$$

Proof — Indeed, since the absolutely continuous function $(x(\cdot), u(\cdot))$ takes its values into $\text{Graph}(R_U^\varphi)$, then its derivative $(x'(\cdot), u'(\cdot))$ belongs almost everywhere to the contingent cone

$$T_{\text{Graph}(R_U^\varphi)}(x(t), u(t)) := \text{Graph}(DR_U^\varphi(x(t), u(t)))$$

We then replace $x'(t)$ by $f(x(t), u(t))$.

The converse holds true because equation (8) makes sense only if $(x(t), u(t))$ belongs to the graph of R_U^φ. □

The question arises whether we can construct selection procedures of the control component of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 3.2 (Dynamical Closed Loops) *We shall say that a selection g of the contingent derivative from the φ-regulation map R_U^φ in the direction f defined by*

$$\forall \ (x, u) \in \text{Graph}(R_U^\varphi), \ g(x, u) \in DR_U^\varphi(x, u)(f(x, u)) \tag{9}$$

is a dynamical closed loop.

The system of differential equations

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) = g(x(t), u(t)) \end{cases} \tag{10}$$

is called the associated closed loop differential system.

Therefore, a dynamical closed loop being given, solutions to system of ordinary differential equations (10) (if any) are smooth viable state-control pairs of the initial control problem (2).

Such solutions do exist when g is continuous (and if such is the case, they will be continuously differentiable). But they also may exist when g is no longer continuous, as is the case of slow solutions (see [14,3,4,6]) closed loop controls. This is the case for instance when $g(x,u)$ is the element of minimal norm in $DR_U^{\varphi}(x,u)(f(x,u))$.

In both cases, we need to assume that the right-hand side of this system is lower semicontinuous with closed convex images. This happens when we posit the following condition:

Definition 3.3 *We shall say that a control system (U,f) is* φ-dynamically regular *if*

$$\begin{cases} i) & \text{the domains of } U \text{ and } R_U^{\varphi} \text{ coincide} \\ ii) & \text{the } \varphi\text{-regulation map } R_U^{\varphi} \text{ is sleek} \\ iii) & \sup_{(x,u)\in \mathrm{Graph}(R_U^{\varphi})} \|DR_U^{\varphi}(x,u)\| < +\infty \end{cases} \tag{11}$$

Indeed, assumptions (11)ii) and iii) imply that the set-valued map $(x,u,v) \rightsquigarrow DR_U^{\varphi}(x,u,v)$ is lower semicontinuous (see [5] for more details).

Then we begin by deducing from Michael's Theorem (see [1]) the existence of continuously differentiable viable state-control solutions.

Theorem 3.4 *Let us assume that the graph of U is closed and that f is continuous and has linear growth. If the control system (U,f) is φ-dynamically regular, then there exists a continuous dynamical closed loop. The associated closed-loop differential system regulates continuously differentiable viable state-control solutions.*

Since we do not know constructive ways to built continuous dynamical closed loops, we shall investigate whether some explicit dynamical closed loop provides closed loop differential systems which do possess solutions.

The simplest example of dynamical closed loop control is the map g_{φ}° associating with each state-control pair (x,u) the element of minimal norm of $DR_U^{\varphi}(x,u)(f(x,u))$.

Definition 3.5 (Heavy Viable Solutions) *We denote by $g_{\varphi}^{\circ}(x,u)$ the element of minimal norm of $DR_U^{\varphi}(x,u)(f(x,u))$. We shall say that the solutions to the associated closed loop differential system*

$$\begin{cases} i) & x'(t) = f(x(t),u(t)) \\ ii) & u'(t) = g_{\varphi}^{\circ}(x(t),u(t)) \end{cases}$$

are heavy viable solutions *to the control system* (U, f).

Theorem 3.6 (Heavy Viable Solutions) *Let us assume that the graph of* U *is closed and that* f *is continuous and has linear growth. If the control system* (U, f) *is* φ*-dynamically regular, then for any initial state-control* (x_0, u_0) *in Graph(*R_U^φ*),* there exists a heavy viable solution to the control system (2).

Remark — If for some $t_f > 0$, $u(t_f)$ is a punctuated equilibrium, then $u(t) = u_{t_f}$ for all $t \geq t_f$ and $x(t)$ *remains in the viability cell* $N^0(u(t_f))$ *for all* $t \geq t_f$. □

The reason why this theorem holds true is that the minimal selection is obtained through the *selection procedure* of a set-valued map $F : X \rightsquigarrow Y$ we are about to describe.

Let $F : X \rightsquigarrow Y$ be a set-valued map with closed convex values. The projection of 0 onto the closed convex set $F(x)$ is the element $u := m(F(x)) \in F(x)$ such that

$$\|u\|^2 + \sigma(-F(x), u) = \sup_{y \in F(x)} < u - 0, u - y > \leq 0 \tag{12}$$

If we introduce the set-valued map $S_F : X \rightsquigarrow Y$ defined by

$$u \in S_F(x) \quad \text{if and only if} \quad \|u\|^2 + \sigma(-F(x), u) \leq 0 \tag{13}$$

then we observe that the graph of the minimal selection is equal to:

$$\text{Graph}(m(F)) = \text{Graph}(F) \cap \text{Graph}(S_F)$$

Therefore, the minimal selection is obtained through a general selection procedure defined as follows (see [3,4]):

Definition 3.7 (Selection Procedure) *Let* Y *be a Banach space. A* selection procedure of a set-valued map $F : X \rightsquigarrow Y$ *is a set-valued map* $S_F : X \rightsquigarrow Y$ *satisfying*

$$\begin{cases} i) & \forall\, x \in \text{Dom}(F), \ \ S(F(x)) := S_F(x) \cap F(x) \neq \emptyset \\ ii) & \textit{the graph of } S_F \textit{ is closed} \end{cases}$$

We can easily provide other examples of selection procedures through optimization thanks to the Maximum Theorem.

Proposition 3.8 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with compact values. Let V : Graph$(F) \mapsto \mathbf{R}$ be continuous. Then the set-valued map S_F defined by:*

$$S_F(x) := \{y \in Y \mid V(x,y) \leq \inf_{y' \in F(x)} V(x,y')\}$$

is a selection procedure *of F. Consequently, if the graph of F is also closed, so is the graph of the selection $S(F)$ equal to:*

$$S(F(x)) = \{y \in F(x) \mid V(x,y) \leq \inf_{y' \in F(x)} V(x,y'))\}$$

For simplicity, we set

$$G_\varphi(x,u) := DR_U^\varphi(x,u)(f(x,u))$$

Theorem 3.9 *We posit the assumptions of Theorem 1.7. Let S_{G_φ} be a selection procedure of the set-valued map G_φ with convex values. Then, for any initial state $(x_0, u_0) \in$ graph(U), there exists a viable state-control solution starting at (x_0, u_0) to the associated closed loop system of differential inclusions*

$$\left\{ \begin{array}{lll} i) & x'(t) = & f(x(t),u(t)) \\ ii) & u'(t) & \in G_\varphi(x(t),u(t)) \cap S_{G_\varphi}(x(t),u(t)) \end{array} \right. \quad (14)$$

In particular, if for any $(x,u) \in$ Graph(U), the intersection

$$G_\varphi(x,u) \cap S_{G_\varphi}(x,u) = \{s(DR_U^\varphi(x,u)(f(x,u)))\}$$

is a singleton, then there exists a viable state-control solution starting at (x_0, u_0) to the associated closed loop differential system

$$\left\{ \begin{array}{lll} i) & x'(t) & = f(x(t),u(t)) \\ ii) & u'(t) & = s(DR_U^\varphi(x(t),u(t))(f(x(t),u(t)))) \end{array} \right.$$

Proof — We shall replace the system of differential inclusions (8) by the system of differential inclusions

$$\left\{ \begin{array}{lll} i) & x'(t) & = f(x(t),u(t)) \\ ii) & u'(t) & \in S_{G_\varphi}(x(t),u(t)) \end{array} \right. \quad (15)$$

Since the convex selection procedure S_{G_φ} has a closed graph and convex values, the right-hand side is upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. It remains to check that GraphR_U^φ is still a viability domain for this new system of differential inclusions. Indeed, by construction, we know that there exists an element w in the intersection of $G_\varphi(x,u)$ and $S_{G_\varphi}(x,u)$. This means that the pair $(f(x,u),w)$ belongs to $f(x,u) \times S_{G_\varphi}(x,u)$ and that it also belongs to

$$\text{Graph}(G_\varphi) \ := \ T_{\text{Graph}(R_U^\varphi)}(x,u)$$

Therefore, we can apply the Viability Theorem. For any initial state-control (x_0,u_0), there exists a solution $(x(\cdot),u(\cdot))$ to the new system of differential inclusions which is viable in Graph(R_U^φ). Consequently, for almost all $t>0$, the pair $(x'(t),u'(t))$ belongs to the contingent cone to the graph of R_U^φ at $(x(t),u(t))$, which is the graph of the contingent derivative $DR_U^\varphi(x(t),u(t))$. In other words,

$$\text{for almost all } t>0, \ \ u'(t) \ \in \ G_\varphi(x(t),u(t))$$

We thus deduce that for almost all $t>0$, $u'(t)$ belongs to the selection $S(G_\varphi)(x(t),u(t))$ of the set-valued map $G_\varphi(x(t),u(t))$. Hence, the state-control pair is a solution to the system of differential inclusions (14). □

4 Heavy Viable Solutions on Smooth Viability Domains

Consider the case when K is a smooth viability domain defined by

$$K := A^{-1}(0)$$

where $A : X \mapsto Y$ is a twice continuously differentiable map such that $A'(x)$ is surjective for every $x \in A^{-1}(0)$.

Since $T_K(x) = \ker A'(x)$, we deduce that the regulation map is equal to

$$R_K(x) = \{u \in U(x) \mid A'(x)f(x,u) = 0\}$$

We begin by computing its contingent derivative:

Proposition 4.1 *Assume that $A'(x) \in \mathcal{L}(X,Y)$ is surjective whenever $A(x) = 0$, that the graph of U is sleek and that for any $y \in Y$ and $v \in X$, the subsets*

$$DU(x,u)(v) \cap (A'(x)f'_u(x,u))^{-1}(y - A''(x)(f(x,u),v) - A'(x)f'_x(x,u)v)$$

are not empty. Then the contingent derivative of the regulation map is equal to

$$\begin{cases} DR_U(x,u)(v) = DU(x,u)(v) \cap \\ -(A'(x)f'_u(x,u))^{-1}(A''(x)(f(x,u),v) - A'(x)f'_x(x,u)v) \end{cases}$$

when $A'(x)v = 0$ and $DR_U(x,v) = \emptyset$ if not. In particular, if $U(x) \equiv Z$, then it is sufficient to assume that $A'(x)f'_u(x,u)\cdot$ is surjective and we have in this case

$$DR_U(x,u)(v) = -(A'(x)f'_u(x,u))^{-1}(A''(x)(f(x,u),v) - A'(x)f'_x(x,u)v)$$

when $A'(x)v = 0$ and $DR_U(x,v) = \emptyset$ if not.

Proof — The graph of R_U can be written as the subset of pairs $(x,u) \in \text{Graph}(U)$ such that $C(x,u) := (A(x), A'(x)f(x,u)) = 0$. We apply [5, Theorem 4.3.3.], whcih states that since the graph of U is closed and sleek, the transversality condition

$$C'(x,u)T_{\text{Graph}(U)}(x,u) = C'(x,u)\text{Graph}(DU(x,u)) = Y \times Y$$

implies that the contingent cone to this closed subset is the set of elements $(v,w) \in \text{Graph}(DU(x,u))$ satisfying

$$\begin{cases} C'(x,u)(v,w) = \\ (A'(x)v, A'(x)f'_u(x,u)w + A'(x)f'_x(x,u)v + A''(x)(f(x,u),v)) = 0 \end{cases}$$

But the surjectivity of $A'(x)$ and the non emptiness of the intersection imply this transversality condition. □

Therefore, the set-valued map G defined by

$$G(x,u) := DR_U(x,u)(f(x,u))$$

is equal to right-hand

$$\begin{cases} G(x,u) = DU(x,u)(f(x,u)) \cap \\ -(A'(x)f'_u(x,u))^{-1}(A''(x)(f(x,u),f(x,u)) - A'(x)f'_x(x,u)f(x,u)) \end{cases}$$

When we take $U(x) \equiv Z$, we have explicit formulas for computing the dynamical closed loop yielding heavy solutions.

Corollary 4.2 *Assume that $U(x) \equiv Z$, that the regulation map*

$$R(x) := \{u \in Z \mid A'(x)f(x,u) = 0\}$$

has non empty values, that $A'(x)$ is surjective whenever $x \in A^{-1}(0)$ and that $A'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ is surjective whenever $u \in R(x)$.

Then there exist heavy solutions viable in K, which are the solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star A'(x)^\star \\ & (A'(x)f'_u(x,u)f'_u(x,u)^\star A'(x)^\star)^{-1} A'(x) f'_x(x,u) f(x,u) \end{cases}$$

Proof — The element $g(x,u) \in G(x,u)$ of minimal norm is the unique solution to the quadratic minimization problem with equality constraints:

$$\inf_{A'(x)f'_u(x,u)w = -A'(x)f'_x(x,u)f(x,u) - A''(x)(f(x,u),f(x,u))} \|w\|^2$$

It is equal to

$$g(x,u) = -f'_u(x,u)^\star A'(x)^\star (A'(x)f'_u(x,u)f'_u(x,u)^\star A'(x)^\star)^{-1}$$
$$(A'(x)f'_x(x,u)f(x,u) + A''(x)(f(x,u),f(x,u)))$$

because the linear operator $B := A'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ is surjective[13].

Example: Heavy viable solutions in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ is an affine subspace, with $Ax = Lx - y$ where $L \in \mathcal{L}(X,Y)$ is surjective.

Let us assume that

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \text{ such that } Lf(x,u) = 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ \ Lf'_u(x,u) \text{ is surjective} \end{cases}$$

Then, for any initial state $x_0 \in K$ and initial velocity u_0 satisfying $Lf(x_0,u_0) = 0$, there exists a heavy viable solution of the control problem, obtained as a solution to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star L^\star (Lf'_u(x,u)f'_u(x,u)^\star L^\star)^{-1} L f'_x(x,u) f(x,u) \end{cases}$$

[13] Recall that the unique element which minimizes $x \mapsto \|x\|$ under the constraint $Bx = y$, where $B \in \mathcal{L}(X,Y)$ is surjective, is equal to B^+y, where $B^+ = B^\star(BB^\star)^{-1}$ denotes the *orthogonal right-inverse of B.*

When $Y := \mathbf{R}$ and $K := \{x \in X \mid \ < p, x >= y\}$ is an hyperplane, the above assumption becomes

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \mid \ < p, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ \ f'_u(x,u)^* p \neq 0 \end{cases}$$

and heavy viable solutions are solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{< p, f'_x(x,u), f(x,u) >}{\|f'_u(x,u)^* p\|^2} f'_u(x,u)^* p \end{cases}$$

Example: Heavy solutions viable in the sphere. Let $L \in \mathcal{L}(X,X)$ be a symmetric positive-definite linear operator, with which we associate $A(x) :=< Lx, x > -1$ and the viability subset

$$K \ := \ \{x \in X \mid \ < Lx, x >= 1\}$$

We assume that

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \mid \ < Lx, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ \ f'_u(x,u)^* Lx \neq 0 \end{cases}$$

Then there exist heavy viable solutions in the sphere, which are solutions to the system of differential equations

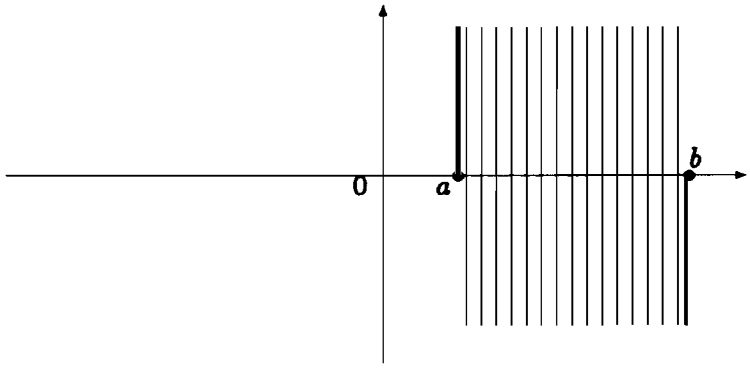
$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{f'_u(x,u)^* Lx}{\|f'_u(x,u)^* Lx\|^2} \\ & (< Lf(x,u), f(x,u) > + < Lx, f'_x(x,u) f(x,u) >) \end{cases}$$

5 Application: Global Contingent Sets

Definition 5.1 *Let $K \subset X$ be a closed subset of a finite dimensional vector-space X and $c > 0$ be a positive constant. We shall denote by $T^c_K(x)$ the subset of elements $v \in T_K(x)$ such that there exists a measurable function $x''(\cdot)$ bounded by c satisfying*

$$\forall\, t \geq 0,\ \ x + tv + \int_0^t (t-\tau) x''(\tau) d\tau \ \text{ is viable in } K$$

Figure 3: The Graph of $T_{[a,b]}(\cdot)$



We introduce the Peano F from $X \times X$ to itself defined by $F(x,v) := \{v\} \times cB$. The functions $t \mapsto x(t) := x(0) + tv(0) + \int_0^t (t-\tau)x''(\tau)d\tau$ where $\|x''(\tau)\| \leq c$ are solutions to the differential inclusion $\|x''(t)\| \leq c$, as well as solutions to the differential inclusion

$$(x'(t), v'(t)) \in F(x(t), v(t))$$

We remark at once that *the graph of the set-valued map* T_K^c *is the viability kernel of* Graph(T_K) *for the set-valued map* $(x,v) \rightsquigarrow \{v\} \times cB$.

Observe that $0 \in T_K^c(x)$ for all $x \in K$.

Example

We can check easily that for $K := [0,1]$, the contingent cone $T_K(x)$ is defined by

$$T_K(x) = \begin{cases} \mathbf{R}_+ & \text{if } x = 0 \\ \mathbf{R} & \text{if } x \in]0,1[\\ \mathbf{R}_- & \text{if } x = 1 \end{cases}$$

and the global contingent set is equal to

$$\forall\, x \in [0,1],\ T_K^c(x) = \left[-\sqrt{cx}\,,\ \sqrt{c(1-x)}\right] \quad \square$$

We deduce from the properties of the viability kernels the following statements.

Proposition 5.2 *The graph of the set-valued map $x \rightsquigarrow T^c_K(x)$ is closed. Let $K^\sharp := \limsup_{n\to\infty} K_n$ denote the (Kuratowski) upper limit of a sequence of closed subsets K_n. Then the (Kuratowski) upper limit of the graphs of $T^c_{K_n}$ is contained in the graph of $T^c_{K^\sharp}$.*

Proof — It follows from the fact that the viability kernel of a closed subset is closed and that the (Kuratowski) upper limit of a sequence of closed viability domains is a viability domain.

Let us consider any element (x, v) of the (Kuratowski) upper limit of the sequence of viability kernels $\text{Viab}(\text{Graph}(T_{K_n}))$. Then (x, v) is the limit of a subsequence (x_n, v_n) of elements of $\text{Viab}(\text{Graph}(T_{K_n}))$, so that there exist solutions $x_n(\cdot)$ to the differential inclusion $\|x''\| \leq c$ satisfying the initial conditions

$$x_n(0) = x_n \ \& \ x'_n(0) = v_n$$

and converging to some function $x(\cdot)$ satisfying $x(0) = x$ and $x'(0) = v$. Since $x_n(t) \in K$ for all $t \geq 0$, then $x(t) \in K^\sharp$ for all $t \geq 0$. Therefore, $x'(t) \in T_{K^\sharp}(x(t))$. Hence, the pair $(x(t), x'(t))$ is a solution which is viable in $\text{Graph}(T_{K^\sharp})$ and consequently, $(x, v) \in \text{Viab}(\text{Graph}(T_{K^\sharp}))$. □

Obviously, if $c_1 \leq c_2$, then $T^{c_1}_K \subset T^{c_2}_K$. Also, we deduce from the upper semicontinuity of the solution map that for any $\varepsilon > 0$, there exists $\eta > 0$ such that $T^{c_2}_K \subset T^{c_1}_K + \varepsilon(B \times B)$.

We also observe that

$$\forall\, x \in K,\ \forall\, v \in T^c_K(x),\ DT^c_K(x, v)(v) \cap cB \neq \emptyset$$

Proposition 5.3 *Let $A \in \mathcal{L}(X, Y)$ be a linear operator and $K \subset X$, $M \subset Y$ be closed subsets. Then, setting $d := c\|A\|$ and $L := \overline{A(K)}$,*

$$\forall\, x \in K,\ A\left(T^c_K(x)\right) \subset T^d_L(Ax)$$

and thus

$$\forall x \in A^{-1}(M),\ T^c_{A^{-1}(M)}(x) \subset A^{-1}\left(T^{c\|A\|}_M(Ax)\right)$$

If we assume furthermore that A is surjective, then there exists a constant $\rho > 0$ such that

$$\forall x \in A^{-1}(M),\ A^{-1}\left(T^{c\rho}_M(Ax)\right) \subset T^c_{A^{-1}(M)}(x)$$

Proof — Let $v \in T^c_K(x)$. Then there exists a solution $x(\cdot)$ to $\|x''\| \leq c$ viable in K and satisfying $(x(0), x'(0)) = (x, v)$. Then $y(t) := A(x(t))$ is solution to the differential inclusion $y'(t) = w(t)$ and $w'(t) \in cA(B) \subset c\|A\|B$, viable in $A(K)$, such that $(y(0), y'(0)) = (A(x), A(v))$.

The second statement follows by taking $K := A^{-1}(M)$.

To prove the last one, consider $w \in T_M(y)$ and a viable solution

$$y(t) := y + tw + \int_0^t (t-\tau)y''(\tau)d\tau$$

Since A is surjective, there exists a constant $\rho > 0$ and solutions x and v to the equations $Ax = y$ and $Av = w$ satisfying inequalities $\|x\| \leq \rho\|y\|$ and $\|v\| \leq \rho\|w\|$. By the Measurable Selection Theorem, there exists a measurable selection $z(\cdot)$ to the equation $Az(\tau) = y''(\tau)$ satisfying inequality $\|z(\tau)\| \leq \rho\|y''(\tau)\| \leq \rho c$.

Then $x(t) := x + tv + \int_0^t (t-\tau)z(\tau)d\tau$ is a solution to the differential inclusion $\|x''\| \leq \rho c$ which is viable in $A^{-1}(M)$. □

6 The Modified Zero Dynamics Algorithm

The zero dynamics algorithm has been devised to obtain the viability kernel of closed subsets defined by equality constraints, i.e., subsets of the form $K := h^{-1}(0)$ where h is a map from X to a finite dimensional vector-space Y. It is shown to converge for linear control systems (see [7,27]) and for smooth nonlinear control systems (see [9,10,11,13]. In this framework, viability property is called *controlled invariance* and the restriction of the control system to the viability kernel is called *zero dynamics*).

In the general case, let us consider a closed subset K of the domain of a set-valued map $F : X \rightsquigarrow X$.

We start with $K_0 := K$ and we construct

$$K_1 := \mathrm{Dom}(R_{K_0}) \text{ where } R_{K_0}(x) := F(x) \cap T_K(x)$$

Since the viability kernel $\mathrm{Viab}_F(K)$ is contained in K and since $T_L(x) \subset T_K(x)$ whenever $K \subset L$, we infer that $\mathrm{Viab}_F(K) \subset K_1$

Assume that a decreasing sequence of subsets K_i satisfying $\mathrm{Viab}_F(K) \subset K_i \subset K_{i-1} \subset K$ has been defined up to n. We then set

$$R_{K_n}(x) := F(x) \cap T_{K_n}(x)$$

define $K_{n+1} := \mathrm{Dom}(R_{K_n})$ and we observe that $\mathrm{Viab}_F(K) \subset K_{n+1}$.

Therefore

$$\mathrm{Viab}_F(K) \subset \bigcap_{n=0}^{\infty} K_n$$

The problem is to show that equality holds true. Several requirements have to be met to solve the problem. The first one is that the subsets K_n should be closed. The second one is that the (Kuratowski) upper limit of the contingent cones $T_{K_n}(x)$ is contained in the contingent cone to the (Kuratowski) upper limit of the subsets K_n (which, in this case, is the intersection of the decreasing sequence of the subsets K_n).

These conditions are not met for finding the viability kernel of $K := [0,1] \times \mathbf{R}$ for the system $F(x,v) := \{v\} \times cB$ since

$$K_0 = \{0\} \times \mathbf{R}_+ \cup]0,1[\times \mathbf{R} \cup \{1\} \times \mathbf{R}_-$$

, $K_1 = K_0$ and since the viability kernel is the graph of $T_K^c(\cdot)$.

Thanks to Proposition 5.2, by replacing the contingent cones $T_K(x)$ by the subsets $T_K^c(x)$ in the *structure algorithm*, we can prove that the modified version converges to a closed viability domain.

Let us set $K_0^c := K$. For defining $K_1^c \subset K_0^c$, we introduce the set-valued map R_0^c defined by $R_0^c(x) := F(x) \cap T_{K_0^c}^c(x)$ and set $K_1^c := \mathrm{Dom}(R_0^c)$.

If the subsets K_i^c have been defined up to n, we set

$$R_n^c(x) := F(x) \cap T_{K_n^c}^c(x)$$

and we defined

$$K_{n+1}^c := \mathrm{Dom}(R_n^c)$$

Proposition 6.1 *Assume that K is compact and that $F : K \rightsquigarrow X$ is upper semicontinuous with nonempty closed values. Then either K_i^c is empty for some step i or $K_\infty := \bigcap_{i=1}^{\infty} K_i^c$ is a nonempty closed viability domain of F:*

$$\forall\, x \in K_\infty,\ \ F(x) \cap T_{K_\infty}^c(x) \neq \emptyset$$

Proof — First, since the graph of R_i^c is the intersection of the graph of F and the graph of $T_{K_i^c}^c$ which are both closed, it is also closed. Furthermore, the subset K_i^c is closed since $F(K)$ is compact (If $x_n \in K_i^c$ converges to x, the sequence of elements $v_n \in F(x_n) \cap T_{K_{i-1}^c}^c(x_n)$ lying in a compact set, a subsequence (again denoted by) v_n converges to some v. Since the graphs of F and $T_{K_{i-1}^c}(\cdot)$ are closed, we infer that $v \in F(x) \cap T_{K_{i-1}^c}^c(x)$, i.e., that x belongs to K_i^c). Then the K_i^c's form a decreasing sequence of closed subsets of a compact subset. Either one of the K_i^c's is empty or the intersection K_∞ is not empty. In this case, let x be chosen in K_∞. For any n, there exists $v_n \in F(x) \cap T_{K_n^c}^c(x)$. Since the v_n's remain in the compact subset $F(K)$, a subsequence (again denoted) v_n converges to some v. Since (x, v_n) belongs to the graph of $T_{K_n^c}^c$, we know that (x, v) belongs to the graph of $T_{K_\infty}^c$, because K_∞ is the (Kuratowski) upper limit of the decreasing sequence of the subsets K_i^c. Hence v belongs to $F(x) \cap T_{K_\infty}(x)$. □

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