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APPROXIMATIONS TO DIFFERENTIAL INCLUSIONS BY DISCRETE INCLUSIONS

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Foreword

This report is devoted to second order discrete approximations to differential inclusions. The approximations are of the form of discrete inclusions with right-hand sides, which are explicitly described for some classes of differential inclusions. In the cases of linear differential inclusions or of differential inclusions with strongly convex right-hand sides, the approximating discrete inclusions are analogs of certain second order Runge-Kutta schemes.

The approach can serve as a tool for numerical treatment of uncertain dynamical system and optimal control problems.

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Approximations to differential inclusions by discrete inclusions

V.M. Veliov

1. Introduction

In this paper we investigate the problems of approximation of a differential inclusion by discrete inclusions. This problem will be stated more precisely in the further lines.

Consider the differential inclusion

$$\dot{x} \in F(x,t), \;\; x(t_0) \in X_0, \;\; t \in [t_0, T], \tag{1.1}$$

where $x \in \mathbf{R}^n$, $F: \mathbf{R}^n \times [t_0,T] \rightrightarrows \mathbf{R}^n$ ($\rightrightarrows$ indicates that F is multivalued), $X_0 \subset \mathbf{R}^n$. The interval $[t_0, T]$ is fixed. Denote by $X_{[t_0,T]}$ the trajectory bundle of (1.1) on $[t_0,T]$, i.e.,

$$X_{[t_0,T]} = \left\{ x(\cdot);\; x(\cdot) \text{is absolutely continuous and satisfies (1.1)for a.e. } t \in [t_0,T] \right\}$$

and by $X(T)$ the attainability domain of (1.1) on $[t_0,T]$, i.e.,

$$X(T) = \left\{ x(T); x(\cdot) \in X_{[t_0,T]} \right\}.$$

Along with the inclusion (1.1) consider a family of discrete inclusions, parametrized by the integer N:

$$x_{k+1} \in \mathcal{F}(x_k, k, N) \tag{1.2}$$

where $\mathcal{F}(\cdot,k,N) : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ for every $N \geq 1$ and $k = 0, \cdots N-1$. Define the trajectory bundle and the attainability domain of (1.2) (for fixed N) on $[t_0,T]$ as follows:

$$X^N_{[0,N]} = \left\{ (x_0, \ldots, x_N);\; \text{(1.2) is fulfilled for } k = 0, \ldots, N-1 \right\},$$

$$X^N(N) = \left\{ x_N;\;\; \text{there is } (x_0, \ldots, x_N) \in X^N_{[0,N]} \right\}$$

The trajectory $(x_0, \ldots x_N)$ of (1.2) will be denoted for brevity by $\bar{x}$.

In the sequel we shall consider the approximating properties of (1.2) with respect to (1.1), associating every x_k with the moment $t_k = t_0 + kh$ of the time scale of (1.1) (here and further $h = (T - t_0) \,/\, N$). This point of view motivates the following definition of a distance between the trajectory bundles of (1.1) and (1.2), related to the Hausdorf metric:

$$\rho(X_{[t_0,T]}, X^N_{[0,N]}) = \max \left\{ \sup_{x(\cdot) \in X_{[t_0,T]}} \; \inf_{\bar{x} \in X^N_{[0,N]}} \| x(\cdot) - \bar{x} \| \;,\; \sup_{\bar{x} \in X^N_{[0,N]}} \; \inf_{x(\cdot) \in X_{[t_0,T]}} \| \bar{x} - x(\cdot) \| \right\},$$

where

$$\|x(\cdot) - \bar{x}\| = \max\left\{|x(t_i) - x_i| \ ;i=0,\ldots,N\right\}.$$

Similarly, $\rho(X(T),X^N(N))$ will denote the Hausdorf distance between the two sets indicated as arguments of $\rho(\cdot,\cdot)$.

Definition 1. The discrete inclusion (1.2) (in fact, the family of inclusions (1.2)) provides a s-th order approximation to the trajectory bundle of (1.1) if there is a constant c, such that

$$\rho(X_{[t_0,T]},X^N_{[0,N]}) \leq c/N^s$$

for all $N \geq 1$.

Definition 2. The discrete inclusion (1.2) provides a s-th order approximation to the attainability domain of (1.1) if there is a constant c, such that

$$\rho(X(T),X^N(N)) \leq c/N^s$$

for all $N \geq 1$.

The aim of this paper is to present an approach for constructing discrete inclusions of the type of (1.2), providing approximations in the sense of definitions 1 and 2 to a differential inclusion, and we shall concentrate, especially, on second order approximations.

The natural way to construct approximating discrete inclusions is to apply some difference scheme for discretization of differential equations to the differential inclusion. The simplest one is the Euler scheme, which leads to the discrete inclusion

$$x_{k+1} \in \mathcal{F}(x_k,k,N) = x_k + hF(x_k,t_k), \quad k=0,\ldots,N-1, \tag{1.3}$$

where as above, $h = (T - t_0)/N$, $t_k = t_0 + kh$. The fact that every condensation point of a sequence (when h goes to zero) of discrete trajectories is a trajectory of the differential inclusion, is exploited by a great number of authors and in several different contexts. The result of A. Panasyuk and V. Panasyuk [12] implies that the Euler scheme provides an approximation to the attainability domain. Estimations for this approximation are obtained in M. Nikol'skii [11] and in A. Dontchev and E. Farkhi [3]. The latter paper shows that the Euler scheme provides first order approximation to both the trajectory bundle and the attainability domain. In a slightly different setting, a convergence result for the

Euler scheme is contained also in P. Wolenski [18]. The Euler approximation in the more complicated case with state constraints is investigated in A. Kurzhanski and A. Filippova [7]. K. Taubert [15] applies multistep schemes and proves corresponding (one sided) convergence results. These results are extended in H.-D. Niepage and W. Wendt [10], where multistep and Runge-Kutta schemes for differential inclusions are investigated by means of the unified approach presented there. The results are also of the type that the condensation points of discrete trajectories (when the step-length goes to zero) are trajectories of the differential inclusion.

It does not seem reasonable to expect that applying a higher order discretization scheme (say, of Runge - Kutta type) we shall come to a discrete inclusion with higher than first order accuracy. The reason is that if we restrict ourselves to consider only those trajectories which have (uniformly) enough smoothness to ensure higher order approximation by a discrete scheme (for instance, uniformly bounded second order averaged moduli of smoothness, see B. Sendov and V. Popov [14]) then both the trajectory bundle and the attainability domain will essentially reduce. For this reason we shall not try to apply formally some discretization scheme to (1.1) and then to study its convergence, but instead we shall construct discrete approximations of the type of (1.2) by taking into account the local expantion of the attainability domain of (1.1). The inclusion (1.1) is supposed to be in a more specific form, namely $F(x,t) = f(x,t) + g(x,t)\,U$, where U is a convex compact set in $\mathbf{R}^r$, $g(x,t)$ is a $(n \times r)$ -matrix and $f(x,t) \in \mathbf{R}^n$. In Sections 3.1, 3.2 and 3.3 the cases when U is an interval in $\mathbf{R}^1$, a coordinate polyhedron in $\mathbf{R}^r$ and a strongly convex set in $\mathbf{R}^n$ are successively considered. In all of these cases we reduce the construction of a second order approximation of the type of (1.2) to the approximation of certain simple integrals of multivalued mappings. The latter is explicitly found in the cases mentioned above, which gives as a result corresponding discrete inclusions, providing second order approximation to the trajectory bundle. In the single valued case (when (1.1) is a differential equation) the so obtained discrete inclusions are also single valued and coincide with a second order Runge-Kutta scheme. Nevertheless, in the multivalued case the approximating discrete inclusion differ from those which can be obtained from (1.1) by a formal application of this Runge-Kutta scheme.

In Section 4, it turns out that in the case of a linear inclusion $(F(x,t) = A(t)x + B(t)U\ )$ with polyhedral right-hand side, a certain Runge-Kutta scheme provides a second order approximation to the attainability domain, but only first order approximation to the trajectory bundle. This, namely, is the motivation of the two definitions given above.

In Section 5 we present as applications some second order discrete approximations to control constrained optimal control problems. Some bibliography in this direction is included there. Section 6 deals with the problem of approximation of a given function by a trajectory of a differential inclusion.

2. An auxiliary result

Throughout the paper we shall assume the following.

Basic assumption. Let Δ be an open interval containing $[t_0, T]$ and $S \subset \mathbf{R}^n$ be an open set, containing X_0. Let F be convex and compact valued mapping defined on $S \times \Delta$. We suppose that F is measurable in t for every fixed x and Lipschitz continuous in x, uniformly in t:

$$\rho(F(x_1,t),\ F(x_2,t)) \leq L \mid x_1 - x_2 \mid \text{ for every } t \in \Delta, \quad x_1, x_2 \in S.$$

Moreover,

$$|F(x,t)| \leq m(t) \text{ for every } t \in \Delta \text{ and } x \in S,$$

where $m(\cdot)$ is a L_1 - function.

We suppose also, that the attainability domain $X(t)$ of (1.1) is nonempty for $t \in [t_0, T]$ and $X(t) \subset \text{int } S_0(t)$, $t \in [t_0, T]$, where $S_0(\cdot): \Delta \rightrightarrows \mathbf{R}^n$ is a Hausdorff continuous compact valued mapping, such that $S_0(t) \subset S$ for $t \in [t_0, T]$.

The following is a direct consequence of the Carathéodory type existence theorem (see e.g. Filippov a [5] or J.-P. Aubin and A. Cellina [1]).

Lemma 1. There is $\kappa > 0$, such that for every $t \in [t_0, T]$ and $x_0 \in S_0(t)$ and for every selection $f(x,\tau) \in F(x,\tau)$ defined for $\tau \in [t - \kappa \ \ t + \kappa]$ and $x \in S$, which is continuous in x and measurable in t, the solution of the equation

$$\dot{x} = f(x,\tau) \quad x(t) = x_0$$

exists on $[t - \kappa \ \ t + \kappa]$ and does not abandon the set S.

Denote by $X(x;\ t_1, t_2)$ the attainability domain of (1.1) on $[t_1, t_2]$, starting from x at the moment t_1. By Lemma 1, when $t_1, t_2, \in [t_0, T]$, $|t_2 - t_1| < \kappa$ and $x \in S_0(t_1)$, the set $X(x; t_1, t_2)$ is nonempty.

Definition 3. The discrete inclusion (1.2) provides a s-th order local approximation to (1.1) in the tube $S_0(\cdot)$, if there is a constant c, such that

$$\rho(X(x;t_k,t_{k+1}),\ F(x,k,N)) \leq c/N^s$$

for every sufficiently large N, $k=0,\ldots,N-1$ and $x \in S_0(t_k)$.

Proposition 1. Let the basic assumptions be fulfilled. Let, in addition, the discrete inclusion (1.2) provide s-th order local approximation to (1.1) in the tube $S_0(\cdot)$ $(s > 1)$. Then (1.2) provides $(s - 1)$ -th order approximation to both the trajectory bundle and the attainability domain of (1.1).

Proof. We shall sketch the proof which is enough standard. From the compactness of $X_{[t_0,T]}$ in the uniform metric it follows that there is $\sigma > 0$, such that $x(t) + \mathcal{B}(\sigma) \subset S_0(t)$ for every $x(\cdot) \in X_{[t_0,T]}$ and $t \in [t_0,T]$. (Here and further $\mathcal{B}(\sigma)$ denotes the ball with radius σ, centered at the origin of the respective space). We can suppose that h is so small, that $h \leq \kappa$ (trom Lemma 1) and $h^{s-1}c \exp\left((T-t_0)L\right) / L < \sigma$.

Take an arbitrary $x(\cdot) \in X_{[t_0,T]}$. We shall define a trajectory $\bar{x}$ of (1.2) in the following way.

Take $x_0 = x(t_0)$. Let x_k be already defined so that $x_k \in S_0(t_k)$. Consider the equation

$$\dot{y} = f(y,s), \quad y(t_k) = x_k,$$

where $f(y,s) = P_{F(y,s)}\dot{x}(s)$ and P_Y is the projection of x on the convex compact set Y. Since f is continuous in x and measurable in t by Lemma 1 the solution $y(\cdot)$ exists on $[t_k,t_{k+1}]$. Since for a. e. s

$$|\ \dot{x}\ (s) - \dot{y}(s)| \leq \rho(F(x(s),s),\ F(y(s),s)) \leq L|x(s) - y(s)|,$$

we conclude by the Grunwall inequality that

$$|x(t_{k+1}) - y(t_{k+1})| \leq e^{hL}|x(t_k) - x_k|$$

Since $y(t_{k+1}) \in X\ (x_k\ ;t_k,\ t_{k+1})$ there is $x_{k+1} \in \mathcal{F}\ (x_k,k,N)$ (by Definition 3) such that $|x_{k+1} - y(t_{k+1})| \leq ch^s$. Hence

$$|x_{k+1} - x(t_{k+1})\ | \ \leq ch^s + e^{hL}\ |\ x_k - x(t_k)\ |$$

By induction we can see that if h is so small as required above, then $x_{k+1} \in S_0(t_{k+1})$ and

$$|x_{k+1} - x(t_{k+1})| \leq \frac{C}{L}\exp((T - t_0)L)h^{s-1},$$

which completes the first part of the proof.

In a very similar way we can prove that every trajectory of the discrete inclusion (1.2) can be approximated with the same accuracy by a trajectory of (1.1), Q.E.D.

We shall mention, that the constant c in definitions 1 and 2 can be taken to depend only on the constant c , comming from Definition 3 and the Lipschitz constant L (as seen in the proof), if only N is supposed to be sufficiently large.

3. Second order approximations to the trajectory bundle

3.1 The single input case

We shall begin with the single input case in order to present the idea of the approximation in a more clear way. Consider the differential inclusion

$$\dot{x} \in f(x,t) + g(x,t)[0,1], \quad x(t_0) \in X_0, \quad t \in [t_0,T], \tag{3.1}$$

where $x \in \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$, X_0 is a convex and compact subset of $\mathbf{R}^n$.

First we shall mention that we consider for simplicity the interval $[0, 1]$ in the righhand side of (3.1), but the more general case of an interval $[a(t),b(t)]$ can be reduced to $[0,1]$ by taking in (3.1) $f(x,t) + a(t)g(x,t)$ instead of $f(x,t)$ and $(b(t) - a(t))g(x,t)$ instead of $g(x,t)$.

We shall strengthen the basic assumption from Section 2.1, supposing the following:

A. There are an open bounded set $S \subset \mathbf{R}^n$, open interval $\Delta \supset [t_0,T]$ and a Hausdorff continuous mapping $S_0(\cdot) : \Delta \rightrightarrows \mathbf{R}^n$, which is convex and compact valued, $S_0(t) \subset S$ for every $t \in \Delta$ and

A1. f and g are differentiable and the derivatives f'_x, f'_t, g'_x and g'_t are Lipschitz continuous with respect to each of the variables x and t, uniformly in the other variable, in the set $S \times \Delta$;

A2. $X(t)$ is nonempty and is contained in int $S_0(t)$ for every $t \in [t_0,T]$.

Let L be a real which majorates the Lipschitz constants of f,g and their derivatives in $S \times \Delta$, and let M majorates $|f(x,t)|$ and $|g(x,t)|$ when $x \in S$, $t \in \Delta$.

In the sequel we shall denote by $0(\cdot)$ any function $(0,1) -> \mathbf{R}^n$, such that $|0(h)| / h$ is bounded by a constant, and in what follows this constant will depend on L, M and $[t_0,T]$, only.

As in Section 2 we shall denote by $X(x;t_1,t_2)$ the attainability domain of (3.1) on $[t_1,t_2] \subset [t_0,T]$, starting from the point $x \in S_0(t_1)$ at t_1. By Lemma 1 $X(x;t_1,t_2)$ is nonempty when $t_2 - t_1 < \kappa$ and is contained in S. We can suppose that $h \leq \kappa$.

Now, take an arbitrary $t \in [t_0,T-h]$, $x \in S_0(t)$ and $\tilde{x} \in X(x;\, t,\, t+h)$. Then there is a selection $u(\cdot)$ of $[0,1]$ such that

$$\tilde{x} = x + \int_t^{t+h} (f(x(s),s) + g(x(s),s)u(s))ds = x + M(t+h),$$

where $x(\cdot)$ is the corresponding solution of (3.1), $x(t) = x$, and $M(\cdot)$ is defined in an obvious way. Taking into account A1 and A2 we obtain

$$\tilde{x} = x + \int_t^{t+h} (f(x+M(s),s) + g(x+M(s),s)u(s))ds$$

$$=x + \int_t^{t+h} (f(x,s) + f'_x(x,s)M(s))ds + \int_t^{t+h} (g(x,s) + g'_x(x,s)M(s))u(s)ds + O(h^3)$$

$$=x + \int_t^{t+h} (f(x,s) + f'_x(x,t)((s-t)f(x,t) + g(x,t)\int_t^s u(\tau)d\tau))ds$$

$$+ \int_t^{t+h} (g(x,s) + g'_x(x,t)((s-t)f(x,t) + g(x,t)\int_t^s u(\tau)d\tau))u(s)ds + O(h^3)$$

$$=x + \int_t^{t+h} (f(x,t) + (s-t)f'_t(x,t) + (s-t)f'_x(x,t)f(x,t))ds$$

$$+ \int_t^{t+h} ((t+h-s)f'_x(x,t)g(x,t) + g(x,t) + (s-t)g'_t(x,t) + (s-t)g'_x(x,t)f(x,t))u(s)ds$$

$$+g'_x(x,t)g(x,t) \int_t^{t+h}\int_t^{s} u(\tau)d\tau u(s)ds + O(h^3)$$

Introduce the notations

$$\tilde{F}_0 = \tilde{F}_0(x,t,h) = x + hf(x,t) + 0.5h^2(f'_t(x,t) + f'_x(x,t)f(x,t))$$

$$\tilde{G}_0 = \tilde{G}_0(x,t,h) = g(x,t) + hf'_x(x,t)g(x,t)$$

$$\tilde{G}_1 = \tilde{G}_1(x,t) = g'_x(x,t)f(x,t) - f'_x(x,t)g(x,t) + g'_t(x,t)$$

$$\tilde{H}_0 = \tilde{H}_0(x,t) = g'_x(x,t)g(x,t).$$

Then using the equality

$$\int_t^{t+h}\int_t^s u(\tau)d\tau\, u(s)ds = 0.5\left[\int_t^{t+h} u(s)ds\right]^2$$

we get

$$\tilde{x} - \tilde{F}_0 + 0(h^3) = \int_t^{t+h} (\tilde{G}_0 + (s-t)\tilde{G}_1)u(s)ds + 0.5\tilde{H}_0(\int_t^{t+h} u(s)ds)^2.$$

Denote by $R = R(x,t,h)$ the set of points in the right-hand side of the above equality, corresponding to all measurable selections $u(\cdot)$ of $[0, 1]$. Thus, we have proven so far that

$$X(x;t,t+h) \subset \tilde{F}_0(x,t,h) + R(x,t,h) + \mathcal{B}\ (ch^3), \tag{3.2}$$

where $\mathcal{B}(\tau)$ is the ball with radius σ centered at the origin, and c is an appropriate constant. Observe, that c is not only independent of h,t and x, but it can be taken to depend only on L and M.

From Lemma 1 it follows that the "inverse" inclusion to (3.2) also holds. Actually, if $y \in \tilde{F}_0(x,t,h) + R(x,t,h)$, we can use in (3.1) the selection $u(\cdot)$, corresponding to y, with $x(t) = x$, and repeating the same argument to verify that $|x(t+h) - y| \le ch^3$ with the same constant c. Hence,

$$\rho(X(x;t,t+h),\ \tilde{F}_0(x,t,h) + R(x,t,h)) \le ch^3 \tag{3.3}$$

for every $h>0$, $t \in [t_0, T-h]$ and $x \in S_0(t)$. Now, let us tackle the set R. Obviously R can be presented in the form

$$R = \alpha\tilde{G}_0 + \beta\tilde{G}_1 + 0.5\alpha^2\tilde{H}_0;\quad (\alpha,\beta) \in \Omega, \tag{3.4}$$

where (changing the variable of integration)

$$\Omega = \left\{(\alpha,\beta);\ \alpha = \int_0^h u(s)ds,\ \beta = \int_0^h su(s)ds,\ u \in L_\infty(0,h),\ u(s) \in [0,1]a.e.\right\}.$$

Fortunately, the set Ω can be exactly found. Using the obvious fact that every point from the boundary $\partial\Omega$ corresponds to a piece-wise constant $u(\cdot)$ taking only the values 0 and 1 and having only one jumping point, we easily calculate that

$$\Omega = \left\{(\alpha,\beta);\ \alpha \in [0,h],\ \beta \in [0.5\alpha^2,\ h\alpha - 0.5\alpha^2]\right\} \tag{3.5}$$

Using (3.3) - (3.5) and replacing α by $h\alpha$ and β by $2h^2\beta$, we obtain

$$\rho(X(x;t,t+h),\ \left\{\tilde{F}_0 + \alpha h \tilde{G}_0 + 0.5\beta h^2 \tilde{G}_1 + 0.5\alpha^2 h^2 H_0;\right. \tag{3.6}$$

$$\left.\alpha \in [0,1], \beta \in [\alpha^2, 2\alpha - \alpha^2]\right\}) \leq ch^3$$

Now we shall get rid of the derivatives in $\tilde{F}_0,\dots,\tilde{H}_0$, replacing them with finite difference (obviously first order approximation of the derivatives is enough). This leads to the new notations

$$F_0 = F_0(x,t,h) = f(x,t) + f(p(x,t,h),t+h) \quad (= 2(\tilde{F}_0 - x) / h + 0(h^2))$$

$$G_0 = G_0(x,t,h) = 2(g(x,t) + f(q(x,t,h),t) - f(x,t)) \quad (= 2\tilde{G}_0 + 0(h^2)) \tag{3.7}$$

$$G_1 = G_1(x,t,h) = g(p(x,t,h),t+h) - g(x,t) - f(q(x,t,h),t) + f(x,t) \quad (= h\tilde{G}_1 + 0(h^2))$$

$$H_0 = H_0(x,t,h) = g(q(x,t,h),t) - g(x,t) \quad (= h\tilde{H}_0 + 0(h^2)),$$

where

$$p = p(x,t,h) = x + hf(x,t)$$

$$q = q(x,t,h) = x + hg(x,t). \tag{3.8}$$

In these notations (3.6) can be rewritten as

$$\rho(X(x;\ t,t+h),\left\{x + 0.5h(F_0 + \alpha G_0 + \beta G_1 + \alpha^2 H_0);\right. \tag{3.9}$$

$$\left.\alpha \in [0,1],\ \alpha \in [\alpha^2, 2\alpha - \alpha^2]\right\}) \leq ch^3$$

where c possibly differs from the constant in (3.6), but has the same property mentioned after (3.2). Now, define the set

$$\mathcal{F}(x,k,N) = \Big\{x + 0.5h(F_0(x,t_k,h) + \alpha G_0(x,t_k,h) + \beta G_1(x,t_k,h) \tag{3.10}$$

$$+ \alpha^2 H_0(x,t_k,h); \alpha \in [0,1],\ \beta \in [\alpha^2, 2\alpha - \alpha^2])\Big\}$$

and consider the discrete inclusion

$$x_{k+1} \in \mathcal{F}(x_k,k,N), \quad x_0 \in X_0, \quad k = 0,\dots,N-1. \tag{3.11}$$

Theorem 1. Under the assumptions A the discrete inclusion (3.11) with $\mathcal{F}$ given by (3.10), (3.7) and (3.8), provides a second order approximation to both the trajectory bundle and the attainability domain of the differential inclusion (3.1).

The assertion of the theorem follows directly from (3.9) and Proposition 1.

Observe that the constant c in definitions 1 and 2 can be estimated making use only of the constants L,M and $T - t_0$, if N is supposed to be sufficiently large.

In the particular case when (3.1) is single valued, i.e., $g(x,t) \equiv 0$ we have

$$\mathcal{F}(x,k,N) = x + 0.5h(f(x,t_k) + f(p(x,t_k,h),t_{k+1})$$

and (3.11) is just a second order Runge-Kutta formula. Nevertheless, in the multivalued case the definition (3.10) of $F(x,k,N)$ is not a result of a formal application of this Runge-Kutta formula to (3.1). To make clear the difference, let us apply the above Runge-Kutta formula to (3.1), but taking a particular selection $u(\cdot)$ of $[0, 1]$. After some transformations we come to the discretization

$$x_{k+1} = x_k + 0.5h(F_0(x_k,t_k,h) + 0.5(u(t_k) + u(t_{k+1}))G_0(x_k,t_k,h) + \tag{3.12}$$

$$+ u(t_{k+1})G_1(x_k,t_k,h) + u(t_k)u(t_{k+1})H_0(x_k,t_k,h))$$

Let us neglect for simplicity the term H_0 in (3.10) (if g is independent of x, then H_0 is actually equal to zero). There are different possible interpretations of (3.12). If in (3.1) we consider only selections $u(\cdot)$ which are constant at every interval $[t_k,t_{k+1}]$, then $u(t_k) = u(t_{k+1})$ and (3.12) corresponds to (3.10) with $\beta = \alpha$ in the right-hand side. This means that the set of trajectories generated by (3.12) is not enough reach to approximate $X_{[t_0,T]}$ of order 2 (this will be seen by an example in Selection 4).

If we admit arbitrary (it is enough piece-wise linear) selection of [0, 1] in (3.1) (as it is done in [10]), then $u(t_k)$ and $u(t_{k+1})$ can be rewritten in the form (again in the case of $H_0 = 0$)

$$x_{k+1} \in x_k + 0.5h(F_0 + \{\alpha G_0 + \beta G_1;\ \alpha \in [0,1],\ \beta \in [\max\{0, 2\alpha - 1\}, \min\{2\alpha, 1\}\ ]\ \}\ ] \tag{3.13}$$

Comparing with (3.10) we see that the right-hand side in (3.13) is essentially larger than in (3.10) (the difference is $O(h^2)$) and what can be concluded from here for the discrete inclusion (3.13) is that it provides approximation of order one to the trajectory bundle of (3.1) (this also can be seen by an example).

Often in the discrete approximations of optimal control problems the value of $u(\cdot)$ at the right side of the internal $[t_k, t_{k+1}]$ is taken to be just the value of $u(\cdot)$ at the left side of the next interval $[t_{k+1}, t_{k+2}]$. In this case the difference between (3.10) and (3.12) is not well seen in one step. But even the example $\dot{x} = u$ in the one dimensional case shows that the accuracy of this approximation is not better than $O(h)$.

3.2 Second order approximation in the multi-input case.

We shall extend the approach presented in the preceding section to differential inclusion of the type

$$\dot{x} \in f(x,t) + \sum_{i=1}^{r} g_i(x,t)[0,1], \quad x(t_0) \in X_0, \quad t \in [t_0, T] \tag{3.14}$$

where $x \in \mathbf{R}^n$, $f,g : \mathbf{R}^n \times \mathbf{R}^1 \to \mathbf{R}^n$. As mentioned in the previous section, also included here is the case of intervals $[a_i(t), b_i(t)]$ in the right-hand side of (3.14).

We shall suppose that the assumptions A from section 3.1 are fulfilled (what is required for g here concerns $g_1, \ldots, g_r$). In order to prevent some technical complications we shall introduce the following additional assumption, restricting the interaction between different g_i.

$$A3. \quad [g_i, g_j](x,t) = 0 \quad \text{for } i \neq j$$

here

$$[g_i, g_j] = \frac{\partial g_j}{\partial x} g_i - \frac{\partial g_i}{\partial x} g_j$$

is the Lie bracket of g_i and g_j with respect to x.

Similarly, as in Section 3.1, we can verify that for every point $x \in S(t)$, the set $X(x;\ t,t+h)$ is approximated in Hausdorff sense by the set of all points

$$\tilde{x} = x + hf(x,t) + 0.5h^2(f'_t(x,t) + f'_x(x,t)f(x,t)) +$$

$$\sum_{i=1}^{r} \int_t^{t+h} (g_i(x,t) + hf'_x(x,t)g_i(x,t) + (s-t)([f,g_i](x,t) + g'_{it}(x,t)))u_i(s)ds + \tag{3.15}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} g'_{ix}(x,t)g_j(x,t) \int_t^{t+h}\int_t^{s} u_j(\tau)d\tau u_i(s)ds,$$

corresponding to measurable selections $u_i(\cdot)$ of $[0,1]$. Denoting

$$v_i(s) = \int_t^s u_i(\tau)d\tau$$

we have for $i \neq j$

$$g'_{ix}\, g_j \int_t^{t+h}\int_t^{s} u_j(\tau)d\tau u_i(s)ds + g'_{ix}\, g_i \int_t^{t+h}\int_t^{s} u_i(\tau)d\tau u_j(s)ds =$$

$$= g'_{ix}\, g_j \int_t^{t+h} v_j(s)dv_i(s) + g'_{jx}\, g_i \int_t^{t+h} v_j(s)dv_j(s) \tag{3.16}$$

$$= g'_{ix}\, g_j v_j(t+h)v_i(t+h) + [g_i,\, g_j] \int_t^{t+h} v_i(s)dv_j(s) = g'_{ix}\, g_j v_i(t+h)v_j(t+h)$$

Hence, denoting $\alpha_i = v_i(t+h)$ and $\beta_i = \int_t^{t+h} (s-t)u_i(s)ds$ it remains to repeat the argument from Section 3.1. We shall formulate the final result, using the following notations, similar to (3.7) and (3.8):

$$F_0(x,t,h) = f(x,t) + f(p(x,t,h),\ (t+h)$$

$$G_0^i(x,t,h) = 2(g_i(x,t) + f(q_i(x,t,h),\ t) - f(x,t))$$

$$G_1^i(x,t,h) = g_i(p(x,t,h) - g_i(x,t) - f(q_i(x,t,h),\ t) + f(x,t) \tag{3.17}$$

$$H_0^{ij}(x,t,h) = g_i(q_j(x,t,h),t) - g_i(x,t),$$

where $i,j = 1,...,r$ and

$$p(x,t,h) = x + hf(x,t), \quad q_i(x,t,h) = x + hg_i(x,t). \tag{3.18}$$

Define the set

$$\mathcal{F}(x,k,N) = \Big\{ x + 0.5h(F_0(x,t_k,h) + \sum_{i=1}^{r} \alpha_i G_0^i(x,t_k,h) + \sum_{i=1}^{r} \beta_i G_1^i(x,t_k,h) + \tag{3.19}$$

$$+ \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\, \alpha_j H_0^{ij}(x,t_k,h));\ \alpha_i \in [0,1],\ \beta_i \in [\alpha_i^2, 2\alpha_i - \alpha_i^2], i=1,...,r \Big\}$$

and consider the discrete inclusion

$$x_{k+1} \in \mathcal{F}(x_k,k,N), \quad x_0 \in X_0, \quad k=0,...,N-1. \tag{3.20}$$

Theorem 2. Under the assumptions $A1-A3$ the discrete inclusion (3.20) with $\mathcal{F}$ given by (3.19), (3.17) and (3.18) provides a second order approximation to both the trajectory bundle and the attainability domain of (3.14).

3.3 Second order approximation in the strongly convex case

In this section we shall consider the case where the right-hand side of the differential inclusion is strongly convex in the state space. It turns out that the strongly convex case is remarkably different, compared with the previously considered polyhedral cases. The reason is, that in the third order local approximation of the attainability domain it is enough to use constant inputs only as it will be shown below, in contrast to the polyhedral case where at least one jump is needed to ensure third order local approximation. This reflects in the fact that in the approximating discrete inclusion which will be obtained below, the right-hand side is parametized only by one parameter, instead of α and β in the previous cases. This discrete inclusion turns out to be just the one which can be obtained by the formal discretization of the differential inclusion using a second-order Runge-Kutta formula.

The differential inclusion will be supposed to be in the form

$$\dot{x} \in f(x,t) + G(x,t)U, \quad x(t_0) \in X_0, \quad t \in [t_0,T], \tag{3.21}$$

where $x \in \mathbf{R}^n$, $f(x,t) \in \mathbf{R}^n$, $G(x,t)$ is $(n \times n)$ - matrix and U is a time invariant strongly convex set in $\mathbf{R}^n$.

We shall remind that the strong convexity of U means, that there is a constant $\mu > 0$, such that the inclusions $u_1, u_2 \in U$ imply

$$0.5(u_1 + u_2) + v \in U \text{ for every } v \in \mathbf{R}^n,\ |v| \leq \mu |u_1 - u_2|^2,$$

i.e., with every two points u_1 and u_2, U contains a ball centered at $(u_1 + u_2) / 2$ with a radius proportional to $|u_1 - u_2|^2$.

The rows of $G(x,t)$ will be denoted by $g_1, \ldots, g_n$.

We shall suppose that conditions $A1-A3$ are fulfilled. In addition we shall introduce the following assumption, which prevents the flattening of the set $G(x,t)U$.

$A4$. U is strongly convex and rank $G(x,t) = n$ for every $t \in [t_0, T]$ and $x \in S_0(t)$.

We consider a time-invariant set U, but this is not quite a restrictive assumption. Actually, if $U(t)$ is an elipsoid given by

$$U(t) = \left\{ u \in \mathbf{R}^n; \ < Q(t)u, u > \ \leq 1 \right\},$$

where $Q(t)$ is a strictly positive definite symetric matrix with Lipschitz continuous derivative, then we can replace $U(t)$ with the unit ball, taking $G(x,t)Q^{-1}(t)$ instead of $G(x,t)$. Observe that this transformation does not affect the property $A3$ of G.

Using (3.15) and (3.16) as in Section 3.2 we see that given $t \in [t_0, T-h]$ and $x \in S_0(t)$, the set $X(x;\ t, t+h)$ differs only of order $0(h^3)$ (in the Hausdorf metric) from the set of points

$$\tilde{x} = \tilde{F}_0(x,t,h) + \int_t^{t+h} (\tilde{G}_0(x,t,h) + (s-t)\tilde{G}_0(x,t,h))u(s)\,ds + \tag{3.22}$$

$$+ 0.5\tilde{H}_0(x,t,h)\left(\int_t^{t+h} u(s)\,ds,\ \int_t^{t+h} u(s)\,ds \right)$$

corresponding to all measurable selections $u(\cdot)$ of U. Here we use the following notations: $\tilde{F}_0$ is exactly as in Section 3.1,

$$\tilde{G}_0 = G(x,t) + hf'_x(x,t)G(x,t),$$

$$\tilde{G}_1 = [f,G](x,t) + G'_t(x,t),$$

(by definition $[f,G]$ is the matrix with columns $[f,g_i]$)

$$\tilde{H}_0(v,w) = \sum_{i=1}^{n}\sum_{j=1}^{n} g'_{ix}(x,t)g_i(x,t)v_iw_j$$

Define

$$\tilde{\mathcal{F}}(x,t,h) = \left\{\tilde{F}_0 + h\tilde{G}_0u + 0.5h^2\tilde{G}_1u + 0.5h^2\tilde{H}_0(u,u);\;\; u\in U\right\},$$

that is the set of points in (3.22), corresponding to constant selections $u\in U$. We shall prove below that if $\mathcal{F}(x,t,h)$ is the set of points defined by (3.22), then

$$\varphi(\tilde{\mathcal{F}}(x,t,h),\;\hat{\mathcal{F}}(x,t,h)) \le ch^3, \tag{3.23}$$

where the constant c does not depend on $t\in[t_0,T]$, $x\in S_0(t)$ and h. This means that we can replace $\hat{F}$ by $\tilde{F}$ and getting rid of the derivatives in the definition of $\tilde{F}$ we come to the discrete inclusion

$$x_{k+1}\in\mathcal{F}(x_k,k,N),\quad x_0\in X_0,\quad k=0,\ldots N-1, \tag{3.24}$$

where

$$\mathcal{F}(x_k,k,N) = \Big\{x + 0.5h((F_0(x,t_k,h) + G_0(x,t_k,H))u + G_1(x,t_k,h)u + \tag{3.25}$$

$$+ H_0(x,t_k,h)(u,u));\quad u\in U\Big\}$$

Here we use the notations (3.17) and (3.18) in a matrix form: G_0 and G_1 are the matrices with columns G_0^i and G_1^i respectively, and H_0 is the bilinear mapping defined by

$$H_0(u,v) = \sum_{i=1}^{r}\sum_{j=1}^{r} H_0^{ij}u_iv_j.$$

Theorem 3. Under the assumptions $A1-A4$ the discrete inclusion (3.24) with $\mathcal{F}$ given by (3.25), (3.17) and (3.18), provides a second order approximation to both the trajectory bundle and the attainability domain of (3.21).

Proof. It remains to prove only the inequality (3.23). Denote by $\delta^*(l|\,Y)$ the support function of the bounded set of $Y\subset\mathbf{R}^n$ at l, i.e.

$$\delta^*(l|\,Y) = \sup_{y \in Y} <l,y>.$$

First we shall prove that there is c_1, such that

$$\delta^*(l|\,\hat{\mathcal{F}}) \le \delta^*(l|\,\tilde{\mathcal{F}}) + c_1 h^3 \tag{3.26}$$

for every $l \in \mathbf{R}^n, |l| = 1$ and then, that there is c_2, such that

$$\rho(\tilde{\mathcal{F}},\ co\tilde{\mathcal{F}}) \le c_2 h^3, \tag{3.27}$$

which together with $\tilde{\mathcal{F}} \subset \hat{\mathcal{F}}$ imply (3.23) and complete the proof of the theorum.

From the definition of $\tilde{G}_0$ and assumption $A4$ it follows that there is $\alpha_0 > 0$ such that $|G_0^* l| \ge \alpha_0$ for every $l \in \mathbf{R}^n, |l| = 1$ and for all sufficiently small h, uniformly in $t \in [t_0, T]$ and $x \in S_0(t)$.

We may suppose, in addition, that h is so small that

$$0.5 \sup_{|u| = |v| = 1} |H_0(u,v)|\ |U|h + \|G_1^*\|h < \alpha_0 \,/\, 2. \tag{3.28}$$

Take an arbitrary vector $l \in \mathbf{R}^n$, $|l| = 1$, and let $u(\cdot)$ be a measurable selection of U, such that

$$\delta^*(l|\hat{F}) = \ < l,\ \tilde{F}_0 + \int_t^{t+h} (\tilde{G}_0 + (s-t)\tilde{G}_1)u(s)ds + 0.5\tilde{H}_0\Big(\int_t^{t+h} u(s)ds,\ \int_t^{t+h} u(s)ds\Big) > \tag{3.29}$$

($u(\cdot)$ exists because of the continuity of the functional in the right-hand side of (3.29) with respect to $u(\cdot)$ in the L_2 weak topology). Define the matrix $H_1(l)$ by

$$<H_1(l)u,v> = 0.5<l,\ \tilde{H}_0(u,v)>$$

and let

$$v(t) = \int_t^{t+s} u(s)ds.$$

then (3.29) can be rewritten as

$$\delta^*(l|\hat{F}) = <l,\tilde{F}_0> + <\tilde{G}_0^* l, v(h)> + <H_1(l)v(h),v(h)> + \int_t^{t+h} <(s-t)\tilde{G}_1^* l, u(s)>ds.$$

Hence, $u(\cdot)$ satisfies the following necessary condition (the maximum principle):

$$u(s) = \arg\max_{u \in U} \langle \tilde{G}_0^* l + (t-s)\tilde{G}_1^* l + H_1(l)v(h),\ u \rangle \tag{3.30}$$

Since $|v(h)| \le |U|h$ we conclude from (3.28) that

$$|\varphi(s)| = |\tilde{G}_0^* l + (t-s)\tilde{G}_1^* l + H_1(l)v(h)| \ge \alpha_0 / 2 \tag{3.31}$$

and $u(s)$ is uniquely defined. Let us estimate the difference $|u(s) - u(t)|$. From the definition of the strong convexity of U

$$p_\tau = 0.5(u(t) + u(s)) + \mu \frac{\varphi(\tau)}{|\varphi(\tau)|} |u(t) - u(s)|^2 \in U \text{ for every } \tau \in [t,\ t+h].$$

Hence

$$\langle \varphi(\tau),\ p_\tau - u(\tau) \rangle \le 0$$

which yields

$$\langle \varphi(\tau),\ 2u(\tau) - u(t) - u(s) \rangle \ge 2\mu |\varphi(\tau)|\ |u(t) - u(s)|^2.$$

Setting successively $\tau = t$ and $\tau = s$ and summing the corresponding inequalities, we get

$$\langle \varphi(t) - \varphi(s),\ u(t) - u(s) \rangle \ge 2\mu(\ |\varphi(t)| + |\varphi(s)|)\ |u(t) - u(s)|^2,$$

$$|u(t) - u(s)| \le (2\mu\alpha_0)^{-1} |\varphi(t) - \varphi(s)|$$

Taking into account (3.31) we obtain

$$|u(t) - u(s)| \le (2\mu\alpha_0)^{-1}\ |G_1^* l| h.$$

Using the last inequality and (3.30), we estimate

$$\delta^*(l|\hat{F}) - \delta^*(l|\tilde{F}) \le$$

$$\le \int_t^{t+h} \langle \varphi(s), u(s) \rangle ds - h \langle l, \tilde{G}_0 u(t) + 0.5h\tilde{G}_1 u(t) + 0.5hH_0(u(t), u(t)) \rangle$$

$$= \int_t^{t+h} \langle \varphi(s), u(s) - u(t) \rangle ds + h \langle H_1(l)(v(h) - hu(t)), u(t) \rangle$$

$$\le \int_t^{t+h} \langle \varphi(s) - \varphi(t),\ u(s) - u(t) \rangle ds + h\|H_1(l)\| |v(h) - hu(t)|\ |U|$$

$$\le (2\mu\alpha_0)^{-1}\ |G_1^* l|^2\ h^3 + h\|H_1(l)\|\ |U| (2\mu\alpha_0)^{-1}\ |G_1^* l| h^2 = c_1 h^3$$

Thus we proved (3.26).

Now, let us prove (3.27). Take arbitrary $y_1, y_2 \in \tilde{\mathcal{F}}$ and $\alpha \in (0,1)$ and consider the distance ρ between $y = \alpha y_1 + (1-\alpha) y_2$ and $\tilde{\mathcal{F}}$. Let

$$y_i = \tilde{F}_0 + h\tilde{G}_0 u_i + 0.5h^2 \tilde{G}_1 u_i + 0.5h^2 \tilde{H}_0(u_i, u_i),\ u_i \in U,\ i = 1,2.$$

Denote $u = \alpha u_1 + (1-\alpha) u_2$. From the strong convexity of U it follows that $u + v \in U$ if

$$|v| \leq 4\mu\alpha(1-\alpha)\ |u_1 - u_2|^2. \tag{3.32}$$

Then for every v satisfying (3.32) we have

$$\rho \leq \mid y - \tilde{F}_0 - h\tilde{G}_0(u+v) - 0.5h^2\tilde{G}_1(u+v) - 0.5h^2\tilde{H}_0(u+v, u+v) \mid .$$

Using the identity

$$\tilde{H}_0(\alpha u_1 + (1-\alpha)u_2,\ \alpha u_1 + (1-\alpha)u_2) = \alpha\tilde{H}_0(u_1,u_1) + (1-\alpha)\tilde{H}_0(u_2,u_2) -$$

$$- \alpha(1-\alpha)\tilde{H}_0(u_1 - u_2, u_1 - u_2)$$

we obtain

$$\rho \leq \mid 0.5h^2\tilde{H}_0(u,u) - 0.5h^2\tilde{H}_0(u+v,u+v) + 0.5h^2\alpha(1-\alpha)\tilde{H}_0(u_1-u_2,\ u_1-u_2) - h\tilde{G}_0 v - 0.5h^2\tilde{G}_1 v \mid$$

There is a constant d, such that $|\tilde{H}_0(p,q)| \leq d$ for ever $p,q \in \mathbf{R}^n, |p| = |q| = 1$ and $t \in [t_0, T], x \in S_0(t)$ (we remind that $\tilde{H}_0$ depends on x and t). From the property of $\tilde{G}_0$ it follows that when v varies according to (3.32), $h\tilde{G}_0 v$ covers a ball with a radius $h\alpha_0 4\mu\alpha(1-\alpha)|u_1 - u_2|^2$.

Since

$$|\tilde{H}_0(u_1 - u_2, u_1 - u_2)| \ \leq\ d|u_1 - u_2|^2$$

we conclude that there is v, $|v| \leq \dfrac{|U|^2}{2\alpha_0} h$, such that

$$\rho \leq \mid 0.5h^2\tilde{H}_0(u,u) - 0.5h^2\tilde{H}_0(u + v,\ u + v) - 0.5h^2\tilde{G}_1 v \mid ,$$

which can be estimated by $c_2 h^3$, because of the inequality for $|v|$, and c_2 can be found independent of $t \in [t_0, T]$, $x \in S_0(t)$ and h.

The proof is now complete.

Remark. The inclusion (3.24) and (3.25) can be formally obtained applying the Runge-Kutta scheme mentioned in Section 3.1 to the inclusion (3.21), and considering the selection $u(\cdot)$ constant on every interval $[t_k,t_{k+1}]$. In this sense, the differential inclusions with strongly convex right-hand side have better behaviour (than in the polyhedral case) with respect to second orde discretizations. This is connected with the fact, that in the strongly convex case the set of trajectories, corresponding to continuous selections, generates the whole attainability domain.

4. Second order approximations to the attainability domain

The results from Section 3 concern also the approximation of the attainability domain. However, this is a more specific problem than the approximation of the trajectory bundle, and the difference between the two problems turns out to be essential. We saw in Section 3 that excepting the strongly convex case, the discrete inclusions providing second order approximation to the trajectory bundle ar more complicated that those which can be obtained by formally applying a second order Runge-Kutta scheme to the differential inclusion. In particular, even in the linear case with polyhedral constraints, the right-hand side of the approximating discrete inclusion is not described by linear constraints (because of the quadratic relationship between the 'free' parameters α and β). Nevertheless, we shall see in this section, that we can get rid of this nonlinearity, and in fact, that the formal analog to a second order Runge-Kutta formula provides second order approximation to the attainability domain (but not to the trajectory bundle) of a linear differential inclusion with polyhedral right-hand side (see also V. Veliov [17]).

Consider the inclusion

$$\dot{x} \in A(t)x + B(t)U, \;\; x(t_0) \in X_0, \;\; t \in [t_0, T], \tag{4.1}$$

where $x \in \mathbf{R}^n$, $A(t)$ and $B(t)$ are $(n \times n)$ and $(n \times r)$ – matrices, correspondingly, and $U \subset \mathbf{R}^n$.

Assumptions:

B1) $A(\cdot)$ and $B(\cdot)$ have Lipschitz continuous derivatives;

B2) X_0 is convex and compact;

B3) U is a convex and compact polyhedron (i.e., an intersection of finite number of half spaces, which is compact).

Given the integer N we define the matrices

$$\tilde{A}_k(h) = I + 0.5h(A(t_k) + A(t_{k+1})) + 0.5h^2 A(t_{k+1})^2$$

$$\tilde{B}_k(h) = 0.5h(B(t_k) + B(t_{k+1})) + 0.5h^2 A(t_{k+1})B(t_{k+1})$$

where as above $= (T - t_0) / N, \quad t_k = t_0 + kh, \quad k = 0,\ldots,N-1.$

Theorem 4. Let the assumptions $B1-B3$ be fulfilled. Then the discrete inclusion

$$x_{k+1} \in \tilde{A}_k(h)x_k + \tilde{B}_k(h)U, \quad x_0 \in X_0, \quad k = 0,\ldots,N-1 \tag{4.2}$$

provides a second order approximation to the attainability domain of (4.1).

The statement of the theorem can be reformulated in the following way. If we set $X_0^N = X_0$ and successively

$$X_{k+1}^N = \tilde{A}_k(h)X_k^N + \tilde{B}_k(h)U, \quad k = 0,\ldots,N-1. \tag{4.3}$$

then there is a constant c, such that

$$\rho(X_N^N, X(T)) \leq ch^2 \tag{4.4}$$

(as above $X(T)$ is the attainability domain of (4.1) on $[t_0,T]$).

Proof. Obviously

$$\tilde{A}_k(h) = I + hA(t_k) + 0.5h^2(\dot{A}(t_k) + (A(t_k))^2) + O(h^3) = \Phi(t_{k+1},t_k) + O(h^3)$$

$$\tilde{B}_k(h) = hB(t_{k+1}) + 0.5h^2(-\dot{B}(t_{k+1}) + A(t_{k+1})B(t_{k+1})) + O(h^3) \tag{4.5}$$

$$= \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s)B(s)ds + O(h^3)$$

where $\Phi(t,s)$ is the fundamental matrix solution of (4.1), normalized at $t = s$. From (4.3) we obtain

$$X_N^N = \left\{\tilde{A}_{N-1}(h)\ldots\tilde{A}_0(h)X_0 + \sum_{i=0}^{N-1} \tilde{A}_{N-1}(h)\ldots\tilde{A}_{i+1}(h)\tilde{B}_i(h)u_i;\ u_i \in U\right\},$$

which using (4.5) and the semigroup property of Φ gives

$$X_N^N = \left\{ \Phi\,(T,t_0)X_0 + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \Phi\,(T,s)B(s)ds u_i;\ u_i \in U \right\} + 0(h^2)$$

Hence, using the Couchy formula for (4.1) we obtain the following estimation of the difference between the support functions of $X(T)$ and X_N^N:

$\delta^*(l \mid X(T)) - \delta^*(l \mid X_N^N) =$

$$\int_{t_0}^{T} \delta^*(B^*(s)\ \Phi^*(T,s)l \mid U)ds - \sum_{i=0}^{N-1} \delta^*(\int_{t_i}^{t_{i+1}} B^*(s)\ \Phi^*(T,s)lds \mid U) + 0(h^2)$$

We can replace

$$\int_{t_i}^{t_{i+1}} B^*(s)\ \Phi^*(T,s)lds = hB^*(t_i + 0.5h)\ \Phi^*(T,t_i + 0.5h)l + 0(h^2),$$

since the subintegral function is differentiable and its derivative is Lipschitz continuous. Denote

$$l(t) = B^*(t)\ \Phi^*(T,t)l. \tag{4.5}$$

Then

$$\delta^*(l \mid X(T)) - \delta^*(l \mid X_N^N) = \int_{t_0}^{T} \delta^*(l(s) \mid U)ds - h\sum_{i=0}^{N-1} \delta^*(l((i+0.5)h) \mid U) + 0(h^2).$$

What remains to prove now, is that the rectangular formula for numerical integration has accuracy $0(h^2)$ when applied to the function $\delta^*(l(\cdot) \mid U)$, and moreover, that this accuracy $0(h^2)$ is uniform with respect to all functions of the form of (4.5), when $|l| = 1$. This will imply (4.4).

In B. Sendov and V. Popov [14] it is proved that if $\varphi(\cdot)$ is absolutely continuous, the error of the rectangular formula can be estimated by

$$ch^2 \bigvee_{t_0}^{T} \dot{\varphi}(\cdot),$$

where $\bigvee$ means the variation and the constant c does not depend on $\varphi(\cdot)$. Thus we shall complete the proof by the following result.

Lemma 2. Let $l(\cdot) : [t_0,T] \to \mathbf{R}^r$ be differentiable and let $\dot{l}(\cdot)$ be of bounded variation. Let U be a compact convex polyhedron in $\mathbf{R}^r$. Then

$$\varphi(t) = \delta^*(l(t) \mid U)$$

is absolutely continuous and

$$\bigvee_{t_0}^{T} \varphi(\cdot) \leq c \bigvee_{t_0}^{T} \dot{l}(\cdot),$$

where c is independent of $l(\cdot)$.

Proof. Denote

$$U(t) = \left\{ u \in U;\ < l(t), u > \ = \varphi(t) \right\}.$$

$$\mathcal{E} = \left\{ e;\ |e| = 1,\ e - \text{ colinear to some edge of } U \right\}.$$

It is easy to prove that for every $s,t \in [t_0,T]$ and $u \in U(t)$, $v \in U(s)$ there is a relation

$$v = u + \sum_{i=1}^{p} \alpha_i e_i$$

where $e_i \in \mathcal{E}$, $<l(t_i),e_i> = 0$ for some $t_i \in [s,t]$, $i=1,\ldots p$, and $|\alpha_i|$ are bounded by a constant c, depending only on U (but not on $l(\cdot)$).

It is standard to prove that $\varphi(\cdot)$ is Lipschitz continuous and hence its derivative exits almost everywhere. It is well known that

$$\partial\varphi(t) = [\inf_{u \in U(t)} < \dot{l}(t), u > ,\ \sup_{u \in U(t)} < \dot{l}(t), u >]$$

($\partial\varphi$ is the subdifferential of φ) and hence

$$\dot{\varphi}(t) = \sup_{u \in U(t)} < \dot{l}(t), u >$$

where it exists. Thus it remains to estimate the variation of the above function. Taking again arbitrary $t_1,\ldots,t_p \in [t_0,T]$ we have

$$\sum_{i=1}^{p} | \dot{\varphi}(t_{i+1}) - \dot{\varphi}(t_i) | = \sum_{i=1}^{p} | <\dot{l}(t_{i+1}),u_{i+1}> - <\dot{l}(t_i),u_i> | \tag{4.7}$$

$$\leq \sum_{i=1}^{p} | <\dot{l}(t_{t+1} - \dot{l}(t_i),u_i> | + \sum_{i=1}^{p} | <\dot{l}(t_{i+1}),\sum_{j=1}^{p_i} \alpha_{ij} e_{ij} > |,$$

where as above $u_i \in U(t_i)$, but in addition

$$<\dot{l}(t_i),u_i> = \sup_{u \in U(t_i)} <\dot{l}(t_i),u>,$$

$|\alpha_{ij}| \leq c$ and $<l(t_{ij}),\ e_{ij}> = 0$ for some $t_{ij} \in [t_i, t_{i+1}]$. Hence we estimate (4.7) by

$$||U|| \bigvee_0^T \dot{l}(\cdot) + c\sum_{i=1}^{p} \sum_{l \in \mathcal{E}_i} < \dot{l}(t_{i+1}),\ e>, \tag{4.8}$$

where $\mathcal{E}_i$ the subset of $\mathcal{E}$, consisting of these e, for which $<l(t),e>$ vanishes at some point $t_i(e) \in [t_i, t_{i+1}]$. The second term in (4.8) can be written as

$$c\sum_{l \in \mathcal{E}} \sum_{i \in I(l)} | <\dot{l}(t_{i+1}),e> | ,$$

where $I(e)$ is the set of those i, such that $<l(t),e>$ vanishes somewhere in $[t_i, t_{i+1}]$. Fix an arbitrary $e \in \mathcal{E}$ and take two neighboring i and i' from $I(e)$. Since $<l(t),e>$ vanishes in $[t_i, t_{i+1}]$ and in $[t_{i'}, t_{i'+1}]$, then $<\dot{l}(t),e>$ vanishes at some point $\tilde{t}_i \in [t_i, t_{i'+1}]$. Then

$$\sum_{i \in I(e)} | <\dot{l}(t_{i+1}),\ e> | = \sum_{i \in I(e)} | <\dot{l}(t_{i+1}),\ e> - <\dot{l}(\tilde{t}_i),\ e> |$$

$$\leq \sum_{i \in I(e)} | \dot{l}(t_{i+1}) - \dot{l}(\tilde{t}_i) | \leq 2\bigvee_{t_0}^{T} \dot{l}(\cdot),$$

which proves the lemma, since $\mathcal{E}$ is a finite set.

The following example shows that the discrete inclusion (4.2) does not provide a second order approximation to the trajectory bundle of (4.1).

Example.

$$\dot{x} = y, \quad x(0) = 0 \tag{4.9}$$

$$\dot{y} = u, \quad y(0) = 0, \quad u \in [0,1], \quad t \in [0,1]$$

The discrete inclusion (4.2) is now

$$x_{k+1} = x_k + hy_k + 0.5h^2 u, \quad k = 0,\ldots,N-1 \tag{4.10}$$

$$y_{k+1} = y_k + hu, \quad u \in [0,1].$$

From Theorem 4 we know that

$$\rho(X(1),\, X^N(N)) \leq ch^2.$$

Nevertheless, it is easily seen that

$$\rho(X_{[0,1]},\, X^N_{[0,N]}) \geq h \,/\, 8,$$

which means that (4.10) provides only a first order approximation to the trajectory bundle of (4.3).

5. Second order discrete approximations to optimal control problems

In this section we shall apply some of the preceding results to obtain second order discrete approximations to some optimal control problems with control constraints.

A great number of papers are devoted to the problems of how to discretize an optimal control problem so that the solution of the discrete (finite - dimensional) problem to converge in some sense to the solution of the original one (see e.g. A. Dontchev [2] and B. Mordukhovič [9] and the bibliography there). If there are no constraints on the control and the state variables, then discrete approximations with higher accuracy than $O(h)$ are developed for various optimal control problems and by different approaches (W. Hager [6], G. Redien [13], F. Mathis and G. Redian [8], K. Teo [16]).

Applying the approximations developed in the previous sections one can obtain second order approximations for some classes of optimal control problems with control constraints.

First, consider the problem

$$\min \left\{ \varphi(x(T)) + \int_{t_0}^{T} (f_0(x(t),t) + g_0(x(t),t)u(t))\,dt \right\} \tag{5.1}$$

$$\dot{x} = f(x,t) + g(x,t)u, \quad x(t_0) \in X_0 \tag{5.2}$$

$$u(t) \in [0,1], \tag{5.3}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^1$, f and g satisfy conditions A from Section 3.1, f_0, g_0 and φ also satisfy condition A1, X_0 is a convex compact set.

We consider the single-input case only for notational simplicity. As already mentioned the case of more general control constraints $[a(t),b(t)]$ with $\dot{a}(\cdot)$ and $\dot{b}(\cdot)$ being Lipschitz continuous, can be reduced to (5.3) by change of the control variable.

Introducing the new variable y by

$$\dot{y} = f_0(x(t),t) + g_0(x(t), t)u, \quad y(t_0) = 0 \tag{5.4}$$

we can replace (5.1) by

$$J(u(\cdot)) = \varphi(x(T)) + y(T) \rightarrow \min$$

which is a minimization problem over the attainability domain of (5.2), (5.4) and (5.3). Applying Theorem 1 and taking into account the specificity of the right-hand side of (5.2) and (5.4) we come to the following discrete relations

$$x_{k+1} = x_k + 0.5h(F_0(x_k,k) + \alpha_k G_0(x_k,k) + \beta_k G_1(x_k,k) + \alpha_k^2 H_0(x_k,k))$$

$$y_{k+1} = y_k + 0.5h(F_0^0(x_k,k) + \alpha_k G_0^0(x_k,k) + \beta_k G_1^0(x_k,k) + \alpha_k^2 H_0^0(x_k,k))$$

where F_0, G_0, G_1 and H_0 are defined by (3.7) in Section 3.1 and F_0^0, G_0^0, G_1^0 and H_0^1 are defined by exactly the same formulae, but applied to f_0 and g_0 instead of f and g (p and q remain unchanged).

Now we can approximate the problem (5.1) - (5.3) by the following discrete problem

$$\min_{\bar{x},\bar{\alpha},\bar{\beta}} \left\{ \varphi(x_N) + 0.5h \sum_{i=0}^{N-1} (F_0^0(x_k,k) + \alpha_k G_0^0(x_k,k) + \beta_k G_1^0(x_k,k) + \alpha_k^2 H_0^0(x_k,k)) \right\} \tag{5.5}$$

subject to

$$x_{k+1} = x_k + 0.5h(F_0(x_k,k) + \alpha_k G_0(x_k,k) + \beta_k G_1(x_k,h) + \alpha_k^2 H_0(x_k,k)), \quad x_0 \in X_0 \tag{5.6}$$

$$\alpha_k \in [0,1], \; \beta_k \in [\alpha_k^2, \; 2\alpha_k - \alpha_k^2], \quad k=0,\ldots,N-1 \tag{5.7}$$

Theorum 5. Both problems (5.1) - (5.3) and (5.5) - (5.7) have solutions. If $\hat{J}$ and $\hat{J}_N$ are the optimal values of the objective functions of the respective problems, then

$$|\hat{J} - \hat{J}_N| \leq \text{const} \, / \, N^2.$$

In fact, problem (5.5) - (5.7) provides much more information than the approximation of $\hat{J}$. If $\bar{x},\bar{\alpha},\bar{\beta}$ is a solution (or even a ϵ - solution) of (5.5) - (5.7), then one can immediately reconstruct from α_k,β_k a piece-wise constant control $u(\cdot)$ on $[t_k,t_{k+1}]$ with at most one switching point in each of these intervals, which when applied to (5.2) results in a trajectory $x(\cdot)$, such that

$$|x(t_i) - x_i| \leq \text{ const } / N^2.$$

In particular

$$J(u(\cdot)) \leq \hat{J} + \text{ const } / N^2 \, (+ \epsilon).$$

Let us compare the discrete problem (5.5) - (5.7) with the Euler discretization of (5.1) - (5.3). In order to attain accuracy $O(h^2)$ by the Euler discretization one need $N \sim 1 / h^2$, while in (5.5) - (5.7) $N \sim 2 / h$. But in the same time, in the second order discretization there appeared N new constraints (5.7), which are quadratic. Thus (5.5) - (5.7) is a nonlinear problem, even in case of a linear problem (5.1) - (5.3).

The result from Section 5 can be also applied in an obvious way to obtain a second order approximization to the problem

$$\min \left\{ \varphi(x(T)) + \int_{t_0}^{T} [C(t)x(t) + D(t)u(t)]\,dt \right\}$$

$$\dot{x} = A(t)x + B(t)u, \quad x_0 \in X_0$$

$$u(t) \in U - \text{ convex and compact polyhedron.}$$

The discretized problem is with linear constraints and the accuracy in $x(T)$ is $O(h^2)$. Similar discretization is studied also in E. Farkhi [4], but the estimate of the convergence obtained there depends on the second order averaged modulus of smoothness of the solution $u(\cdot)$.

6. Application to a problem of approximation by trajectories of a differential inclusion

In this section we shal consider the following problem. Let

$$\dot{x}\in F(x,t), \quad x(t_0)\in X_0, \quad t\in[t_0,T] \tag{6.1}$$

be a given differential inclusion in $\mathbf{R}^n$ and $\tilde{x}(\cdot)$ be an absolutely continuous function $[t_0,T] \to \mathbf{R}^n$. Following [3] we define the discrepancy

$$d(\tilde{x}(\cdot)) = \mathrm{dist}(\tilde{x}(t_0),\, X_0) + \int_{t_0}^{T} \mathrm{dist}(\dot{\tilde{x}}(t),\, F(\tilde{x}(t),\, t))\,dt, \tag{6.2}$$

which is a measure of how much $\tilde{x}(\cdot)$ fails to be a trajectory of (6.1).

In [3] it is developed a numerical procedure based on the Euler discretization formula, which gives a sequence $x_0,\dots,x_N$, having the properties:

1) there is a trajectory $x(\cdot)$ of (6.1), such that

   $\max_{i=0,\dots,N} |x(t_i)-x_i| \le c \,/\, N;$

2) $\max_{i=0,\dots,N} |\tilde{x}(t_i) - x_i| \le d(\tilde{x}(\cdot)) + c \,/\, N.$

where c is a constant.

On the basis of the results from sections 2 and 3 one can replace $c \,/\, N$ with $c \,/\, N^2$ in 1) and 2).

Suppose the following.

B1. For every $x\in\mathbf{R}^n$ and $t\in[t_0,T]$ the set $F(x,t)$ is nonempty convex and compact; $F(\cdot,t)$ is locally Lipschitzian, uniformly in $t\in[t_0,T]$; $F(x,\cdot)$ is Hausdorff continuous.

B2. There exists constants M and a, such that

$$|F(x,t)| \le M|x| + a$$

for every $x\in\mathbf{R}^n$ and $t\in[t_0,T]$.

It is obvious that B1 and B2 imply the basic assumption from Section 2, for appropriate S and $S_0(\cdot)$. Let

$$x_{k+1} \in \mathcal{F}(x_k,k,h), \quad x_0\in X_0, \quad k = 0,\dots,N-1 \tag{6.3}$$

be a discrete inclusion which provides a third order local approximation to (6.1) in the tube $S_0(\cdot)$. Define a particular trajectory of (6.3) setting

$$x_{k+1} = P_{\mathcal{F}(x_k,k,h)}\tilde{x}(t_{k+1}), \;\; x_0 = P_{X_0}\tilde{x}(t_0), \tag{6.4}$$

where $P_Y x$ is the projection of x on Y.

The following result is a direct consequence of Proposition 1 and [Corallary, 3].

Theorem. The sequence $x_0,...,x_N$ defined by (6.4) satisfies the properties 1) and 2) with $c \;/\; N^2$ instead of $c \;/\; N$ in the right-hand sides. Moreover, the constant c can be found independently of $\tilde{x}(\cdot)$.

We shall mention that if the 'function' $\tilde{x}(\cdot)$ is known only at the points $t_0,...,t_N$, it is not a trivial problem to estimate the discrepancy $d(\tilde{x}(\cdot))$, because the subintegral function in (6.2) is not known. Theorem 6 means, that $\max_i|\tilde{x}(t_i) - x_i|$ with x_i given by (6.4) is a lower estimate of $d(\tilde{x}(\cdot))$ with accuracy $0(h^2)$. To obtain it we need the discrete inclusion (6.3) with 'local accuracy' $0(h^3)$. Such discrete inclusions were constructed in Section 3.

Let us consider the simplest case when (6.1) is in the form

$$\dot{x} \in f(x,t) + g(x,t)[0,1], \quad x(t_0) \in X_0, \quad t \in [t_0,T] \tag{6.5}$$

Then the discrete inclusion (3.11) with $\mathcal{F}$ given by (3.10) provides a third order local approximization to (6.5). In order to construct the sequence $\{x_k\}$ from (6.3) we have to solve at every step the problem

$$\min_{\alpha,\beta}\left\{h \;\; \| \; \alpha G_0 + \beta G_1 + \alpha^2 H_0 \; \| \; ^2 - <\alpha G_0 + \beta G_1 + \alpha^2 H_0, \;\; \tilde{x}(t_{k+1}) - x_k - 0.5hF_0>\right\}$$

subject to

$$\alpha \in [0,1], \quad \beta \in [\alpha^2 \;, 2\alpha - \alpha^2].$$

This problem is explicitly solvable at least when $H_0 = 0$ which happens whenever $g(x,t)$ does not depend on x. In this case the right-hand side of (6.4) can be written by an explicit formula.

We shall mention also that when the values of α and β are already known, then the corresponding selection $u(\cdot)$ of $[0,1]$ can be found in an obvious way as a piece-wise constant function, having one switching point in every interval $[t_k,t_{k+1}]$ (see Section 3.1).

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